

CHERN CLASSES OF AUTOMORPHIC VECTOR BUNDLES AND THE REDUCTIVE BOREL-SERRE COMPACTIFICATION

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ABSTRACT. The Hecke invariants in the cohomology of the reductive Borel-Serre compactification of a locally symmetric space are computed in terms of the compact dual symmetric space. This is used to show that the Chern classes of the canonical topological extensions of automorphic vector bundles to the reductive Borel-Serre compactification of a Shimura variety have motivic properties in Hodge, de Rham, and Galois realizations.

The *reductive Borel-Serre (RBS) compactification* \overline{M} of a noncompact arithmetic locally symmetric space $M = \Gamma \backslash D$ was introduced by Zucker [Z1] by modifying the earlier construction of Borel and Serre. It has since played a central role in several important developments related to automorphic forms and the cohomology of arithmetic groups, e.g. the topological trace formula [GHM, GKM, GM] and the theory of \mathcal{L} -modules and vanishing theorems [Sa1, Sa2]. When M is a connected Shimura variety (i.e. D is Hermitian and Γ congruence), the RBS compactification interpolates between the two known algebraic compactification methods: It has much milder singularities than the *minimal compactification* of Baily-Borel [BB] and is canonical, unlike the smooth *toroidal compactifications* of Mumford et al. [AMRT]. Although \overline{M} is not an algebraic variety, it is motivic [AZ]. In particular, its cohomology has a mixed Hodge structure, an algebraic de Rham k -structure, and an l -adic $\text{Gal}(\overline{k}/k)$ representation for each l (cf. [Z3, N2, AZ], see §2 for a review; here k is the number field of definition of M).

The main results here are the following:

(1) We compute the invariants and coinvariants in the cohomology of the RBS compactification of an arithmetic locally symmetric space under its natural symmetries, the Hecke correspondences. The invariants are canonically a direct summand isomorphic to the cohomology of the compact dual (Theorem 1).

(2) When M is a Shimura variety we show that there is a good theory of Chern classes of *automorphic vector bundles* on the RBS compactification. Automorphic vector bundles on M extend naturally to topological vector bundles on \overline{M} ([GT, Z2]) and their Chern classes span the invariants in cohomology. We show (using (1)) that these *topological* Chern classes look *motivic*, i.e. in Hodge, de Rham, and Galois realizations of $H^*(\overline{M})$ they have the properties one would expect if \overline{M} were an algebraic variety over k and automorphic vector bundles on M extended algebraically to \overline{M} , respecting fields of rationality (Theorem 2).

In the rest of this introduction we state these two results precisely (0.1, 0.2), briefly discuss their proofs and related results in the literature (0.3), and discuss what happens for the minimal compactification (0.4), where analogues of (1) and (2) fail, but for rather interesting reasons.

0.1. Invariants in cohomology. Let G be a connected reductive group over \mathbb{Q} , $K_\infty \subset G(\mathbb{R})$ a maximal compact subgroup, A_G the maximal \mathbb{Q} -split torus in the centre of G , and

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$A_\infty = A_G(\mathbb{R})^0$. We assume that A_G is also the maximal \mathbb{R} -split torus in the centre of G . For a compact open subgroup $K \subset G(\mathbb{A}_f)$ the double coset space

$$M_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty^0 A_\infty K$$

is a finite union of locally symmetric spaces like $\Gamma \backslash D$ where $D = G(\mathbb{R})^0 / K_\infty^0 A_\infty$ and $\Gamma \subset G(\mathbb{Q})$ is a congruence subgroup. The RBS construction applied to each component gives a compact topological space \overline{M}_K in which M_K is open and dense. The direct limit

$$H^*(\overline{\mathcal{M}}, \mathbb{C}) = \varinjlim_K H^*(\overline{M}_K, \mathbb{C})$$

is an admissible $G(\mathbb{A}_f)$ -module, where $g \in G(\mathbb{A}_f)$ acts by pullback by the isomorphism $\cdot g : \overline{M}_{gKg^{-1}} \rightarrow \overline{M}_K$ induced by right translation by g .

