

LEFSCHETZ PROPERTIES FOR NONCOMPACT ARITHMETIC BALL QUOTIENTS

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ABSTRACT. We prove a Lefschetz property for restriction of the cohomology of noncompact congruence ball quotients to ball quotients of smaller dimension.

INTRODUCTION

Fix an imaginary quadratic number field $E \subset \mathbb{C}$ and a vector space V of dimension $n + 1 \geq 3$ over E . Let $h : V \times V \rightarrow E$ be a Hermitian form with respect to the conjugation of E/\mathbb{Q} such that $h \otimes_{\mathbb{Q}} \mathbb{R}$ is of signature $(n, 1)$ on $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^{n+1}$. A congruence arithmetic group Γ in the isometry group $SU(h)$ acts properly discontinuously on the unit ball \mathbb{B} in \mathbb{C}^n and the quotient

$$M = \Gamma \backslash \mathbb{B}$$

is a quasiprojective variety defined over an abelian extension of E . The variety M contains subvarieties of a similar kind: An E -rational subspace $W \subset V$ of dimension $m + 1$ on which h is nondegenerate and indefinite gives a \mathbb{Q} -subgroup $H = SU(h|_W) \subset G$. For $\Gamma_H = \Gamma \cap H$ we have a ball quotient $M_H = \Gamma_H \backslash \mathbb{B}_H$ where \mathbb{B}_H is the unit ball in \mathbb{C}^m . There is a morphism of varieties

$$M_H \longrightarrow M$$

which is finite onto a closed subvariety of codimension $n - m$. In particular, if $\dim W = m + 1 = n$ then $M_H \rightarrow M$ is finite onto a divisor.

In [24] Oda proved an injectivity statement for restriction of cohomology in degree one to subvarieties like M_H . He showed that there are finitely many subspaces W_1, \dots, W_s of the type above and with dimension 2 such that for $H_i = SU(h|_{W_i})$, the direct sum of pullback maps in degree one $H^1(M, \mathbb{Q}) \rightarrow \bigoplus_j H^1(M_{H_j}, \mathbb{Q})$ to the s modular curves M_{H_1}, \dots, M_{H_s} is injective. He raised the natural question of what happens in higher degrees; this is (partially) answered by the following:

Theorem 0.1. *There exist subspaces W_1, \dots, W_s in V of dimension n such that*

$$H^i(M, \mathbb{Q}) \longrightarrow \bigoplus_j H^i(M_{H_j}, \mathbb{Q}) \tag{0.1}$$

is injective for $i \leq n - 3$. More generally, for any $m < n$ there are subspaces W_1, \dots, W_s of dimension $m + 1$ such that (0.1) is injective for $i \leq m - 2$.

I do not know what happens in degrees $n - 2$ and $n - 1$, or degrees $m - 1$ and m in the more general statement. (The theorem as stated does not quite cover Oda's result if $n = 2, 3$ but the method of proof does so.)

The theorem can be reformulated using Hecke correspondences. For $g \in G(\mathbb{Q})$ one has a finite correspondence $(a, b) : \Gamma \cap g^{-1}\Gamma g \backslash \mathbb{B} \rightrightarrows M$ on M where a is the covering map and

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b is induced by translation by g on \mathbb{B} . Write $C_g^* = a_*b^* : H^i(M, \mathbb{Q}) \rightarrow H^i(M, \mathbb{Q})$ for the induced endomorphism. Fix an E -rational subspace $U \subset V$ with $h|_U$ positive definite and let $W = U^\perp$, $H = SU(h|_W)$ and $M_H = \Gamma_H \backslash \mathbb{B}_H$. We show that for a nonzero class $\alpha \in H^i(M, \mathbb{Q})$ of degree $i \leq \dim W - 2$, there exists a $g \in G(\mathbb{Q})$ such that $C_g^*(\alpha)$ pulls back nontrivially to M_H (Thm 3.19). This implies the theorem above.

In the years since Oda's result [24], there has been a lot of work on Lefschetz properties (i.e. the injectivity of maps like (0.1)) for general arithmetic quotients, using both geometric and automorphic techniques [21, 31, 9, 15, 30, 4, 6, 5, 7]. (One motivation was the general conjecture of Langlands, Kottwitz, and Arthur on the Galois representations appearing in the cohomology of Shimura varieties, which is now largely proven for many Shimura varieties related to moduli problems, in particular, for those attached to unitary groups of the type we consider, by comparing trace formulas.) The work on Lefschetz properties following Oda's [24] has generally been for compact quotients, or for compactly supported (e.g. cuspidal) cohomology classes on noncompact quotients. (The exceptions to this rule I am aware of are [31], [4, §6], [5, §9].) In particular, no generalization of Oda's original result was known.

For *compact* ball quotients (e.g. those arising from a Hermitian form with respect to an imaginary quadratic extension E of a totally real field F which is of signature $(n, 1)$ at one real place of F and definite at the others), Venkataramana [30] proved the injectivity of (0.1) in degrees $i \leq n - 1$, confirming a conjecture of Harris and Li [15]. The essential point is that a linear combination of the divisors $M_{H_i} \rightarrow M$ gives a particular ample class, the hyperplane class in the canonical projective embedding of M . The Lefschetz property then follows from the hard Lefschetz theorem for the cohomology of M .

In the proof of Theorem 0.1 we use this idea of [30] from the compact case and combine it with the study of compactifications. The starting point is to note that if M^* is the minimal (i.e. Satake-Baily-Borel) compactification of M , which simply adds cusps, then $H^i(M) = IH^i(M^*)$ for $i \leq n - 1$ where $IH^i(M^*)$ is the i th intersection cohomology group ([12, 13]). The variety M also has a canonical smooth compactification \overline{M} which resolves the singularities of M^* . Elementary arguments using the explicit geometry of \overline{M} at infinity (cf. §2), or the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [2], can be used to show that there is a canonical and Hecke-equivariant decomposition

$$H^*(\overline{M}) = IH^*(M^*) \oplus J^*$$

where $IH^*(M^*)$ has its natural Hecke action (cf. Theorem 2.6 and the more detailed version Theorem 3.5 in which we identify the other summands explicitly as induced modules). This result is of some independent interest and should admit a generalization to all Shimura varieties (cf. Remark 3.7). Using this decomposition, classes in $H^i(M)$ for $i \leq n - 1$ can be considered as classes on \overline{M} in a canonical way. Theorem 0.1 is then proved by studying the Hecke-invariants in the summands in this decomposition and adapting the methods of [30] to the map $H^*(\overline{M}) \rightarrow H^*(\overline{M}_H)$. There is also a Lefschetz property for the cohomology of compactifications: There exist subspaces W_i of dimension $m + 1$ so that the map

$$H^i(\overline{M}) \longrightarrow \bigoplus_j H^i(\overline{M}_{H_j}, \mathbb{Q})$$

is injective in degrees $i \leq m$ (i.e. with no loss of degrees) (Theorem 3.17). Aside from one simple analytic input, namely the semisimplicity of $IH^*(M^*)$ as a Hecke-module (available via its relation to L^2 cohomology), the proofs of these results are entirely geometric (and

elementary). We note that this analytic input (Prop. 3.8) is the only place where it is necessary to assume that Γ is a congruence subgroup.

As in the compact case [30], the same methods give a nonvanishing result about cup products in $H^*(M, \mathbb{Q})$ (cf. Theorem 3.21): If $\alpha \in H^i(M_\Gamma), \beta \in H^j(M_\Gamma)$ are such that $i + j \leq n - 2$ then $\alpha \cdot C_g^*(\beta) \neq 0$ for some $g \in G(\mathbb{Q})$. Once again, I do not know what happens if the sum of degrees is $n - 1$ or n .

In [22] we study similar Lefschetz properties for a certain subspace of $H^*(M)$ in a more general setting, namely a locally symmetric “subvariety” of a general locally symmetric variety. Using slightly different methods we prove a weaker version of Theorem 0.1 for inclusions of the form $SO(2, n - 1) \subset SO(2, n)$. (In fact, a generalization of the decomposition theorem mentioned above to this setting (Remark 3.7) would allow the methods of this paper to be extended to that situation also.)

Bergeron has informed me that forthcoming joint work of his with Clozel would prove the spectral gap property (conjectured in [6]) for unitary groups, using the (expected) extension to unitary groups of Arthur’s endoscopic classification of automorphic representations (analogously to [7] for orthogonal groups). This would allow the application of the Burger-Sarnak method in the noncompact case by arguments sketched in [4, 5]. Presumably this would yield the optimal version of the Lefschetz property.

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1. BALL QUOTIENTS AND THEIR NATURAL COMPACTIFICATIONS

In this section we review the geometry (over the complex numbers) of the ball quotients we treat and their compactifications, specifically the minimal (i.e. Satake-Baily-Borel) compactification M_Γ^* and the smooth (toroidal, or in this case, toric) compactification \overline{M}_Γ . In doing so we are making the construction of [1] explicit in the case of ball quotients. The special case $n = 2$ (Picard modular surfaces) is also treated in [11, §§1,5] and [20, §§1,2] and shows the main geometric features. An elegant basis-free description of the general case (i.e. $n \geq 2$) is contained in [17, §4].

1.1. Arithmetic ball quotients. As in the introduction, $E = \mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$ is an imaginary quadratic field with ring of integers \mathcal{O}_E , and V is an E -vector space of dimension $n + 1 \geq 3$. Write $\lambda \mapsto \bar{\lambda}$ for the conjugation of E over \mathbb{Q} (which is just complex conjugation since $E \subset \mathbb{C}$). Let $h : V \times V \rightarrow E$ be a Hermitian form with respect to the conjugation, i.e. nondegenerate bilinear form such that $h(\lambda v, \mu w) = \lambda \bar{\mu} h(v, w)$ and $h(v, w) = \overline{h(w, v)}$ for $v, w \in V$. We assume that $h_{\mathbb{R}}$ on $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R} = V \otimes_E \mathbb{C} \cong \mathbb{C}^{n+1}$ is of signature $(n, 1)$. Consider the semisimple \mathbb{Q} -algebraic group

$$G = SU(h)$$

of isometries of (V, h) with determinant one. Thus for a \mathbb{Q} -algebra A one has

$$G(A) = \{g \in GL(V \otimes_{\mathbb{Q}} A) : h(gv, gw) = h(v, w) \text{ for } v, w \in V \otimes_{\mathbb{Q}} A\} \cap SL(V \otimes_{\mathbb{Q}} A).$$

By results of Kneser (see [28, 10.1.6(iv)]), h is isotropic over \mathbb{Q} , so that G has \mathbb{Q} -rank one.

By assumption, in a suitable basis for $V_{\mathbb{R}}$ the form h is given by $h(z, w) = -z_0\bar{w}_0 + z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$, so that $G(\mathbb{R}) \cong SU(n, 1)$. The group $G(\mathbb{R})$ acts transitively on the domain of h -negative complex lines in $V_{\mathbb{R}}$

$$\mathbb{B} := \{\ell \in \mathbb{P}(V_{\mathbb{R}}) \mid h|_{\ell} < 0\}$$

which is biholomorphic to the unit ball in \mathbb{C}^n . The isotropy subgroup of any line $\ell \in \mathbb{B}$ is a maximal compact subgroup (isomorphic to $S(U(n) \times U(1))$), so that \mathbb{B} is the (Hermitian) symmetric space of $G(\mathbb{R})$. The projective space $\mathbb{P}(V_{\mathbb{R}}) \cong \mathbb{C}\mathbb{P}^n$ is homogeneous for $G(\mathbb{C}) = SL(V_{\mathbb{R}}) \cong SL(n+1, \mathbb{C})$ and the embedding $\mathbb{B} \hookrightarrow \mathbb{P}(V_{\mathbb{R}})$ is $G(\mathbb{R})$ -equivariant.

An \mathcal{O}_E -stable lattice $L \subset V$ gives a subgroup $\text{Stab}_{G(\mathbb{Q})}(L) = \{\gamma \in G(\mathbb{Q}) : \gamma L \subset L\}$. For an ideal $\mathfrak{a} \subset \mathcal{O}_E$ we have the principal congruence subgroup

$$\Gamma(\mathfrak{a}) = \{\gamma \in \text{Stab}_{G(\mathbb{Q})}(L) \mid (\gamma - \text{Id})L \subset \mathfrak{a}L\}.$$

An arithmetic subgroup of $G(\mathbb{Q})$ is one which is commensurable with $\text{Stab}_{G(\mathbb{Q})}(L)$. For Γ arithmetic, the quotient

$$M_{\Gamma} = \Gamma \backslash \mathbb{B}$$

is a complex space. Let us assume that Γ is neat (i.e. the subgroup of \mathbb{C}^{\times} generated by eigenvalues of elements of $\Gamma \subset GL(V_{\mathbb{R}})$ is torsion-free), which can always be achieved by intersecting with a subgroup $\Gamma(\mathfrak{a})$ for \mathfrak{a} with $|\mathcal{O}_E/\mathfrak{a}|$ large enough. Then M_{Γ} is a smooth noncompact complex manifold.

1.2. Minimal compactification. Satake showed how to compactify M_{Γ} using the embedding $\mathbb{B} \subset \mathbb{P}(V_{\mathbb{R}})$. The boundary of \mathbb{B} in this embedding consists of the h -isotropic lines in $V_{\mathbb{R}}$. Let

$$\mathbb{B}^* = \mathbb{B} \sqcup \{E\text{-rational } h\text{-isotropic lines in } V_{\mathbb{R}}\}.$$

The reduction theory for Γ on \mathbb{B} gives a natural topology, the Satake topology, for which the evident action of $G(\mathbb{Q})$ on \mathbb{B}^* is continuous. The minimal compactification of M^* is the quotient

$$M_{\Gamma}^* = \Gamma \backslash \mathbb{B}^*,$$

which is compact and Hausdorff. The boundary consists of finitely many points (“cusps”), indexed (bijectively) by Γ -equivalence classes of E -rational h -isotropic lines. The assignment $\ell \leftrightarrow \text{Stab}_G(\ell)$ is a bijection between isotropic lines in \mathbb{B}^* and proper rational parabolic subgroups. Thus the cusps of M_{Γ}^* are also in bijection with Γ -conjugacy classes of rational parabolics; choosing a parabolic P gives a bijection of the cusps with $\Gamma \backslash G(\mathbb{Q})/P(\mathbb{Q})$.

Let \mathbb{L} denote the restriction of the tautological bundle of $\mathbb{P}(V_{\mathbb{R}})$ to \mathbb{B} . \mathbb{L} is $G(\mathbb{R})$ -equivariant and so descends to a line bundle, denoted \mathbb{L}_{Γ} , on M_{Γ} for each Γ . (It is easy to see that $\mathbb{L}_{\Gamma}^{\otimes n+1}$ is the canonical bundle of M_{Γ} .) The Baily-Borel theory shows that \mathbb{L}_{Γ} extends to a line bundle \mathcal{L}_{Γ} on M_{Γ}^* for which the graded algebra $R^* = \bigoplus_{k \geq 0} \Gamma(M_{\Gamma}^*, \mathcal{L}_{\Gamma}^{\otimes k})$ is finitely generated and $M_{\Gamma}^* = \text{Proj}(R^*)$. This defines a canonical projective variety structure on M_{Γ}^* and hence a canonical quasi-projective structure on M_{Γ} . The minimal compactification M_{Γ}^* is a normal variety with an isolated singularity at each cusp.

1.3. Stabilizer of a cusp. We will recall the structure of the stabilizer of a cusp. This is a special case of the “five-factor decomposition” of [1, Chp. III]. The case of ball quotients is discussed in a basis-free way in [17, 4.2], but we will give a more concrete description. If $n = 2$ this is also contained in [11, §1] and [20, §1].

The following notation is useful: For \mathbb{Q} -subgroups $J \triangleleft H < G$ let $\Gamma_H := \Gamma \cap H(\mathbb{R})$ and $\Gamma_{H/J} := \Gamma_H/\Gamma_J$. (This could be slightly misleading in case H/J has a natural lift in G .) Since Γ is neat these are torsion-free arithmetic subgroups of their respective groups.

Let $\ell \subset V$ be an isotropic line giving a cusp in \mathbb{B}^* . The stabilizer $P_\ell = \text{Stab}_G(\ell)$ is a \mathbb{Q} -parabolic subgroup of G . Its unipotent radical W_ℓ is a Heisenberg group of dimension $2n - 1$, i.e. it is an extension

$$1 \longrightarrow U_\ell \longrightarrow W_\ell \longrightarrow V_\ell \longrightarrow 1$$

where U_ℓ is the centre of W_ℓ , $U_\ell \cong \mathbb{G}_a$, and $V_\ell = W_\ell/U_\ell$ is abelian with $V_\ell(\mathbb{R}) \cong \mathbb{R}^{2n-2}$. We write $\mathfrak{w}_\ell, \mathfrak{u}_\ell, \mathfrak{v}_\ell = \mathfrak{w}_\ell/\mathfrak{u}_\ell$ for the Lie algebras of the groups of real points. The adjoint action of the \mathbb{Q} -split centre $A_\ell \cong \mathbb{G}_m$ of P_ℓ/W_ℓ is by a character χ on \mathfrak{v}_ℓ which generates the character group $X^*(A_\ell) \cong \mathbb{Z}$, and the action is by χ^2 on \mathfrak{u}_ℓ . The group P_ℓ/W_ℓ is an almost-direct product $M_\ell A_\ell$ where M_ℓ is the kernel of the adjoint action on \mathfrak{u}_ℓ , $M_\ell(\mathbb{R})$ is compact, and any lift of M_ℓ in P_ℓ centralizes U_ℓ . By our neatness assumption $\Gamma_{P_\ell} = \Gamma_{W_\ell}$ and this is an extension

$$1 \longrightarrow \Gamma_{U_\ell} \longrightarrow \Gamma_{W_\ell} \longrightarrow \Gamma_{V_\ell} \longrightarrow 1$$

with Γ_{U_ℓ} and Γ_{V_ℓ} free abelian of rank one and rank $2n - 2$, respectively. The commutator pairing

$$\Psi : \Gamma_{V_\ell} \times \Gamma_{V_\ell} \rightarrow \Gamma_{U_\ell}$$

(i.e. $\Psi(\gamma_1, \gamma_2) = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$) is alternating and nondegenerate. Extending scalars to \mathbb{R} gives the Lie bracket $\wedge^2 \mathfrak{v}_\ell \rightarrow \mathfrak{u}_\ell$, which we note is also invariant under the natural action of $P_\ell/W_\ell \cong M_\ell A_\ell$.

