VV LECTURE 2: TOPOLOGY

Problem 1. Let X and Y be two path connected topological spaces. Show that any homotopy equivalence $f : X \to Y$ induces an isomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

Problem 2. Using the Van-Kampen theorem compute the fundamental group of the Klein bottle, real projective space and of any closed surface of genus ≥ 1.

Problem 3. Show that a map $f : \mathbb{S}^1 \to X$ is homotopic to a constant map if and only if f extends to a map $\mathbb{D}^2 \to X$.

Problem 4. Show that the fundamental group of the Klein bottle is non-abelian. Hence the Klein bottle is not homeomorphic to the real projective plane or to the torus.

Problem 5. Show that the circle and the figure eight are not homotopy equivalent.

Problem 6. Calculate the fundamental group of the real projective plane minus a point.

Problem 7. If you identify two Möbius bands along their circle boundary, recognize the resulting space.