The cohomology of the compact symmetric space \check{D} dual to D is naturally identified with the ring of $G(\mathbb{R})$ -invariant differential forms on $G(\mathbb{R}) / K_\infty^0 A_\infty$. Including invariant forms in the C^∞ de Rham model of [N1] gives a map

$$\theta : H^*(\check{D}, \mathbb{C}) \rightarrow H^*(\overline{\mathcal{M}}, \mathbb{C}).$$

Using analytic methods from [F1], we show:

Theorem 1. *The mapping $\theta : H^*(\check{D}, \mathbb{C}) \rightarrow H^*(\overline{\mathcal{M}}, \mathbb{C})$ is an isomorphism onto the $G(\mathbb{A}_f)$ -invariants and induces an isomorphism onto the $G(\mathbb{A}_f)$ -coinvariants. In particular, the invariants are naturally a direct summand in $H^*(\overline{\mathcal{M}}, \mathbb{C})$.*

0.2. Chern classes of automorphic vector bundles. Assume now that G (as in 0.1) is part of a (motivic) Shimura datum (G, X) , i.e. X is a $G(\mathbb{R})$ -conjugacy class of homomorphisms $\mathbb{C}^\times \rightarrow G(\mathbb{R})$, and Deligne's axioms hold (cf. 2.1). Then each component of $X = G(\mathbb{R}) / K_\infty^0 A_\infty$ is a Hermitian symmetric domain. The Shimura variety at level $K \subset G(\mathbb{A}_f)$ is

$$M_K = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty^0 A_\infty K.$$

This is a quasiprojective complex variety with a canonical model over the reflex field $E = E(G, X)$ (cf. [BB, Mi]).

In this setting the homomorphism θ of Theorem 1 has another description. The compact dual \check{D} is a flag variety for $G(\mathbb{C})$, and any $G(\mathbb{C})$ -homogeneous vector bundle $\check{\mathcal{V}}$ on \check{D} gives an *automorphic vector bundle* \mathcal{V}_K on M_K . When M_K is noncompact the underlying topological vector bundle of \mathcal{V}_K extends naturally to a topological vector bundle $\overline{\mathcal{V}}_K$ on \overline{M}_K (see [GT, 9.2] or [Z2, 1.9]) and we consider the Chern classes

$$c_i(\overline{\mathcal{V}}_K) \in H^{2i}(\overline{M}_K, \mathbb{Q}).$$

Then $\theta(c_i(\check{\mathcal{V}})) = (-1)^i c_i(\overline{\mathcal{V}}_K)$. (In the compact case this is essentially Hirzebruch proportionality; in general it follows from results of Zucker [Z2], see 3.6.) Since the cohomology of \check{D} is generated by Chern classes of homogeneous bundles, we see that θ is Betti rational. (In contrast, when $G = GL(n)$, θ is likely to be highly transcendental [B].)

The cohomology of \overline{M}_K has a mixed realization (in the sense of [D] or [J]): The Betti cohomology $H_B^*(\overline{M}_K) = H^*(\overline{M}_K, \mathbb{Q})$ is part of a mixed Hodge structure, there is an algebraic de Rham cohomology E -vector space $H_{dR}^*(\overline{M}_K/E)$ with a Hodge filtration F and a comparison isomorphism $H_{dR}^*(\overline{M}_K/E) \otimes_E \mathbb{C} = H^*(\overline{M}_K, \mathbb{C})$, and for each prime l there is an l -adic $\text{Gal}(\overline{\mathbb{Q}}/E)$ -representation $H_l^*(\overline{M}_K)$ with a comparison isomorphism $H_B^*(\overline{M}_K) \otimes_{\mathbb{Q}} \mathbb{Q}_l = H_l^*(\overline{M}_K)$. In each case the weights are like those of a complete variety. These structures are recalled in detail in §2, following [N2, NV] or [AZ]. We then have:

Theorem 2. *The Chern classes $c_i(\overline{\mathcal{V}}_K) \in H^{2i}(\overline{M}_K, \mathbb{Q})$ of canonical topological extensions of automorphic vector bundles to the RBS compactification have the following properties:*

(i) *They are of type (i, i) in the mixed Hodge structure, i.e. $c_i(\overline{\mathcal{V}}_K)$ belongs to $F^i H^{2i}(\overline{M}_K, \mathbb{C}) \cap F^i H^{2i}(\overline{M}_K, \mathbb{C})$.*

(ii) *If the homogeneous vector bundle $\check{\mathcal{V}}$ is L -rational for $E \subset L \subset \mathbb{C}$ then the classes $c_i^{dR}(\overline{\mathcal{V}}_K) := (2\pi\sqrt{-1})^i c_i(\overline{\mathcal{V}}_K) \in H^{2i}(\overline{M}_K, \mathbb{C})$ are de Rham L -rational and belong to $F^i H_{dR}^{2i}(\overline{M}_K/L)$.*