Let us illustrate these facts in a matrix representation convenient for calculations. Fix a nonzero vector $e \in \ell$. Choose an isotropic vector f with $h(e, f) = 1$, so that e and f span a hyperbolic plane. There is an orthogonal Witt decomposition $(V, h) = \left(Ee + Ef, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right) \oplus (V_0, h|_{V_0})$ where $(V_0, h|_{V_0})$ is anisotropic (see e.g. [28, 7.9]). Replacing f by $\sqrt{d}f$ and choosing a basis v_1, \dots, v_{n-1} for V_0 in which $h|_{V_0}$ is diagonal gives an E -basis $e, v_1, \dots, v_{n-1}, f$ for V in which h is given by $h(v, w) = {}^t \bar{v} J w$ where

$$J = \begin{pmatrix} & & \sqrt{d}^{-1} \\ & J_0 & \\ -\sqrt{d}^{-1} & & \end{pmatrix} \quad (1.1)$$

for an anisotropic diagonal form J_0 in $n - 1$ variables. In this matrix representation $P_\ell = \text{Stab}_G(\ell)$ is the intersection of $SU(h)$ with the upper triangular Borel subgroup in $SL(V)$. The unipotent radical W_ℓ consists of matrices of the form

$$\begin{pmatrix} 1 & \sqrt{d} {}^t \bar{b} & * \\ & I_{n-1} & b \\ & & 1 \end{pmatrix}$$

and its centre U_ℓ consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & * \\ & I_{n-1} & 0 \\ & & 1 \end{pmatrix}.$$

The group of matrices

$$t_\lambda = \begin{pmatrix} \lambda & & \\ & (\lambda^{-1}\bar{\lambda})I_{n-1} & \\ & & \bar{\lambda}^{-1} \end{pmatrix} \quad (1.2)$$

gives a torus $T_\ell = R_{E/\mathbb{Q}}\mathbb{G}_m$ which is an almost-direct product of a \mathbb{Q} -split torus \tilde{A}_ℓ (a lift of A_ℓ in P_ℓ) and the norm one torus $T_\ell^1 := \ker(\text{Nm} : R_{E/\mathbb{Q}}\mathbb{G}_m \rightarrow \mathbb{G}_m)$. On the level of \mathbb{Q} -points $\tilde{A}_\ell(\mathbb{Q}) = \{t_\lambda : \lambda \in \mathbb{Q}^\times\} = \left\{ \begin{pmatrix} \lambda & & \\ & 1 & \\ & & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{Q}^\times \right\}$. A lift \tilde{M}_ℓ of M_ℓ in P_ℓ is given by the group of matrices

$$t_\mu \begin{pmatrix} 1 & & \\ & g_0 & \\ & & 1 \end{pmatrix} \quad (g_0 \in SU(J_0), \mu \in T_\ell^1).$$

Thus $\tilde{M}_\ell\tilde{A}_\ell$ is a Levi subgroup in P_ℓ .

The following fact will be used repeatedly in the sequel:

Lemma 1.1. *The choice of $e \in \ell$ fixes an isomorphism of \mathbb{Q} -algebraic groups*

$$\xi_e : U_\ell \rightarrow \mathbb{G}_a$$

such that $\xi_{\lambda e} = m_{\text{Nm}(\lambda)^{-1}} \circ \xi_e$ where $m_\alpha : \mathbb{G}_a \rightarrow \mathbb{G}_a$ is multiplication by $\alpha \in \mathbb{Q}^\times$.

Proof. Complete e to an E -basis $e, v_1, \dots, v_{n-1}, f$ in which h is given by (1.1). For a \mathbb{Q} -algebra A an element of $U_\ell(A) \subset SL(V \otimes_{\mathbb{Q}} A)$ in this basis is a matrix in $SL(n+1, A \otimes E)$ with $(1, n)$ entry in A . This defines an isomorphism $\xi_A : U_\ell(A) \rightarrow A$ which may, a priori, depend on the choice of f, v_1, \dots, v_{n-1} . Suppose that $e' = \lambda e \in \ell$ (with $\lambda \in E^\times$) and f' is another isotropic vector such that $h|_{Ee'+Ef'} = \begin{pmatrix} & \sqrt{d}^{-1} \\ -\sqrt{d}^{-1} & \end{pmatrix}$ in the basis e', f' . Then $e \mapsto e', f \mapsto f'$ defines an isometry of hyperbolic planes $(Ee+Ef, h) \rightarrow (Ee'+Ef', h)$. By the Witt decomposition this extends to an isometry of (V, h) , i.e. there is $g \in SU(h)(\mathbb{Q})$ restricting to the isometry $Ee+Ef \rightarrow Ee'+Ef'$. Since $ge = e' = \lambda e$, we have $g \in P(\mathbb{Q})$ and in the matrix representation associated with $e, v_1, \dots, v_{n-1}, f$ we can write $g = w \begin{pmatrix} 1 & & \\ & g_0 & \\ & & 1 \end{pmatrix} t_\lambda$ for $w \in W_\ell(\mathbb{Q})$ and $g_0 \in SU(J_0)$. Thus in the basis $e', v_1, \dots, v_{n-1}, f'$ the new isomorphism $\xi'_A : U_\ell(A) \cong A$ is $\xi_A \circ \text{Int}(g^{-1})$ where $g^{-1} = t_\lambda^{-1} \begin{pmatrix} 1 & & \\ & g_0^{-1} & \\ & & 1 \end{pmatrix} w^{-1}$. Since $t_\lambda^{-1} \begin{pmatrix} 1 & 0 & u \\ & I_{n-1} & 0 \\ & & 1 \end{pmatrix} t_\lambda = \begin{pmatrix} 1 & 0 & \text{Nm}(\lambda)^{-1}u \\ & I_{n-1} & 0 \\ & & 1 \end{pmatrix}$ we have $\xi'_A = m_{\text{Nm}(\lambda)^{-1}} \circ \xi_A$. (Changing the basis of V_0 obviously has no effect on the isomorphism $U_\ell \cong \mathbb{G}_a$, so we have ignored it.) In particular ξ_A depends only on $e \in \ell$ and not on how it is completed to a basis. The various ξ_A define the isomorphism $\xi_e : U_\ell \rightarrow \mathbb{G}_a$. \square

Remark 1.2. In [17, 4.2] one finds a natural identification $U_\ell(\mathbb{R}) = \sqrt{-1}(\bar{\ell}_{\mathbb{R}} \otimes_{\mathbb{C}} \ell_{\mathbb{R}})$ and the \mathbb{Q} -rational structure is given by $\sqrt{-1}(\bar{\ell} \otimes_E \ell)$. (Here $\bar{\ell}$ means the same underlying \mathbb{Q} -vector space with the conjugate action of E .) The choice of $e \in \ell$ fixes the generator $\sqrt{-1}(\bar{e} \otimes e) \in U(\mathbb{Q})$, and hence an isomorphism $U_\ell(\mathbb{Q}) \cong \mathbb{Q}$. This is the same as our ξ_e ; the dependence on e found above is clear in this description.

1.4. **Smooth compactification.** The variety M_Γ^* has a canonical desingularization

$$\pi : \overline{M}_\Gamma \rightarrow M_\Gamma^*$$

which is a smooth compactification of M_Γ in which the exceptional divisor

$$D_\Gamma := \overline{M}_\Gamma - M_\Gamma = \bigsqcup_{\ell \in \Gamma \backslash (\mathbb{B}^* - \mathbb{B})} D_{\Gamma, \ell}$$

is smooth and each component $D_{\Gamma, \ell}$ is a complex abelian variety. This is a special case of the general construction of toroidal compactifications of Ash, Mumford, Rapoport, Tai [1]. It will suffice for our purposes to have a natural description of a neighbourhood of the divisor $D_{\Gamma, \ell}$ in the Hausdorff topology (equivalently, of its normal bundle) and to compute the first Chern class of the normal bundle as a class in $H^2(D_{\Gamma, \ell}, \mathbb{Z})$. We will describe the construction of \overline{M}_Γ (as a complex manifold) in enough detail to make this computation in 1.5.

A basic role is played by the Siegel domain picture of \mathbb{B} . A basis-independent treatment for ball quotients for all n is recalled in [17, 4.2] and treatments of the case $n = 2$ (which show the main features and are close to ours below) in [11, 1.3–1.6] and [20, §2]. We recall this picture, leaving some minor details to be verified (as is easily done in the matrix representation).

Let $\ell \subset V$ be an isotropic line, $P_\ell = \text{Stab}_G(\ell)$ and W_ℓ, U_ℓ etc. be as before. Consider the domain in $\mathbb{P}(V_\mathbb{R})$ given by:

$$\mathbb{B}(\ell) := U_\ell(\mathbb{C}) \cdot \mathbb{B} = \mathbb{P}(V_\mathbb{R}) - \mathbb{P}(\ell_\mathbb{R}^\perp).$$

As in 1.3, choose a basis $e, v_1, \dots, v_{n-1}, f$ in which $e \in \ell$ and h is given by a matrix of the form (1.1). Then $\mathbb{B}(\ell)$ consists of lines spanned by vectors $v \in V$ which in this basis have a nontrivial f -component (equivalently, $v \in V$ with $h(e, v) \neq 0$). The map

$$\begin{aligned} \mathbb{C} \times V_0 &\longrightarrow \mathbb{B}(\ell) \\ (z, w) &\mapsto \mathbb{C}(ze + w + f) \end{aligned} \tag{1.3}$$

is then a biholomorphism. In the coordinates (z, w) the action of $u \in U_\ell(\mathbb{C})$ is by translation by $\xi_e(u)$ in the first coordinate (ξ_e as in Lemma 1.1). In these coordinates,

$$\mathbb{B} \cong \{(z, w) \in \mathbb{C} \times V_0 : 2\sqrt{|d|} \text{Im}(z) > h(w, w)\}.$$

For $t > 0$ the domain

$$\mathbb{B}_t \cong \{(z, w) \in \mathbb{C} \times V_0 : 2\sqrt{|d|} \text{Im}(z) > h(w, w) + t\}$$

is invariant under $\Gamma_{P_\ell} = \Gamma_{W_\ell}$. A basic result of the reduction theory of Γ on \mathbb{B}^* is that for $t \gg 0$, the induced map

$$\Gamma_{W_\ell} \backslash \mathbb{B}_t \hookrightarrow M_\Gamma$$

is injective with image a deleted neighborhood of the cusp of M_Γ^* given by ℓ . Thus to compactify in the direction ℓ it is enough to construct a partial compactification of $\Gamma_{W_\ell} \backslash \mathbb{B}_t$.

Let d_e be the unique positive rational number such that $\xi_e(\Gamma_{U_\ell}) = d_e \mathbb{Z}$ where $\xi_e : U_\ell(\mathbb{Q}) \cong \mathbb{Q}$ is as in Lemma 1.1. Consider the diagram:

$$\Gamma_{U_\ell} \backslash \mathbb{B}_t \hookrightarrow \Gamma_{U_\ell} \backslash \mathbb{B}(\ell) \xrightarrow{\cong} (d_e \mathbb{Z} \backslash \mathbb{C}) \times V_0 \xrightarrow{\cong} \mathbb{C}^\times \times V_0$$

where the second map is induced by (1.3) and the third map is induced by $(z, w) \mapsto (q = \exp(2\pi iz/d_e), w)$. In the coordinates (q, w) this realizes $\Gamma_{U_\ell} \backslash \mathbb{B}_t$ as

$$\Gamma_{U_\ell} \backslash \mathbb{B}_t \cong \{(q, w) \in \mathbb{C} \times V_0 : 0 < |q| < r_w\}$$

where r_w depends on w (and on t, d_e). Thus $\Gamma_{U_\ell} \backslash \mathbb{B}_t$ is a punctured disc bundle with respect to the projection to V_0 , the radius varying with $w \in V_0$. Filling in the zero section and dividing by $\Gamma_{V_\ell} = \Gamma_{W_\ell} / \Gamma_{U_\ell}$ defines a smooth partial compactification

$$\Gamma_{W_\ell} \backslash \mathbb{B}_t \hookrightarrow \overline{\Gamma_{W_\ell} \backslash \mathbb{B}_t}$$

in which the boundary added is the complex torus

$$D_{\Gamma, \ell} = \Gamma_{W_\ell} U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell) \cong \Gamma_{V_\ell} \backslash V_0.$$

This defines the compactification in the direction of the cusp ℓ . Repeating the construction at each cusp defines the compactification \overline{M}_Γ as a compact complex manifold. The results of [1, Ch. IV] prove that \overline{M}_Γ is naturally a smooth complex projective variety containing M_Γ as a Zariski-dense open subset and there is a unique projective morphism $\pi : \overline{M}_\Gamma \rightarrow M_\Gamma^*$ extending the identity.

Remark 1.3. Here is a slightly different description of the boundary divisor, closer to [1]. The torus $\mathbb{G}_m = \Gamma_{U_\ell} \backslash U_\ell(\mathbb{C})$ acts on $\Gamma_{W_\ell} \backslash \mathbb{B}(\ell)$ making it a $\Gamma_{U_\ell} \backslash U_\ell(\mathbb{C})$ -torsor over $\Gamma_{W_\ell} U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$. Define

$$\overline{\Gamma_{W_\ell} \backslash \mathbb{B}(\ell)} := \Gamma_{W_\ell} \backslash \mathbb{B}(\ell) \times_{\mathbb{G}_m} \mathbb{A}^1.$$

The partial compactification of the deleted neighborhood defined above is then

$$\overline{\Gamma_{W_\ell} \backslash \mathbb{B}_t} = \text{interior of the closure of } \Gamma_{W_\ell} \backslash \mathbb{B}_t \text{ in } \overline{\Gamma_{W_\ell} \backslash \mathbb{B}(\ell)}.$$

The projection $\overline{\Gamma_{W_\ell} \backslash \mathbb{B}(\ell)} \rightarrow \Gamma_{W_\ell} U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$ then gives the natural identification $D_{\Gamma, \ell} = \Gamma_{W_\ell} U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$.

Remark 1.4. The action of $W_\ell(\mathbb{R}) U_\ell(\mathbb{C})$ on $\mathbb{B}(\ell)$ is simply transitive, so the choice of basepoint $\ell_0 \in \mathbb{B}(\ell)$ gives an identification $W_\ell(\mathbb{R}) U_\ell(\mathbb{C}) \rightarrow \mathbb{B}(\ell)$ by the orbit map $w \mapsto w \ell_0$. With a basis e, v_i, f chosen as above, there is a natural basepoint $\ell_0 = \mathbb{C} f \in \mathbb{B}(\ell)$ (corresponding to $(0, 0) \in \mathbb{C} \times V_0$ under (1.3)). The orbit map then descends to an isomorphism

$$V_\ell(\mathbb{R}) \rightarrow U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$$

which is $(M_\ell A_\ell)(\mathbb{R})$ -equivariant for the conjugation action on $V_\ell(\mathbb{R})$ and the action via the lift $(\tilde{M}_\ell \tilde{A}_\ell)(\mathbb{R})$ on $U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$. (This is easily checked in the matrix representation above.) The map is also equivariant for the translation action of $V_\ell(\mathbb{R})$ on itself and the action on $U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$ induced by the action of $W_\ell(\mathbb{R})$, so that it gives an isomorphism

$$\Gamma_{V_\ell} \backslash V_\ell(\mathbb{R}) \rightarrow \Gamma_{W_\ell} U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell) = D_{\Gamma, \ell}$$

giving another description of $D_{\Gamma, \ell}$ (though without its complex structure).

Remark 1.5. The theory of arithmetic compactifications of ball quotients (i.e. compactification over E or abelian extensions) is contained in the general theory for Shimura varieties developed in Pink's thesis [26]. A finer theory over rings of integers (with some primes depending on the level inverted) exists for $n = 2$ in Larsen's thesis (cf. [3] for a summary). We will not use étale cohomology or reduction to positive characteristic in the sequel so we do not go into this.

1.5. Chern class of the normal bundle of the boundary divisor. Fix an isotropic line $\ell \subset V$, let $P_\ell = \text{Stab}_G(\ell)$ and let W_ℓ, V_ℓ, U_ℓ etc. be as in 1.3. Next we compute the Chern class of the normal bundle of $D_{\Gamma, \ell}$, which lies in $H^2(D_{\Gamma, \ell}, \mathbb{Z})$. Choose $e \in \ell$. Let d_e be as above, i.e. the unique positive rational number such that $\xi_e(\Gamma_{U_\ell}) = d_e \mathbb{Z}$. Let $\Psi_e := \xi_e \circ \Psi : \Gamma_{V_\ell} \times \Gamma_{V_\ell} \rightarrow \mathbb{Q}$ where Ψ is the commutator. Both d_e and Ψ_e depend on e , but the alternating form

$$c_{\Gamma, \ell} := -\frac{1}{d_e} \Psi_e : \Gamma_{V_\ell} \times \Gamma_{V_\ell} \rightarrow \mathbb{Z} \quad (1.4)$$

does not. (By Lemma 1.1, $\Psi_{\lambda e} = \text{Nm}(\lambda)^{-1} \Psi_e$ and $d_{\lambda e} = \text{Nm}(\lambda)^{-1} d_e$ for $\lambda \in E^\times$.) Since $D_{\Gamma, \ell}$ is a complex torus with fundamental group Γ_{V_ℓ} , we have (cf. [18, p.17]):

$$H^2(D_{\Gamma, \ell}, \mathbb{Z}) = \text{Hom}(\wedge^2 \Gamma_{V_\ell}, \mathbb{Z}) = \wedge^2 \text{Hom}(\Gamma_{V_\ell}, \mathbb{Z}).$$

and so $c_{\Gamma, \ell} = -\frac{1}{d_e} \Psi_e$ gives an element of $H^2(D_{\Gamma, \ell}, \mathbb{Z})$ independent of the choice of $e \in \ell$.

