LEFSCHETZ PROPERTIES FOR NONCOMPACT ARITHMETIC BALL QUOTIENTS II

ARVIND N. NAIR

Abstract. The Lefschetz property and the nonvanishing result for cup products in the cohomology of congruence ball quotients proved in [6] are improved.

Introduction

Let $E \subset \mathbb{C}$ be an imaginary quadratic field and let (V, h) be a Hermitian space with respect to $E/\mathbb{Q}$ of dimension $n+1 \geq 3$ and signature $(n, 1)$ over the reals. A congruence subgroup Γ of $G = \mathrm{SU}(h)$ acts properly discontinuously on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ and the quotient $M = \Gamma\backslash\mathbb{B}$ is a noncompact algebraic variety. If $W \subset V$ is a subspace of dimension $m+1$ on which h is nondegenerate and indefinite, then the standard embedding of $H = \mathrm{SU}(h|_W)$ in G gives an embedding of the m-ball $\mathbb{B}_H \subset \mathbb{B}$, and the quotient $M_H = \Gamma \cap H\backslash\mathbb{B}_H$ admits a morphism $M_H \longrightarrow M$ which is finite onto its image. Letting $\mathrm{H}^*(\cdot)$ denote cohomology with complex coefficients, we have:

Theorem 1. *For each $m < n$, there exist subspaces $W_1, \ldots, W_s$ in V of dimension $m+1$ such that the sum of pullback maps*

$$\mathrm{H}^i(M) \longrightarrow \bigoplus_j \mathrm{H}^i(M_{H_j}) \tag{0.1}$$

is injective in degrees $i < m$ and for $i = m$ is injective on the interior cohomology $\mathrm{H}^m_!(M) = \mathrm{im}(\mathrm{H}^m_c(M) \to \mathrm{H}^m(M))$ (which includes the cuspidal cohomology) and on the subspace $\mathrm{H}^{m,0}(M) \oplus \mathrm{H}^{0,m}(M)$.

The first part of the statement improves [6, Theorem 0.1] by one degree, i.e. we prove the injectivity on $\mathrm{H}^i(M)$ in degrees $i < m$ instead of $i < m-1$. The second part (for $i = m$) is contained in [6], but was not stated there explicitly.

For $g \in G(\mathbb{Q})$ let C_g^* be the action in cohomology of the Hecke operator associated with the double coset $\Gamma g \Gamma \subset G(\mathbb{Q})$. The following improves [6, Theorem 3.21], which assumes $i + j < n - 1$:

Theorem 2. *If $\alpha \in \mathrm{H}^i(M)$ and $\beta \in \mathrm{H}^j(M)$ are nonzero and either $i + j < n$ or one of them lies in $\mathrm{H}^*_!(M)$, then $\alpha \cdot C_g^*(\beta) \neq 0$ for some $g \in G(\mathbb{Q})$.*

Theorems 1 and 2 seem to be the best one can do by geometric means: The question of injectivity on the full cohomology in the case $i = m > 1$ or of nonvanishing of the cup product in case $i + j = n$ seems to require arithmetic methods to resolve, see Remarks 1.13 and 2.6. This is in contrast to the compact case [9].

The proof of Theorem 1 is a small variation on [6]: The Satake (or Baily-Borel or minimal) compactification M^* is projective and $\partial M^* = M^* - M$ consists of finitely many cusps. The decomposition $M \overset{j}{\hookrightarrow} M^* \overset{i}{\hookleftarrow} \partial M^*$ gives an exact sequence

$$0 \longrightarrow \mathrm{H}^i_!(M) \longrightarrow \mathrm{H}^i(M) \longrightarrow \mathrm{H}^i(i^* Rj_* \mathbb{C})$$

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where the first term is interior cohomology and the third term is the cohomology "at infinity". (It can be identified with the cohomology of the Borel-Serre boundary or with the cohomology of the link of ∂M^*.) The sequence is functorial for $H \subset G$ because $M_H \to M$ extends to $M_H^* \to M^*$. Now Corollary 3.16 of [6] implies the injectivity of (0.1) on the interior cohomology $\mathrm{H}^i_!(M)$ for $i \leq m$. An elementary argument (Prop. 1.8) using Kostant's theorem [5] then deals with the cohomology at infinity for $i < m$, proving the theorem. (We also use this argument to correct a minor error in the proof of the Lefschetz property for toroidal compactifications in [6, Theorem 3.17], cf. Remark 1.10.)

The proof of Theorem 2 uses similar arguments.

The writing of this note was prompted by the realization that if one is only interested in restriction or products in the cohomology of M, as opposed to in the cohomology of the toroidal compactification $\overline{M}$, one does not need the full strength of the results of [6]. In particular, Theorem 2.6 of [6], which canonically realizes the intersection cohomology $\mathrm{IH}^*(M^*)$ of the minimal compactification as a Hecke-summand of $\mathrm{H}^*(\overline{M})$, can be avoided for this problem. In fact, the toroidal compactification can be completely dispensed with; the minimal compactification and its intersection cohomology play the main role. (Theorem 2.6 of [6] remains of independent interest and is indispensable to treat $\mathrm{H}^*(\overline{M}) \to \mathrm{H}^*(\overline{M}_H)$, as in loc. cit..) Rather than rework the arguments of [6] here to reflect this, we simply use Cor. 3.16 of loc. cit. as discussed above. The approach using only the minimal compactification will be used elsewhere to treat noncompact orthogonal Shimura varieties, where the noncanonical nature of the toroidal compactification is an issue.

The reader is referred to the introduction of [6] for a discussion of earlier work by many authors on Lefschetz properties for Shimura subvarieties of Shimura varieties. We simply note here that the study of such properties was begun by Oda in [7] in precisely the case of noncompact ball quotients at hand (for $i = 1$), and a key idea used in [6] was adapted from the compact case treated by Venkataramana [9].

Bergeron and Clozel have given a different approach to Theorems 1 and 2 using spectral methods. More precisely, they give a completely different approach to injectivity on interior cohomology (i.e. our Theorem 1.1 below) and use a similar argument at infinity (cf. the forthcoming revised version of the preprint [2]).

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1. Proof of Theorem 1

1.1. **Direct limits and restriction maps.** We briefly recall the setup of [6].

Fix $E = \mathbb{Q}(\sqrt{d})$, (V, h), and $G = \mathrm{SU}(h)$ as in the introduction or [6, §1]. The ball quotients $M_\Gamma = \Gamma\backslash\mathbb{B}$ for congruence $\Gamma \subset G(\mathbb{Q})$ form an inverse system and one takes the direct limit

$$\mathrm{H}^i(\mathcal{M}) = \varinjlim_\Gamma \mathrm{H}^i(M_\Gamma)$$

(with complex coefficients, cf. [6, §2.1]) over congruence subgroups. For $H \subset G$ given by a $m+1$-dimensional subspace $W \subset V$ on which h is indefinite and nondegenerate, we have a natural embedding $H \subset G$ by letting H act trivially on $W^\perp$. This gives an embedding $\mathbb{B}_H \subset \mathbb{B}$ and morphisms $M_{H,\Gamma_H} \to M_\Gamma$ for all Γ. These give a restriction map

$$\mathrm{Res} : \mathrm{H}^i(\mathcal{M}) \longrightarrow I_H^G\, \mathrm{H}^i(\mathcal{M}_H)$$

where I_H^G is an exact induction functor from smooth $H(\mathbb{Q})$-modules to smooth $G(\mathbb{Q})$-modules. Recall from [6, 3.1] that a $G(\mathbb{Q})$-module is smooth if every vector is fixed by a congruence subgroup of $G(\mathbb{Q})$. If U is a smooth $H(\mathbb{Q})$-module then the induced module $I_H^G U$ consists of functions $f : G(\mathbb{Q}) \to U$ such that (1) $f(gh) = h^{-1} \cdot f(g)$ and (2) f is left-invariant by a congruence subgroup of $G(\mathbb{Q})$. Then

I_H^G takes smooth modules to smooth modules and is right adjoint to restriction. The map Res has the following concrete description: For $\alpha \in \mathrm{H}^i(M_\Gamma)$ we have $\mathrm{Res}(\alpha)(g) = (g\cdot)^*\alpha|_{M_{H,\Gamma'_H}}$ where $\Gamma' = \Gamma \cap g^{-1}\Gamma g$ and $g\cdot : M_{\Gamma'} \to M_\Gamma$ is induced by translation by g on $\mathbb{B}$. So Theorem 1 is implied by:

Theorem 1.1. *The map* $\mathrm{Res} : \mathrm{H}^k(\mathcal{M}) \to I_H^G\,\mathrm{H}^k(\mathcal{M}_H)$ *is injective in degrees* $k < m$ *and on the subspace* $\mathrm{H}_!^m(\mathcal{M})$ *in degree* $k = m$.