(iii) *If the homogeneous vector bundle $\check{\mathcal{V}}$ is L -rational for $E \subset L \subset \overline{\mathbb{Q}}$ then for each l , the action of $\text{Gal}(\overline{\mathbb{Q}}/L)$ on $c_i(\overline{\mathcal{V}}_K) \in H_l^{2i}(\overline{M}_K)$ is by χ_l^{-i} , where χ_l is the l -adic cyclotomic character.*

When M_K is compact (i) is immediate from the algebraicity of automorphic vector bundles ([BB, §10]), and (ii) and (iii) follow from Harris's theory [H1] of canonical models for automorphic vector bundles. When M_K is noncompact, there is no a priori reason to expect that the *topologically* defined Chern classes $c_i(\overline{\mathcal{V}}_K)$ should have *motivic* properties. More surprising perhaps is that the proof of these geometric properties uses the *analytic* input of Theorem 1. The use of Hecke eigenvalues to characterize the Chern classes is the main novelty here (see further remarks on the proof in 0.3).

Theorem 2 and the description of θ give (for the correct E -structure on \check{D}):

Corollary 1. *θ gives an isomorphism of mixed realizations from $H^*(\check{D})$ onto the summand of invariants in $H^*(\overline{M})$.*

A motivic version of the corollary should hold, i.e. the motive of \overline{M}_K from [AZ] should have the motive of \check{D} as a direct summand characterized by Hecke invariance properties, but this seems well out of reach even in the compact case. The problem of giving a motivic meaning to $\overline{\mathcal{V}}_K$ in the noncompact case (e.g. by defining Chern classes in the cycle groups of the motive of \overline{M}_K) seems quite interesting.

Theorem 2 can be used, together with methods of [N2], to show that the invariants $H_c^*(\mathcal{M})^{G(\mathbb{A}_f)}$ of the cohomology of the Shimura variety is a mixed Tate realization.

0.3. On the proofs. When M_K is compact (i.e. $\overline{M}_K = M_K$) Theorem 1 follows from the semisimplicity of $H^*(\mathcal{M}, \mathbb{C})$ as a $G(\mathbb{A}_f)$ -module and the description of invariants in terms of the constant representation, both of which follow from Matsushima's formula (see 1.2). In the noncompact case $H^*(\overline{M}, \mathbb{C})$ is not $G(\mathbb{A}_f)$ -semisimple and does not satisfy Poincaré duality. The proof of Theorem 1 uses the C^∞ de Rham models for $H^*(\overline{M}_K, \mathbb{C})$ and its Poincaré dual theory $W^d H^*(\overline{M}_K, \mathbb{C}) = H_{\dim M - *}(M_K, \mathbb{C})$ from [N1], the fundamental results of Franke [F1] relating cohomology and automorphic forms, and a filtration on the space of automorphic forms from [F1, §6]. (The proof takes up §1; for an outline see 1.2. For the relation to the computation of the coinvariants of $H^*(M_K)$ in [F2] see Remark 1.6.3.)

Theorem 2 is related to (and depends on) several results in the literature. The minimal compactification M_K^* , the RBS compactification $p : \overline{M}_K \rightarrow M_K^*$, and any smooth toroidal compactification $\pi : M_K^\Sigma \rightarrow M_K^*$ are related by a commutative diagram (for $? = B, dR, l$):

$$\begin{array}{ccc} H_?^*(M_K^*) & \xrightarrow{\pi^*} & H_?^*(M_K^\Sigma) \\ & \searrow p^* & \nearrow \gamma^* \\ & H_?^*(\overline{M}_K) & \end{array}$$

(See [GT] for $? = B$ and §2 for an explanation using [NV, N2] or [AZ]). Mumford [Mu] (see also Harris [H2]) showed that an automorphic vector bundle \mathcal{V}_K has a canonical extension

