

MOTIVIC AND AUTOMORPHIC ASPECTS OF THE REDUCTIVE BOREL-SERRE COMPACTIFICATION

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The reductive Borel-Serre compactification M^{rbs} of a noncompact locally symmetric space M was introduced in [Z2] and has since played a fundamental role in work on cohomology of arithmetic groups and locally symmetric spaces. This paper is a survey of two aspects of its cohomology, motivic (when M has an algebraic structure) and automorphic, and their interaction.

It is a well-known idea that when M is a locally symmetric variety M^{rbs} behaves like a partial resolution of the singularities of the minimal (or Satake-Baily-Borel) compactification M^{bb} . Although M^{rbs} is not an algebraic variety, its cohomology looks motivic: It carries a mixed Hodge structure [Z4], underlies a Voevodsky motive [AZ], and is part of a mixed realization [N2]. In §2 we outline the approach of [N2] to the mixed realization of $H^*(M^{rbs})$ based on Morel's weight truncations in categories of mixed sheaves constructed by Saito and on a comparison between weight truncations and the weighted complexes of [GHM]. A special case of the comparison puts a mixed realization on the cohomology of M^{rbs} . We also discuss motivic results of Ayoub and Zucker [AZ] and Vaish [V].

The cohomology of M^{rbs} is also automorphic, in a precise sense: There are C^∞ de Rham models for its cohomology and homology in terms of relative Lie algebra cohomology of spaces of functions on the group (by specializing the main result of [N1]). For homology we can further reduce to a space of automorphic forms [F1]. In §3 we review these results and describe some consequences drawn in [N3]. The results in §3 work for any locally symmetric space, algebraic or not.

In §4 we describe a result of [N3] which exploits both the motivic and automorphic results on M^{rbs} to show that the Chern classes of the natural topological extensions to M^{rbs} of automorphic vector bundles on a locally symmetric variety M behave as they would if M^{rbs} were an algebraic variety over k and automorphic vector bundles extended algebraically to M^{rbs} .

Much of the interest in M and its compactifications comes from the hope of relating the mixed motives appearing in the cohomology of M to the special values of L -functions (see e.g. [H]). It follows from the result described in §4 that the summand of cohomology with trivial Hecke eigenvalues consists of mixed Tate realizations. In [N4] we treat this summand using automorphic methods from [F2] (see Remark 4); the RBS compactification plays an important role in this.

It should be obvious how much Steve Zucker's work (both on locally symmetric spaces and in Hodge theory) has influenced the matters discussed here and it is an honour to contribute to this volume for him. My own interest in Steve's "favorite space" M^{rbs} was rekindled by a discussion with him at Banff in May 2008 where he told me about [Z4]. I realized soon after that Morel's work [Mo] gives a simple and natural approach to such questions. The main result of [N2] was worked out during a stay at the IAS Princeton in the Fall of 2008 and this later led to [NV, N3, V].

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1. MIXED REALIZATIONS AND M^{rbs}

After recalling the categories of mixed sheaves of relevance to us from Saito's [S1, S2, S3] and Morel's method of weight truncations from [Mo], we sketch how the cohomology M^{rbs} carries a natural mixed realization, following [N2]. We also mention the motivic result of Ayoub-Zucker [AZ] and some further developments due to Vaish [V].

1.1. Some categories of mixed sheaves. Let k be a number field in \mathbb{C} and Var/k the category of varieties (i.e. separated schemes of finite type) and morphisms defined over k . The algebraic closure of k in \mathbb{C} is denoted $\bar{\mathbb{Q}}$.

The cohomology of an algebraic variety over k carries extra structures: The singular cohomology $H^*(X(\mathbb{C}), \mathbb{Q})$ with its increasing weight filtration W and decreasing Hodge filtration F on $H^*(X(\mathbb{C}), \mathbb{C})$ is a mixed Hodge structure. If X is smooth the algebraic de Rham cohomology $\mathbb{H}^*(X, \Omega_{X/k}^\bullet)$ with its weight and Hodge filtrations is a bifiltered k -vector space. For each prime l the l -adic etale cohomology $H^*(X \times_k \bar{\mathbb{Q}}, \mathbb{Q}_l)$ is a continuous l -adic $\text{Gal}(\bar{\mathbb{Q}}/k)$ -representation with a Galois-stable filtration by Frobenius weights. There are comparison isomorphisms $H^*(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_l \cong H^*(X \times_k \bar{\mathbb{Q}}, \mathbb{Q}_l)$ and $H^*(X(\mathbb{C}), \mathbb{C}) \otimes \mathbb{C} \cong \mathbb{H}^*(X, \Omega_{X/k}^\bullet) \otimes_k \mathbb{C}$ relating filtrations.

This leads to the notion (due to Deligne and Jannsen [J]) of a *mixed realization* over k . This is a collection $(H_B, H_{dR}, (H_l)_l, I, (I_l)_l)$ where the Betti part $H_B = (H_{\mathbb{Q}}, W \text{ on } H_{\mathbb{Q}}, F \text{ on } H_{\mathbb{C}})$ is a \mathbb{Q} -mixed Hodge structure, the de Rham part $H_{dR} = (H_k, W, F)$ is a bifiltered k -vector space, and for each prime l we have \mathbb{Q}_l -vector space H_l with a continuous representation of $\text{Gal}(\bar{\mathbb{Q}}/k)$ and a Galois-stable filtration W . The map $I : H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow H_k \otimes_k \mathbb{C}$ is an isomorphism identifying the W and F filtrations, and for each l , $I_l : H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_l \rightarrow H_l$ is an isomorphism identifying the W filtrations. Mixed realizations over k form an abelian category \mathcal{MR}_k and $X \mapsto H^*(X) = (H_B^*(X), H_{dR}^*(X/k), (H_l^*(X))_l)$ with $H_B^*(X) = (H^*(X(\mathbb{C}), \mathbb{Q}), W, F)$, $H_{dR}^*(X/k) = (\mathbb{H}^*(X, \Omega_{X/k}^\bullet), W, F)$, and $H_l^*(X) = (H^*(X \times_k \bar{\mathbb{Q}}, \mathbb{Q}_l), W)$ defines a functor from smooth varieties to \mathcal{MR}_k . (We will drop the data $I, (I_l)_l$ from the notation when there is no ambiguity.)