Recall that the interior cohomology is $\mathrm{H}_!^k(M_\Gamma) = \mathrm{im}(\mathrm{H}_c^k(M_\Gamma) \to \mathrm{H}^k(M_\Gamma))$ and, in the limit,

$$\mathrm{H}_!^k(\mathcal{M}) = \mathrm{im}(\mathrm{H}_c^k(\mathcal{M}) \to \mathrm{H}^k(\mathcal{M})).$$

Clearly the map Res above restricts to $\mathrm{Res} : \mathrm{H}_!^i(\mathcal{M}) \to I_H^G\,\mathrm{H}_!^i(\mathcal{M}_H)$.

1.2. **Cohomology at infinity.** Recall (e.g. from [6, §1]) that $\mathbb{B}$ is identified with the space of h-negative $\mathbb{C}$-lines in $V_\mathbb{R}$ and the Satake extension of $\mathbb{B}$ is

$$\mathbb{B}^* = \mathbb{B} \sqcup \{h\text{-isotropic } E\text{-lines } \ell \subset V\}.$$

The minimal (i.e. minimal Satake) compactification is $M_\Gamma^* = \Gamma\backslash\mathbb{B}^*$; this is a complex projective variety. There are complementary inclusions

$$M_\Gamma \xhookrightarrow{j_\Gamma} M_\Gamma^* \xhookleftarrow{i_\Gamma} \partial M_\Gamma^* .$$

The cohomology at infinity at level Γ is

$$\mathrm{H}^i(i_\Gamma^* Rj_{\Gamma *}\mathbb{C}) = \bigoplus_{\ell \in \Gamma\backslash\mathbb{B}^*-\mathbb{B}} \mathrm{H}^i(i_{\Gamma,\ell}^* Rj_{\Gamma *}\mathbb{C})$$

(with the obvious notation $i_{\Gamma,\ell} : \{\ell\} \hookrightarrow M_\Gamma^*$) which can be nonzero for $1 \leq i \leq 2n-1$, and we take the limit

$$\mathrm{H}^i(\mathcal{L}) := \varinjlim_\Gamma \mathrm{H}^i(i_\Gamma^* Rj_{\Gamma *}\mathbb{C}).$$

This is an induced module, described as follows: Choose a line $\ell \in \mathbb{B}^* - \mathbb{B}$ and let $P_\ell = \mathrm{Stab}_G(\ell)$. Recall from [6, §1] that the unipotent radical W_ℓ of P_ℓ is two-step unipotent with centre $U_\ell \cong \mathbb{G}_a$ and $V_\ell = W_\ell/U_\ell$ is abelian (in fact, $V_\ell \cong R_{E/\mathbb{Q}}\mathbb{G}_a^{n-1}$). The Lie algebras of $W_\ell(\mathbb{R}), U_\ell(\mathbb{R}), V_\ell(\mathbb{R})$ are denoted $\mathfrak{w}_\ell, \mathfrak{u}_\ell, \mathfrak{v}_\ell$ and their complexifications $\mathfrak{w}_{\ell,\mathbb{C}}, \mathfrak{u}_{\ell,\mathbb{C}}, \mathfrak{v}_{\ell,\mathbb{C}}$. The Lie algebra cohomology groups $\mathrm{H}^i(\mathfrak{w}_{\ell,\mathbb{C}})$ are P_ℓ-modules, via the action induced by the coadjoint action on the complex $\wedge^*\mathfrak{w}_{\ell,\mathbb{C}}^*$ computing them. The action in cohomology factors through the Levi quotient P_ℓ/W_ℓ. The same assertions hold for $\wedge^i\mathfrak{v}_{\ell,\mathbb{C}}^*$ and $\wedge^i\mathfrak{u}_{\ell,\mathbb{C}}^*$.

Lemma 1.2. *There is a natural isomorphism* $\mathrm{H}^i(\mathcal{L}) = I_{P_\ell}^G\,\mathrm{H}^i(\mathfrak{w}_{\ell,\mathbb{C}})$.

Proof. For the chosen isotropic line $\ell \in V$ the limit

$$T_\ell^i := \varinjlim_\Gamma \mathrm{H}^i(i_{\Gamma,\ell}^* Rj_{\Gamma *}\mathbb{C})$$

is a $P_\ell(\mathbb{Q})$-module in the obvious way. The link of the cusp $\{\ell\}$ in M_Γ^* is the nilmanifold $\Gamma \cap W_\ell(\mathbb{R})\backslash W_\ell(\mathbb{R})$ and the van Est theorem identifies its cohomology, using the complex of invariant forms, with $\mathrm{H}^*(\mathfrak{w}_{\ell,\mathbb{C}})$, and the $P(\mathbb{Q})$-action is induced by the coadjoint action. This gives a $P_\ell(\mathbb{Q})$-equivariant identification $T_\ell^i = \mathrm{H}^i(\mathfrak{w}_{\ell,\mathbb{C}})$. It follows formally from this, exactly as in the proof of [6, Lemma 3.3], that $\mathrm{H}^i(\mathcal{L}) = I_{P_\ell}^G\,\mathrm{H}^i(\mathfrak{w}_{\ell,\mathbb{C}})$. □

Lemma 1.3. *There are natural short exact sequences*

$$0 \longrightarrow \mathfrak{u}_{\ell,\mathbb{C}}^* \otimes \wedge^{k-2}\mathfrak{v}_{\ell,\mathbb{C}}^* \longrightarrow \wedge^k\mathfrak{v}_{\ell,\mathbb{C}}^* \longrightarrow \mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}}) \longrightarrow 0$$

for $k < n$ *and*

$$0 \longrightarrow \mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}}) \longrightarrow \mathfrak{u}_{\ell,\mathbb{C}}^* \otimes \wedge^{k-1}\mathfrak{v}_{\ell,\mathbb{C}}^* \longrightarrow \wedge^{k+1}\mathfrak{v}_{\ell,\mathbb{C}}^* \longrightarrow 0$$

for $k \geq n$.