\mathcal{V}_K^Σ to a vector bundle on M_K^Σ and generalized Hirzebruch proportionality using the Chern classes $c_k(\mathcal{V}_K^\Sigma)$. Goresky and Pardon [GP] showed that there are classes

$$c_i^{GP}(\mathcal{V}_K) \in H^{2i}(M_K^*, \mathbb{C})$$

refining Mumford's classes, i.e. $\pi^*(c_i^{GP}(\mathcal{V}_K)) = c_i(\mathcal{V}_K^\Sigma)$ for any $\pi : M_K^\Sigma \rightarrow M_K^*$. This is remarkable, because \mathcal{V}_K need not admit an extension (not even topologically) as a vector bundle over M_K^* ; we discuss these classes further in 0.4 below. Zucker [Z2] showed that $c_i(\overline{\mathcal{V}}_K) = p^*(c_i^{GP}(\mathcal{V}_K))$, so that $\gamma^*(c_i(\overline{\mathcal{V}}_K)) = c_i(\mathcal{V}_K^\Sigma)$. Theorem 2 then follows from Theorem 1, results of Harris [H2] on the E -rationality of $\check{\mathcal{V}} \mapsto \mathcal{V}_K^\Sigma$, and the following fact:

Proposition 1. *The map γ^* is injective on the invariants in $H^*(\overline{M}_K)$.*

Note that the splitting property provided by Theorem 1 is the key to proving the properties of $c_i(\overline{\mathcal{V}}_K)$ in $H^{2i}(\overline{M}_K)$ and not simply in its top weight quotient $\mathrm{Gr}_{2i}^W H^{2i}(\overline{M}_K) = \gamma^*(H^{2i}(\overline{M}_K))$.

0.4. Minimal compactification. What happens on the minimal compactification? For intersection cohomology the answer is straightforward (see 4.1), so consider the cohomology of M_K^* . There is a surjection of mixed realizations

$$H^*(M_K^*) \longrightarrow H^*(\check{D})$$

taking $c_i^{GP}(\mathcal{V}_K)$ to $(-1)^i c_i(\check{\mathcal{V}})$ (e.g. by Theorem 1). The Goresky-Pardon classes $c_i^{GP}(\mathcal{V}_K)$ are defined via Chern forms of certain explicit connections (from [H1]) and a patching construction for connections, so that the relations between them, their Betti or de Rham rationality properties, their Hodge types, and their Galois properties are all unclear. (See [GP, 1.6] for these questions.) The lack of an automorphic description for $H^*(M_K^*, \mathbb{C})$ makes an approach like that for \overline{M}_K difficult.

In fact, the situation is subtle. The surjections above give a surjection

$$H^*(\mathcal{M}^*)^{G(\mathbb{A}_f)} \longrightarrow H^*(\check{D})$$

where $H^*(\mathcal{M}^*) := \varinjlim_K H^*(M_K^*)$. In general, this is *not* an isomorphism and is *not* split in mixed realizations: In the example $G = Sp(6)$ one finds that $H^6(\mathcal{M}^*)^{G(\mathbb{A}_f)}$ contains a nontrivial extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$. (The extension class can be computed in terms of $\zeta(3)$ in the mixed Hodge realization. Similar examples exist for $Sp(2g)$, $g \geq 3$.) The conjectural picture one gets is the following: The Goresky-Pardon construction gives a natural splitting of $H^*(\mathcal{M}^*, \mathbb{C})^{G(\mathbb{A}_f)} \rightarrow H^*(\check{D}, \mathbb{C})$ compatible with the Hodge filtration, but there is no motivic splitting of $H^*(\mathcal{M}^*)^{G(\mathbb{A}_f)} \rightarrow H^*(\check{D})$. For a more detailed discussion of the example and what we expect to be true in general see 4.3.

A different perspective, at least for Shimura varieties of Hodge type, comes from *p-adic Hodge theory* in the recent work of Scholze and others (see [Sc1, Sc2]). For such Shimura varieties, the limits $M_{K^p} \sim \varprojlim_{K_p} M_{K_p K^p, \mathbb{C}_p}^{\mathrm{ad}}$ and $M_{K^p}^* \sim \varprojlim_{K_p} M_{K_p K^p, \mathbb{C}_p}^{*, \mathrm{ad}}$ of the adic spaces associated with the varieties $M_{K_p K^p}$ and $M_{K_p K^p}^*$ are perfectoid. (Here $K^p \subset G(\mathbb{A}_f^p)$ is a fixed tame level and $K_p \subset G(\mathbb{Q}_p)$ shrink to the identity.) There is a period map $\pi_{\mathrm{HT}} : M_{K^p} \rightarrow \check{D}_{\mathbb{C}_p}^{\mathrm{ad}}$ to the (adic space of the) flag variety which extends to the minimal compactification at infinite level, i.e. to

$$\pi_{\mathrm{HT}} : M_{K^p}^* \longrightarrow \check{D}_{\mathbb{C}_p}^{\mathrm{ad}},$$