Lemma 1.6. *The normal bundle of $D_{\Gamma, \ell}$ in \overline{M}_Γ has Chern class $c_{\Gamma, \ell}$.*

Proof. By construction, the normal bundle of $D_{\Gamma, \ell}$ in \overline{M}_Γ is naturally isomorphic to the line bundle associated with the \mathbb{C}^\times -torsor $\Gamma_{W_\ell} \backslash \mathbb{B}(\ell) \rightarrow D_{\Gamma, \ell}$. Here $\Gamma_{W_\ell} \backslash \mathbb{B}(\ell)$ is naturally a $\Gamma_{U_\ell} \backslash U_\ell(\mathbb{C})$ -torsor and becomes a \mathbb{C}^\times -torsor using the isomorphisms

$$\Gamma_{U_\ell} \backslash U_\ell(\mathbb{C}) \cong d_e \mathbb{Z} \backslash \mathbb{C} \cong \mathbb{C}^\times$$

(the first induced by ξ_e and the second by $z \mapsto q = \exp(2\pi iz/d_e)$). The torsor becomes trivial when lifted to $V_0 = U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$ and a standard computation (cf. e.g. [18, I.2] or [26, Lemma 3.19]) shows that the cohomology class of the $\Gamma_{U_\ell} \backslash U_\ell(\mathbb{C})$ -torsor in $H^2(\Gamma_{V_\ell}, \Gamma_{U_\ell}) = \text{Hom}(\wedge^2 \Gamma_{V_\ell}, \Gamma_{U_\ell})$ is given by $-\Psi$. The isomorphism $\Gamma_{U_\ell} \backslash U_\ell(\mathbb{C}) \cong \mathbb{C}^\times$ induces $\Gamma_{U_\ell} \rightarrow \mathbb{Z}$ by $\Gamma_{U_\ell} \xrightarrow{\xi_e} d_e \mathbb{Z} \xrightarrow{1/d_e} \mathbb{Z}$, so that the lemma follows. \square

Remark 1.7. Note that Ψ_e is positive definite on V_0 , so that the complex torus $D_{\Gamma, \ell}$ is polarizable, i.e. is an abelian variety and its conormal bundle in \overline{M}_Γ is ample.

1.6. We note the behaviour of the constructions above with respect to change of level. Let $\Gamma' \subset \Gamma$ be a subgroup of finite index. The finite morphism $p : M_{\Gamma'} \rightarrow M_\Gamma$ extends naturally to finite morphisms of both types of compactifications, namely to $p : M_{\Gamma'}^* \rightarrow M_\Gamma^*$ and $p : \overline{M}_{\Gamma'} \rightarrow \overline{M}_\Gamma$ and restricts to $p : D_{\Gamma'} \rightarrow D_\Gamma$. (We will use the same notation for all these maps as it will cause no confusion.)

An immediate consequence of Lemma 1.6 (with the notation there) is:

Lemma 1.8. *For $\Gamma' \subset \Gamma$ of finite index we have $c_{\Gamma', \ell} = \frac{1}{[\Gamma_{U_\ell} : \Gamma'_{U_\ell}]} p^*(c_{\Gamma, \ell})$.*

This lemma is also easy to see topologically: Locally in the Hausdorff topology near a point of $D_{\Gamma', \ell}$ the map p looks like the product of the self-map $z \mapsto z^{[\Gamma'_{U_\ell} : \Gamma_{U_\ell}]}$ of a unit disc (in the “normal” or U_ℓ -directions) with the (restriction of) the étale map $D_{\Gamma', \ell} \rightarrow D_{\Gamma, \ell}$. (This is clear from the Siegel domain picture (1.3).) Thus $p : \overline{M}_{\Gamma'} \rightarrow \overline{M}_\Gamma$ ramifies along the divisor at infinity, with local ramification index along $D_{\Gamma', \ell}$ given by $[\Gamma'_{U_\ell} : \Gamma_{U_\ell}]$. This implies that the normal bundles are related by p^* , up to taking a power $[\Gamma'_{U_\ell} : \Gamma_{U_\ell}]$.

2. COHOMOLOGY OF BALL QUOTIENTS AND THEIR COMPACTIFICATIONS

In this section we prove that the direct limit

$$H^i(\overline{\mathcal{M}}) = \varinjlim_{\Gamma} H^i(\overline{M}_{\Gamma})$$

of cohomology groups of smooth compactifications has a canonical $G(\mathbb{Q})$ -invariant decomposition in which one summand is the direct limit

$$IH^*(\mathcal{M}^*) = \varinjlim_{\Gamma} IH^*(M_{\Gamma}^*)$$

of intersection cohomology of minimal compactifications and the other summands are explicitly described in terms of the boundary divisor D_{Γ} (Theorem 2.6). This is proved using a purely geometric statement (Prop. 2.2) which is a consequence of the decomposition theorem of [2], but for which we give an elementary proof. The direct limit here can be taken over any family of arithmetic subgroups which is closed under $G(\mathbb{Q})$ -conjugation and finite intersections and whose intersection is the identity (see 2.2).

All varieties are complex algebraic varieties, all cohomology groups are singular cohomology with complex coefficients and intersection cohomology is always of middle perversity. The arguments work as well with mixed Hodge structures modulo keeping track of Tate twists (and the references given below are adequate for this). If we choose models over a number field it is also possible to work with étale cohomology (but the references we give do not cover this situation).

2.1. Decomposition theorem. The decomposition theorem of [2] relates the intersection cohomology of a (singular) variety to the cohomology of a resolution. There is no canonical decomposition in general but in the situation at hand (a resolution of an isolated singularity with smooth exceptional divisor), we will show that there is a way to fix a decomposition with good properties. In this special situation, the required results reduce to the hard Lefschetz theorem on the exceptional divisor, so we prove them directly in Prop. 2.2 and then apply them to ball quotients in 2.3. In Remark 2.5 we comment on how these also follow from the decomposition theorem of [2].

The following lemma gives a simple description of intersection cohomology in a very special case. (For our purposes this may even be taken as the definition.)

Lemma 2.1. *Let X be a normal variety of pure dimension n with isolated singularities and let U be the smooth locus of X . Then there are natural isomorphisms*

$$IH^{i+n}(X) = \begin{cases} H^{i+n}(U) & i < 0 \\ \text{im}[H_c^n(U) \rightarrow H^n(U)] & i = 0 \\ H_c^{i+n}(U) & i > 0 \end{cases}$$

compatible with duality.

Proof. This is elementary (e.g. it follows from [12, 6.1]). □

Proposition 2.2. *Suppose that we have a diagram of complex algebraic varieties*

$$\begin{array}{ccc} & & Y \\ & & \downarrow \pi \\ U & \xrightarrow{j} & X \end{array}$$

where

- X is normal, projective, purely n -dimensional with isolated singularities
- U is the smooth locus of X
- Y is a smooth projective compactification of U dominating X
- $Z := \pi^{-1}(X - U)$ is a smooth divisor in Y .

Then there is a canonical decomposition

$$H^{i+n}(Y) = IH^{i+n}(X) \oplus \begin{cases} H^{i+n}(Z) & i \geq 0 \\ H_Z^{i+n}(Y) & i < 0 \end{cases}$$

with the following properties: (1) The restriction $H^*(Y) \rightarrow H^*(U)$ factors through the projection $H^*(Y) \rightarrow IH^*(X)$. (2) The restriction $H^*(Y) \rightarrow H^*(Z)$ has the obvious description on the second summand, namely the identity on $H^{i+n}(Z)$ for $i \geq 0$ and the adjunction map $H_Z^{i+n}(Y) \rightarrow H^{i+n}(Z)$ for $i < 0$. (3) The decomposition is compatible with Poincaré duality in the sense that the pairing $H^{i+n}(Y) \times H^{-i+n}(Y) \rightarrow \mathbb{C}$ is the sum of the pairing $IH^{i+n}(X) \times IH^{-i+n}(X) \rightarrow \mathbb{C}$ and the pairing $H_Z^{i+n}(Y) \times H^{-i+n}(Z) \rightarrow \mathbb{C}$.

The existence of *some* decomposition as in the proposition is well-known (cf. e.g. [23]) or is easily deduced from the decomposition theorem of [2] (cf. Remark 2.1), but our aim is to *fix* one (in fact, it is canonical) and we will need to use some information which comes out of the proof. Before giving the proof we recall some standard facts.

Suppose that Z is a closed subset of a smooth compact n -dimensional variety Y (e.g. as in the geometric setting of the proposition). We write $H_Z^*(Y)$ for the relative cohomology $H^*(Y, Y-Z)$ and $res : H^*(Y) \rightarrow H^*(Z)$ and $adj : H_Z^*(Y) \rightarrow H^*(Y)$ for the natural maps. We will use the cap product

$$H_Z^i(Y) \times H_j(Y) \xrightarrow{\cap} H_{j-i}(Z)$$

and the cup product

$$H_Z^i(Y) \times H^j(Z) \xrightarrow{\cup} H_Z^{i+j}(Y)$$

which refine the usual products. (See [25, Appendix B] and the references therein.)

- (1) By Poincaré-Lefschetz duality ([25, App. B, Theorem B.31]) cap product with the fundamental class $[Y] \in H_{2n}(Y)$ gives an isomorphism

$$\cap[Y] : H_Z^i(Y) \rightarrow H_{2n-i}(Z)$$

for all i . Thus there is a nondegenerate duality pairing $H_Z^i(Y) \times H^{2n-i}(Z) \rightarrow \mathbb{C}$. (This is the pairing referred to in (3) of the proposition. It is the same as the cup product $H_Z^i(Y) \times H^{2n-i}(Z) \rightarrow H_Z^{2n}(Y) \cong H^{2n}(Y) \cong \mathbb{C}$.) res and adj are dual under this pairing.

- (2) Suppose Z is a closed subvariety of pure dimension d with irreducible components Z_i . Each component has a fundamental class $[Z_i] \in H_{2d}(Z)$. Let

$$c := \sum_i (\cap[Y])^{-1}([Z_i]) \in H_Z^{2n-2d}(Y). \quad (2.1)$$

When Z is smooth, c is the *Thom class* and cupping with c gives the Thom isomorphism

$$H^{i-2(n-d)}(Z) \cong H_Z^i(Y) \quad (2.2)$$

for all i ([25, Theorem B.70]).

(3) Suppose that $d = n - 1$, i.e. Z has codimension one. Under the mappings

$$H_Z^2(Y) \xrightarrow{adj} H^2(Y) \xrightarrow{res} H^2(Z) \quad (2.3)$$

the class c is mapped to the cohomology cycle class of Z in $H^2(Y)$ and then to the first Chern class of the normal bundle of Z in Y . (See the proof of [25, Theorem B.70] for the relation of the Thom class and the Euler class of the normal bundle, which in this case (real codimension two) is the same as the first Chern class. See [25, 2.6] for a Hodge-theoretic version.)

We will use the same notation c for all three classes in (3), i.e. for the class in $H_Z^2(Y)$ defined by (2.1) and also for its images in $H^2(Y)$ and $H^2(Z)$. The precise meaning will always be clear from the context.

Proof. Consider the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^{i+n-1}(U) & \longrightarrow & H_Z^{i+n}(Y) & \xrightarrow{adj} & H^{i+n}(Y) & \longrightarrow & H^{i+n}(U) & \longrightarrow & \cdots \\ & & & & \cong \downarrow & & \downarrow res & & & & \\ & & & & H^{i+n-2}(Z) & \xrightarrow{\cdot c} & H^{i+n}(Z) & & & & \end{array} \quad (2.4)$$

The top row is the usual cohomology long exact sequence for the pair (Y, U) . The left vertical map is the (inverse) Thom isomorphism. The right vertical map is the restriction. The bottom map is cupping with the first Chern class $c \in H^2(Z)$ of the normal bundle of Z and makes the square commutative. Since Z is contracted to a finite set of points in X , its conormal bundle in Y is ample (see [14]). Then c has the hard Lefschetz property on $H^*(Z)$, so that

$$\cdot c : H^{i+n-2}(Z) \longrightarrow H^{i+n}(Z)$$

is injective for $i \leq 0$ and an isomorphism for $i = 0$. The long exact sequence then gives a short exact sequence for each $i < 0$:

$$0 \longrightarrow H_Z^{i+n}(Y) \longrightarrow H^{i+n}(Y) \longrightarrow H^{i+n}(U) \longrightarrow 0. \quad (2.5)$$

The (Poincaré) dual short exact sequences are:

$$0 \longrightarrow H_c^{i+n}(U) \longrightarrow H^{i+n}(Y) \longrightarrow H^{i+n}(Z) \longrightarrow 0 \quad (2.6)$$

for $i > 0$. Notice that by the hard Lefschetz property and Thom isomorphism the cup product

$$\cdot c^i : H_Z^{-i+n}(Y) \longrightarrow H^{i+n}(Z) \quad (2.7)$$

is an isomorphism for $i > 0$. In the middle dimension (i.e. $i = 0$) we have dual short exact sequences which are the rows of the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_Z^n(Y) & \longrightarrow & H^n(Y) & \longrightarrow & \text{im}[H^n(Y) \rightarrow H^n(U)] & \longrightarrow & 0 \\ & & & & \parallel & & & & \\ 0 & \longrightarrow & \text{im}[H_c^n(U) \rightarrow H^n(Y)] & \longrightarrow & H^n(Y) & \longrightarrow & H^n(Z) & \longrightarrow & 0. \end{array} \quad (2.8)$$

The composition $H_Z^n(Y) \rightarrow H^n(Y) \rightarrow H^n(Z)$ in this diagram is an isomorphism (by the Thom isomorphism and hard Lefschetz on Z for c), so that the restriction from Y to U induces an isomorphism

$$\text{im}[H_c^n(U) \rightarrow H^n(Y)] \longrightarrow \text{im}[H^n(Y) \rightarrow H^n(U)].$$

This map factors through $\text{im}[H_c^n(U) \rightarrow H^n(U)]$ and induces isomorphisms

$$\text{im}[H_c^n(U) \rightarrow H^n(Y)] \cong \text{im}[H_c^n(U) \rightarrow H^n(U)] \cong \text{im}[H^n(Y) \rightarrow H^n(U)] \quad (2.9)$$

since $\text{im}[H_c^n(U) \rightarrow H^n(U)] \rightarrow \text{im}[H^n(Y) \rightarrow H^n(U)]$ is trivially injective. Note that via these isomorphisms the duality pairing of $\text{im}[H_c^n(U) \rightarrow H^n(Y)]$ and $\text{im}[H^n(Y) \rightarrow H^n(U)]$ translates to the obvious self-duality of $\text{im}[H_c^n(U) \rightarrow H^n(U)]$. Thus the short exact sequences in (2.8) split each other, once the identification $H_Z^n(Y) \cong H^n(Z)$ has been made.

Let us split the short exact sequences in (2.5), (2.6) above to get a decomposition of $H^*(Y)$. (Of course, being sequences of vector spaces these are certainly split, but we want to *fix* a splitting with good properties, so that in our application to ball quotients these will have Hecke-equivariance properties.)

For $i > 0$ consider the diagram

$$\begin{array}{ccccc} H^{i+n}(Z) & \xleftarrow{\cdot c^i} & H_Z^{-i+n}(Y) & \xrightarrow{\text{adj}} & H^{-i+n}(Y) & \xrightarrow{\cdot c^i} & H^{i+n}(Y) \\ & & & & \searrow & \swarrow & \\ & & & & & & s_i \end{array} \quad (2.10)$$

where s_i is defined by:

$$s_i := (\cdot c^i)^{-1} \circ \text{adj} \circ (\cdot c^i).$$

Here $(\cdot c^i)^{-1}$ is the inverse of the hard Lefschetz isomorphism $\cdot c^i : H_Z^{-i+n}(Y) \rightarrow H^{i+n}(Z)$. The map s_i is a section of res , so it splits (2.6).

For $i < 0$ one takes the dual mapping to s_i , which is defined by the dual diagram

$$\begin{array}{ccccc} H_Z^{i+n}(Y) & \xrightarrow{\cdot c^{-i}} & H^{-i+n}(Z) & \xleftarrow{\text{res}} & H^{-i+n}(Y) & \xleftarrow{\cdot c^{-i}} & H^{i+n}(Y) \\ & & & & \searrow & \swarrow & \\ & & & & & & s_i \end{array} \quad (2.11)$$

and

$$s_i := (\cdot c^{-i})^{-1} \circ \text{res} \circ (\cdot c^{-i}).$$

where $(\cdot c)^{-i}$ inverts the hard Lefschetz isomorphism $H_Z^{i+n}(Y) \rightarrow H^{i+n}(Z)$. This splits (2.5).

For $i = 0$ the same composition in (2.10), i.e. $H^n(Z) \cong H_Z^n(Y) \rightarrow H^n(Y)$ splits the map $H^n(Y) \rightarrow H^n(Z)$ in the bottom row of (2.8) and so by duality we get a splitting of the top row of (2.8). On the other hand (2.8) is also split by the composition $H^n(Y) \rightarrow H^n(Z) \cong H_Z^n(Y)$. That these two splittings of (2.8) agree follows from the fact that the isomorphism $H_Z^n(Y) \cong H^n(Z)$ is self-dual. We will denote the splitting by s_0 . Thus the decomposition of $H^n(Y)$ is compatible with duality.