Proof. Let $\pi : \overline{M}_\Gamma \to M^*_\Gamma$ be the toroidal compactification at level Γ and

$$M_\Gamma \xhookrightarrow{\bar{j}_\Gamma} \overline{M}_\Gamma \xhookleftarrow{\bar{i}_\Gamma} D_\Gamma = \textstyle\coprod_{\ell \in \Gamma\backslash \mathbb{B}^* - \mathbb{B}} D_{\Gamma,\ell} \tag{1.1}$$

the decomposition into an open subset and the complement. (Here $D_\Gamma = \overline{M}_\Gamma - M_\Gamma$.) Applying $R\pi_*$ to the standard distinguished triangle $\bar{i}_{\Gamma !}\bar{i}^!_\Gamma \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow R\bar{j}_{\Gamma *}\mathbb{C} \xrightarrow{+1}$ for (1.1) and taking stalks at the point $\{\ell\} \in \partial M^*_\Gamma$ gives a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^k_{D_{\Gamma,\ell}}(\overline{M}_\Gamma) \xrightarrow{adj} \mathrm{H}^k(D_{\Gamma,\ell}) \longrightarrow \mathrm{H}^k(i^*_{\Gamma,\ell} Rj_{\Gamma *}\mathbb{C}) \longrightarrow \cdots \tag{1.2}$$

Under the Thom isomorphism $\mathrm{H}^k_{D_{\Gamma,\ell}}(\overline{M}_\Gamma) \cong \mathrm{H}^{k-2}(D_{\Gamma,\ell})(-1)$, the map adj is simply cupping with the first Chern class of the normal bundle. Since the conormal bundle is ample this has the hard Lefschetz property (see [6, §2, Remark 2.4]), so that the long exact sequence breaks up into short exact sequences. In [6, Lemma 3.3] we gave $P(\mathbb{Q})$-equivariant identifications

$$\varinjlim_\Gamma \mathrm{H}^k(D_{\Gamma,\ell}) = \wedge^k \mathfrak{v}^*_{\ell,\mathbb{C}}$$
$$\varinjlim_\Gamma \mathrm{H}^k_{D_{\Gamma,\ell}}(\overline{M}_\Gamma) = \mathfrak{u}^*_{\ell,\mathbb{C}} \otimes \wedge^{k-2} \mathfrak{v}^*_{\ell,\mathbb{C}}.$$

Putting these two facts together gives the lemma. □

Remark 1.4. The sequence (1.2) is one of rational mixed Hodge structures by the theory of mixed Hodge modules [8]. The fact that the short exact sequences in the lemma come from geometry means that they are compatible with weights. Since $\wedge^k \mathfrak{v}^*_{\ell,\mathbb{C}} = \mathrm{H}^k(D_{\Gamma,\ell})$ and $\mathrm{H}^k_{D_{\Gamma,\ell}}(\overline{M}) \cong \mathrm{H}^{k-2}(D_{\Gamma,\ell})(-1)$ are pure of weight k (because D_Γ is smooth and projective), we see that the cohomology at infinity $\mathrm{H}^k(\mathcal{L})$ is pure of weight k if $k < n$ and pure of weight $k+1$ if $k \geq n$.

Remark 1.5. The short exact sequences in the lemma can also be seen from the Hochschild-Serre sequence for $\mathfrak{u}_{\ell,\mathbb{C}} \subset \mathfrak{w}_{\ell,\mathbb{C}}$ which has E_2 term $\mathrm{H}^p(\mathfrak{v}_{\ell,\mathbb{C}}, \mathrm{H}^q(\mathfrak{u}_{\ell,\mathbb{C}})) = \wedge^p \mathfrak{v}^*_{\ell,\mathbb{C}} \otimes \wedge^q \mathfrak{u}^*_{\ell,\mathbb{C}}$ and d_2 differential induced by the Lie bracket $\mathfrak{u}^*_{\ell,\mathbb{C}} \hookrightarrow \wedge^2 \mathfrak{v}^*_{\ell,\mathbb{C}}$. Since $\dim \mathfrak{u}_{\ell,\mathbb{C}} = 1$ the spectral sequence degenerates at E_3 and gives the lemma. The relation with the geometric picture is given by [6, Lemma 1.6], which relates the Chern class and the Lie bracket.

1.3. **Proof of Theorem 1.1.** Now suppose we are in the situation of §1.1. For each level Γ this gives a morphism $M_{H,\Gamma_H} \to M_\Gamma$ which extends to Satake compactifications

$$M^*_{H,\Gamma_H} \longrightarrow M^*_\Gamma$$

(see e.g. the description in §1 of [6]). This induces a morphism

$$\mathrm{H}^k(i^*_\Gamma Rj_{\Gamma *}\mathbb{C}) \longrightarrow \mathrm{H}^k(i^*_{H,\Gamma_H} Rj_{H*}\mathbb{C})$$

where $M_H \xhookrightarrow{j_{H,\Gamma}} M^*_{H,\Gamma_H} \xhookleftarrow{i_{H,\Gamma_H}} \partial M^*_H$. In the limit one has an $H(\mathbb{Q})$-equivariant map $\mathrm{H}^k(\mathcal{L}) \to \mathrm{H}^k(\mathcal{L}_H)$, which gives (via Frobenius reciprocity) a $G(\mathbb{Q})$-map

$$\mathrm{Res}_\infty : \mathrm{H}^k(\mathcal{L}) \longrightarrow I^G_H \mathrm{H}^k(\mathcal{L}_H).$$

The proof of Theorem 1.1 is based on the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^k_!(\mathcal{M}) & \longrightarrow & \mathrm{H}^k(\mathcal{M}) & \longrightarrow & \mathrm{H}^k(\mathcal{L}) \\ & \downarrow & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}_\infty} \\ 0 \longrightarrow & I^G_H\, \mathrm{H}^k_!(\mathcal{M}_H) & \longrightarrow & I^G_H\, \mathrm{H}^k(\mathcal{M}_H) & \longrightarrow & I^G_H\, \mathrm{H}^k(\mathcal{L}_H) \end{array} \tag{1.3}$$

with exact rows. The first row comes from the long exact sequence $\cdots \to \mathrm{H}^k_c(M_\Gamma) \to \mathrm{H}^k(M_\Gamma) \to \mathrm{H}^k(i^*_\Gamma R j_{\Gamma *}\mathbb{C}) \to \cdots$ and the second is similar.

Proposition 1.6. *The map* $\mathrm{Res} : \mathrm{H}^k_!(\mathcal{M}) \to I^G_H \mathrm{H}^k_!(\mathcal{M}_H)$ *is injective for* $k \le m$.

Proof. This is a straighforward consequence of [6, Corollary 3.16]. First note that

$$\mathrm{Gr}^W_k \mathrm{H}^k_c(\mathcal{M}) = \mathrm{im}(\mathrm{H}^k_c(\mathcal{M}) \to \mathrm{IH}^k(\mathcal{M}^*)) = \mathrm{im}(\mathrm{H}^k_c(\mathcal{M}) \to \mathrm{H}^k(\overline{\mathcal{M}}))$$

where $\mathrm{H}^k(\overline{\mathcal{M}}) = \varinjlim_\Gamma \mathrm{H}^k(\overline{M}_\Gamma)$ is the limit of cohomology of the toroidal compactification. (The equality of the extreme terms is part of Deligne's mixed Hodge theory for open smooth varieties [3]. By the decomposition theorem [1, 8], the map $\mathrm{H}^k_c(\mathcal{M}) \to \mathrm{H}^k(\overline{\mathcal{M}})$ factors through $\mathrm{IH}^k(\mathcal{M}^*)$, so that the second equality holds.)