Relative versions of mixed Hodge structures and mixed realizations have been defined by M. Saito [S1, S2, S3]. The category $\text{MHM}(X)$ of algebraic mixed Hodge modules ([S1], especially §4) on a complex algebraic variety X is an abelian category which for $X = \text{Spec}(\mathbb{C})$ is the category of graded-polarizable rational mixed Hodge structures and which for X smooth contains admissible variations of mixed Hodge structure on X . The objects of $\text{MHM}(X)$ are triples $((M, W, F), (K, W), \alpha)$ where (M, W, F) is a bifiltered regular holonomic D_X -module, (K, W) is a filtered algebraically constructible perverse \mathbb{Q}_X -sheaf, and $\alpha : \text{DR}(M, W) \rightarrow (K, W) \otimes \mathbb{C}$ is a filtered quasiisomorphism. These are subject to complicated inductively defined conditions. (The theory rests, in the end, on [Z1] over curves.) Remembering only the underlying perverse sheaf K defines a faithful exact functor *rat* to the category $\text{P}(X)$ of algebraically constructible perverse sheaves on X . This derives to a functor *rat* : $D^b\text{MHM}(X) \rightarrow D_c^b(\mathbb{Q}_X)$ to the derived category of complexes of \mathbb{Q}_X -sheaves with algebraically constructible cohomology sheaves (because $D^b(\text{P}(X))$ is equivalent to $D_c^b(\mathbb{Q}_X)$ by [B]).

For a k -variety X , the category $\mathcal{M}(X)$ of mixed realizations over X is an abelian category defined in [S2, 1.8] and [S3, 1.1] by enriching mixed Hodge modules on $X(\mathbb{C})$: An object in $\mathcal{M}(X)$ consists of $((M_k, W, F), (K, W), (K_l, W)_l, \alpha, (\alpha_l)_l)$ where (M_k, F) is a regular holonomic D_X -module with finite increasing filtration W , (K, W) is a filtered perverse $\mathbb{Q}_{X(\mathbb{C})}$ -sheaf, and for each l , (K_l, W) is a filtered perverse l -adic sheaf on X , and α and α_l are comparison isomorphisms. We refer to [S3, 1.1] and [S2, 1.8] for more details. For $X = \text{Spec}(k)$ we get mixed realizations in \mathcal{MR}_k with graded-polarizable Betti part. Remembering $((M_k, W, F) \otimes \mathbb{C}, (K, W), \alpha)$ gives a faithful exact functor $\mathcal{M}(X) \rightarrow \text{MHM}(X(\mathbb{C}))$ and remembering only K gives a faithful exact functor For to perverse sheaves. The induced functor

$$For : D^b \mathcal{M}(X) \longrightarrow D_c^b(\mathbb{Q}_{X(\mathbb{C})})$$

on derived categories factors through *rat*.

Both $\text{MHM}(\cdot)$ and $\mathcal{M}(\cdot)$ are examples of theories of mixed sheaves in the sense of [S2]. It then follows that there is a functorial formalism: For $f : X \rightarrow Y$ in Var/k there are functors $f_*, f^*, f_!, f^!$ between the derived categories $D^b \text{MHM}(X(\mathbb{C}))$ and $D^b \text{MHM}(Y(\mathbb{C}))$ (resp. between $D^b \mathcal{M}(X)$ and $D^b \mathcal{M}(Y)$) compatible under *rat* (resp. under *For*) with the usual functors between $D_c^b(\mathbb{Q}_{X(\mathbb{C})})$ and $D_c^b(\mathbb{Q}_{Y(\mathbb{C})})$, and these satisfy all the usual identities. (See [S1] for $\text{MHM}(\cdot)$ and [S3, Theorem 1.2] and [S2] for $\mathcal{M}(\cdot)$.) Thus for $K \in D^b \mathcal{M}(X)$ and $a_X : X \rightarrow \text{Spec}(k)$ the structure morphism, the object

$$\mathbb{H}^i(a_{X*}K) \in \mathcal{M}(\text{Spec}(k)) \tag{1}$$

gives a mixed realization with rational vector space underlying the Betti part given by $\mathbb{H}^i(X, For(K))$.

There is a unit object $\mathbb{Q}^{\mathcal{M}}$ in $\mathcal{M}(\text{Spec}(k))$ with $For(\mathbb{Q}^{\mathcal{M}}) = \mathbb{Q}$. For $X \in Var/k$ with structure morphism $a_X : X \rightarrow \text{Spec}(k)$ the object $\mathbb{Q}_X^{\mathcal{M}} = a_X^* \mathbb{Q}^{\mathcal{M}}$ (“the constant sheaf on X ”) in $D^b \mathcal{M}(X)$ has $For(\mathbb{Q}_X^{\mathcal{M}}) = \mathbb{Q}_{X(\mathbb{C})}$. Taking $\mathbb{H}^i(a_{X*} \mathbb{Q}_X^{\mathcal{M}})$ gives a mixed realization

$$\mathbb{H}^i(X) := (\mathbb{H}_B^i(X), \mathbb{H}_{dR}^i(X/k), (\mathbb{H}_l^i(X))_l)$$

with Betti part $\mathbb{H}^i(X(\mathbb{C}), \mathbb{Q})$ and this defines a functor $Var/k \rightarrow \mathcal{MR}_k$ extending the earlier functor from smooth varieties. (Note that the underlying k -vector space of $\mathbb{H}_{dR}^i(X/k)$ will not usually be $\mathbb{H}^i(X, \Omega_{X/k}^\bullet)$ if X is singular.) Homology becomes a mixed realization by duality and intersection cohomology becomes a mixed realization using $IC_X = (j_! \mathbb{Q}_U^{\mathcal{M}}[\dim X])[-\dim X]$ in (1) where $j : U \hookrightarrow X$ is an open dense smooth subset. We denote it $\mathbb{IH}^*(X) = (\mathbb{IH}_B^*(X), \mathbb{IH}_{dR}^*(X), (\mathbb{IH}_l^*(X))_l)$.