Using Lemma 2.1 we have produced a decomposition

$$H^{i+n}(Y) = IH^{i+n}(X) \oplus \begin{cases} H^{i+n}(Z) & i \geq 0 \\ H_Z^{i+n}(Y) & i < 0 \end{cases}$$

as in the proposition. It remains to verify the properties claimed. The restriction $H^*(Y) \rightarrow H^*(U)$ factors through the projection $H^*(Y) \rightarrow IH^*(X)$ because the other summands are in the image of $H_Z^*(Y) \rightarrow H^*(Y)$ and hence vanish on restriction to U . The restriction $H^*(Y) \rightarrow H^*(Z)$ has the obvious description on the second summand. (If $i \geq 0$ this map is the identity on $H^{i+n}(Z)$ by construction. If $i \leq 0$ this is also clear.) The compatibility with Poincaré duality is straightforward. \square

Remark 2.3. The second term in $H^n(Y)$ can be written as either $H^n(Z)$ or $H_Z^n(Y)$ as $\text{adj} : H_Z^n(Y) \rightarrow H^n(Z)$ is a natural isomorphism.

Remark 2.4. In the situation of ball quotients of 1.4, where each component of $Z = D_\Gamma$ is an explicitly given abelian variety and the class $c = c_\Gamma \in H^2(D_\Gamma)$ is given in terms of the commutator (see 1.4), the hard Lefschetz property is elementary.

Remark 2.5. We indicate the relation of the proposition with the decomposition theorem, which is proved in the étale setting over finite fields in [2, §5] and transferred to complex varieties using the dictionary of [2, §6]. The facts mentioned in this remark will not be used in the sequel so we do not give all details. (The reader familiar with [2] will easily supply these, and this would give another proof of Prop. 2.2.)

If X is a variety and $\pi : Y \rightarrow X$ a resolution of singularities (so that Y is smooth and π is a projective morphism) the decomposition theorem [2, Theorem 6.2.5, cf. also Theorem 5.4.5] gives an isomorphism

$$R\pi_*\mathbb{C}_Y[n] \cong \bigoplus_k {}^pH^k(R\pi_*\mathbb{C}_Y[n])[-k] \quad (2.12)$$

in the derived category of constructible complexes of sheaves on X . Here ${}^pH^k$ is the k th perverse cohomology functor (for middle perversity). Assume the hypotheses of Proposition 2.2. ⁽¹⁾ The summands can be explicitly computed as follows: Let $i_x : \{x\} \hookrightarrow X$ for a singular point $x \in X - U$ and let $Z_x := \pi^{-1}(x)$. The summands in (2.12) for $k \neq 0$ are supported on the singular locus: There are natural isomorphisms

$${}^pH^k(R\pi_*\mathbb{C}_Y[n]) = \begin{cases} \bigoplus_{x \in X-U} i_{x*}H^{k+n}(Z_x) & k > 0 \\ \bigoplus_{x \in X-U} i_{x*}H_{Z_x}^{k+n}(Y) & k < 0. \end{cases} \quad (2.13)$$

The $k = 0$ summand is canonically a direct sum

$${}^pH^0(R\pi_*\mathbb{C}_Y[n]) = j_{!*}\mathbb{C}_U[n] \oplus \bigoplus_{x \in X-U} i_{x*}H^n(Z_x) \quad (2.14)$$

where $j_{!*}\mathbb{C}_U[n]$ is the intermediate extension (so that its i th hypercohomology gives $IH^{i+n}(X)$). Taking hypercohomology in (2.12) gives a decomposition of $H^i(Y, \mathbb{C})$ as in Prop. 2.2.

In general, there is no canonical isomorphism (2.12). In [10], Deligne showed how to fix an isomorphism given a Lefschetz operator on $R\pi_*\mathbb{C}_Y[n]$, which, by definition, is a homomorphism $L : R\pi_*\mathbb{C}_Y[n] \rightarrow R\pi_*\mathbb{C}_Y[n+2]$ in the derived category such that $L^k : {}^pH^{-k}(R\pi_*\mathbb{C}_Y[n]) \rightarrow {}^pH^k(R\pi_*\mathbb{C}_Y[n])$ is an isomorphism for all $k > 0$. (Remember here that we are using \mathbb{C} -coefficients and ignoring Tate twists.) The Chern class of a line bundle $c_1(\mathcal{L}) \in H^2(Y, \mathbb{C}) = \text{Hom}_{D_c^b(\mathbb{C}_X)}(\mathbb{C}, \mathbb{C}[2])$ gives, by functoriality, an operator $L : R\pi_*\mathbb{C}_Y[n] \rightarrow R\pi_*\mathbb{C}_Y[n+2]$. If \mathcal{L} is relatively ample for π then, by the relative hard Lefschetz theorem [2, Théorème 6.2.10], L is an example of a Lefschetz operator.

In fact, [10] gives more than one way to fix an isomorphism (2.12) using a Lefschetz operator, but we will use the one of [10, §3], which has the following characterization: Given an isomorphism (2.12) the Lefschetz operator L can be written as a sum of its homogeneous components: $L = \sum_{k \leq 2} L^{(k)}$. ($L^{(k)} = 0$ for $k > 2$ because perverse sheaves form a t -structure.) Then there is a unique isomorphism in which

$$(\text{ad } L^{(2)})^{1-d} L^{(d)} = 0 \quad (\text{for } d \leq 1). \quad (2.15)$$

(This is [10, Prop. 3.5].) This decomposition is self-dual with respect to Verdier duality and has properties (1) and (2) in the proposition because it is sheaf-theoretic.

¹In fact in this remark one may drop the assumption that the exceptional divisor is smooth; it is enough to assume that it has simple normal crossings and smooth components.

The decomposition of $H^*(Y)$ we have produced in Prop. 2.2 is the one given by [10, §3] by using $\mathcal{O}(-Z)$ (which restricts on Z to the conormal bundle) as the relatively ample line bundle (and then taking hypercohomology). This can be verified by first rephrasing the decomposition of Prop. 2.2 in sheaf-theoretic terms and then checking that the relations of [10, Prop. 3.5] hold for the Lefschetz operator given by $c_1(\mathcal{O}(-Z)) = -c \in H^2(Y)$.

Note that taking stalks at a point $x \in X - U$ in a decomposition (2.12) gives, by the base change theorem for the proper morphism π , a decomposition

$$H^{k+n}(Z_x) = H^k(R\pi_*\mathbb{C}_Y[n])_x = \begin{cases} H^{k+n}(Z_x) & k \geq 0 \\ H^k(j_*\mathbb{C}_U[n])_x \oplus H_{Z_x}^{k+n}(Y) & k < 0. \end{cases} \quad (2.16)$$

For the decomposition satisfying (2.15), this gives (for $k < 0$) the decomposition into primitive and nonprimitive cohomology with respect to L , i.e. the first summand $H^k(j_*\mathbb{C}_U[n])_x$ is the primitive cohomology

$$\ker(H^{k+n}(Z_x) \xrightarrow{L^{-k}} H^{-k+n}(Z_x))$$

(remember that $\dim Z_x = n - 1$) and the other summand is $L(H^{k+n-2}(Z_x))$. When L comes from a line bundle \mathcal{L} the homomorphism $H^*(Z_x) \rightarrow H^{*+2}(Z_x)$ is cupping with $c_1(\mathcal{L})|_{Z_x} \in H^2(Z_x)$ and so these are the usual primitive and nonprimitive cohomology with respect to the class $c_1(\mathcal{L})|_{Z_x}$.

2.2. Direct limits in cohomology. We turn to the setting of ball quotients of §1.

For the rest of §2 we fix a family Σ of arithmetic subgroups of $G(\mathbb{Q})$ which is closed under finite intersections and $G(\mathbb{Q})$ -conjugation, and such that $\bigcap_{\Gamma \in \Sigma} \Gamma = \{e\}$. (The two main examples we have in mind are the family of all arithmetic subgroups of $G(\mathbb{Q})$ and the family of all congruence subgroups of $G(\mathbb{Q})$.) All arithmetic groups will be assumed to be in Σ and the notation $\varinjlim_{\Gamma} \dots$ will mean a direct limit over $\Gamma \in \Sigma$.

For $\Gamma' \subset \Gamma$ there is a finite covering $M_{\Gamma'} \rightarrow M_{\Gamma}$. Consider the direct limit

$$H^i(\mathcal{M}) := \varinjlim_{\Gamma} H^i(M_{\Gamma}).$$

For $g \in G(\mathbb{Q})$, left translation by g^{-1} on \mathbb{B} descends to an isomorphism $g^{-1} \cdot : M_{g\Gamma g^{-1}} \rightarrow M_{\Gamma}$. This defines an action of $G(\mathbb{Q})$ on the direct limit $H^i(\mathcal{M})$ where g acts by pullback by $(g^{-1} \cdot)^* : H^i(M_{\Gamma}) \rightarrow H^i(M_{g\Gamma g^{-1}})$. (The inverse is required to make this a left action.) The transition maps in this direct limit are injective. (If $\Gamma' \subset \Gamma$ is a normal subgroup of finite index, then the covering $p : M_{\Gamma'} \rightarrow M_{\Gamma}$ is Galois, so that $p^* : H^i(M_{\Gamma}) \rightarrow H^i(M_{\Gamma'})^{\Gamma}$ is an isomorphism.)

The variety M_{Γ}^* is normal with isolated singularities, so by Lemma 2.1 we have:

$$IH^i(M_{\Gamma}^*) = \begin{cases} H^i(M_{\Gamma}) & i < n \\ \text{im}[H_c^n(M_{\Gamma}) \rightarrow H^n(M_{\Gamma})] & i = n \\ H_c^i(M_{\Gamma}) & i > n. \end{cases} \quad (2.17)$$

The following general properties of intersection cohomology are evident in our situation from this description:

- (i) there is a nondegenerate pairing $IH^i(M_{\Gamma}^*) \times IH^{2n-i}(M_{\Gamma}^*) \rightarrow \mathbb{C}$
- (ii) $IH^*(M_{\Gamma}^*)$ is a module over $H^*(M_{\Gamma}^*)$ and there are natural $H^*(M_{\Gamma}^*)$ -module maps

$$H^*(M_{\Gamma}^*) \rightarrow IH^*(M_{\Gamma}^*) \rightarrow H^*(M_{\Gamma})$$

- (iii) for $\Gamma' \subset \Gamma$ of finite index there is a natural pullback $IH^i(M_{\Gamma}^*) \rightarrow IH^i(M_{\Gamma'}^*)$

The pullback maps in (iii) are injective (as follows from the fact that for $\Gamma' \subset \Gamma$ normal and of finite index we have $H^i(M_\Gamma) = H^i(M_{\Gamma'})^\Gamma$ and $H_c^i(M_\Gamma) = H_c^i(M_{\Gamma'})^\Gamma$). Consider the direct limit

$$IH^i(\mathcal{M}^*) := \varinjlim_\Gamma IH^i(M_\Gamma^*).$$

The isomorphism $g^{-1} \cdot : M_{g\Gamma g^{-1}} \rightarrow M_\Gamma$ extends to an isomorphism $g^{-1} \cdot : M_{g\Gamma g^{-1}}^* \rightarrow M_\Gamma^*$. This gives an action of $G(\mathbb{Q})$ on $IH^i(\mathcal{M}^*)$ where g acts by pullback by $(g^{-1} \cdot)^* : IH^i(M_\Gamma^*) \rightarrow IH^i(M_{g\Gamma g^{-1}}^*)$.

The direct limit of the cohomology of smooth compactifications is

$$H^i(\overline{\mathcal{M}}) := \varinjlim_\Gamma H^i(\overline{M}_\Gamma).$$

The transition maps are injective because for $\Gamma' \subset \Gamma$ a normal subgroup of finite index, $H^i(\overline{M}_{\Gamma'})^\Gamma = H^i(\overline{M}_\Gamma)$. (Indeed, for the projection $p : \overline{M}_{\Gamma'} \rightarrow \overline{M}_\Gamma$ we have $p_! p^*(\alpha) = \alpha \cup p_!(1) = |\Gamma/\Gamma'| \alpha$. So the pullback p^* is injective. Averaging over Γ/Γ' gives a map $H^i(\overline{M}_{\Gamma'}) \rightarrow H^i(\overline{M}_\Gamma)$ which splits p^* .) It follows that $H^i(\overline{M}_\Gamma) = H^i(\overline{\mathcal{M}})^\Gamma$ for any Γ . The isomorphism $g^{-1} \cdot : M_{g\Gamma g^{-1}} \rightarrow M_\Gamma$ extends to an isomorphism $g^{-1} \cdot : \overline{M}_{g\Gamma g^{-1}} \rightarrow \overline{M}_\Gamma$ and pullback by these isomorphisms defines a $G(\mathbb{Q})$ action on $H^i(\overline{\mathcal{M}})$ by ring automorphisms.

It will also be necessary to consider the direct limit

$$H^i(\mathcal{D}) := \varinjlim_\Gamma H^i(D_\Gamma)$$

for $0 \leq i \leq 2n - 2$ with pullbacks as transition maps. Similarly, we will need the direct limit

$$H_{\mathcal{D}}^i(\overline{\mathcal{M}}) := \varinjlim_\Gamma H_{D_\Gamma}^i(\overline{M}_\Gamma)$$

for $2 \leq i \leq 2n$. Here the transition maps come from the morphism of pairs $p : (\overline{M}_{\Gamma'}, M_{\Gamma'}) \rightarrow (\overline{M}_\Gamma, M_\Gamma)$ for $\Gamma' \subset \Gamma$, since $H_{D_\Gamma}^i(\overline{M}_\Gamma) = H^i(\overline{M}_\Gamma, M_\Gamma)$. Both direct limits are $G(\mathbb{Q})$ -modules in an obvious way. It will be useful to know that in the limits $H^i(\mathcal{D})$, $H_{\mathcal{D}}^i(\overline{\mathcal{M}})$ the modules at a finite stage can be recovered from the direct limit by taking invariants, i.e. that we have

$$\begin{aligned} H^i(\mathcal{D})^\Gamma &= H^i(D_\Gamma) \\ H_{\mathcal{D}}^i(\overline{\mathcal{M}})^\Gamma &= H_{D_\Gamma}^i(\overline{M}_\Gamma) \end{aligned}$$

(For $H^i(\mathcal{D})$ one can argue as follows: If $\Gamma' \subset \Gamma$ is normal of finite index then taking Γ -invariants in the long exact sequence

$$\cdots \longrightarrow H_c^i(M_{\Gamma'}) \longrightarrow H^i(\overline{M}_{\Gamma'}) \longrightarrow H^i(D_{\Gamma'}) \longrightarrow H_c^{i+1}(M_{\Gamma'}) \longrightarrow \cdots$$

must give the long exact sequence at level Γ (because $H^i(\overline{M}_{\Gamma'})^\Gamma = H^i(\overline{M}_\Gamma)$ and $H_c^i(M_{\Gamma'})^\Gamma = H_c^i(M_\Gamma)$ as we have seen earlier). Thus $H^i(D_{\Gamma'})^\Gamma = H^i(D_\Gamma)$. The proof for $H_{D_{\Gamma'}}^i(\overline{M}_{\Gamma'})^\Gamma = H_{D_\Gamma}^i(\overline{M}_\Gamma)$ is similar, using the dual sequence.)

Thus in each of the direct limits $H^*(\mathcal{M})$, $IH^*(\mathcal{M}^*)$, $H^*(\overline{\mathcal{M}})$, $H^*(\mathcal{D})$, and $H_{\mathcal{D}}^*(\overline{\mathcal{M}})$, the groups at level Γ can be recovered by taking Γ -invariants, and these spaces of invariants are finite-dimensional. (It follows that the $G(\mathbb{Q})$ -action induces, in the space of Γ -invariants, an action of the Hecke algebra at level Γ .)

2.3. Decomposition of the direct limit $H^*(\overline{\mathcal{M}})$. For any neat Γ the results of 2.1 apply to the situation

$$\begin{array}{ccc} & & \overline{M}_\Gamma \\ & & \downarrow \\ M_\Gamma & \hookrightarrow & M_\Gamma^* \end{array}$$

to give a decomposition

$$H^{i+n}(\overline{M}_\Gamma) = IH^{i+n}(M_\Gamma^*) \oplus \begin{cases} H^{i+n}(D_\Gamma) & i \geq 0 \\ H_{D_\Gamma}^{i+n}(\overline{M}_\Gamma) & i < 0. \end{cases} \quad (2.18)$$

We then have:

Theorem 2.6. *The decompositions (2.18) induce a $G(\mathbb{Q})$ -invariant decomposition*

$$H^{i+n}(\overline{\mathcal{M}}) = IH^{i+n}(\mathcal{M}^*) \bigoplus \begin{cases} H^{i+n}(\mathcal{D}) & i \geq 0 \\ H_{\mathcal{D}}^{i+n}(\overline{\mathcal{M}}) & i < 0 \end{cases} \quad (2.19)$$

in the limit over all arithmetic subgroups of $G(\mathbb{Q})$ in Σ .

Proof. If $\Gamma' \subset \Gamma$ is of finite index there is a map of pairs

$$p : (\overline{M}_{\Gamma'}, M_{\Gamma'}) \longrightarrow (\overline{M}_\Gamma, M_\Gamma).$$

Each term in this decomposition at level Γ has a pullback p^* to the corresponding term at level Γ' , and these are compatible with the cohomology long exact sequences of the pairs (by the naturality of the sequence). Our first task is to prove that the decompositions (2.18) at different levels are compatible, so that we have a decomposition of the direct limit. In this proof we will usually drop subscripts Γ, Γ' etc. to lighten the notation. So we use the notation \overline{M}, D, c etc. and \overline{M}', D', c' etc. for objects at level Γ' .