Now for all k one has a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}^W_k \mathrm{H}^k_c(\mathcal{M}) & \hookrightarrow & \mathrm{H}^k(\overline{\mathcal{M}}) \\ \downarrow & & \downarrow \overline{\mathrm{Res}} \\ \mathrm{Gr}^W_k \mathrm{H}^k_c(\mathcal{M}_H) & \hookrightarrow & \mathrm{H}^k(\overline{\mathcal{M}}_H) \end{array}$$

By Proposition 3.15 of [6], if $0 \ne \alpha \in \mathrm{H}^k(\overline{\mathcal{M}})$ we have that $\overline{\mathrm{Res}}(\alpha) \ne 0$ if $\alpha \cdot e^{n-m} \ne 0$ where e is the canonical Lefschetz class coming from the Baily-Borel projective embedding of M^*_Γ. Since $\mathrm{Gr}^W_k \mathrm{H}^k_c(\mathcal{M}) = \mathrm{im}(\mathrm{H}^i_c(\mathcal{M}) \to \mathrm{IH}^i(\mathcal{M}^*))$ is stable under $\cdot e$ and $\cdot e$ has the hard Lefschetz property on $\mathrm{IH}^*(\mathcal{M}^*)$, we conclude that $\overline{\mathrm{Res}}$ is injective on the subspace $\mathrm{Gr}^W_k \mathrm{H}^k_c(\mathcal{M})$ for $k \le m$.

Now consider the sequence

$$\cdots \longrightarrow \mathrm{H}^{k-1}(\mathcal{L}) \longrightarrow \mathrm{H}^k_c(\mathcal{M}) \longrightarrow \mathrm{H}^k(\mathcal{M}) \longrightarrow \cdots$$

In degrees $k \le n$ the first term is pure of weight $k-1$ (by Lemma 1.3), so taking Gr^W_k gives an injection $\mathrm{Gr}^W_k \mathrm{H}^k_c(\mathcal{M}) \hookrightarrow \mathrm{H}^k(\mathcal{M})$. Since $\mathrm{H}^k_!(\mathcal{M})$ is pure of weight k (indeed, this follows from the definition because $\mathrm{H}^k_c(\mathcal{M})$ has weights $\le k$ and $\mathrm{H}^k(\mathcal{M})$ has weights $\ge k$), it follows that

$$\mathrm{Gr}^W_k \mathrm{H}^k_c(\mathcal{M}) = \mathrm{H}^k_!(\mathcal{M}) \text{ for } k \le n \tag{1.4}$$

and the proposition follows. □

Remark 1.7. On $\mathrm{H}^{m,0}(\mathcal{M}) \oplus \mathrm{H}^{0,m}(\mathcal{M})$ injectivity follows from the criterion of [6, Prop. 3.15] and the fact that the map $\mathrm{H}^{m,0}(\overline{\mathcal{M}}) \to \mathrm{H}^{m,0}(\mathcal{M})$ is an isomorphism (by mixed Hodge theory [3]).

Given Proposition 1.6 and the diagram (1.3), Theorem 1.1 follows from:

Proposition 1.8. *The map* $\mathrm{Res}_\infty : \mathrm{H}^k(\mathcal{L}) \to I^G_H \mathrm{H}^k(\mathcal{L}_H)$ *is injective for* $k < m$.

This is proved in §1.6, after recalling Kostant's description of $\mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}})$ in §1.4 and §1.5.

Proposition 1.8 also implies:

Corollary 1.9. *If* $\ell \subset W$ *then the map* $\mathrm{Res} : I^G_P(\wedge^k \mathfrak{v}^*_{\ell,\mathbb{C}}) \to I^G_H I^H_{P_H}(\wedge^k \mathfrak{v}^*_{\ell,H,\mathbb{C}})$ *is injective for* $k < m$.

Proof. For $k < m$ there is a commutative diagram (by Lemma 1.3)

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathfrak{u}^*_{\ell,\mathbb{C}} \otimes \wedge^{k-2}\mathfrak{v}^*_{\ell,\mathbb{C}} & \longrightarrow & \wedge^k \mathfrak{v}^*_{\ell,\mathbb{C}} & \longrightarrow & \mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}}) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & I^P_{P_H}(\mathfrak{u}^*_{\ell,\mathbb{C}} \otimes \wedge^{k-2}\mathfrak{v}^*_{\ell,H,\mathbb{C}}) & \longrightarrow & I^P_{P_H}(\wedge^k \mathfrak{v}^*_{\ell,H,\mathbb{C}}) & \longrightarrow & I^P_{P_H} \mathrm{H}^k(\mathfrak{w}_{\ell,H,\mathbb{C}}) & \longrightarrow 0 \end{array} \tag{1.5}$$

By Proposition 1.8 the third vertical map is injective for all $k < m$. An induction on k gives that the first two vertical maps are injective for $k < m$. This proves the corollary since $I_P^G \, I_{P_H}^P = I_H^G \, I_{P_H}^H$. □

Remark 1.10. This fixes a minor error in the proof of Theorem 3.17 of [6], the injectivity of $\overline{\mathrm{Res}} : \mathrm{H}^i(\overline{\mathcal{M}}) \to I_H^G \, \mathrm{H}^i(\overline{\mathcal{M}}_H)$ for $i \leq m$. The proof there (correctly) reduces the theorem to the statement that $\wedge^{i-2}\mathfrak{v}_{\ell,\mathbb{C}}^* \to I_{P_H}^P \wedge^{i-2} \mathfrak{v}_{\ell,H,\mathbb{C}}^*$ is injective for $i \leq m$. This follows from the corollary since $i-2 < m$. (The error in [6] is in the proof of the injectivity of $\wedge^{i-2}\mathfrak{v}_{\ell,\mathbb{C}}^* \to I_{P_H}^P \wedge^{i-2} \mathfrak{v}_{\ell,H,\mathbb{C}}^*$: It occurs on the line after formula (3.9) of loc. cit., where we assert that the exterior powers $\wedge^{i-2}\mathfrak{v}_\ell^*$ are irreducible because $\mathfrak{v}_\ell^*$ is the standard representation of $\mathrm{SU}(J_0)$. This is true, but we actually need to look at $\mathfrak{v}_{\ell,\mathbb{C}}^* = (\mathfrak{v}_{\ell,\mathbb{C}}^*)^{1,0} + (\mathfrak{v}_{\ell,\mathbb{C}}^*)^{0,1}$, which is a sum of two irreducibles, and its exterior powers.)

1.4. **Kostant's theorem.** We recall results of [5]. Fix a complex semisimple Lie group $G_\mathbb{C}$, maximal torus $T \subset G_\mathbb{C}$ and Borel subgroup $B \supset T$, and let ρ denote the half-sum of positive roots and let $W = W(T, G_\mathbb{C})$ be the Weyl group of T in $G_\mathbb{C}$. Let $P = LN$ be the standard Levi decomposition of a standard parabolic subgroup of $G_\mathbb{C}$ and let $\mathfrak{n}$ be the Lie algebra of N. The Weyl group of L is a subgroup $W_L \subset W$, and we let W^P be the set of minimal length coset representatives of $W_L \backslash W$. For each $w \in W^P$ there is the associated set of positive roots

$$\Phi(w) = \{\alpha \in \Phi(T, G_\mathbb{C}) : \alpha > 0, w^{-1}\alpha < 0\}$$

which has cardinality $\ell(w)$ (the length of w). For $w \in W^P$ the weights $w(\rho) - \rho$ are dominant for L and distinct.