The Tate object is $\mathbb{Q}^{\mathcal{M}}(1) = \mathbb{H}_2(\mathbb{P}^1) = (\mathbb{Q}_B(1), \mathbb{Q}_{dR}(1), (\mathbb{Q}_l(1))_l)$, where $\mathbb{Q}_B(1)$ is the usual Tate Hodge structure of weight -2 and $\mathbb{Q}_l(1)$ is one-dimensional with Galois action by the l -adic cyclotomic character $\chi_l : \text{Gal}(\bar{\mathbb{Q}}/k) \rightarrow \mathbb{Z}_l^*$. For $n \geq 0$ set $\mathbb{Q}^{\mathcal{M}}(n) = \mathbb{Q}^{\mathcal{M}}(1)^{\otimes n}$ and $\mathbb{Q}^{\mathcal{M}}(-n) = \mathbb{Q}^{\mathcal{M}}(-1)^{\otimes n}$. For an object $K \in \mathcal{M}(X)$ we let $K(n) = K \otimes \mathbb{Q}_X^{\mathcal{M}}(n)$ where $\mathbb{Q}_X^{\mathcal{M}}(n) = a_X^* \mathbb{Q}^{\mathcal{M}}(n)$.

1.2. Truncation by weights. A beautiful idea of S. Morel [Mo] is to define truncation functors using t -structures coming from the theory of weights. Morel’s construction was originally for mixed l -adic complexes on schemes over finite fields, but works mutatis mutandis in any reasonable category of sheaves in which one has weight filtrations with the correct properties, in particular in any theory of mixed sheaves in the sense of Saito [S2]. We work with $\mathcal{M}(\cdot)$.

For $a \in \mathbb{Z}$, Morel defines a t -structure $({}^wD^{\leq a}, {}^wD^{> a})$ on $D^b\mathcal{M}(X)$ as follows: ${}^wD^{\leq a}$ (resp. ${}^wD^{> a}$) is the full subcategory of complexes K such that $H^i(K) \in \mathcal{M}(X)$ has weights $\leq a$ for all i (resp. $> a$ for all i).⁽¹⁾ This defines a t -structure with heart ${}^wD^{\leq a} \cap {}^wD^{> a} = \{0\}$ and the corresponding truncation functors are denoted $w_{\leq a}, w_{> a}$. These t -structures have many nice properties, in particular the subcategories ${}^wD^{\leq a}, {}^wD^{> a}$ are stable under translation and triangulated and the truncation functors commute with translation.

If $X = \bigsqcup_{S \in \mathcal{S}} S$ is a stratification of X by equidimensional varieties over k , and $\underline{d} : \mathcal{S} \rightarrow \mathbb{Z} \cup \{\pm\infty\}$ is a function on the set of strata, then the standard machinery of [BBD] allows us to glue the t -structures $({}^wD^{\leq \underline{d}(S)}, {}^wD^{> \underline{d}(S)})$ on $D^b\mathcal{M}(S)$ for $S \in \mathcal{S}$ to get a t -structure $({}^wD^{\leq \underline{d}}, {}^wD^{> \underline{d}})$ on $D^b\mathcal{M}(X)$. If $i_S : S \hookrightarrow X$ is the locally closed immersions of a stratum $S \in \mathcal{S}$ then a complex K belongs to ${}^wD^{\leq \underline{d}}$ (resp. ${}^wD^{> \underline{d}}$) if $i_S^*K \in {}^wD^{\leq \underline{d}(S)}$ (resp. $i_S^*K \in {}^wD^{> \underline{d}(S)}$) for all $S \in \mathcal{S}$. The associated truncation functors are denoted $w_{\leq \underline{d}}, w_{> \underline{d}}$.

There are two canonical functions $\mathcal{S} \rightarrow \mathbb{Z} \cup \{\pm\infty\}$ associated with any equidimensional stratified variety $X = \bigsqcup_{S \in \mathcal{S}} S$: The constant function $\underline{d} \equiv n = \dim X$ and the function \underline{dim} defined by $\underline{dim}(S) := \dim S$. For the former, Morel [Mo] proves the following remarkable formula: If $j : U \hookrightarrow X$ is the inclusion of an open dense smooth subset then

$$w_{\leq n} j_* \mathbb{Q}_U^{\mathcal{M}}[n] = j_* \mathbb{Q}_U^{\mathcal{M}}[n]. \quad (2)$$

So one recovers the intersection complex $IC_X = w_{\leq n} j_* \mathbb{Q}_U^{\mathcal{M}}$ as a weight truncation.

In [NV] V. Vaish and I found that the glued t -structure given by the function \underline{dim} defined by $\underline{dim}(S_i) = \dim S_i$ has particularly nice properties. (The interest in \underline{dim} came from Theorem 2 below, so we are inverting the chronology here.) In particular, the following was proven:

Theorem 1. ([NV, Proposition 4.1.2]) *Suppose that U is smooth of dimension n , $j : U \hookrightarrow X$ is an open immersion as a Zariski-dense subset of an irreducible variety, and $\pi : Y \rightarrow X$ is a proper morphism from Y smooth such that $\pi|_{\pi^{-1}(U)}$ is an isomorphism. Assume X is given a stratification such that $j_! \mathbb{Q}_U^{\mathcal{M}}[n], j_* \mathbb{Q}_U^{\mathcal{M}}[n]$, and $\pi_* \mathbb{Q}_Y^{\mathcal{M}}[n]$ are constructible. Then*

$$w_{\leq \underline{dim}} j_* \mathbb{Q}_U^{\mathcal{M}}[n] = w_{\leq \underline{dim}} j_! \mathbb{Q}_U^{\mathcal{M}}[n] = w_{\leq \underline{dim}} \pi_* \mathbb{Q}_Y^{\mathcal{M}}[n]$$

in $D^b\mathcal{M}(X)$.