Recall (from (2.3) and the remarks before the proof of Prop. 2.2) that the Chern class c admits a refinement in $H_D^2(\overline{M})$ (and similarly for c'). Consider the diagram relating levels Γ and Γ' :

$$\begin{array}{ccccc} H^0(D') & \xrightarrow{\cong} & H_{D'}^2(\overline{M}') & \longrightarrow & H^2(D') \\ \uparrow & & \uparrow p^* & & \uparrow p^* \\ H^0(D) & \xrightarrow{\cong} & H_D^2(\overline{M}) & \longrightarrow & H^2(D) \end{array} \quad (*)$$

(The horizontal maps in the square marked $(*)$ are Thom isomorphisms and it does not commute.) The class c is the image of $1 \in H^0(D)$ under the maps indicated (by (2.1), (2.2)), and similarly for c' . By Lemma 1.8, for each component D'_i of D' we have the relation $p^*(c)|_{D'_i} = \lambda_i c'|_{D'_i}$ in $H^2(D')$ for some scalars $\lambda_i \in \mathbb{Q}^\times$. By the diagram one has the same relation

$$p^*(c)|_{D'_i} = \lambda_i c'|_{D'_i} \quad (2.20)$$

in $H_{D'}^2(\overline{M}')$ for the refined classes.

To prove the compatibility of decompositions at levels Γ and Γ' it is enough to check that the splittings s_i, s'_i defined in the proof of Prop. 2.2 (specifically by the diagrams (2.10) and (2.11) at each level), are compatible under pullback by p . The case $i = 0$ is immediate from

the naturality of the maps adj, adj' , so we are left with $i \neq 0$. First consider s_i for $i > 0$. Consider the diagram relating the splittings at levels Γ' and Γ :

$$\begin{array}{ccccc}
& & \xrightarrow{s'_i} & & \\
H^{i+n}(D') & \xleftarrow{\cdot c'^i} & H_{D'}^{-i+n}(\overline{M}') & \xrightarrow{\cdot c'^i \circ adj'} & H^{i+n}(\overline{M}') \\
\uparrow p^* & & \uparrow p^* & & \uparrow p^* \\
(1) & & (2) & & \\
H^{i+n}(D) & \xleftarrow{\cdot c^i} & H_D^{-i+n}(\overline{M}) & \xrightarrow{\cdot c^i \circ adj} & H^{i+n}(\overline{M}) \\
& & \xrightarrow{s_i} & &
\end{array} \tag{2.21}$$

The squares (1) and (2) do *not* commute in general because $p^*(c) \neq c'$. Nevertheless, the outer square formed by s_i and s'_i *does* commute, as we will now show. For this it will be enough to concentrate attention on a single component of D , which we will continue to denote by D . If D'_1, \dots, D'_r are the components of D' lying over D then it is enough to show that $p^* \circ s_i = s'_i \circ (p|_{D'_k})^*$, i.e. that the diagram

$$\begin{array}{ccccc}
& & \xrightarrow{s'_i} & & \\
H^{i+n}(D'_k) & \xleftarrow{\cdot c'^i} & H_{D'_k}^{-i+n}(\overline{M}') & \xrightarrow{\cdot c'^i \circ adj'} & H^{i+n}(\overline{M}') \\
\uparrow & & \uparrow & & \uparrow \\
(1)_k & & (2)_k & & \\
H^{i+n}(D) & \xleftarrow{\cdot c^i} & H_D^{-i+n}(\overline{M}) & \xrightarrow{\cdot c^i \circ adj} & H^{i+n}(\overline{M}) \\
& & \xrightarrow{s_i} & &
\end{array} \tag{2.22}$$

commutes for each k . Recall that we have the relation $p^*(c)|_{D'_k} = \lambda_k c'|_{D'_k}$ in $H_{D'}^2(\overline{M}') = \bigoplus_k H_{D'_k}^2(\overline{M}')$ for some scalars $\lambda_k \in \mathbb{Q}^\times$ by (2.20). Thus $(1)_k$ commutes up to the scalar λ_k^i in the sense that

$$p^* \circ (\cdot c^i) = \lambda_k^i (\cdot c'^i) \circ p^*. \tag{2.23}$$

The square $(2)_k$ also commutes up to the scalar λ_k^i . To see this consider the following diagram:

$$\begin{array}{ccccc}
H_{D'_k}^{-i+n}(\overline{M}') & \xrightarrow{\cdot c'^i \circ adj'} & H^{i+n}(\overline{M}') & \xrightarrow{res'} & H^{i+n}(D'_k) \\
\uparrow & & \uparrow & & \uparrow \\
(2)_k & & & & \\
H_D^{-i+n}(\overline{M}) & \xrightarrow{\cdot c^i \circ adj} & H^{i+n}(\overline{M}) & \xrightarrow{res} & H^{i+n}(D)
\end{array} \tag{2.24}$$

The outer square commutes up to λ_k^i , i.e.

$$p^* \circ res \circ (\cdot c^i) \circ adj = \lambda_k^i res' \circ (\cdot c'^i) \circ adj' \circ p^*.$$

Since res (resp. res') is injective on the image of $c^i \circ adj$ (resp. $c'^i \circ adj'$) (by hard Lefschetz) and the square involving res, res' commutes, this proves that $(2)_k$ commutes up to λ_k^i in the sense that

$$p^* \circ (\cdot c^i) \circ adj = \lambda_k^i (\cdot c'^i) \circ adj' \circ p^*. \tag{2.25}$$

Thus both squares in (2.22) commute up to the *same* scalar λ_k^i . Since the horizontal maps in $(1)_k$ are *inverted* to define s_i, s'_i , these scalars cancel and the outer square (involving s_i, s'_i) in (2.22) commutes for each k . Hence the same is true in (2.21).

This proves that the splitting s_i for $i > 0$ is compatible with change of level. The case of s_i for $i < 0$ is treated in exactly the same way using the diagram (2.11) and the property (2.20) for c, c' , so we omit the argument.

Thus we have a decomposition of the direct limit $H^*(\overline{\mathcal{M}})$ as claimed. It remains to show that it is $G(\mathbb{Q})$ -invariant. If $g^{-1} : \overline{\mathcal{M}}_{g\Gamma g^{-1}} \rightarrow \overline{\mathcal{M}}_\Gamma$ is the isomorphism induced by $g \in G(\mathbb{Q})$ then it relates the normal bundles, so that $(g^{-1})^*(c_\Gamma) = c_{g\Gamma g^{-1}}$. (This also follows from the expression in Lemma 1.6.) \square

Remark 2.7. The proof of Theorem 2.6 works in a more general context than ball quotients. Suppose that we are given a commutative diagram

$$\begin{array}{ccccc}
 & & Y' & & \\
 & & \downarrow \pi' & \searrow p & \\
 U' \hookrightarrow & X' & & & Y \\
 & \searrow & \square & \swarrow q & \downarrow \pi \\
 & & U \hookrightarrow & X &
 \end{array} \tag{2.26}$$

where $(\pi : Y \rightarrow X, U, Z)$ and $(\pi' : Y' \rightarrow X', U', Z')$ are as in Prop. 2.2, p, q are finite, and the square marked \square is Cartesian. We use c, c' for the Chern classes of normal bundles of Z, Z' . Assume further that

- (1) $p|_{U'} = q|_{U'} : U' \rightarrow U$ is étale, $p^{-1}(Z) = Z'$ and $p|_{Z'}$ is étale.
- (2) for each component Z'_i of Z' there is a nonzero scalar λ_i such that

$$p^*(c)|_{Z'_i} = \lambda_i c'|_{Z'_i}. \tag{2.27}$$

(In the setting of ball quotients $p : \overline{\mathcal{M}}_{\Gamma'} \rightarrow \overline{\mathcal{M}}_\Gamma$, (2) holds by Lemma 1.8.) Then the proof shows that the decompositions of $H^*(Y)$ and $H^*(Y')$ produced in Prop. 2.2 are compatible under p^* .

Remark 2.8. We explain why Theorem 2.6 (rather, the abstracted version of Remark 2.7) holds from the point of view of the decomposition theorem of [2] (continuing Remark 2.5). Once again, we leave some details to be checked by the reader familiar with [2]. In a geometric setup $\pi : Y \rightarrow X$ as in Prop. 2.2 and given a Lefschetz operator we will always use Deligne's decomposition from Remark 2.5 in which the relations (2.15) hold.

If $\pi : Y \rightarrow X$ is as in Prop. 2.2 and L is a Lefschetz operator then replacing L by a nonzero scalar multiple (e.g. if L comes from a line bundle \mathcal{L} then one might replace \mathcal{L} by a power) does not change the decomposition because the relations (2.15) still hold. More generally, if L_1, L_2 are Lefschetz operators satisfying

$$(*) \text{ for each } x \in X - U, L_1|_{U \sqcup \{x\}} \text{ is a nonzero scalar multiple of } L_2|_{U \sqcup \{x\}}$$

then L_1 and L_2 give the same decomposition. (Indeed, if α_1, α_2 are the two isomorphisms $R\pi_*\mathbb{C}_Y[n] \rightarrow \bigoplus_k {}^pH^k(R\pi_*\mathbb{C}_Y[n])[-k]$ then $A := \alpha_1 \circ \alpha_2^{-1}$ is a matrix (A_{ij}) with

$$A_{ij} : {}^pH^j(R\pi_*\mathbb{C}_Y[n])[-j] \longrightarrow {}^pH^i(R\pi_*\mathbb{C}_Y[n])[-i].$$

Since $A|_{U \sqcup \{x\}} = \alpha_1 \circ \alpha_2^{-1}|_{U \sqcup \{x\}} = id$ for any $x \in X - U$ (because of $(*)$), it follows from (2.13) and (2.14) that $A_{ij} = 0$ if $i \neq j$ and $A_{ii} = id$ for $i \neq 0$. A_{00} respects the decomposition (2.14) and is the identity on each summand. (For the summands supported on $X - U$ this follows from $(*)$, and $A_{00}|_{j_* \mathbb{C}_U[n]} \in \text{End}(j_* \mathbb{C}_U[n])$ is the functorial extension of $A_{00}|_U = id$.) Thus $A = id$ and $\alpha_1 = \alpha_2$.)

Now suppose we are in the situation of Remark 2.7, viz. a diagram (2.26) with the properties mentioned there, including (2.27). The pullback in cohomology by $p : Y' \rightarrow Y$ is induced by the canonical morphism of sheaves $\mathbb{C}_Y \rightarrow p_* \mathbb{C}_{Y'}$ or, applying $R\pi_*$, by the morphism ρ in the following diagram:

$$\begin{array}{ccc} R\pi_* \mathbb{C}_Y[n] & \xrightarrow{\rho} & q_* R\pi'_* \mathbb{C}_{Y'}[n] \\ \alpha \downarrow & & \downarrow q_* \alpha' \\ \bigoplus_k {}^p H^k(R\pi_* \mathbb{C}_Y[n])[-k] & \xrightarrow{\bigoplus_k {}^p H^k(\rho)[-k]} & \bigoplus_k {}^p H^k(q_* R\pi'_* \mathbb{C}_{Y'}[n])[-k] \end{array} \quad (2.28)$$

The rest of the diagram is as follows: The line bundles $\mathcal{O}(-Z), \mathcal{O}(-Z')$ give Lefschetz operators L, L' on $R\pi_* \mathbb{C}_Y[n]$ and $R\pi'_* \mathbb{C}_{Y'}[n]$, respectively, and hence isomorphisms α as in the diagram and $\alpha' : R\pi'_* \mathbb{C}_{Y'}[n] \rightarrow \bigoplus_k {}^p H^k(R\pi'_* \mathbb{C}_{Y'}[n])[-k]$. Since q is finite, q_* is t -exact for the perverse t -structure, i.e. ${}^p H^k \circ q_* = q_* \circ {}^p H^k$, so that $q_* \alpha'$ is as in the diagram. The lower horizontal arrow is self-explanatory. We will show that the diagram commutes, which implies Theorem 2.6.

There is a second Lefschetz operator L'' on $R\pi'_* \mathbb{C}_{Y'}$, given by $p^* \mathcal{O}(-Z)$ (which is relatively ample for π' since p is finite). By the previous paragraph, L' and L'' give the same decomposition of $R\pi'_* \mathbb{C}_{Y'}[n]$ (they satisfy $(*)$ by assumption (2.27)), so we will replace L' by L'' in the discussion. By duality (using the smoothness of Y, Y' and finiteness of p), the morphism $\mathbb{C}_Y \rightarrow p_* \mathbb{C}_{Y'}$ is split, i.e. there is a decomposition $p_* \mathbb{C}_{Y'} = \mathbb{C}_Y \oplus F$ and hence

$$q_* R\pi'_* \mathbb{C}_{Y'}[n] = R\pi_* \mathbb{C}_Y[n] \oplus R\pi_* F[n].$$

The Lefschetz operator L'' , coming by pullback from Y , respects this decomposition and agrees with L on the first summand. Thus on the first factor it is Deligne's decomposition for L , i.e. the two decompositions agree. So (2.28) commutes.

Note that in the decomposition (2.16) of $H^*(Z_x)$ these abstract considerations boil down to the simple fact that the primitive cohomology and Lefschetz decomposition do not change if the class $c_1(\mathcal{L})$ is multiplied by a scalar.

3. INVARIANTS, COINVARIANTS, AND RESTRICTION

In this section we first describe the ‘‘extra’’ summands in the decomposition of $H^*(\overline{\mathcal{M}})$ in Theorem 2.6 as induced modules (Theorem 3.5). This is used to compute the $G(\mathbb{Q})$ -invariants and coinvariants in the direct limit (Prop. 3.11, Cor. 3.12). With these results in hand we are able to adapt the methods of [30] to treat restriction maps, proving Lefschetz properties (Theorems 3.17 and 3.19) and a cup product property (Thm 3.21).

In 3.1-3.3 the family Σ is either the family of arithmetic subgroups of a given group or the family of congruence subgroups. From 3.4 onwards, Σ is the family of congruence subgroups, i.e. all arithmetic subgroups are congruence and limits such as $H^(\overline{\mathcal{M}}), IH^*(\mathcal{M}^*)$ etc. are understood to be over the family of congruence subgroups.*

3.1. Induced modules. We need some elementary facts about duality and induction in a slightly nonstandard context. This context allows us to work with \mathbb{Q} -points of groups and avoid the machinery of adeles; the reader should keep in mind that because of this in many standard formulas below g gets replaced by g^{-1} (as we already saw in the discussion in 2.2).

Let H be a \mathbb{Q} -algebraic group. A vector in an $H(\mathbb{Q})$ -module is Σ -smooth if its stabilizer is an arithmetic subgroup in Σ . An $H(\mathbb{Q})$ -module V is Σ -smooth if every vector in V is smooth and Σ -admissible if it is Σ -smooth and the space of invariants V^Γ under any arithmetic subgroup $\Gamma \subset H(\mathbb{Q})$ in Σ is finite-dimensional. The *contragredient* V^\sim of a Σ -smooth $H(\mathbb{Q})$ -module V is the subspace of Σ -smooth vectors in the linear dual $V^* = \text{Hom}(V, \mathbb{C})$ with the action $({}^t g)^{-1}$.

Lemma 3.1. *Let V, W be Σ -smooth $G(\mathbb{Q})$ -modules. Then*

- (i) $\text{Hom}_{G(\mathbb{Q})}(V, W^\sim) = \text{Hom}_{G(\mathbb{Q})}(W, V^\sim)$
- (ii) $V \cong (V^\sim)^\sim$ if V is Σ -admissible.

Proof. Let $G(\mathbb{Q})_\Sigma$ be the completion of $G(\mathbb{Q})$ with respect to the uniform topology in which a fundamental system of neighborhoods of the identity are given by arithmetic subgroups. The closure $\bar{\Gamma}^\Sigma$ in $G(\mathbb{Q})_\Sigma$ of an arithmetic group $\Gamma \in \Sigma$ is profinite, hence compact. (These give a fundamental system of neighborhoods for the topology on $G(\mathbb{Q})_\Sigma$, which is locally compact.) A Σ -smooth $G(\mathbb{Q})$ -module V is canonically a module for $G(\mathbb{Q})_\Sigma$. Thus for an arithmetic group $\Gamma \in \Sigma$, one has a decomposition $V = V^{\bar{\Gamma}^\Sigma} \oplus V' = V^\Gamma \oplus V'$ where $V^{\bar{\Gamma}^\Sigma} = V^\Gamma$ by density. It follows that $(V^\Gamma)^* = (V^*)^\Gamma = (V^\sim)^\Gamma$. (i) and (ii) follow directly from these equalities. \square

Let H be a subgroup of G . For a Σ -smooth $H(\mathbb{Q})$ -module W the induced module $I_H^G(W)$ is the space of functions

$$I_H^G(W) := \left\{ f : G(\mathbb{Q}) \rightarrow W \text{ s.t. } \begin{array}{l} f(gh) = h^{-1} \cdot f(g) \text{ for } h \in H(\mathbb{Q}), g \in G(\mathbb{Q}) \\ f \text{ is left-invariant under an arithmetic} \\ \text{subgroup of } G(\mathbb{Q}) \text{ in } \Sigma \end{array} \right\}$$

with the action of $G(\mathbb{Q})$ by left translations, i.e. $(g \cdot f)(x) = f(g^{-1}x)$. It is Σ -smooth.

Lemma 3.2. *The functor I_H^G is right adjoint to restriction, i.e. there is Frobenius reciprocity: For Σ -smooth modules V, W ,*

$$\text{Hom}_{G(\mathbb{Q})}(V, I_H^G(W)) = \text{Hom}_{H(\mathbb{Q})}(V, W).$$

It takes Σ -admissible modules to Σ -admissible modules if $\Gamma \backslash G(\mathbb{Q})/H(\mathbb{Q})$ is finite for all $\Gamma \in \Sigma$ (e.g. if H is a parabolic subgroup).