The Lie algebra cohomology $\mathrm{H}^*(\mathfrak{n}) = \mathrm{H}^*(\mathfrak{n}, \mathbb{C})$ is the cohomology of $\wedge^*\mathfrak{n}^*$ with the Lie algebra differential. The natural P-module structure on $\wedge^*\mathfrak{n}^*$ descends to an $L \cong P/N$-module structure in cohomology. For a dominant weight $\mu \in X^*(T)$ let E_μ^L denote the irreducible finite-dimensional algebraic representation of L with highest weight μ. Then by [5, Theorem 5.14] there is a multiplicity-free decomposition of L-modules

$$\mathrm{H}^k(\mathfrak{n}) = \bigoplus_{w \in W^P, \ell(w) = k} E_{w(\rho)-\rho}^L. \tag{1.6}$$

Kostant also identified a highest weight vector in each summand above. Let $\mathfrak{n}^-$ be the nilradical of the Lie algebra of the parabolic subgroup opposite to P. The Killing form gives isomorphisms $\mathfrak{n}^- \cong \mathfrak{n}^*$ and hence $\wedge^i\mathfrak{n}^- \cong \wedge^i\mathfrak{n}^*$. For $w \in W^P$ choose a vector $e_{-\phi}$ in the root space of $-\phi$ for each $\phi \in \Phi(T, G_\mathbb{C})$ and form

$$e_w := \wedge_{\phi \in \Phi(w)} e_{-\phi} \in \wedge^{\ell(w)}\mathfrak{n}^-. \tag{1.7}$$

Under the identification of $\mathfrak{n}^*$ with $\mathfrak{n}^-$ this form is closed and its cohomology class is a highest weight vector for the summand $E_{w(\rho)-\rho}^L$ in (1.6). (See [5, Theorem 5.14].) A lowest weight vector is given by $\wedge_{\phi \in \Phi(w)} e_{-w_0^L(\phi)}$, where w_0^L is the longest element of $W_L \subset W$ and we choose. (See [5, Remark 8.2].) In fact, taking the sum of the L-submodules of $\wedge^*\mathfrak{n}^-$ generated by the e_w as w runs over W^P gives (using the identification $\wedge^*\mathfrak{n}^* \cong \wedge^*\mathfrak{n}^-$) a canonical L-equivariant inclusion $\mathrm{H}^*(\mathfrak{n}) \subset \wedge^*\mathfrak{n}^*$ inducing the identity in cohomology and compatible with products (see [5, Theorem 5.7]).

1.5. We now make §1.4 explicit for the situation of relevance to us. So $G = \mathrm{SU}(h)$ as in §1.1 and $P_\ell = \mathrm{Stab}_G(\ell)$ for an h-isotropic E-line $\ell \subset V$. This gives the complex group $G_\mathbb{C} = SL(V_\mathbb{R}) \cong SL(n+1, \mathbb{C})$ and $P_\ell(\mathbb{C})$ is the parabolic stabilizing the flag $\{0\} \subset \ell_\mathbb{R} \subset \ell_\mathbb{R}^\perp \subset V_\mathbb{R}$ of complex subspaces of dimension $0, 1, n-1, n$. We will be interested in $\mathfrak{w}_{\ell,\mathbb{C}}$-cohomology in degrees $k < n$. Before applying Kostant's results we summarize some facts about the structure of P_ℓ for use in §1.6 below.

Choose a nonzero vector $e_1 \in \ell$ and choose an isotropic vector $f \in V$ such that $U = Ee_1 + Ef$ is a hyperbolic plane in V. There is an orthogonal decomposition $V = U \perp V_0$ for a subspace $V_0 \subset V$ on which h is nondegenerate and anisotropic. As in [6, §1.3], let $e_{n+1} = \sqrt{d}\, f$ and choose a basis $e_2, \dots, e_n$ of V_0 in which $h|_{V_0}$ is diagonal. In the basis $e_1, \dots, e_{n+1}$ the form h is given by $h(v, w) = {}^t\bar{v}Jw$ for

$$J = \begin{pmatrix} & & \sqrt{d}^{-1} \\ & J_0 & \\ -\sqrt{d}^{-1} & & \end{pmatrix} \tag{1.8}$$

where J_0 is an anisotropic diagonal form in $n-1$ variables and $\bar{J}_0 = J_0$. This gives an identification $G = \mathrm{SU}(J)$ in which the parabolic subgroup $P_\ell = \mathrm{Stab}_G(\ell)$ consists of the block-upper-triangular matrices of block size $1, n-1, 1$. A Levi subgroup is given by

$$L_\ell = \left\{ g = \begin{pmatrix} \lambda_1 & & \\ & g_0 & \\ & & \lambda_{n+1} \end{pmatrix} : \begin{array}{c} \bar{\lambda}_1 \lambda_{n+1} = 1 \\ g_0 \in \mathrm{U}(J_0) \\ \det(g) = 1 \end{array} \right\} \subset R_{E/\mathbb{Q}}\mathbb{G}_m \times \mathrm{U}(J_0)$$

and this contains the obvious subgroup $\mathrm{SU}(J_0)$. (We will confuse the Levi quotient and the Levi subgroup L_ℓ below as it will not matter.)

Now we apply Kostant's theorem in §1.4 to the situation $G_\mathbb{C} = G(\mathbb{C}), P = P_\ell(\mathbb{C})$, and $\mathfrak{n} = \mathfrak{w}_{\ell,\mathbb{C}}$. The choice of basis $e_1, \dots, e_{n+1}$ fixes an identification $G(\mathbb{C}) = SL(V_\mathbb{R}) = SL(n+1, \mathbb{C})$. Fix the Borel $B \subset G(\mathbb{C})$ of upper triangular matrices (with respect to which $P_\ell(\mathbb{C})$ is standard) and the maximal torus of diagonal matrices

$$T = \{t = diag(t_1, \dots, t_{n+1}) : t_1 \dots t_{n+1} = 1\}.$$

The Weyl group $W = W(T, G(\mathbb{C}))$ is the symmetric group $\mathfrak{S}_{n+1}$ on $n+1$ letters, acting on T by permutations of the t_i. A set of simple positive roots of T in $G(\mathbb{C})$ is given by

$$\alpha_1(t) = t_1/t_2, \quad \alpha_2(t) = t_2/t_3, \quad \dots, \quad \alpha_n(t) = t_n/t_{n+1}$$

and W is generated by the n reflections $s_1, \dots, s_n$ where s_i is the reflection in $\alpha_i = t_i/t_{i+1}$, i.e. exchanges t_i and t_{i+1}. The Weyl group W_{L_ℓ} of the Levi $L_\ell(\mathbb{C})$ is the symmetric group on $n-1$ letters generated by $s_2, \dots, s_{n-1}$. The set W^{P_ℓ} of minimal length representatives for the cosets $W_{L_\ell}\backslash W$ is easily listed: For each $k < n$ there are $k+1$ elements of W^{P_ℓ} of length k, namely:

$$\begin{aligned} w_{k,k} &= s_1 s_2 \cdots s_k \\ w_{k,k-1} &= s_1 s_2 \cdots s_{k-1} s_n \\ &\vdots \\ w_{k,1} &= s_1 s_n \cdots s_{n-(k-2)} \\ w_{k,0} &= s_n s_{n-1} \cdots s_{n-(k-1)} \end{aligned} \tag{1.9}$$

Note that since $k < n$ there is always a gap in the indices, so that the "first part" of the word $w_{k,r}$ (i.e. $s_1 \cdots s_r$) and the "second part" (i.e. $s_n s_{n-1} \cdots s_{n-(k-r-1)}$) commute. (We will not need the elements of length $k \geq n$, but we note that they are $\{w_0^{L_\ell} w_{2n-1-k,r} w_0 : 0 \leq r \leq 2n-1-k\}$ where w_0 (resp. $w_0^{L_\ell}$) is the longest element of W (resp. of W_{L_ℓ}).)

The sets of positive roots $\Phi(w)$ are easily enumerated for the elements of W^{P_ℓ} with length $< n$: If $w_{k,r} = s_1 \dots s_r s_n \dots s_{n-(k-r-1)}$ is a length k representative where $0 \leq r \leq k < n$, then

$$\Phi(w_{k,r}) = \Phi_1(w_{k,r}) \sqcup \Phi_2(w_{k,r}) \tag{1.10}$$

where

$$\Phi_1(w_{k,r}) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_r\} \tag{1.11}$$

$$\Phi_2(w_{k,r}) = \{\alpha_{n-(k-r-1)} + \cdots + \alpha_n, \ldots, \alpha_{n-1} + \alpha_n, \alpha_n\}. \tag{1.12}$$

(By convention $\Phi_2(w_{k,k}) = \Phi_1(w_{k,0}) = \emptyset$.) Kostant's theorem gives an L_ℓ-isotypic decomposition

$$\mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}}) = \bigoplus_{0 \leq r \leq k} E^{L_\ell}_{w_{k,r}(\rho)-\rho} \tag{1.13}$$

for $k < n$.