and is $G(\mathbb{A}_f^p)$ -equivariant (for the trivial action on the target). (See [Sc2, Theorem 16.1] or [Sc1, Theorem III.3.17] for a precise statement and proofs.) Automorphic vector bundles are simply pulled back by π_{HT} , hence extend to the limit $M_{K^p}^*$. This would give a beautiful explanation for the existence of Chern classes in the invariants in the cohomology $\varinjlim_{K^p} H^*(M_{K^p}^* \times_E \overline{\mathbb{Q}}, \mathbb{Q}_l)$. The period map is transcendental, so that while the classes will have the right Hodge-Tate properties, the splitting given by π_{HT}^* need not be Galois-equivariant. This fits with the picture above that there is a natural analytic splitting but no motivic one.

0.5. The contents of the various sections are as follows: §1 recalls the necessary results about automorphic forms and gives the proof of Theorem 1. §2 reviews the mixed realizations in the cohomology of the RBS compactification for Shimura varieties. §3 discusses automorphic vector bundles and their canonical extensions and contains the proofs of Proposition 1 and Theorem 2. §4 discusses the situation on other compactifications.

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1. (CO)INVARIANTS IN COHOMOLOGY

In this section we prove Theorem 1.6.2, which implies Theorem 1 of the introduction. We will write $H^*(-)$ for singular cohomology with complex coefficients in this section.

1.1. Locally symmetric spaces and the compact dual. Fix a connected reductive \mathbb{Q} -group G and a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$. We assume that the \mathbb{Q} -split part A_G of the central torus of G is also the \mathbb{R} -split part and set $A_\infty := A_G(\mathbb{R})^0$. For $K \subset G(\mathbb{A}_f)$ a compact open subgroup let

$$M_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_\infty K_\infty^0 K.$$

Fix a component D of $G(\mathbb{R}) / K_\infty^0 A_\infty$. The set of double cosets $\text{Stab}_{G(\mathbb{Q})}(D) \backslash G(\mathbb{A}_f) / K$ is finite (by strong approximation). If $\{g_i\}$ is a set of representatives then

$$M_K = \bigsqcup_i \Gamma_i \backslash D \quad \text{for } \Gamma_i = \text{Stab}_{G(\mathbb{Q})}(D) \cap g_i K g_i^{-1}.$$

Thus for K small enough (e.g. neat in the sense of [P, 0.6]), M_K is a finite disjoint union of smooth locally symmetric spaces, compact if and only if G is \mathbb{Q} -anisotropic.

Let $G_0 := G^{\text{der}}(\mathbb{R})^0$ and $K_{0,\infty} := K_\infty \cap G_0 = K_\infty^0 \cap G_0$; this is maximal compact in G_0 . Then D is identified with the symmetric space $G_0 / K_{0,\infty}$ of G_0 and for K small enough, $M_K = \bigsqcup_i \Gamma_i \backslash D$ for $\Gamma_i \subset G^{\text{der}}(\mathbb{Q})$. The *compact dual* symmetric space is

$$\tilde{D} = G_0^c / K_{0,\infty}$$

where G_0^c is a compact real form of G_0 containing $K_{0,\infty}$. Its cohomology is identified with the ring of G_0 -invariant (equivalently, $G(\mathbb{R})^0$ -invariant) differential forms on D , or with the $G(\mathbb{R})$ -invariant differential forms on $G(\mathbb{R}) / K_\infty^0 A_\infty$.

The Lie algebras of $G(\mathbb{R})$ and G_0 are denoted \mathfrak{g} and \mathfrak{g}_0 , respectively.

1.2. Outline of the argument. First consider the compact case. The homomorphism $\theta : H^*(\check{D}) \rightarrow H^*(\mathcal{M})$ is induced by the inclusion of the constant functions in all smooth K_∞ -finite functions on $G(\mathbb{Q})A_\infty \backslash G(\mathbb{A})$. Matsushima's formula gives a direct sum decomposition as $G(\mathbb{A}_f)$ -modules

$$H^*(\mathcal{M}) = \varinjlim_K H^*(M_K) = \bigoplus_{\pi = \pi_f \otimes \pi_\infty} m(\pi) \pi_f \otimes H^*(\mathfrak{g}, K_\infty^0 A_\infty, \pi_\infty).$$

This is an algebraic direct sum, over irreducible $\pi = \pi_f \otimes \pi_\infty$ appearing in the Hilbert space direct sum decomposition of $L^2(G(\mathbb{Q})A_\infty \backslash G(\mathbb{A}))$, and the multiplicities $m(\pi)$ are finite. The $G(\mathbb{A}_f)$ -invariants come from the constant representation, for which the multiplicity is one, so that the invariants are

$$H^*(\mathcal{M})^{G(\mathbb{A}_f)} = H^*(\mathfrak{g}, K_\infty^0 A_\infty, \mathbb{C}) = H^*(\mathfrak{g}_0, K_{0,\infty}, \mathbb{C}) = H^*(\check{D}).$$

Invariants and coinvariants are isomorphic because $H^*(\mathcal{M})$ is $G(\mathbb{A}_f)$ -semisimple.

