

Mumbai Math Circle

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Prime Numbers:

What are prime numbers?

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How to check if a number is prime or not a prime ?

How many prime numbers are there?

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How to show this?
Euclid's Argument:



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“Euclid replied there is no royal road to geometry.”

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This implies there is at least 1 more prime.

Fermat number

For $n = \{0, 1, 2, \dots\}$

$$F(n) := 2^{2^n} + 1$$

3, 5, 17, 257, 65537, 4294967297, 18446744073709551617, ...

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Fermat conjectured that all Fermat numbers are prime.

Leonhard Euler in 1732 showed that

$$F(5) = 2^{2^5} + 1 = 2^{32} + 1 = 4294967297 = 641 \times 6700417.$$

Euler proved that every factor of $F(n)$ must have the form $k2^{n+1} + 1$.

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Exercise: Prove it.



Pierre de Fermat (1607 1665)

He is best known for his Fermat's principle for light propagation and his Fermat's Last Theorem in number theory.

$$a^n + b^n = c^n$$

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$\gcd(a, b) = \gcd(b, a) = \gcd(a - b, b)$ if $a > b$

Observe that

$$(F(0) \times F(1) \times \cdots \times F(n-1)) \times F(n) = F(n+1) - 2$$

Proof:

$$\begin{aligned} \left(\prod_{k=0}^{n-1} F(k) \right) \times F(n) &= (F(n) - 2) \times F(n) \\ &= (2^{2^n} - 1)(2^{2^n} + 1) \\ &= 2^{2^{n+1}} - 1 \end{aligned}$$

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Thus, every $F(n)$ is a prime or it has a new prime factor.

Exercise: Consider the number $2^p - 1$ where p is a prime. Show that all its prime factors are greater than p .

Exercise: Consider any polynomial $P(x)$. Show that the sequence $P(0), P(1), \dots$ cannot be only prime numbers.

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Proof: We consider a simpler and smaller sum

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots\right) + \left(\frac{1}{32} + \dots\right) \dots () \dots$$

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$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots = ?$$

We give a proof by Erdős.

Paul Erdős (Hungarian: 1913 –1996)

Erdős published around 1,500 mathematical papers during his lifetime.



Suppose

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots = M$$

(p_i 's are in increasing order) Then there must be a k such that

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Let $P_s = p_1, p_2, \dots, p_k$ be called small primes and $P_b := p_{k+1}, p_{k+2}, \dots$ be called big primes.

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Clearly $N_b + N_s = N$.

We will show that for a suitable N , $N_s + N_b < N$, a contradiction.

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Total choices number of for square free part is at most 2^k .

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This implies $N_s \leq 2^k \sqrt{N}$.

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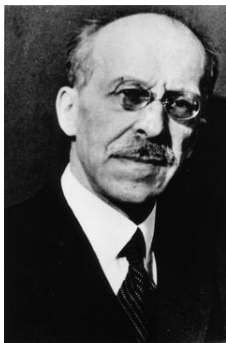
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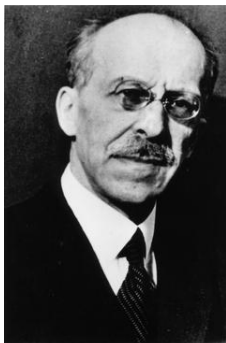
For contradiction find N such that $2^k \sqrt{N} < \frac{N}{2}$.



I. Schur



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Let $P(x)$ be a non-constant polynomial with integer coefficients. Then $\{P(i) : i \in \mathbb{N}\}$ has infinitely many prime divisors.
(see Lemma 3 in <https://mast.queensu.ca/~murty/poly2.pdf>)

Christian Elsholtz: Prime divisors of thin sequences, Amer. Math. Monthly 119 (2012), 331–333

Let $S := \{s_1, s_2, \dots\} \subset \mathbb{Z}$ be a sequence of integers such that

- 1** No integer appears more than c times.
- 2** S has subexponential growth i.e. $|s_n| < 2^{2^{f(n)}}$ where $\frac{f(n)}{\log_2 n} \rightarrow 0$. ($f : \mathbb{N} \rightarrow \mathbb{R}$)

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Let $S := \{s_1, s_2, \dots\} \subset \mathbb{Z}$ be a sequence of integers such that

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Theorem: If a set S is “almost” injective and of sub-exponential growth (the above two conditions) then the set of prime numbers P_S that divide a member of S is infinite.

The two conditions are clearly required. Suppose the first condition does not hold. Then consider the sequence $S := \{2, 4, 4, 8, 8, 8, 8, 16, \dots\}$.

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$$S := \{2, 4, 4, 8, 8, 8, 8, 16, \dots\}.$$

If the second condition does not hold then the sequence $\{2^i 3^j\}$, $i, j \in \mathbb{N}$ arranged in increasing order has $\frac{f(n)}{\log_2 n} \sim \frac{1}{2}$

Without loss of generality assume $f(n)$ is increasing.

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Otherwise redefine it as $g(n) := \max_{i \leq n} f(i)$.

Suppose $P_S = \{p_1, p_2, \dots, p_k\}$.

Then $s_n = \epsilon_n p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where $\epsilon_n = \{-1, 0, +1\}$ and $\alpha_i \geq 0$

This implies

$$2^{\alpha_1 + \alpha_2 + \dots + \alpha_k} \leq |s_n| \leq 2^{2^{f(n)}}$$

for $s(n) \neq 0$.

$$\Rightarrow 0 \leq \alpha_i \leq \alpha_1 + \alpha_2 + \dots + \alpha_k \leq 2^{f(n)} \text{ for } 1 \leq i \leq k$$

.

$$\#\{|s(n)| \neq 0 \text{ and } n \leq N\} \leq (2^{f(N)} + 1)^k \leq 2^{(f(N)+1)k}.$$

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$\log_2(N - c) - \log_2(2c) \leq k(f(N) + 1)$ for all N .

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$$\frac{N - c}{2c} \leq 2^{k(f(N)+1)}$$

$$\log_2(N - c) - \log_2(2c) \leq k(f(N) + 1) \text{ for all } N.$$

Divide both sides by $\log_2 N$. Then LHS goes to 1 and RHS goes to 0.

$$\frac{\log_2(N - c)}{\log_2 N} \rightarrow 1 \text{ as } N \rightarrow \infty$$

and

$$\frac{f(N)}{\log_2 N} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

$$\frac{\log_2(N - c)}{\log_2 N} \rightarrow 1 \text{ as } N \rightarrow \infty$$

and

$$\frac{f(N)}{\log_2 N} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

This is a contradiction.

Frustenberg's proof: (Mercer's note)

Notation: All integers congruent to $r \pmod m$ is denoted by $r + m\mathbb{Z}$ and they are called AP.

Example:

$$3 + 11\mathbb{Z} = \{\dots, -30, -19, -8, 3, 14, 25, 36, \dots\}$$

For $m > 1$, set of integers not divisible (ND) by m are as

$$(1 + m\mathbb{Z}) \cup \dots \cup ((m - 1) + m\mathbb{Z}).$$

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Assertion 2: Finite intersection of finite unions of sets is also a finite union of finite intersections of sets.

Example:

$$(A \cup B \cup C) \cap (D \cup E) \cap (F \cup G) = (A \cap D \cap F) \cup (\dots)$$

Proof: If p_1, p_2, \dots, p_k is the set of all primes then

$$\{-1, 1\} = ND(p_1) \cap ND(p_2) \dots \cap ND(p_k)$$

RHS is either empty or infinite!