VV LECTURE 1: TOPOLOGY

Problem 1. Consider the following two spaces. $X = [-1, 1] \subset \mathbb{R}$ and

$$Y = \{(x, 0) \in \mathbb{R}^2 \mid -1 \leq x \leq 1\} \cup \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}.$$

Show that they are not homeomorphic, without using the point $(0, 0) \in \mathbb{R}^2$.

Problem 2. Show that any contractible space is path connected and any two contractible spaces are homotopy equivalent.

Problem 3. Show that the cylinder $\mathbb{S}^1 \times [0, 1]$ and the Möbius band M are not homeomorphic. Recall that M is the quotient space of $[0, 1] \times [0, 1]$ by identifying $(0, r)$ with $(1, 1 - r)$ for $r \in [0, 1]$. But show that they are homotopy equivalent.

Problem 4. Let $\mathbb{Q} \subset \mathbb{R}$ be the rational numbers with respect to the subspace topology. Show that the inclusion map $\mathbb{Q} \to \mathbb{R}$ is homotopic to any constant map. See if it is true for any map from $\mathbb{Q}$ to $\mathbb{R}$. What if we replace $\mathbb{Q}$ by any other topological space. Identity which property of $\mathbb{R}$ is crucial in your argument and generalize the problem.

Problem 5. Let $X = \mathbb{R}^2 - \{(0, 0)\}$, show that any two continuous maps $[0, 1] \to X$ are homotopic.

Problem 6. Let X be a Hausdorff space and A is a subset of X. Assume that there is a map $X \to A$ whose restriction to A is identity. Show that A is a closed subset of X.

Problem 7. Let $\mathbb{D}_r = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r\}$ for $r > 0$, be the disc of radius r. Show that the inclusion $\mathbb{D}_s \to \mathbb{D}_r$, for $0 \leq s \leq r$, is a homotopy equivalence.