Remark 1.11. The decomposition (1.13) is the Hodge decomposition of the pure Hodge structure on $\mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}})$ coming from geometry (see Remark 1.4). For $k < n$ this holds because the former is multiplicity-free for L_ℓ with $k+1$ summands and the latter is preserved by L_ℓ and also has $k+1$ summands. By duality the same holds for $k \geq n$. (It is easily checked that the summand $E^{L_\ell}_{w_{k,r}(\rho)-\rho}$ in (1.13) is the $(r, k-r) = (|\Phi_1(w_{k,r})|, |\Phi_2(w_{k,r})|)$ Hodge space.) In particular, the Hodge spaces are irreducible representations of the Levi quotient.

1.6. **Proof of Proposition 1.8.** We now put ourselves in the setting of §1.1. Thus we are given a subspace $W \subset V$ of dimension $m+1$ on which h is indefinite and nondegenerate and the corresponding embedding $H = \mathrm{SU}(h|_W) \subset \mathrm{SU}(h) = G$. Fix an isotropic E-line $\ell \subset W$ so that $\mathrm{H}^*(\mathcal{L}) = I^G_{P_\ell}\mathrm{H}^*(\mathfrak{w}_{\ell,\mathbb{C}})$ and $\mathrm{H}^*(\mathcal{L}_H) = I^G_{P_{\ell,H}}\mathrm{H}^*(\mathfrak{w}_{\ell,H,\mathbb{C}})$.

Choose an E-basis $e_1, \ldots, e_{n+1}$ of V as in §1.5. We may assume that W is spanned by $e_1, e_2, \ldots, e_m$ and e_{n+1}. (Indeed, in the choice of basis in §1.5 choose $f \in W$, so that $U = Ee_1 + Ef$ is a hyperbolic subspace of W and let $e_{n+1} = \sqrt{d}\, f$. Then $W = U \perp W_0$ and $V = U \perp V_0$ for h-definite spaces $W_0 \subset V_0$. Now choose $e_2, \ldots, e_m$ spanning W_0 and extend to a basis for V_0.) In the identification $V = E\, e_1 \oplus \cdots \oplus E\, e_{n+1}$ we will consider the $m+1$-dimensional subspaces

$$W_s = (E\, e_1 \oplus \cdots \oplus E\, e_s) \oplus (E\, e_{s+n-m+1} \oplus \cdots \oplus E\, e_{n+1})$$

for $s = 1, \ldots, m$. (So $W_m = W$.) For each of these one has a subgroup

$$H_s = \mathrm{SU}(h|_{W_s}) \subset G$$

using the standard embedding, i.e. H_s is the identity on $W_s^\perp = E\, e_{s+1} \oplus \cdots \oplus E\, e_{s+n-m}$. There are elements $g_1, \ldots, g_m = e \in \mathrm{SU}(J_0)(\mathbb{R}) \subset L_\ell(\mathbb{R})$ such that $W_s = g_s W_m = g_s W$ and hence $H_s = g_s H g_s^{-1}$. (For this note that over $\mathbb{R}$ we can choose the basis of V_0 so that J_0 is the identity matrix. For $2 \leq i, j \leq n$ it is easy to write an element of $\mathrm{SU}(J_0)(\mathbb{R})$ sending e_i to e_j and e_j to $-e_i$, and which is the identity on the other e_k. Taking suitable products of such elements gives the g_s. Note that the determinant $\det(J_0) \in \mathbb{Q}^\times / N_{E/\mathbb{Q}} E^\times$ is an obstruction to doing this with $g_s \in L_\ell(\mathbb{Q})$.)

For each $1 \leq s \leq m$ a maximal torus in $H_s(\mathbb{C}) = SL(W_{s,\mathbb{R}})$ is given by

$$T_{H_s} = \{t \in T : t_{s+1} = \cdots = t_{s+n-m} = 1\}.$$

A set of simple roots for T_{H_s} in $H_s(\mathbb{C})$ is

$$\Big\{\alpha_i^{H_s} = t_i/t_{i+1}\Big\}_{1 \leq i \leq s-1} \sqcup \big\{\beta^{H_s} := t_s/t_{s+n-m+1}\big\} \sqcup \Big\{\alpha_i^{H_s} = t_i/t_{i+1}\Big\}_{s+n-m+1 \leq i \leq n}.$$

For each s the Weyl group $W_{H_s} \cong \mathfrak{S}_{m+1}$ is generated by

$$s_1^{H_s}, \ldots, s_{s-1}^{H_s}, \sigma_\beta^{H_s}, s_{s+n-m+1}^{H_s}, \ldots, s_n^{H_s} \tag{1.14}$$

where $s_i^{H_s} = s_i$ exchanges t_i and t_{i+1} and $\sigma_\beta^{H_s}$ exchanges t_s and $t_{s+n-m+1}$. For the parabolic $P_{\ell,H_s} = \mathrm{Stab}_{H_s}(\ell)$ and its Levi L_{ℓ,H_s} as above, the subgroup $W_{L_{\ell,H_s}} \cong \mathfrak{S}_{m-1}$ is generated by the same set of transpositions omitting $s_1^{H_s}$ and $s_n^{H_s}$. The set of minimal length coset representatives for $W_{L_{\ell,H_s}} \backslash W_{H_s}$ is denoted $W^{P_{\ell,H_s}}$ and for each $k < m$ there are $k+1$ representatives of length k, enumerated as in §1.5.

Now suppose that $k < m$ and $w_{k,r} \in W^{P_\ell}$ is a minimal length coset representative as in §1.5, for some $0 \le r \le k$. The roots in $\Phi(w_{k,r})$ are sums of simple roots in

$$\{\alpha_1, \dots, \alpha_r\} \sqcup \{\alpha_{n-(k-r)+1}, \dots, \alpha_n\}. \tag{1.15}$$

Consider the subgroup H_{r+1} above, i.e. H_s for $s = r+1$. On $T_{H_{r+1}}$ one has $\alpha_i|_{T_{H_{r+1}}} = \alpha_i^{H_{r+1}}$ for $i \le r$ and for $i \ge r+n-m+2$, i.e. $i > r+n-m+1$. Since $k < m$, we have $n-(k-r)+1 = n-k+r+1 > n-m+r+1$. So we have that

$$\alpha_i|_{T_{H_{r+1}}} = \alpha_i^{H_{r+1}} \quad \text{ if } \alpha_i \text{ appears in (1.15)}. \tag{1.16}$$

(Of course, if k is small there will be other H_s with this property too.)