Informally, this shows that the object $EC_X := w_{\leq \underline{dim}} j_* \mathbb{Q}_U^{\mathcal{M}}$ can be constructed “from the inside” using $j : U \hookrightarrow X$ or “from the outside” using $\pi : Y \rightarrow X$. This defines a “cohomology theory” $X \mapsto EH^*(X) := \mathbb{H}^*(X, w_{\leq \underline{dim}} j_* \mathbb{Q}_U^{\mathcal{M}})$ lying somewhere between the usual cohomology and intersection cohomology and enjoying a number of pleasant formal properties established in [NV]. Among these properties are ring structure and functoriality for certain morphisms. Whether there is more topological content to the theory seems to be an interesting question.

1.3. Locally symmetric varieties. Let M be a locally symmetric variety and $n = \dim_{\mathbb{C}} M$. We work over the field of definition k of M , a number field in \mathbb{C} ,

¹Contrast this with the property “ K has weights $\leq a$ ” (resp. “ K has weights $\geq a$ ”) of [BBD, S1], which holds if and only if $H^i(K)$ is of weights $\leq i + a$ for all i (resp. of weights $\geq i + a$ for all i).

which can be made quite explicit if M is a Shimura variety. We will not distinguish notationally between the varieties over k and their complex points.

Consider the diagram

$$\begin{array}{ccc} & & M^{rbs} \\ & & \downarrow p \\ M & \xrightarrow{j} & M^{bb} \end{array}$$

where M^{bb} is the minimal (or minimal Satake or Baily-Borel) compactification. The main technical result of [N2] (Theorem 4.3.1 of loc. cit.) compared various weight truncations $w_{\leq \underline{d}j_*} \mathbb{Q}_M^{\mathcal{M}}$ with the pushforward of the weighted complexes of Goresky-Harder-MacPherson [GHM]. This was done to prove that certain spectral sequences appearing in work of Franke [F1] on Borel's conjecture are spectral sequences of mixed realizations (the main result of [N2]). For the special case of the \underline{dim} truncation we get:

Theorem 2. *There is a natural isomorphism $For(w_{\leq \underline{dim}j_*} \mathbb{Q}_M^{\mathcal{M}}) = p_* \mathbb{Q}_{M^{rbs}}$.*

Another special case of [N2] is the constant function $n = \dim M$, when one gets

$$For(w_{\leq n} j_* \mathbb{Q}_M^{\mathcal{M}}) = p_* WC_{M^{rbs}}(\mathbb{Q}_M)$$

where $WC_{M^{rbs}}(\mathbb{Q}_M)$ is the (upper or lower) middle weighted cohomology complex of [GHM]. By the main result of [GHM] $p_* WC_{M^{rbs}}(\mathbb{Q}_M) = (j_{!*} \mathbb{Q}_M[n])[-n]$, so this reduces to Morel's formula (2) (or could be deduced from it).

The proof of Theorem 2 uses several ingredients that had been around for some time. The main technical problem is that one is comparing the functorial image (under For) of a complex defined using a t -structure in $D^b \mathcal{M}(M^{bb})$ to a complex in $D_c^b(\mathbb{Q}_{M^{bb}})$ where there is no similar t -structure available. The key point is to use the local Hecke operator introduced by Looijenga in his proof of Zucker's conjecture and a splitting property in the derived category for its action on each of the objects. For the weighted complexes the splitting property was proved in [GHM]. For weight truncations $w_{\leq \underline{dim}j_*} \mathbb{Q}_M$ the splitting property is established in [N1] using results of Looijenga and Rapoport [LR] and Jordan decomposition for endomorphisms in the derived category.

Corollary 3. *The cohomology of the RBS compactification is part of a mixed realization, i.e. there is a mixed realization*

$$H^*(M^{rbs}) = (H_B^*(M^{rbs}), H_{dR}^*(M^{rbs}/k), (H_l^*(M^{rbs}))_l)$$

in \mathcal{MR}_k with Betti part a mixed Hodge structure on $H^*(M^{rbs}, \mathbb{Q})$.

Now we combine Theorems 1 and 2. Choose a smooth toroidal compactification $\pi : M^\Sigma \rightarrow M^{bb}$ and consider the following diagram:

$$\begin{array}{ccc} & & M^\Sigma \\ & \swarrow \gamma & \searrow \pi \\ M^{rbs} & & M^{bb} \\ & \searrow p & \end{array} \quad (3)$$

Theorem 1 gives identities

$$w_{\leq \underline{dim}j_*} \mathbb{Q}_M = w_{\leq \underline{dim}j_{!*}} \mathbb{Q}_M = w_{\leq \underline{dim}\pi_*} \mathbb{Q}_{M^\Sigma}. \quad (4)$$

The consequences are summarized in the following commutative diagram in $\mathcal{M}(\mathrm{Spec}(k))$:

$$\begin{array}{ccc}
 & & \mathrm{H}^*(M^\Sigma) \\
 & \nearrow \gamma^* & \\
 \mathrm{H}^*(M^{rbs}) & \xrightarrow{\rho} & \mathrm{IH}^*(M^{bb}) \\
 & \nwarrow p^* & \uparrow \\
 & & \mathrm{H}^*(M^{bb}) \\
 & & \nearrow \pi^*
 \end{array}
 \tag{5}$$