Proof. For the first part we follow the usual proof of Frobenius reciprocity: Evaluation at the identity defines a map $e : I_H^G(W) \rightarrow W$. If $\Phi \in \text{Hom}_{G(\mathbb{Q})}(V, I_H^G(W))$ then $e \circ \Phi \in \text{Hom}_{H(\mathbb{Q})}(V, W)$. The inverse map is defined by $\phi \mapsto \Phi$ where $\phi \in \text{Hom}_{H(\mathbb{Q})}(V, W)$ defines $\Phi : V \rightarrow I_H^G(W)$ by $\Phi(v)(g) = \phi(g^{-1} \cdot v)$. The second part follows in the usual way from the finiteness of $\Gamma \backslash G(\mathbb{Q})/H(\mathbb{Q})$. \square

3.2. Decomposition and induced modules. We use the induced module construction to identify the extra summands in Theorem 2.6 as $G(\mathbb{Q})$ -modules. From now until the end of the section we are in the situation of ball quotients of §1 and the notation fixed there will be used freely.

Lemma 3.3. *Fix an isotropic line $\ell \in \mathbb{B}^*$ and let $P_\ell = \text{Stab}_G(\ell)$, $\mathbf{u}_\ell, \mathbf{v}_\ell$ be as in 1.3. Then as $G(\mathbb{Q})$ -modules*

$$H^i(\mathcal{D}) = I_{P_\ell}^G(\wedge^i \mathbf{v}_{\ell, \mathbb{C}}^*)$$

and

$$H^i(\overline{\mathcal{M}}) = I_{P_\ell}^G(\wedge^{i-2} \mathbf{v}_{\ell, \mathbb{C}}^* \otimes \mathbf{u}_{\ell, \mathbb{C}}^*)$$

where the actions on $\wedge^i \mathbf{v}_{\ell, \mathbb{C}}^*$ and $\wedge^{i-2} \mathbf{v}_{\ell, \mathbb{C}}^* \otimes \mathbf{u}_{\ell, \mathbb{C}}^*$ are induced by the coadjoint action.

Proof. The boundary divisor D_Γ is a disjoint union of components indexed by Γ -equivalence classes of isotropic lines:

$$D_\Gamma = \bigsqcup_{\lambda \in \Gamma \backslash (\mathbb{B}^* - \mathbb{B})} D_{\Gamma, \lambda}.$$

For an isotropic line λ , let

$$R_\lambda^i := \varinjlim_\Gamma H^i(D_{\Gamma, \lambda}).$$

This has an action of $P_\lambda(\mathbb{Q})$ as follows: For $q \in P_\lambda(\mathbb{Q})$ the isomorphism $q^{-1} : D_{q\Gamma q^{-1}} \rightarrow D_\Gamma$ takes $D_{q\Gamma q^{-1}, \ell}$ to $D_{\Gamma, \ell}$ and we let $q \in P_\lambda(\mathbb{Q})$ act on the direct limit by $(q^{-1})^*$ to get a left action. We will first show that $H^i(\mathcal{D}) \cong I_{P_\ell}^G(R_\ell^i)$ as $G(\mathbb{Q})$ -modules and then compute R_ℓ^i as a $P_\ell(\mathbb{Q})$ -module.

Frobenius reciprocity applied to the obvious $P_\ell(\mathbb{Q})$ -module map $H^i(\mathcal{D}) \rightarrow R_\ell^i$ by

$$\alpha = (\alpha_\Gamma)_\Gamma \mapsto (\alpha_\Gamma|_{D_{\Gamma, \ell}})_\Gamma$$

gives a $G(\mathbb{Q})$ -module map

$$H^i(\mathcal{D}) \longrightarrow I_{P_\ell}^G(R_\ell^i). \quad (3.1)$$

We will use the following observations to prove that this is an isomorphism:

(1) For each Γ there is a natural identification

$$H^i(D_{\Gamma, \lambda}) = (R_\lambda^i)^{\Gamma \cap P_\lambda}. \quad (3.2)$$

(Indeed, for $\Gamma' \subset \Gamma$ the morphism $D_{\Gamma', \lambda} \rightarrow D_{\Gamma, \lambda}$ is a Galois covering with group $\Gamma'_{V_\lambda} / \Gamma_{V_\lambda}$, so that $H^i(D_{\Gamma', \lambda})^{\Gamma'_{V_\lambda}} = H^i(D_{\Gamma, \lambda})$. Since Γ_{U_λ} acts trivially,

$$H^i(D_{\Gamma', \lambda})^{\Gamma \cap P_\lambda} = H^i(D_{\Gamma', \lambda})^{\Gamma_{V_\lambda}} = H^i(D_{\Gamma, \lambda})$$

This gives (3.2).)

(2) If $\lambda = g\ell$ then $P_\lambda = gP_\ell g^{-1}$ and pullback by the isomorphism $g : D_{g^{-1}\Gamma g, \ell} \rightarrow D_{\Gamma, \lambda}$ induces an isomorphism $(g \cdot)^* : R_\lambda^i \rightarrow R_\ell^i$. At level Γ this restricts to an isomorphism

$$(R_\lambda^i)^{\Gamma \cap P_\lambda} \longrightarrow (R_\ell^i)^{g^{-1}\Gamma g \cap P_\ell} \quad (3.3)$$

with inverse $(g^{-1} \cdot)^*$. This is compatible with (3.2) in the sense that

$$\begin{array}{ccc} H^i(D_{\Gamma, \lambda}) & \xrightarrow{(g \cdot)^*} & H^i(D_{g^{-1}\Gamma g, \ell}) \\ \downarrow & & \downarrow \\ (R_\lambda^i)^{\Gamma \cap P_\lambda} & \xrightarrow{(g \cdot)^*} & (R_\ell^i)^{g^{-1}\Gamma g \cap P_\ell} \end{array} \quad (3.4)$$

commutes.

Fix a level Γ and choose representatives $\{g_i\}_{i=1}^r$ for the double cosets $\Gamma \backslash G(\mathbb{Q}) / P_\ell(\mathbb{Q})$. The isomorphism

$$I_{P_\ell}^G(R_\ell^i)^\Gamma = \bigoplus_j (R_\ell^i)^{g_j^{-1}\Gamma g_j \cap P_\ell}$$

is given by $f \mapsto (f(g_1), \dots, f(g_r))$. Under (3.1) a class $\alpha \in H^i(\mathcal{D})^\Gamma$ maps to the element with j -component $(g_j^{-1} \cdot \alpha)_\Gamma|_{D_{\Gamma,\ell}} = (g_j \cdot)^*(\alpha)|_{D_{\Gamma,\ell}}$ (as in the proof of Frobenius reciprocity in Lemma 3.2). Restricting to the g_j -summand of

$$H^i(\mathcal{D})^\Gamma = H^i(D_\Gamma) = \bigoplus_j H^i(D_{\Gamma, g_j \ell})$$

gives the composition in (3.4), which is an isomorphism with the summand $(R_\ell^i)^{g_j^{-1}\Gamma g_j \cap P_\ell}$ of $I_{P_\ell}^G(R_\ell^i)^\Gamma$. Thus $H^i(\mathcal{D})^\Gamma \rightarrow I_{P_\ell}^G(R_\ell^i)^\Gamma$ is an isomorphism for each Γ and (3.1) is an isomorphism.

Next we compute the $P_\ell(\mathbb{Q})$ -module R_ℓ^i . (Since each $D_{\Gamma,\lambda}$ is an abelian variety, $R_\lambda^i = \wedge^i R_\lambda^1$ so one has only to compute R_ℓ^1 .) Let $p \in P_\ell(\mathbb{Q})$ and $\Gamma' \subset \Gamma \cap p\Gamma p^{-1}$. The action of p on R_ℓ^i is computed by comparing pullbacks by

$$\begin{aligned} e \cdot : D_{\Gamma',\ell} &\rightarrow D_{\Gamma,\ell} \\ p^{-1} \cdot : D_{\Gamma',\ell} &\rightarrow D_{\Gamma,\ell} \end{aligned}$$

Recall from the construction of \overline{M}_Γ described in §1.4 that there is a natural identification

$$D_{\Gamma,\ell} = \Gamma_{W_\ell} U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$$

and the morphism $p^{-1} \cdot$ is induced by left translation by p^{-1} on $U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$. The action of elements of $W_\ell(\mathbb{Q})$ on R_ℓ^i is trivial (for $u \in U_\ell(\mathbb{Q})$ we have $u^{-1} \cdot = e \cdot$ and for elements of $V_\ell(\mathbb{Q})$ this holds because the cohomology is computed by invariant forms). So the action is one of $P_\ell(\mathbb{Q})/W_\ell(\mathbb{Q}) = (M_\ell A_\ell)(\mathbb{Q})$ and can be computed using any lift of $M_\ell A_\ell$ in P , e.g. the one denoted $\tilde{M}_\ell \tilde{A}_\ell$ in 1.3 fixed by the choice of basis in loc. cit. Recall from Remark 1.4 that the basepoint in $\mathbb{B}(\ell)$ given by the basis gives an identification $V_\ell(\mathbb{R}) \rightarrow U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$ which intertwines the conjugation action of $(M_\ell A_\ell)(\mathbb{R})$ with the action of $(\tilde{M}_\ell \tilde{A}_\ell)(\mathbb{R})$ on $U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$. Using the induced isomorphism $\Gamma_{V_\ell} \backslash V_\ell(\mathbb{R}) \rightarrow \Gamma_{W_\ell} U_\ell(\mathbb{C}) \backslash \mathbb{B}(\ell)$ we are reduced to comparing, for $p \in (M_\ell A_\ell)(\mathbb{Q})$, the two pullbacks

$$\begin{aligned} (e \cdot)^* : H^i(\Gamma'_{V_\ell} \backslash V_\ell(\mathbb{R})) &\longrightarrow H^i(\Gamma_{V_\ell} \backslash V_\ell(\mathbb{R})) \\ \text{Ad}(p^{-1})^* : H^i(\Gamma'_{V_\ell} \backslash V_\ell(\mathbb{R})) &\longrightarrow H^i(\Gamma_{V_\ell} \backslash V_\ell(\mathbb{R})). \end{aligned}$$

The natural isomorphisms $H^i(\Gamma_{V_\ell} \backslash V_\ell(\mathbb{R})) = \wedge^i \mathfrak{v}_{\ell,\mathbb{C}}^*$ given (e.g.) by invariant differential forms are compatible with change of level (i.e. so that $(e \cdot)^* = id$) and in this identification $\text{Ad}(p^{-1})^*$ is the transpose of the action of p^{-1} on $\wedge^i \mathfrak{v}_{\ell,\mathbb{C}}$, i.e. the coadjoint action of p .

This completes the proof that $H^i(\mathcal{D}) = I_{P_\ell}^G(\wedge^i \mathfrak{v}_{\ell,\mathbb{C}}^*)$. Next consider $H^i_{\mathcal{D}}(\overline{\mathcal{M}})$. The arguments above, mutatis mutandis, show that there is a natural $G(\mathbb{Q})$ -isomorphism $H^i_{\mathcal{D}}(\overline{\mathcal{M}}) = I_{P_\ell}^G(S_\ell^i)$ where

$$S_\ell^i := \varinjlim_{\Gamma} H^i_{D_{\Gamma,\ell}}(\overline{M}_\Gamma).$$

(The analogue of (3.2) required follows from (3.2) by the Thom isomorphism.) So it remains to prove that $S_\ell^i = \mathfrak{u}_{\ell,\mathbb{C}}^* \otimes \wedge^{i-2} \mathfrak{v}_{\ell,\mathbb{C}}^*$ as a $P_\ell(\mathbb{Q})$ -module. We first note that S_ℓ^* is (under cup product) a free graded $R_\ell^* = \wedge^* \mathfrak{v}_{\ell,\mathbb{C}}^*$ -module generated in degree two by S_ℓ^2 . (Indeed, by the Thom isomorphism (2.2) this is true at each level Γ .) So it suffices to produce a

natural isomorphism $S_\ell^2 = \mathfrak{u}_{\ell, \mathbb{C}}^*$. The natural map $S_\ell^2 \rightarrow R_\ell^2 = \wedge^2 \mathfrak{v}_{\ell, \mathbb{C}}^*$ is injective, because $H_{D_{\Gamma, \ell}}^2(\overline{M}_\Gamma) \hookrightarrow H^2(D_{\Gamma, \ell})$ for all Γ . (The image of the natural generator of $H_{D_{\Gamma, \ell}}^2(\overline{M}_\Gamma) \cong H_{2n-2}(D_{\Gamma, \ell}) \cong \mathbb{C}[D_{\Gamma, \ell}]$ is the Chern class of the normal bundle, which is nonzero.) On the other hand dualizing the Lie bracket $[\cdot, \cdot] : \mathfrak{v}_\ell \times \mathfrak{v}_\ell \rightarrow \mathfrak{u}_\ell$ gives an injective map $\delta : \mathfrak{u}_\ell^* \hookrightarrow \wedge^2 \mathfrak{v}_\ell^*$. Choosing $e \in \ell$ fixes a generator $x_e \in \mathfrak{u}_\ell$ and a dual generator $x_e^* \in \mathfrak{u}_\ell^*$. Then $\delta(x_e^*) = \Phi_e$ (by definition, cf. 1.5). Thus the image of $H_{D_{\Gamma, \ell}}^2(\overline{M}_\Gamma)$ in $H^2(D_{\Gamma, \ell}) \cong \wedge^2 \mathfrak{v}_{\ell, \mathbb{C}}^*$, which has dimension one, contains a multiple of Φ_e (namely, the Chern class $-d_e^{-1}\Phi_e$ of loc. cit.) and hence coincides with $\delta(\mathfrak{u}_{\ell, \mathbb{C}}^*) \subset \wedge^2 \mathfrak{v}_{\ell, \mathbb{C}}^*$. This gives a natural (and $P(\mathbb{Q})$ -equivariant) isomorphism $S_\ell^2 \cong \mathfrak{u}_{\ell, \mathbb{C}}^*$. \square

Lemma 3.4. *The $G(\mathbb{Q})$ -modules $H^i(\mathcal{D})$ and $H_{\mathcal{D}}^{2n-i}(\overline{\mathcal{M}})$ are contragredient to each other.*

Proof. For each Γ there is the nondegenerate Poincaré-Lefschetz duality pairing $H^i(D_\Gamma) \times H_{D_\Gamma}^{2n-i}(\overline{M}_\Gamma) \rightarrow \mathbb{C}$ which factors as the cup product $H^i(D_\Gamma) \times H_{D_\Gamma}^{2n-i} \rightarrow H_{D_\Gamma}^{2n}$ followed by the natural isomorphisms $H_{D_\Gamma}^{2n}(\overline{M}_\Gamma) = H^{2n}(\overline{M}_\Gamma) = \mathbb{C}$. (Here $H^{2n}(\overline{M}_\Gamma) = \mathbb{C}$ is fixed either by noting that \overline{M}_Γ , being an algebraic variety, is oriented (or explicitly as in 3.3 below).) In the limit this gives a pairing

$$H^i(\mathcal{D}) \times H_{\mathcal{D}}^{2n-i}(\overline{\mathcal{M}}) \rightarrow \mathbb{C}$$

to the trivial representation. (The isomorphism $H^{2n}(\overline{\mathcal{M}}) = \mathbb{C}$ comes from compatibility of orientations at different levels, either as in 3.3 below or because transition morphisms are algebraic.) Taking Γ -invariants gives back the pairing at finite level. Thus for any Γ the natural map $H^i(\mathcal{D}) \rightarrow (H_{\mathcal{D}}^{2n-i}(\overline{\mathcal{M}}))^*$ gives an isomorphism

$$\begin{aligned} H^i(\mathcal{D})^\Gamma &= H^i(D_\Gamma) \cong H_{D_\Gamma}^{2n-i}(\overline{M}_\Gamma)^* = (H_{\mathcal{D}}^{2n-i}(\overline{\mathcal{M}})^\Gamma)^* \\ &= (H_{\mathcal{D}}^{2n-i}(\overline{\mathcal{M}})^*)^\Gamma = (H_{\mathcal{D}}^{2n-i}(\overline{\mathcal{M}})^\sim)^\Gamma. \end{aligned}$$

(To go from the first line to the second here we have used that $(V^\Gamma)^* = (V^*)^\Gamma$ for a smooth module V , as shown in the proof of Lemma 3.1.) This proves the lemma. \square

Theorem 3.5. *There is a $G(\mathbb{Q})$ -invariant decomposition*

$$H^*(\overline{\mathcal{M}}) = IH^*(\mathcal{M}^*) \oplus \mathcal{J}^*$$

where

$$\mathcal{J}^{i+n} := \begin{cases} I_P^G(\wedge^{i+n} \mathfrak{v}_{\ell, \mathbb{C}}^*) & i \geq 0 \\ I_P^G(\wedge^{i+n-2} \mathfrak{v}_{\ell, \mathbb{C}}^* \otimes \mathfrak{u}_{\ell, \mathbb{C}}^*) & i < 0. \end{cases}$$

The Poincaré duality pairing on $H^(\overline{\mathcal{M}})$ is the direct sum of the Poincaré duality pairing on $IH^*(\mathcal{M}^*)$ and the pairing $\mathcal{J}^{i+n} \times \mathcal{J}^{-i+n} \rightarrow \mathbb{C}$ coming from Lemma 3.4. The restriction $H^*(\overline{\mathcal{M}}) \rightarrow H^*(\mathcal{M})$ factors through the projection $H^*(\overline{\mathcal{M}}) \rightarrow IH^*(\mathcal{M}^*)$ and the restriction $H^*(\overline{\mathcal{M}}) \rightarrow H^*(\mathcal{D})$ is the obvious map on the second summand.*

Proof. This follows from Theorem 2.6 and Lemmas 3.3 and 3.4. \square

Remark 3.6. The map $\wedge^{n-2} \mathfrak{v}_\ell^* \otimes \mathfrak{u}_\ell^* \rightarrow \wedge^n \mathfrak{v}_\ell^*$ induced by the map $\mathfrak{u}_\ell^* \rightarrow \wedge^2 \mathfrak{v}_\ell^*$ dual to the bracket is a P -equivariant isomorphism. Thus the summand for $i = 0$ is also described as $I_P^G(\wedge^{n-2} \mathfrak{v}_\ell^* \otimes \mathfrak{u}_\ell^*)$.