Now consider the irreducible summand of $\mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}})$ in (1.13) corresponding to $w_{k,r}$ for some $0 \le r \le k < m$. A highest weight vector is given by

$$e_{w_{k,r}} = \wedge_{\phi \in \Phi(w_{k,r})} e_{-\phi}$$

where we have chosen a root vector $e_{-\phi}$ for each absolute root. For each s, every root space of H_s is also a root space of G, so we will choose the same vector $e_{-\phi}^{H_s} := e_{-\phi}$ to span it. Recall that $w_{k,r} = s_1 \cdots s_r s_n s_{n-1} \cdots s_{n-(k-r-1)}$. Let

$$w_{k,r}^{H_{r+1}} := s_1^{H_{r+1}} \cdots s_r^{H_{r+1}} s_n^{H_{r+1}} s_{n-1}^{H_{r+1}} \cdots s_{n-(k-r-1)}^{H_{r+1}}.$$

This obviously belongs to $W^{P_{\ell,H_{r+1}}}$ and has length k. (Note that since $k < m$ we have $n-(k-r-1) > n-m+r+1$ and so if $s = r+1$ this is $\ge s+n-m+1$, so that the reflection $\sigma_\beta^{H_s}$ in (1.14) does not appear.) The corresponding summand of $\mathrm{H}^k(\mathfrak{w}_{\ell,H_{r+1}})$ has highest weight vector $e^{H_{r+1}}_{w_{k,r}^{H_{r+1}}}$. With this notation, one has:

Lemma 1.12. *For $k < m$ and $0 \le r \le k$ the restriction of $e_{w_{k,r}} \in \mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}})$ to $\mathfrak{w}_{\ell,H_{r+1},\mathbb{C}}$ is $e^{H_{r+1}}_{w_{k,r}^{H_{r+1}}} \in \mathrm{H}^k(\mathfrak{w}_{\ell,H_{r+1},\mathbb{C}})$.*

Proof. Under the identification given by Killing forms, for a root ϕ of H_{r+1}, the restriction of $e_{-\phi}$ to $\mathfrak{w}^*_{\ell,H_{r+1},\mathbb{C}}$ is $e_{-\phi}^{H_{r+1}}$. The lemma then follows from (1.16). □

We will now finish the proof of Proposition 1.8 by showing that

$$\mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}}) \longrightarrow I_{P_{\ell,H}}^{P_\ell} \mathrm{H}^k(\mathfrak{w}_{\ell,H,\mathbb{C}})$$

is injective for $k < m$. (This suffices by Lemma 1.2.) The $W_\ell(\mathbb{Q})$-action on $\mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}})$ is trivial, so that this map factors through $\left(I_{P_{\ell,H}}^{P_\ell} \mathrm{H}^k(\mathfrak{w}_{\ell,H,\mathbb{C}})\right)^{W_\ell(\mathbb{Q})} = I_{L_{\ell,H}}^{L_\ell} \mathrm{H}^k(\mathfrak{w}_{\ell,H,\mathbb{C}})$, i.e. we consider the map of $L_\ell(\mathbb{Q})$-modules

$$\mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}}) \longrightarrow I_{L_{\ell,H}}^{L_\ell} \mathrm{H}^k(\mathfrak{w}_{\ell,H,\mathbb{C}}). \tag{1.17}$$

Now since $\mathrm{H}^k(\mathfrak{w}_{\ell,\mathbb{C}})$ is an algebraic representation of $L_\ell(\mathbb{C})$, and $L_\ell(\mathbb{Q})$ is Zariski-dense in $L_\ell(\mathbb{C})$, the kernel of (1.17) is an $L_\ell(\mathbb{C})$-submodule. Since the decomposition into irreducibles (1.13) is L_ℓ-isotypic, it is enough to prove that (1.17) is nonzero on each irreducible summand for $k < m$, i.e. to show that for each such summand the restriction to $\mathfrak{w}_{\ell,H',\mathbb{C}}$ for some conjugate H' of H is nonzero. This follows from the lemma: For the summand $E^{L_\ell}_{w_{k,r}(\rho)-\rho}$ and $H' = H_{r+1} = g_{r+1} H g_{r+1}^{-1}$ the highest weight vector $e_{w_{k,r}}$ is nonzero on restriction to $\mathfrak{w}_{\ell,H_{r+1},\mathbb{C}}$. □

This completes the proof of Proposition 1.8 and proves Theorem 1.1.

Remark 1.13. When $i = m$ there are, in general, classes in $\mathrm{H}^m(\mathcal{M})$ (recall that the dimension of $\mathcal{M}$ is $n > m$) which restrict nontrivially to the boundary. Constructing such classes is a nontrivial problem since they are necessarily square-integrable, and therefore, since $m < n$, residual. (For some explicit examples when $n = 2$ and $i = 1$, i.e. noncuspidal square-integrable classes on Picard modular surfaces which survive at infinity, see §3.3 of Harder's paper [4].) By weights, $\mathrm{Res}_\infty = 0$ in degree $i = m$ (cf. Remark 1.4), so the question of whether these classes survive under Res is a global problem and cannot be treated locally on a compactification by geometric methods.

2. Proof of Theorem 2

2.1. **Cup products.** Recall from [6, 3.7] that the cup product gives a map

$$\mathrm{Cup} : \mathrm{H}^i(\mathcal{M}) \otimes \mathrm{H}^j(\mathcal{M}) \longrightarrow I_{\Delta G}^{G\times G}\, \mathrm{H}^{i+j}(\mathcal{M})$$

where ΔG is the diagonal copy of G in $G \times G$. (This was denoted Res in loc. cit..) Theorem 2 of the introduction is implied by:

Theorem 2.1. *If $i+j < n$ and $0 \neq \alpha\otimes\beta \in \mathrm{H}^i(\mathcal{M})\otimes\mathrm{H}^j(\mathcal{M})$ then $\mathrm{Cup}(\alpha\otimes\beta) \neq 0$. If $i + j = n$ and $0 \neq \alpha \otimes \beta \in \mathrm{H}^i_!(\mathcal{M}) \otimes \mathrm{H}^j(\mathcal{M})$ then $\mathrm{Cup}(\alpha \otimes \beta) \neq 0$.*

This will follow from two propositions which will be proved below.

Proposition 2.2. *If $i+j \leq n$ and $0 \neq \alpha\otimes\beta \in \mathrm{H}^i_!(\mathcal{M})\otimes\mathrm{H}^j(\mathcal{M})$ then $\mathrm{Cup}(\alpha\otimes\beta) \neq 0$.*

Proof. Restricting the cup product on $\mathrm{H}^*(\overline{\mathcal{M}})$ gives a map

$$\mathrm{Cup} : \mathrm{Gr}^W_i \mathrm{H}^i_c(\mathcal{M}) \otimes \mathrm{H}^j(\overline{\mathcal{M}}) \longrightarrow I_{\Delta G}^{G\times G} \mathrm{Gr}^W_{i+j}\mathrm{H}^{i+j}_c(\mathcal{M}).$$

By the same argument used in the proof of Theorem 3.21 of [6], one has that if $0 \neq \alpha\otimes\beta \in \mathrm{Gr}^W_i \mathrm{H}^i_c(\mathcal{M})\otimes\mathrm{H}^j(\overline{\mathcal{M}})$ with $i+j \leq n$, then $\mathrm{Cup}(\alpha\otimes\beta) \neq 0$. By (1.4) we are done since $\mathrm{H}^j(\overline{\mathcal{M}}) \twoheadrightarrow \mathrm{H}^j(\mathcal{M})$ for $j < n$ (this is because $\mathrm{H}^j(\mathcal{M})$ is pure of weight j for $j < n$ and by mixed Hodge theory [3] the pure weight part of the cohomology of a smooth variety comes from any smooth compactification). (The case $j = n$ of the proposition is true trivially.) ☐

The cup product in the cohomology of the link induces a map

$$\mathrm{Cup}_\infty : \mathrm{H}^i(\mathcal{L}) \otimes \mathrm{H}^j(\mathcal{L}) \longrightarrow I_{\Delta G}^{G\times G}\, \mathrm{H}^{i+j}(\mathcal{L}).$$

The following will be proved in §2.2 below.