The maps ρ and γ^* are given by (4) and $\iota : \mathrm{IH}^*(M^{bb}) \hookrightarrow \mathrm{H}^*(M^\Sigma)$ is given by any homomorphism $j_! \mathbb{Q}_{M_k}^{\mathcal{M}}[n] \rightarrow \pi_* \mathbb{Q}_{M_k^\Sigma}^{\mathcal{M}}[n]$ coming from the decomposition theorem ([S2, 6.10]). (That $\pi^* = \gamma^* \circ p^*$ recovers the result of [GT] in rational cohomology. Thus, although no γ extending the identity exists in (3), cohomologically it looks like it does.) The morphisms other than ι are canonical and Hecke-equivariant in the appropriate sense (e.g. for π^* , γ^* one keeps in mind that a Hecke operator goes from $\mathrm{H}^*(M^\Sigma) \rightarrow \mathrm{H}^*(M^{\Sigma'})$ for some Σ').

We also have the following consequence of the formalism:

$$\mathrm{Gr}_i^W \mathrm{H}^i(M^{rbs}) = \mathrm{im}[\mathrm{H}^i(M^{rbs}) \rightarrow \mathrm{IH}^i(M^{bb})] = \mathrm{im}[\mathrm{H}^i(M^{rbs}) \rightarrow \mathrm{H}^i(M^\Sigma)]$$

which is analogous to standard formulas in mixed Hodge theory.

1.4. Motivic results. The categories above depend on realizations. A more geometric construction is Ayoub's triangulated category of (etale) motivic sheaves, which we denote here by $\mathrm{DM}(X)$; then $\mathrm{DM}(\mathrm{Spec}(k)) = \mathrm{DM}_k$ is Voevodsky's triangulated category of motives over k . (See [A] for a survey of motivic sheaves; $\mathrm{DM}(X)$ is the category $\mathbf{DA}^{\mathrm{et}}(X; \mathbb{Q})$ of loc. cit.) For a morphism $f : X \rightarrow Y$ in Var/k there are functors f_* , f^* , $f_!$, $f^!$ between $\mathrm{DM}(X)$ and $\mathrm{DM}(Y)$ satisfying the usual adjunctions.

The result of Ayoub and Zucker [AZ] gives an object $\mathbb{E}_{M^{bb}}$ in $\mathrm{DM}(M^{bb})$ which, on pushforward to $\mathrm{Spec}(k)$, gives an object of DM_k . Using the Betti realization constructed by Ayoub, this is shown to realize to $R\Gamma(M^{rbs}, \mathbb{Q}) = \bigoplus_k \mathrm{H}^k(M^{rbs}, \mathbb{Q})[-k]$. This shows that M^{rbs} is motivic in the sense of Voevodsky. Unfortunately, although Huber has constructed a functor $\mathrm{DM}_k \rightarrow \mathcal{MR}_k$, this is not quite enough to show that $\mathrm{H}^*(M^{rbs})$ is a mixed realization (i.e. Corollary 3) because some compatibilities are unknown (see [N3, 2.5] or [AZ, 4.4, 4.9] for a discussion).

In [V] Vaish generalizes the approach to M^{rbs} using Morel's truncation functor $w_{\leq \dim}$ to categories of motivic sheaves. He defines the truncation $w_{\leq \dim}$ (denoted $w_{\leq \mathrm{Id}}$ in loc. cit.) and develops its formalism in a very general motivic sheaf context, including in positive characteristic. He also shows that $w_{\leq \dim}$ agrees with the relative Artin motive functor ω^0 defined in [AZ] in characteristic zero. In particular, this gives another approach to $\mathbb{E}_{M^{bb}}$ and clarifies the relation of ω^0 with weights. In addition to the formulas $\mathbb{E}_{M^{bb}} = w_{\leq \dim} j_* \mathbf{1}_M = w_{\leq \dim} \pi_* \mathbf{1}_M$ in (4) which are in [AZ] (where $\mathbf{1}_M \in \mathrm{DM}(M)$ is the unit object over M , the analogue of the constant sheaf), he is able to prove the relation with the intersection complex in (4) in the motivic context, as we now explain. The notion of *motivic intersection complex* has been defined by Wildeshaus [W1]. The object IM_X in $\mathrm{DM}(X)$, when it exists, is

unique up to isomorphism (but not up to unique isomorphism). It exists if $X = M^{bb}$ [W2].

Theorem 4. (*Vaish* [V, Theorem 4.1.2]) *If the motivic intersection complex IM_X exists, then $w_{\leq \dim} IM_X = \mathbb{E}_X$. In particular, this holds for $X = M^{bb}$.*

This gives a diagram like (5) in DM_k . (The outer triangle is in [AZ].)

Remark 1. Taken together, the results of [N2] (i.e. Theorem 2) and [V] give a different approach to the main theorem of [AZ]. (The results of [NV] are not required for this, they merely served as the model for the development in [V].) Curiously, while the local Hecke operator plays a key role in our approach (in the proof of Theorem 2), it is not explicitly present in [AZ]. On the other hand, [AZ] uses explicit information about toroidal compactifications, which our approach does not (although some use of them is hidden in our use of [LR]). The approach of [N2, V] is simpler in that much of the burden of proof is shifted to a cohomological setting.

Remark 2. The category $\mathcal{M}_{go}(X)$ of objects of geometric origin is the smallest full subcategory of $\mathcal{M}(X)$ closed under subquotients and containing the objects $H^i(f_* \mathbb{Q}_Y^{\mathcal{M}})(n)$ for any projective morphism $f : Y \rightarrow X$ and all $i, n \in \mathbb{Z}$. There is a functor $DM(X) \rightarrow D^b \mathcal{M}(X)$ and it seems to be expected that this gives an equivalence of the heart of the (conjectured) perverse t -structure on $DM(X)$ with $\mathcal{M}_{go}(X)$.