Remark 3.7. The theorem should admit a generalization to other Shimura varieties, but a nonobvious one. The nonuniqueness of toroidal compactifications of general Shimura varieties suggests looking at the direct limit $H^*(\overline{\mathcal{M}}^{tor}) = \varinjlim_{\Gamma} \varinjlim_{\Sigma} H^*(M_{\Gamma, \Sigma})$ where the inner limit is over all data Σ which can be used to produce a smooth projective toroidal compactification at level Γ . However, a decomposition theorem like the above is not likely to hold for this direct limit: There is no $G(\mathbb{Q})$ -homomorphism $IH^*(\mathcal{M}^*) \rightarrow H^*(\overline{\mathcal{M}}^{tor})$. The solution is to consider a nonalgebraic compactification, namely the excentric Borel-Serre (eBS) compactification, and its intersection cohomology. In [22] we treated a smaller nonalgebraic compactification, the reductive Borel-Serre (rBS) compactification, using Hodge-theoretic methods, some of which can be adapted to the eBS compactification. (In the special case of ball quotients the eBS compactification is \overline{M}_{Γ} and the rBS compactification is M_{Γ}^* . Another simple case is that of Siegel threefolds, where the cohomology of the eBS compactification is that of the Igusa compactification.) We will explore this elsewhere.

3.3. Orientations. We will fix $G(\mathbb{Q})$ -invariant generators for $IH^{2n}(\mathcal{M}^*)$ and $H^{2n}(\overline{\mathcal{M}})$ using the Baily-Borel bundle \mathcal{L}_{Γ} of 1.2. The description there shows that \mathcal{L}_{Γ} pulls back to $\mathcal{L}_{\Gamma'}$ under $M_{\Gamma'} \rightarrow M_{\Gamma}$ so that

$$e := (c_1(\mathcal{L}_{\Gamma}))_{\Gamma} \in \varinjlim_{\Gamma} H^2(M_{\Gamma}^*)$$

defines a class in the direct limit. Moreover, $(g^{-1}\cdot)^*(\mathcal{L}_{\Gamma}) = \mathcal{L}_{g\Gamma g^{-1}}$ under the isomorphism $g^{-1}\cdot : M_{\Gamma}^* \rightarrow M_{g\Gamma g^{-1}}^*$, so that e is $G(\mathbb{Q})$ -invariant. We will use e also for the image in $IH^2(\mathcal{M}^*)$ or $H^2(\overline{\mathcal{M}})$ as the exact meaning will always be clear from the context. The map $c_1(\mathbb{L})^k \mapsto e^k$ defines homomorphisms

$$H^*(\mathbb{P}(V_{\mathbb{R}})) \rightarrow IH^*(\mathcal{M}^*) \tag{3.5}$$

$$H^*(\mathbb{P}(V_{\mathbb{R}})) \rightarrow H^*(\overline{\mathcal{M}}). \tag{3.6}$$

These are both injective. (For (3.6) this follows from the proportionality principle [19, Thm 3.2]: Since $[\overline{M}_{\Gamma}] \cap c_1(\mathcal{L}_{\Gamma})^n \sim [\mathbb{P}(V_{\mathbb{R}})] \cap c_1(\mathbb{L})^n \neq 0$ we see that $e^n \neq 0$. The injectivity of (3.5) follows from that of (3.6) because the pullback $H^*(\mathcal{M}^*) \rightarrow H^*(\overline{\mathcal{M}})$ factors through the canonical homomorphism $H^*(\mathcal{M}^*) \rightarrow IH^*(\mathcal{M}^*)$.) In particular,

$$\mathbf{or} := e^n$$

fixes $G(\mathbb{Q})$ -invariant generators for $IH^{2n}(\mathcal{M}^*)$ and $H^{2n}(\overline{\mathcal{M}})$ which we will also denote by \mathbf{or} .

3.4. Invariants and coinvariants. Theorem 3.5 lets us compute the invariants and coinvariants of $H^*(\overline{\mathcal{M}})$. In the proof of the following proposition (and hence in the sequel) we will need to restrict to congruence subgroups.

Proposition 3.8. *The $G(\mathbb{Q})$ -module $IH^*(\mathcal{M}^*)$ is semisimple and admissible. The invariants (= coinvariants) are given by $H^*(\mathbb{P}(V_{\mathbb{R}})) = \mathbb{C}[e]/(e^{n+1})$.*

Proof. Zucker's conjecture (proved by Looijenga [16] and Saper-Stern [27]) gives an isomorphism of intersection cohomology with L^2 cohomology. Combining this with results of Borel and Casselman [8] gives a natural isomorphism

$$IH^*(\mathcal{M}^*) = \bigoplus_{\pi = \pi_{\infty} \otimes \pi_f} m_{dis}(\pi) \pi_f \otimes H^*(\mathfrak{g}, K_{\infty}, \pi_{\infty}). \tag{3.7}$$

The notation is as follows: \mathfrak{g} is the Lie algebra of $G(\mathbb{R})$, K_∞ is a maximal compact subgroup, $\pi = \pi_\infty \otimes \pi_f$ runs over automorphic representations appearing in the discrete spectrum of $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$, and $m_{dis}(\pi)$ is the multiplicity of π in the discrete spectrum. This isomorphism is equivariant for the action of $G(\mathbb{Q})$ defined earlier on the left and the action on the right induced by the inclusion of $G(\mathbb{Q})$ in $G(\mathbb{A}_f)$. Admissibility now follows from the finiteness of multiplicities and from the admissibility of π_f . Semisimplicity follows from the fact that by density of $G(\mathbb{Q})$ in $G(\mathbb{A}_f)$ (which holds by strong approximation) the modules π_f are irreducible for $G(\mathbb{Q})$. The $G(\mathbb{Q})$ -invariants ($=G(\mathbb{A}_f)$ -invariants) come from the trivial representation (a discrete automorphic representation which is trivial at all finite places is necessarily trivial by strong approximation), which has multiplicity one in the discrete spectrum. Thus $IH^*(\mathcal{M}^*)^{G(\mathbb{Q})} = H^*(\mathfrak{g}, K_\infty, \mathbb{C}) = H^*(\mathbb{P}(V_{\mathbb{R}}))$.

For the coinvariants, note that since (3.7) is an algebraic direct sum it is enough to show that each summand for π_f nontrivial has no coinvariants. By Lemma 3.1(i), the coinvariants of π_f are the invariants of the contragredient as a $G(\mathbb{Q})$ -module (i.e. in the sense of 3.1). By strong approximation, the contragredient as a $G(\mathbb{Q})$ -module of a $G(\mathbb{A}_f)$ -module is its $G(\mathbb{A}_f)$ -contragredient. (Indeed, a vector in the linear dual fixed by a congruence group Γ is fixed by $K = \bar{\Gamma}$, which is an open subgroup of $G(\mathbb{A}_f)$ by strong approximation.) By density of $G(\mathbb{Q})$ in $G(\mathbb{A}_f)$ the contragredient has no invariants. \square

Remark 3.9. In the special case of ball quotients the isomorphism of L^2 and intersection cohomology is elementary and was already proved in [32, §6]. Also, [8] uses Langlands' difficult spectral decomposition of $L^2(\Gamma \backslash G(\mathbb{R}))$, but there are much easier proofs of the spectral decomposition available when the real rank is one.

Remark 3.10. The proof of this proposition is the only place where we have to restrict to congruence subgroups. In particular, if this assumption could be removed here the results of the paper would hold more generally for arithmetic groups. The necessary analytic results do not seem to be available in a nonadelic context.

Proposition 3.11. *The $G(\mathbb{Q})$ -invariants and coinvariants in $H^*(\overline{\mathcal{M}})$ are given by*

$$H^*(\overline{\mathcal{M}})^{G(\mathbb{Q})} = H^*(\overline{\mathcal{M}})_{G(\mathbb{Q})} = H^*(\mathbb{P}(V_{\mathbb{R}})) = \mathbb{C}[e]/(e^{n+1}).$$

Proof. By Prop. 3.8 and Thm 3.5 we are reduced to considering the extra summands \mathcal{J}^* . For $0 < i < 2n$ we have (using Lemmas 3.1, 3.2, and 3.4):

$$\begin{aligned} \mathrm{Hom}_{G(\mathbb{Q})}(\mathbb{C}, I_P^G(\wedge^i \mathfrak{v}_{\ell, \mathbb{C}}^*)) &= \mathrm{Hom}_{P(\mathbb{Q})}(\mathbb{C}, \wedge^i \mathfrak{v}_{\ell, \mathbb{C}}^*) \\ &= \{0\} \\ \mathrm{Hom}_{G(\mathbb{Q})}(I_P^G(\wedge^i \mathfrak{v}_{\ell, \mathbb{C}}^*), \mathbb{C}) &= \mathrm{Hom}_{G(\mathbb{Q})}(\mathbb{C}, I_P^G(\wedge^{2n-i-2} \mathfrak{v}_{\ell, \mathbb{C}}^* \otimes \mathfrak{u}_{\ell, \mathbb{C}}^*)) \\ &= \mathrm{Hom}_{P(\mathbb{Q})}(\mathbb{C}, \wedge^{2n-i-2} \mathfrak{v}_{\ell, \mathbb{C}}^* \otimes \mathfrak{u}_{\ell, \mathbb{C}}^*) \\ &= \{0\} \end{aligned}$$

where we use the fact that (any lift of) $A_\ell(\mathbb{Q})$ acts by the character χ^{a+2b} on $\wedge^a \mathbf{v}_{\ell, \mathbb{C}}^* \otimes \wedge^b \mathbf{u}_{\ell, \mathbb{C}}^*$. Similarly, for $0 < i < 2n$ we have

$$\begin{aligned} \mathrm{Hom}_{G(\mathbb{Q})}(\mathbb{C}, I_P^G(\wedge^{i-2} \mathbf{v}_{\ell, \mathbb{C}}^* \otimes \mathbf{u}_{\ell, \mathbb{C}}^*)) &= \mathrm{Hom}_{P(\mathbb{Q})}(\mathbb{C}, \wedge^{i-2} \mathbf{v}_{\ell, \mathbb{C}}^* \otimes \mathbf{u}_{\ell, \mathbb{C}}^*) \\ &= \{0\} \\ \mathrm{Hom}_{G(\mathbb{Q})}(I_P^G(\wedge^{i-2} \mathbf{v}_{\ell, \mathbb{C}}^* \otimes \mathbf{u}_{\ell, \mathbb{C}}^*), \mathbb{C}) &= \mathrm{Hom}_{G(\mathbb{Q})}(\mathbb{C}, I_P^G(\wedge^{2n-i} \mathbf{v}_{\ell, \mathbb{C}}^*)) \\ &= \mathrm{Hom}_{P(\mathbb{Q})}(\mathbb{C}, \wedge^{2n-i} \mathbf{v}_{\ell, \mathbb{C}}^*) \\ &= \{0\}. \end{aligned}$$

This proves the proposition. \square

Corollary 3.12. *There is a $G(\mathbb{Q})$ -invariant decomposition*

$$H^*(\overline{\mathcal{M}}) = H^*(\mathbb{P}(V_{\mathbb{R}})) \oplus \mathcal{K}^*$$

which is orthogonal with respect to Poincaré duality and in which \mathcal{K}^* has neither invariants nor coinvariants.

3.5. Subvarieties and special cycle classes. Fix a subspace $W \subset V$ of dimension $m+1$ on which h restricts to a form with signature $(m, 1)$. Then

$$H = SU(h|_W) \subset SU(h)$$

is a \mathbb{Q} -subgroup of \mathbb{Q} -rank one with symmetric space the m -ball

$$\mathbb{B}_H = \{ \ell \in \mathbb{P}(W_{\mathbb{R}}) \mid h|_{\ell} < 0 \}$$

contained in \mathbb{B} . For $\Gamma_H = \Gamma \cap H(\mathbb{R})$ and $M_{H, \Gamma_H} = \Gamma_H \backslash \mathbb{B}_H$ we have a morphism

$$j : M_{H, \Gamma_H} \rightarrow M_{\Gamma}.$$

It is well-known (and can be seen easily from the Siegel domain picture in 1.4) that this extends analytically (and hence, since all varieties involved are projective, algebraically) to morphisms $M_H^* \rightarrow M^*$ and

$$\bar{j} : \overline{M}_{H, \Gamma_H} \rightarrow \overline{M}_{\Gamma}.$$

Both extensions are finite onto a closed subvariety of dimension m , i.e. the composition of a finite morphism and a closed immersion.

The generators $\mathbf{or} \in H^{2n}(\overline{M}_{\Gamma})$ and $\mathbf{or}_H \in H^{2m}(\overline{M}_{H, \Gamma_H})$ fixed in 3.3 give a Gysin map

$$\bar{j}_! : H^*(\overline{M}_{H, \Gamma_H}) \rightarrow H^{*+2(n-m)}(\overline{M}_{\Gamma})$$

with the property that

$$\bar{j}_!(\bar{j}^*(\alpha) \cdot \beta) = \alpha \cdot \bar{j}_!(\beta) \quad (\text{for } \alpha \in H^*(\overline{M}_{\Gamma}), \beta \in H^*(\overline{M}_{H, \Gamma_H})).$$

The *special cycle class* is

$$\xi_{\Gamma} := \bar{j}_!(1) \in H^{2c}(\overline{M}_{\Gamma}) \quad (c = n - m).$$

i.e. it is the class in $H^{2c}(\overline{M}_{\Gamma})$ defined by the property

$$\bar{j}^*(\alpha) = \lambda \mathbf{or}_H \iff \xi_{\Gamma} \cdot \alpha = \lambda \mathbf{or} \quad \text{for } \alpha \in H^{2m}(\overline{M}_{\Gamma}).$$

Lemma 3.13. *If $\Gamma' \subset \Gamma$ is normal of finite index, then $\frac{1}{|\Gamma/\Gamma'|} \sum_{\gamma \in \Gamma/\Gamma'} \gamma^*(\xi_{\Gamma'}) = \xi_{\Gamma}$.*

Proof. The averaged class is Γ -invariant, hence belongs to $H^{2c}(\overline{M}_{\Gamma})$. But it also has the defining property of ξ_{Γ} so it must be ξ_{Γ} . \square

The inclusion $\mathbb{P}(W_{\mathbb{R}}) \subset \mathbb{P}(V_{\mathbb{R}})$ gives a cycle class

$$[\mathbb{P}(W_{\mathbb{R}})] \in H^{2c}(\mathbb{P}(V_{\mathbb{R}}))$$

which we think of as a class in $H^{2c}(\overline{M}_{\Gamma}) \subset H^{2c}(\overline{M})$ which is $G(\mathbb{Q})$ -invariant (cf. Prop. 3.11). The following is the analogue of a key result of [30] (Theorem 1, proved in §3 of loc. cit.) and the proof is much the same (given Prop. 3.11).

Proposition 3.14. *The $G(\mathbb{Q})$ -submodule of $H^{2c}(\overline{M})$ generated by $\xi_{\Gamma} \in H^{2c}(\overline{M}_{\Gamma})$ has a one-dimensional space of $G(\mathbb{Q})$ -invariants spanned by $[\mathbb{P}(W_{\mathbb{R}})]$.*

Proof. Let L be the $G(\mathbb{Q})$ -module generated by ξ_{Γ} . Since the space of invariants in $H^{2c}(\overline{M})$ has dimension one, it will suffice to show that L contains $[\mathbb{P}(W_{\mathbb{R}})]$. Write $\xi_{\Gamma} = \xi_0 + \xi_1$ with respect to the decomposition of Cor. 3.12. Since $\xi_0 \in H^{2c}(\overline{M}_{\Gamma})$, it will suffice (by Poincaré duality for $\mathbb{P}(V_{\mathbb{R}})$) to show that

$$\xi_0 \cdot \beta = [\mathbb{P}(W_{\mathbb{R}})] \cdot \beta \quad \text{for all } \beta \in H^{2c}(\mathbb{P}(V_{\mathbb{R}})). \quad (3.8)$$

Note that for $\beta \in H^{2m}(\mathbb{P}(V_{\mathbb{R}}))$, we have

$$\xi_{\Gamma} \cdot \beta = \xi_0 \cdot \beta.$$

(Indeed, otherwise $\xi_1 \cdot \beta = \lambda_{\beta}$ **or** would define an invariant linear functional $\beta \mapsto \lambda_{\beta}$ on \mathcal{H}^{2c} in Cor. 3.12. But \mathcal{H}^i has no coinvariants, so this must be zero.) On the other hand there is a diagram

$$\begin{array}{ccc} H^i(\mathbb{P}(V_{\mathbb{R}})) & \longrightarrow & H^i(\overline{M}_{\Gamma}) \\ \downarrow & & \downarrow \bar{j}^* \\ H^i(\mathbb{P}(W_{\mathbb{R}})) & \longrightarrow & H^i(\overline{M}_{H,\Gamma_H}). \end{array}$$

where the horizontal maps were defined in (3.3). This diagram commutes, so that $\bar{j}^*(\beta) = \lambda$ **or** H for $\beta \in H^{2m}(\mathbb{P}(V_{\mathbb{R}}))$ implies that $\beta \cdot [\mathbb{P}(W_{\mathbb{R}})] = \lambda$ **or**. So for every invariant class β , we have

$$\xi_{\Gamma} \cdot \beta = \lambda$$
 or $= \beta \cdot [\mathbb{P}(W_{\mathbb{R}})]$.

This proves (3.8) and hence that $\xi_0 = [\mathbb{P}(W_{\mathbb{R}})]$. □

3.6. Restriction maps and Lefschetz properties. We continue with the notation above, i.e. $W \subset V$ is a subspace on which h restricts indefinitely, and $H = SU(h|_W) \subset G$ in the natural embedding.