Proposition 2.3. *If $i + j < n$ then Cup_∞ is injective on $\mathrm{H}^i(\mathcal{L}) \otimes \mathrm{H}^j(\mathcal{L})$.*

Theorem 2.1 now follows by considering the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}^i_!(\mathcal{M}) \otimes \mathrm{H}^j(\mathcal{M}) & \longrightarrow & \mathrm{H}^i(\mathcal{M}) \otimes \mathrm{H}^j(\mathcal{M}) & \xrightarrow{r\times id} & \mathrm{H}^i(\mathcal{L}) \otimes \mathrm{H}^j(\mathcal{M}) \\
& & & & & \downarrow{\scriptstyle id\times r} \\
& \downarrow & & \downarrow{\scriptstyle \mathrm{Cup}} & & \mathrm{H}^i(\mathcal{L}) \otimes \mathrm{H}^j(\mathcal{L}) \\
& & & & & \downarrow{\scriptstyle \mathrm{Cup}_\infty} \\
0 \longrightarrow & I_{\Delta G}^{G\times G}\, \mathrm{H}^k_!(\mathcal{M}) & \longrightarrow & I_{\Delta G}^{G\times G}\, \mathrm{H}^k(\mathcal{M}) & \longrightarrow & I_{\Delta G}^{G\times G}\, \mathrm{H}^k(\mathcal{L})
\end{array} \tag{2.1}$$

with exact rows. Suppose $0 \neq \alpha\otimes\beta \in \mathrm{H}^i(\mathcal{M})\otimes\mathrm{H}^j(\mathcal{M})$ with $i+j < n$. By Proposition 2.2 we may assume that $r(\alpha) \otimes \beta \neq 0$ in $\mathrm{H}^i(\mathcal{L}) \otimes \mathrm{H}^j(\mathcal{M})$. By Proposition 2.3 we may assume that $r(\alpha) \otimes r(\beta) = 0$, hence $r(\alpha) \otimes \beta \in \mathrm{H}^i(\mathcal{L}) \otimes \mathrm{H}^j_!(\mathcal{M})$, and hence $\alpha \otimes \beta \in \mathrm{H}^i(\mathcal{M}) \otimes \mathrm{H}^j_!(\mathcal{M})$. By Proposition 2.2, $\mathrm{Cup}(\alpha \otimes \beta) = \pm\mathrm{Cup}(\beta \otimes \alpha) \neq 0$.

2.2. **Proof of Proposition 2.3.** We will use Kostant's theorem to prove that

$$\mathrm{H}^i(\mathfrak{w}_{\ell,\mathbb{C}}) \otimes \mathrm{H}^j(\mathfrak{w}_{\ell,\mathbb{C}}) \longrightarrow I_{\Delta P_\ell}^{P_\ell \times P_\ell} \mathrm{H}^{i+j}(\mathfrak{w}_{\ell,\mathbb{C}})$$

is injective if $i + j < n$. This implies Proposition 2.3 since $\mathrm{H}^i(\mathcal{L}) = I_{P_\ell}^G \mathrm{H}^i(\mathfrak{w}_{\ell,\mathbb{C}})$.

The decomposition of $\mathrm{H}^i(\mathfrak{w}_{\ell,\mathbb{C}}) \otimes \mathrm{H}^j(\mathfrak{w}_{\ell,\mathbb{C}})$ is $L_\ell \times L_\ell$-isotypic, so it is enough to show that the cup product $\mathrm{H}^i(\mathfrak{w}_{\ell,\mathbb{C}}) \otimes \mathrm{H}^j(\mathfrak{w}_{\ell,\mathbb{C}}) \to I_{\Delta P_\ell}^{P_\ell \times P_\ell} \mathrm{H}^{i+j}(\mathfrak{w}_{\ell,\mathbb{C}})$ is nonzero on each irreducible summand if $i + j < n$. Let

$$E^{L_\ell}_{w_{i,r}(\rho)-\rho} \otimes E^{L_\ell}_{w_{j,s}(\rho)-\rho}$$

be one such. (The notation is as in §1.5.) The product is induced by the exterior product in $\wedge^* \mathfrak{w}^*_{\ell,\mathbb{C}} \cong \wedge^* \mathfrak{w}^-_{\ell,\mathbb{C}}$.

The longest element w_0^L of $W_{L_\ell} \subset W$ is the permutation which acts on T by leaving t_1, t_{n+1} fixed and by $t_i \mapsto t_{n+2-i}$ for $2 \leq i \leq n$. Then:

$$w_0^L(\alpha_1 + \cdots + \alpha_a) = \alpha_1 + \cdots + \alpha_{n-a} \qquad \text{for } 1 \leq a < n \tag{2.2}$$

$$w_0^L(\alpha_{n-b+1} + \cdots + \alpha_n) = \alpha_{b+1} + \cdots + \alpha_n \qquad \text{for } 1 \leq b < n. \tag{2.3}$$

Lemma 2.4. *If $i + j < n$ then $\Phi(w_{i,r}) \cap w_0^L(\Phi(w_{j,s})) = \emptyset$.*

Proof. Recall that $\Phi(w_{i,r}) = \Phi_1(w_{i,r}) \sqcup \Phi_2(w_{i,r})$ as in §1.5, and similarly for $w_{j,s}$. Note that since $i + j = r + (i - r) + s + (j - s)$ we have that $r + s < n$ and $(i-r)+(j-s) < n$. The fact that $r+s < n$ implies that $\Phi_1(w_{i,r}) \cap w_0^L(\Phi_1(w_{j,s})) = \emptyset$ by (2.2). Similarly, $(i - r) + (j - s) < n$ implies that $\Phi_2(w_{i,r}) \cap w_0^L(\Phi_2(w_{j,s})) = \emptyset$ by (2.3). Furthermore, $\Phi_1(w_{i,r}) \cap w_0^L \Phi_2(w_{j,s}) = \emptyset$ since by (2.3) any element of $w_0^L \Phi_2(w_{j,s})$ involves α_n, while no element of $\Phi_1(w_{i,r})$ does so. Similarly, $\Phi_2(w_{i,r}) \cap w_0^L \Phi_1(w_{j,s}) = \emptyset$ using (2.2). The lemma follows. □

Recall that the lowest weight vector in $E^{L_\ell}_{w_{j,s}(\rho)-\rho}$ is $f := \wedge_{\phi \in \Phi(w_{j,s})} e_{-w_0^L(\phi)} \in \wedge^j \mathfrak{w}^-_{\ell,\mathbb{C}}$. By the lemma $e_{w_{i,r}} \wedge f \neq 0$ in $\wedge^{i+j} \mathfrak{w}^-_{\ell,\mathbb{C}}$. It follows that $e_{w_{i,r}} \cdot f \neq 0$ in cohomology.

This proves the proposition and completes the proof of Theorem 2.1.

Remark 2.5. By the isomorphism $\mathrm{H}^{n,0}(\overline{\mathcal{M}}) \cong \mathrm{H}^{n,0}(\mathcal{M})$ of Remark 1.7, if $0 \neq \alpha \otimes \beta \in \mathrm{H}^i(\mathcal{M}) \otimes \mathrm{H}^{n-i}(\mathcal{M})$ and α, β are both holomorphic (or both anti-holomorphic) then $\mathrm{Cup}(\alpha \otimes \beta) \neq 0$.

Remark 2.6. If $i+j = n$ then the cup product $\mathrm{H}^i(\mathcal{L}) \otimes \mathrm{H}^j(\mathcal{L}) \to \mathrm{H}^{i+j=n}(\mathcal{L})$ vanishes by weights (the source has weight $i + j = n$ and the target has weight $n + 1$, see Remark 1.4). So improving Theorem 2.1 would require a global argument. However, I expect Theorem 2.1 is optimal.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400005, India
E-mail address: `arvind@math.tifr.res.in`