2. AUTOMORPHIC FORMS AND M^{rbs}

The cohomology of the RBS compactification has an analytic interpretation in terms of automorphic forms, thanks to results of [N1] and [F1]. We review this and some of its consequences. The notation is as follows: G is a semisimple group over \mathbb{Q} , $K \subset G(\mathbb{R})$ is a maximal compact subgroup, $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup, \mathfrak{g} is the Lie algebra of $G(\mathbb{R})$ and $U(\mathfrak{g})$ its universal enveloping algebra. The locally symmetric space $M = \Gamma \backslash G(\mathbb{R})/K$ is not assumed to have a complex structure in this section.

2.1. The RBS compactification and automorphic forms. The de Rham theorem allows us to compute the cohomology of M as relative Lie algebra cohomology:

$$H^*(M, \mathbb{C}) = H^*(\mathfrak{g}, K, C^\infty(\Gamma \backslash G(\mathbb{R}))).$$

Here $C^\infty(\Gamma \backslash G(\mathbb{R}))$ is the space of smooth and K -finite functions. We will define certain (\mathfrak{g}, K) -submodules of $C^\infty(\Gamma \backslash G(\mathbb{R}))$ in terms of growth on Siegel sets and use this to compute $H^*(M^{rbs}, \mathbb{C})$.

Fix a minimal \mathbb{Q} -parabolic P_0 and a Levi subgroup $M_0 \subset P_0$, and let N_0 be the unipotent radical and $A_0 \subset M_0$ the maximal \mathbb{Q} -split central torus in M_0 . Let Δ_0 be the set of simple positive roots of A_0 in P_0 and ρ_0 the half-sum of positive roots. Recall that a Siegel set in $G(\mathbb{R}) = P_0(\mathbb{R})K$ has the form $\mathfrak{S} = \omega \cdot A_0(t) \cdot K$ where $\omega \subset (N_0 M_0)$ is a relatively compact subset and $A_0(t) = \{a \in A_0(\mathbb{R})^0 : a^\alpha > t \text{ for all } \alpha \in \Delta_0\}$. For ω large enough and t small enough there are finitely many elements $g_1, \dots, g_r \in G(\mathbb{Q})$ such that $\bigsqcup_{i=1}^r g_i \mathfrak{S}$ forms a coarse fundamental domain for Γ , i.e. $\bigsqcup_{i=1}^r g_i \mathfrak{S} \rightarrow \Gamma \backslash G(\mathbb{R})$ is surjective with finite fibres and bijective onto the complement of a compact set. Fix a norm $\|\cdot\|$ on $\mathfrak{a}_0 = \text{Lie } A_0(\mathbb{R})$.

Let $B = B(\Gamma \backslash G(\mathbb{R}))$ be the subspace of $f \in C^\infty(\Gamma \backslash G(\mathbb{R}))$ for which there exists $N \in \mathbb{N}$ such that for all $1 \leq i \leq r$ and $D \in U(\mathfrak{g})$, there exists $C > 0$ such that

$$|Df(g_i pak)| \leq C(1 + \|\log(a)\|)^N \quad \text{for } pak \in \mathfrak{S}.$$

Informally, we say that f is “uniformly bounded up to logarithmic terms”.

Let $R = R(\Gamma \backslash G(\mathbb{R}))$ be the subspace of $C^\infty(\Gamma \backslash G(\mathbb{R}))$ of functions f such that for $D \in U(\mathfrak{g})$ and $1 \leq i \leq r$, and for all $N \in \mathbb{N}$, there exists $C > 0$ such that

$$|Df(g_i pak)| \leq C(1 + \|\log(a)\|)^{-N} \cdot a^{2\rho_0} \quad \text{for } pak \in \mathfrak{S}.$$

If $f \in B, g \in R$ then fg is L^1 and this defines a (\mathfrak{g}, K) -invariant pairing of B and R .

Let \mathfrak{I} be the ideal of the centre of $U(\mathfrak{g})$ killing the trivial representation. Then

$$\mathfrak{A}_R = \mathfrak{A}_R(\Gamma \backslash G(\mathbb{R})) = \{f \in R : \mathfrak{I}^m f = 0 \text{ for some } m\}$$

is a (\mathfrak{g}, K) -module of automorphic forms. Specializing the main result of [N1] to the cases $\lambda = 0, \lambda = 2\rho_0$ gives:

Theorem 5. ([N1]) *There are natural isomorphisms*

$$H^*(M^{rbs}, \mathbb{C}) = H^*(\mathfrak{g}, K, B)$$

and

$$H_{\dim_{\mathbb{R}} M - *} (M^{rbs}, \mathbb{C}) = H^*(\mathfrak{g}, K, R) = H^*(\mathfrak{g}, K, \mathfrak{A}_R).$$

We remark that replacing R by the submodule of automorphic forms \mathfrak{A}_R relies on the difficult results of Franke [F1]. The map induced by $B \subset R$ is the fundamental class map $H^*(M^{rbs}, \mathbb{C}) \rightarrow H_{\dim_{\mathbb{R}} M - *} (M^{rbs}, \mathbb{C})$.

The constant functions belong to B . The (\mathfrak{g}, K) -cohomology of the trivial representation is the cohomology of the compact dual symmetric space \hat{X} of $X = G/K$, so this gives a map

$$\theta : H^*(\hat{X}, \mathbb{C}) \rightarrow H^*(M^{rbs}, \mathbb{C}). \quad (6)$$

In the compact case this is injective and is an isomorphism onto the Hecke invariants by Matsushima’s formula, see [N3, 1.2]. Using Theorem 5 and structural results of Franke on the space of automorphic forms, we prove in [N3, §1] that:

Theorem 6. ([N3]) *The inclusion of the constant functions $\mathbb{C} \subset B$ induces an isomorphism of $H^*(\hat{X}, \mathbb{C})$ onto the algebra of Hecke-invariants and Hecke-coinvariants in $H^*(M^{rbs}, \mathbb{C})$.*

Note that the algebra of invariants splits off as a direct summand. It is essential here to first work with \mathfrak{A}_R and compute the invariants in $H_*(M^{rbs}, \mathbb{C})$ and then use duality, because B cannot be replaced by a module of automorphic forms to compute cohomology.