The pullback maps j^* and \bar{j}^* are compatible with change of level and so define $H(\mathbb{Q})$ -module maps $j^* : H^i(\mathcal{M}) \rightarrow H^i(\mathcal{M}_H)$ and $\bar{j}^* : H^i(\overline{\mathcal{M}}) \rightarrow H^i(\overline{\mathcal{M}}_H)$. By Frobenius reciprocity one has $G(\mathbb{Q})$ -module maps

$$\begin{aligned} \text{Res} &: H^i(\mathcal{M}) \longrightarrow I_H^G(H^i(\mathcal{M}_H)) \\ \overline{\text{Res}} &: H^i(\overline{\mathcal{M}}) \longrightarrow I_H^G(H^i(\overline{\mathcal{M}}_H)) \end{aligned}$$

given concretely by

$$\begin{aligned} \text{Res}(\alpha)(g) &= j^*(g^{-1} \cdot \alpha) = j^*((g \cdot)^*(\alpha)) \\ \overline{\text{Res}}(\alpha)(g) &= \bar{j}^*(g^{-1} \cdot \alpha) = \bar{j}^*((g \cdot)^*(\alpha)) \end{aligned}$$

(cf. the proof of Lemma 3.2 and the definition of the $G(\mathbb{Q})$ -actions in 2.2). (Res is the map considered in [30], except that in loc. cit. it is taken to have image in $\prod_{g \in G(\mathbb{Q})} H^i(\mathcal{M}_H)$, of

which $I_H^G(H^i(\mathcal{M}_H))$ is naturally a subspace.) These maps are evidently compatible under restriction of cohomology from compactifications to open ball quotients.

Proposition 3.15. *For $\alpha \in H^i(\overline{\mathcal{M}})$, if $\overline{\text{Res}}(\alpha) = 0$ then $\alpha \cdot [\mathbb{P}(W_{\mathbb{R}})] = 0$.*

Proof. This is the analogue of [30, Thm 2], and the proof is essentially that in [30, 4.3]. Let $\alpha \in H^i(\overline{M}_{\Gamma}) \subset H^i(\overline{\mathcal{M}})$ with $\overline{\text{Res}}(\alpha) = 0$. Let $g \in G(\mathbb{Q})$ and choose a subgroup $\Gamma' \subset \Gamma \cap g^{-1}\Gamma g$ of finite index which is normal in Γ . Let $p : M_{\Gamma'} \rightarrow M_{\Gamma}$ be the projection, which is a Galois cover with group Γ/Γ' . Let $\gamma_1, \dots, \gamma_r$ be coset representatives for the subgroup Γ_H/Γ'_H in Γ/Γ' . Then $p^{-1}(j(M_{H,\Gamma_H})) = \bigsqcup_i \gamma_i \cdot j(M_{H,\Gamma'_H})$. i.e. the preimage of $j(M_{H,\Gamma_H})$ in $M_{\Gamma'}$ is the disjoint union of translates of $j(M_{H,\Gamma'_H})$ by the coset representatives. (Here the same symbol j is used for maps at levels Γ and Γ' .) Then $p^{-1}(\bar{j}(\overline{M}_{H,\Gamma_H})) = \bigcup_i \gamma_i \cdot \bar{j}(\overline{M}_{H,\Gamma'_H})$, where we continue to use p for $\overline{M}_{\Gamma'} \rightarrow \overline{M}_{\Gamma}$. Now for $\alpha \in H^i(\overline{M}_{\Gamma})$, $(\gamma_i^{-1})^* g^*(\alpha) \in H^i(\overline{M}_{\Gamma'})$, so that if $\overline{\text{Res}}(\alpha) = 0$ then $(g\gamma_i^{-1})^*(\alpha) = (\gamma_i^{-1})^* g^*(\alpha)$ restricts to zero on $\bar{j}(\overline{M}_{H,\Gamma'_H})$ for each i , i.e.

$$\bar{j}^*((\gamma_i^{-1})^* g^*(\alpha)) = 0$$

for each i . By definition of the cycle class $\xi_{\Gamma'}$, this implies

$$0 = (\gamma_i^{-1})^* g^*(\alpha) \cdot \xi_{\Gamma'} = g^*(\alpha) \cdot \gamma_i^*(\xi_{\Gamma'})$$

(since $G(\mathbb{Q})$ acts by automorphisms of the cup product on $H^*(\overline{\mathcal{M}})$). Summing over i and using Lemma 3.13 we get

$$0 = g^*(\alpha) \cdot \xi_{\Gamma} = \alpha \cdot (g^{-1})^*(\xi_{\Gamma}) = 0 \quad \text{for all } g \in G(\mathbb{Q}).$$

Taking a linear combination as in Prop. 3.14 gives $\alpha \cdot [\mathbb{P}(W_{\mathbb{R}})] = 0$. \square

The canonical decomposition of Theorem 3.5 gives an inclusion $IH^i(\mathcal{M}^*) \subset H^i(\overline{\mathcal{M}})$.

Corollary 3.16. *If W has dimension $m+1$ then $\overline{\text{Res}}$ is injective on the subspace $IH^i(\mathcal{M}^*) \subset H^i(\overline{\mathcal{M}})$ for $i \leq m$.*

Proof. The class $[\mathbb{P}(W_{\mathbb{R}})]$ is a nonzero multiple of e^{n-m} where e (as defined in 3.3) has the hard Lefschetz property on $IH^*(\mathcal{M}^*)$ (by [2]). (In fact, in our situation, the hard Lefschetz property can be checked directly using (3.7) and the description of e in terms of the Killing form.) In particular, $\cdot e^{n-m}$ is injective in degrees $\leq m$. \square

For the cohomology of compactifications we have:

Theorem 3.17. *If W has dimension $m+1$ then*

$$\overline{\text{Res}} : H^i(\overline{\mathcal{M}}) \longrightarrow I_H^G(H^i(\overline{\mathcal{M}}_H))$$

is injective for $i \leq m$.

Proof. Let $\ell \subset W$ be isotropic. In degrees $i \leq m$ the pullback map $\bar{j}^* : H^i(\overline{M}_{\Gamma}) \rightarrow H^i(\overline{M}_{H,\Gamma_H})$ is given on the extra summand indexed by ℓ by the natural map

$$H_{D_{\Gamma},\ell}^i(\overline{M}_{\Gamma}) \longrightarrow H_{D_{H,\Gamma_H},\ell}^i(\overline{M}_{H,\Gamma_H})$$

induced by the inclusion of pairs

$$(\overline{M}_{H,\Gamma_H}, \overline{M}_{H,\Gamma_H} - D_{H,\Gamma_H}, \ell) \subset (\overline{M}_{\Gamma}, \overline{M}_{\Gamma} - D_{\Gamma}, \ell).$$

In the limit these give an $H(\mathbb{Q})$ -module map $H_{\mathcal{D}}^i(\overline{\mathcal{M}}) \longrightarrow H_{\mathcal{D}_H}^i(\overline{\mathcal{M}}_H)$. The associated $G(\mathbb{Q})$ -map

$$H_{\mathcal{D}}^i(\overline{\mathcal{M}}) \longrightarrow I_H^G(H_{\mathcal{D}_H}^i(\overline{\mathcal{M}}_H))$$

is the restriction of $\overline{\text{Res}}$ to the summand $H_{\mathcal{D}}^i(\overline{\mathcal{M}})$. By Cor. 3.16 it suffices to show that this map is injective for $i \leq m$. Using the induced module descriptions of $H_{\mathcal{D}}^i(\overline{\mathcal{M}})$ and $H_{\mathcal{D}_H}^i(\overline{\mathcal{M}}_H)$ (Lemma 3.3), the fact that $I_H^G I_{P_{\ell,H}}^H = I_{P_{\ell,H}}^G = I_{P_{\ell}}^G I_{P_{\ell,H}}^{P_{\ell}}$, and the fact that $\mathfrak{u}_{\ell,H} = \mathfrak{u}_{\ell}$, we are reduced to showing that

$$\wedge^{i-2} \mathfrak{v}_{\ell}^* \longrightarrow I_{P_{\ell,H}}^{P_{\ell}}(\wedge^{i-2} \mathfrak{v}_{\ell,H}^*) \quad (3.9)$$

(induced by $\mathfrak{v}_{\ell,H} \subset \mathfrak{v}_{\ell}$) is injective for $i \leq m$. Notice that $\wedge^i \mathfrak{v}_{\ell}$ is irreducible as a \tilde{M}_{ℓ} -module, and hence as a P -module. (In the notation of 1.3, the subgroup $SU(J_0) \subset \tilde{M}_{\ell}$ acts on \mathfrak{v}_{ℓ} by its standard representation, so that the exterior powers are already irreducible for this subgroup.) So (3.9) is injective if it is nonzero. But (3.9) comes by Frobenius reciprocity from the $P_{\ell,H}(\mathbb{Q})$ -mapping $\wedge^{i-2} \mathfrak{v}_{\ell}^* \rightarrow \wedge^{i-2} \mathfrak{v}_{\ell,H}^*$, which is obviously nonzero if $i-2 \leq \dim \mathfrak{v}_{\ell,H} = 2m$. \square

Remark 3.18. In the compact case, when $m = n-1$, a linear combination of Hecke translates of the cycle class ξ_{Γ} is the class of an ample divisor, namely the class $[\mathbb{P}(W_{\mathbb{R}})] \in H^2(M_{\Gamma})$ (by Prop. 3.14, i.e. Theorem 1 of [30]). I do not know if this is true in the noncompact case. (The projection of ξ_{Γ} to the extra summand in degree 2 is the class of D_{H,Γ_H} in $H^2(D_{\Gamma})$, which can be Hecke-averaged to produce an ample class in D_{Γ} . The projection of ξ_{Γ} to $IH^2(M_{\Gamma}^*)$ can be Hecke-averaged to produce the class $[\mathbb{P}(W_{\mathbb{R}})]$ (by Prop. 3.14).) The desired consequence is in any case available (Theorem 3.17).

For the cohomology of the open ball quotient we have the following result, which implies Theorem 0.1 (as reformulated in the introduction using Hecke correspondences):

Theorem 3.19. *If W has dimension $m+1$ then*

$$\text{Res} : H^i(\mathcal{M}) \longrightarrow I_H^G(H^i(\mathcal{M}_H))$$

is injective for $i \leq m-2$.

Proof. Since $H^i(\overline{\mathcal{M}}) \rightarrow H^i(\mathcal{M})$ is surjective for $i \leq n-1$ (by (2.5)) and factors through the projection $H^i(\overline{\mathcal{M}}) \rightarrow IH^i(\mathcal{M}^*)$, it suffices (by Cor. 3.16) to show that for $0 \neq \alpha \in IH^i(\mathcal{M}^*)$ the class $\overline{\text{Res}}(\alpha)$ survives under restriction $I_H^G(H^i(\overline{\mathcal{M}}_H)) \rightarrow I_H^G(H^i(\mathcal{M}_H))$ to the open ball quotient if $i \leq m-2$. For this we shall use the class $e_H \in H^2(\overline{\mathcal{M}}_H)$ as in 3.3. It is clear from the definition there that $\bar{j}^*(e) = e_H$. Now cupping with e_H has the hard Lefschetz property on $IH^*(\mathcal{M}_H)$ while it acts by zero on the extra summands in the canonical decomposition of $H^*(\overline{\mathcal{M}}_H)$ given by Theorem 3.5 because the class e_H is pulled back from \mathcal{M}_H^* . For $\alpha \in IH^i(\mathcal{M}^*)$ with $i \leq m-2$, there is a $g \in G(\mathbb{Q})$ such that

$$0 \neq \overline{\text{Res}}(e \cdot \alpha)(g) \sim e_H \cdot \overline{\text{Res}}(\alpha)(g).$$

(We have used the invariance of e ; here \sim means up to a nonzero constant). Hence $\overline{\text{Res}}(\alpha)$ has a nonzero projection to the summand $I_H^G(IH^i(\mathcal{M}_H^*))$. But $IH^i(\mathcal{M}_H^*) = H^i(\mathcal{M}_H)$ since $i \leq m-2$, so $\overline{\text{Res}}(\alpha)$ survives in $I_H^G(H^i(\mathcal{M}_H))$. \square

Remark 3.20. The argument shows that if $m = n-1$, Res is injective on the subspace $e \cdot H^{i-2}(\mathcal{M})$ for $i = n-2, n-1$.

3.7. Cup products. The decomposition of $H^*(\overline{\mathcal{M}})$ is not compatible with cup product (intersection cohomology does not in general have a ring structure, neither is there an obvious ring structure on the extra summands). Nevertheless, we can use the same methods to prove a nonvanishing statement for cup products in $H^*(\mathcal{M})$ by looking at the diagonal subgroup $\Delta G \subset G \times G$.

Pullback by the diagonal embedding $M_\Gamma \longrightarrow M_\Gamma \times M_\Gamma$ gives, as earlier, a map

$$\text{Res} : H^*(\mathcal{M}) \otimes H^*(\mathcal{M}) \longrightarrow I_{\Delta G}^{G \times G}(H^*(\mathcal{M}))$$

related to the cup product by

$$\text{Res}(\alpha \otimes \beta)(g, h) = (g \cdot)^*(\alpha) \cdot (h \cdot)^*(\beta) = (h^{-1}g)^*(\alpha) \cdot \beta$$

for $\alpha, \beta \in H^*(\mathcal{M})$. Similarly, the diagonal embedding $\overline{M}_\Gamma \longrightarrow \overline{M}_\Gamma \times \overline{M}_\Gamma$ defines a map

$$\overline{\text{Res}} : H^*(\overline{\mathcal{M}}) \otimes H^*(\overline{\mathcal{M}}) \longrightarrow I_{\Delta G}^{G \times G}(H^*(\overline{\mathcal{M}}))$$

related to the cup product in $H^*(\overline{\mathcal{M}})$. There is an analogue of Prop. 3.15, namely that $\overline{\text{Res}}(\alpha \otimes \beta) \neq 0$ if $(\alpha \otimes \beta) \cdot [\Delta_{\mathbb{P}(V_{\mathbb{R}})}] \neq 0$. (Briefly, the $(G \times G)(\mathbb{Q})$ -invariants in $H^*(\overline{\mathcal{M}}) \otimes H^*(\overline{\mathcal{M}})$ are $H^*(\mathbb{P}(V_{\mathbb{R}})) \otimes H^*(\mathbb{P}(V_{\mathbb{R}}))$. The proof of Prop. 3.14 can be repeated to show that the cycle class of the diagonal in $\overline{M}_\Gamma \times \overline{M}_\Gamma$ generates a $(G \times G)(\mathbb{Q})$ -module with a space of invariants spanned by the class $[\Delta_{\mathbb{P}(V_{\mathbb{R}})}]$. The proof of Prop. 3.15 then goes through with only notational changes.)

Theorem 3.21. *If $\alpha \in H^i(\mathcal{M}), \beta \in H^j(\mathcal{M})$ with $i + j \leq n - 2$ then there exists $g \in G(\mathbb{Q})$ such that $g(\alpha) \cdot \beta \neq 0$.*

Proof. This amounts to showing that $\text{Res}(\alpha \otimes \beta) \neq 0$ for α, β as in the theorem. We will argue as in [30, p. 251] (substituting the analogue of Prop. 3.15 for Thm 2 of loc. cit.) for the map $\overline{\text{Res}}$ and use the same trick as earlier to deduce the result for $\text{Res}(\alpha \otimes \beta)$.

The class of the diagonal $\Delta_{\mathbb{P}(V_{\mathbb{R}})} \subset \mathbb{P}(V_{\mathbb{R}}) \times \mathbb{P}(V_{\mathbb{R}})$ is a sum $[\Delta_{\mathbb{P}(V_{\mathbb{R}})}] = \sum_k L^k \otimes L^{n-k}$ where $L = c_1(\mathbb{L})$, and $L \mapsto e$ under the embedding of $H^*(\mathbb{P}(V_{\mathbb{R}}))$ in $H^*(\overline{\mathcal{M}})$. So for $\alpha \in H^i(\overline{\mathcal{M}}), \beta \in H^j(\overline{\mathcal{M}})$,

$$\alpha \otimes \beta \cdot [\Delta_{\mathbb{P}(V_{\mathbb{R}})}] = \sum_k \alpha \cdot e^k \otimes \beta \cdot e^{n-k}.$$

Now e has the hard Lefschetz property on the summand $IH^*(\mathcal{M}^*)$, so if $\alpha, \beta \in IH^*(\mathcal{M}^*)$ then $\alpha \cdot e^k \neq 0$ if $i + k \leq n$ and $\beta \cdot e^{n-k} \neq 0$ if $j + n - k \leq n$. If $i + j \leq n$ choosing $k = j$ satisfies both these conditions, so that $\alpha \otimes \beta \cdot [\Delta_{\mathbb{P}(V_{\mathbb{R}})}] \neq 0$. By the analogue of Prop. 3.15, $\overline{\text{Res}}(\alpha \otimes \beta) \neq 0$. Thus if $\alpha \in IH^i(\mathcal{M}^*), \beta \in IH^j(\mathcal{M}^*)$ and $i + j \leq n$ we have $\overline{\text{Res}}(\alpha \otimes \beta) \neq 0$.

Now suppose that $i + j \leq n - 2$. Applying the previous argument to $e_2 \cdot \alpha \otimes \beta$ where $e_2 := e \otimes 1 + 1 \otimes e$ gives that $0 \neq \text{Res}(e_2 \cdot \alpha \otimes \beta) \sim e \cdot \text{Res}(\alpha \otimes \beta)$. Since $e \cdot$ is zero on the terms in $H^*(\overline{\mathcal{M}})$ supported at infinity, $\text{Res}(\alpha \otimes \beta) \neq 0$ in $H^*(\mathcal{M})$. \square

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