Remark 3. Another C^∞ de Rham model for $H^*(M^{rbs}, \mathbb{C})$ using L^p cohomology (for large p) was found in [Z3]. It is not clear to me if Theorem 6 can be proven using this model.

2.2. Functoriality of RBS cohomology: A question. Suppose that $\varphi : H \rightarrow G$ is a homomorphism of semisimple \mathbb{Q} -groups with finite kernel. Let $\Gamma \subset G(\mathbb{Q})$ be arithmetic and $\Gamma_H := \varphi^{-1}(\Gamma)$. Let $X_H \subset X$ be the embedding of symmetric spaces induced by φ . For $M_H = \Gamma_H \backslash X_H$ and $M = \Gamma \backslash X$ there is a proper map of locally symmetric spaces

$$\varphi : M_H \longrightarrow M.$$

It is easy to give examples to show that this does *not* extend continuously to RBS compactifications in general. (Morally the reason is clear: Two non- Γ -conjugate parabolics in G might have preimages in H which are Γ_H -conjugate. Then the corresponding stratum of M_H^{rbs} “does not know where to go” in M^{rbs} .) Nevertheless, it is a consequence of Theorem 5 that there is a natural restriction map

$$H^*(M^{rbs}, \mathbb{C}) \longrightarrow H^*(M_H^{rbs}, \mathbb{C}) \quad (7)$$

because functions in $B(\Gamma \backslash G(\mathbb{R}))$ pull back to functions in $B(\Gamma_H \backslash H(\mathbb{R}))$. In the case where both M_H and M are locally symmetric varieties, there is an extension $M_H^{bb} \rightarrow M^{bb}$, and this can be used (by the properties of EC_X alluded to at the end of 1.2) to show that (7) is \mathbb{Q} -rational and even a homomorphism of mixed realizations.

For general M and M_H , the following topological explanation for (7), inspired by [GT], suggests itself: Embed M_H in the $(M_H \times M)^{rbs} = M_H^{rbs} \times M^{rbs}$ by $x \mapsto (x, \varphi(x))$ and let M'_H be its closure. There are maps

$$M_H^{rbs} \xleftarrow{p_1} M'_H \xrightarrow{p_2} M^{rbs}$$

induced by the two projections. *Are the fibres of p_1 contractible?* If so, the resulting pullback map $H^*(M^{rbs}, \mathbb{Z}) \rightarrow H^*(M_H^{rbs}, \mathbb{Z})$ should give (7).

3. AN APPLICATION: CHERN CLASSES

In [N3] we combined the results surveyed in §1 and §2 with results in the literature due to several authors to prove that the Chern classes of automorphic vector bundles on M^{rbs} have motivic properties in mixed realizations. The result is best formulated in the setting of Shimura varieties, so we will only give an approximate statement, referring to [N3] for details.

Recall the map defined in (6) above

$$\theta : H^*(\hat{X}, \mathbb{C}) \rightarrow H^*(M^{rbs}, \mathbb{C})$$

for a general locally symmetric space using the C^∞ de Rham model of Theorem 5 (or the L^p cohomology model of [Z3]). In general this map does not relate the two rational structures (presumably irrational quantities like $\zeta(k)/\pi^k$ for odd k are involved when $G = SL(n)$ thanks to Borel’s theorem).

When M is a locally symmetric variety, however, θ is Betti rational. This is because θ has another description in terms of Chern classes of automorphic vector bundles. Recall that $\hat{X} = G(\mathbb{C})/Q$ where Q is a parabolic with Levi subgroup the complexification $K_{\mathbb{C}}$ of K . A finite-dimensional complex representation $\rho : K \rightarrow GL(V)$ gives an algebraic vector bundle $\hat{\mathcal{V}}$ on \hat{X} . Restricting by the Borel embedding $X \subset \hat{X}$ and dividing by Γ gives an vector bundle \mathcal{V} on M which has a canonical algebraic structure. The underlying topological vector bundle of \mathcal{V}

extends naturally to a topological vector bundle \mathcal{V}^{rbs} on M^{rbs} (see [GT, 9.2] or [Z3, 1.9]). Then θ satisfies

$$\theta(c_i(\hat{\mathcal{V}})) = (-1)^i c_i(\mathcal{V}^{rbs}).$$

In the compact case this is essentially Hirzebruch proportionality; in general it follows from results of Zucker [Z3], see [N3, 3.6]. Since the rational cohomology of \hat{X} is generated by Chern classes of homogeneous bundles, θ is rational.

Since $H^*(M^{rbs})$ has a mixed realization (Corollary 3), it is natural to ask if θ is a homomorphism of mixed realizations. This amounts to asking if the *topologically defined* classes $c_i(\mathcal{V}^{rbs})$ have “motivic” properties. The following is an informal version of the main result of [N3]; we refer to that paper for precise statements:

Theorem 7. *The Chern classes of the topological extensions to M^{rbs} of automorphic vector bundles on M have the properties one would expect if M^{rbs} were an algebraic variety over k and automorphic vector bundles extended algebraically over M^{rbs} respecting fields of definition.*

Thus for example, the Chern class $c_i(\mathcal{V}^{rbs})$ belongs to the correct step of the Hodge filtration on $H^{2i}(M^{rbs}, \mathbb{C})$, and if \mathcal{V} is defined over L/k the action of $\text{Gal}(\mathbb{Q}/L)$ on $c_i(\mathcal{V}^{rbs})$, considered in étale cohomology via the comparison isomorphism, is via the i th power of the cyclotomic character χ_l .

The proof of this theorem combines Theorem 6 with results about automorphic vector bundles on Shimura varieties and their canonical extensions to toroidal compactifications due to Mumford and Harris, and also results due to Goresky-Pardon and Zucker. As a corollary, for the correct de Rham k -structure on \hat{X} (see [N3]), we have:

Corollary 8. *The map $\theta : H^*(\hat{X}) \rightarrow H^*(M^{rbs})$ given by $\theta(c_k(\hat{\mathcal{V}})) = (-1)^k c_k(\mathcal{V})$ is a ring isomorphism of mixed realizations onto the Hecke-invariants in $H^*(M^{rbs})$.*

This can be seen as a generalization of the classical Hirzebruch-Mumford proportionality theorem [Mu] to M^{rbs} in mixed realizations. The construction of a motivic θ (i.e in a context like 1.4) seems like an interesting problem.

Remark 4. The corollary can be used to show that the direct summand $H^*(M)_{\mathcal{I}}$ of generalized Hecke-invariants of the cohomology of $H^*(M)$, i.e. the summand on which the Hecke operators act with the same eigenvalues as in the trivial Hecke-module, is mixed Tate in \mathcal{MR}_k (i.e. the graded pieces for the weight filtration are sums of Tate objects). The extensions in this summand can further be analyzed using methods of automorphic forms from [F2], see [N4].

Remark 5. A remarkable fact is that the analogue of the corollary is *not* true for M^{bb} due to the appearance of nontrivial mixed Tate extensions in $H^*(M^{bb})$ (see the remarks in [N3, 0.4.4.3] and [N4] and [L]).

REFERENCES

- [A] J. Ayoub, A guide to (étale) motivic sheaves, to appear in the proceedings of ICM 2014, available at <http://user.math.uzh.ch/ayoub/>
- [AZ] J. Ayoub and S. Zucker, Relative Artin motives and the reductive Borel-Serre compactification of a locally symmetric variety, *Invent. Math.* 188 (2012), 277-427.
- [B] A. A. Beilinson, On the derived category of perverse sheaves, *K-Theory, Arithmetic and Geometry*, Lecture Notes in Math. 1289 (1987), 27–41.
- [BBD] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, *Astérisque* 100 (1982).

- [F1] J. Franke, Harmonic analysis in weighted L^2 spaces, *Ann. Sci. ÉNS t.* 31 (1998), 181–279.
- [F2] J. Franke, A topological model for some summand of the Eisenstein cohomology of congruence subgroups, *Eisenstein series*, Prog. in Math. 258 (2008), 27–85.
- [GHM] M. Goresky, G. Harder, and R. MacPherson, Weighted cohomology, *Invent. Math.* 116 (1994), 139–213.
- [GT] M. Goresky and Y.-S. Tai, Toroidal and reductive Borel-Serre compactifications of locally symmetric spaces, *Amer. J. Math.* 121 (1999) 1095–1151.
- [H] G. Harder, *Eisensteinkohomologie und die Konstruktion gemischter Motive*, Lecture Notes in Math. 1562, Springer (1993).
- [J] U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Math. 1400, Springer-Verlag (1990).
- [LR] E. Looijenga and M. Rapoport, Weights in the local cohomology of a Baily-Borel compactification, *Complex Geometry and Lie Theory*, Proc. Symp. Pure Math. 53 (1991), 223–260.
- [L] E. Looijenga, Goresky-Pardon Chern classes and extensions which may be mixed Tate, preprint (2015).
- [Mo] S. Morel, Complexes pondérés sur les compactifications de Baily-Borel: Le cas des variétés de Siegel, *J. Amer. Math. Soc.* 21 (2008), 23–61.
- [Mu] D. Mumford, Hirzebruch’s proportionality theorem in the noncompact case, *Invent. Math.* 42 (1977), 239–272.
- [N1] A. N. Nair, Weighted cohomology of arithmetic groups, *Ann. of Math.* 150 (1999), 1–31.
- [N2] A. N. Nair, Mixed motives in Shimura varieties and automorphic forms, preprint (2013), available at the author’s homepage.
- [NV] A. N. Nair and V. Vaish, Weightless cohomology of algebraic varieties, *J. Algebra* 424 (2015), 147–189.
- [N3] A. N. Nair, Chern classes of automorphic vector bundles and the reductive Borel-Serre compactification, preprint (2014), available at the author’s homepage.
- [N4] A. N. Nair, Mixed Tate motives in A_g , in preparation.
- [S1] M. Saito, Mixed Hodge modules, *Publ. Math. RIMS* 26 (1990), 221–333.
- [S2] M. Saito, On the formalism of mixed sheaves, [arXiv:math/0611597v1](https://arxiv.org/abs/math/0611597v1) (2006).
- [S3] M. Saito, Arithmetic mixed sheaves, *Invent. Math.* 144 (2001), 533–569.
- [V] V. Vaish, Motivic weightless complex and the relative Artin motive, *Advances in Math.* 292 (2016), 316–373.
- [W1] J. Wildeshaus, Motivic intersection complex, *Regulators*, Contemp. Math. 571 (2012), 255–276.
- [W2] J. Wildeshaus, Intermediate extension of chow motives of abelian type, [arXiv.math:1211.5327v2](https://arxiv.org/abs/math/1211.5327v2) (2012).
- [Z1] S. Zucker, Hodge theory with degenerating coefficients: L^2 cohomology in the Poincaré metric, *Ann. of Math.* 109 (1979), 415–476.
- [Z2] S. Zucker, L^2 cohomology of warped products and arithmetic groups, *Invent. Math.* 70 (1982), 169–218.
- [Z3] S. Zucker, On the reductive Borel-Serre compactification: L^p cohomology of arithmetic groups (for large p), *Amer. J. Math.* 123 (2001), 951–984.
- [Z4] S. Zucker, On the reductive Borel-Serre compactification III: Mixed Hodge structures, *Asian. J. Math.* 8 (2004), 881–912.

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