In the noncompact case there is no replacement for Matsushima's formula and $H^*(\overline{\mathcal{M}})$ is not semisimple as a $G(\mathbb{A}_f)$ -module. The C^∞ de Rham model for the cohomology groups $H^*(\overline{M}_K)$ and for the Poincaré dual cohomology groups $W^d H^*(\overline{M}_K) = H_{\dim_{\mathbb{R}} M - *}(M_K)$ given by [N1] is the following: There are $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ -modules $B(G)_1 \subset R(G)_1$ of functions on $G(\mathbb{Q})A_\infty \backslash G(\mathbb{A})$ (defined in 1.4 below) such that

$$\begin{aligned} H^*(\overline{M}_K) &= H^*(\mathfrak{g}, K_\infty^0 A_\infty, B(G)_1^K) \\ W^d H^*(\overline{M}_K) &= H^*(\mathfrak{g}, K_\infty^0 A_\infty, R(G)_1^K). \end{aligned} \tag{1.2.1}$$

The space $B(G)_1$ consists of functions bounded up to certain logarithmic terms, cf. 1.4. The inclusions $\mathbb{C} \subset B(G)_1 \subset R(G)_1$ induce $G(\mathbb{A}_f)$ -maps

$$H^*(\check{D}) \longrightarrow H^*(\overline{\mathcal{M}}) \longrightarrow W^d H^*(\overline{\mathcal{M}}) = \varinjlim_K W^d H^*(\overline{M}_K).$$

The fundamental result of Franke [F1] allows us to replace $R(G)_1$ in (1.2.1) by a certain subspace of automorphic forms $\mathfrak{F}in_{\Theta} R(G)_1$ (defined in 1.5). Franke's method of filtering spaces of automorphic forms by conditions on exponents allows to show that the subspace of $\mathfrak{F}in_{\Theta} R(G)_1$ with the same Hecke eigenvalues as the trivial representation is exactly the constants (see Theorem 1.5.1 for the precise statement). This allows us to show that $H^*(\check{D})$ is a direct summand (as a Hecke-module) of $W^d H^*(\overline{\mathcal{M}})$ and that there are no other Hecke-trivial constituents. By duality we conclude that $H^*(\overline{\mathcal{M}})$ contains $H^*(\check{D})$ as a direct summand and has no other Hecke-trivial constituents.

1.3. Notation. We fix $G, A_G, A_\infty, K_\infty, G_0$ as in 1.1 and a good maximal compact subgroup $\mathbb{K} = K_\infty \times \prod_p \mathbb{K}_p$ of $G(\mathbb{A})$, so that $G(\mathbb{A}) = \mathbb{K} P_0(\mathbb{A})$ for any minimal parabolic subgroup $P_0 \subset G$. In addition, we will use the following notation in this section:

Fix a maximal \mathbb{Q} -split torus A_0 in G such that $A_0(\mathbb{R})$ is stable under the Cartan involution of $G(\mathbb{R})$ given by K_∞ . Let M_0 be the centralizer of A_0 in G ; this is a minimal Levi subgroup. For any standard Levi subgroup $M \supset M_0$ the split centre is a torus $A_M \subset A_0$ (so $A_{M_0} = A_0$) and this gives the dual vector spaces $\mathfrak{a}_M = \text{Lie } A_M(\mathbb{R})$ and $\check{\mathfrak{a}}_M = X^*(M) \otimes \mathbb{R} = X^*(A_M) \otimes \mathbb{R}$. Restriction of characters by $M_0 \subset M$ gives an embedding $\check{\mathfrak{a}}_M \subset \check{\mathfrak{a}}_0$. Restriction by $A_M \subset A_0$ gives a projection $\check{\mathfrak{a}}_0 \rightarrow \check{\mathfrak{a}}_M$ inverse to $\check{\mathfrak{a}}_M \subset \check{\mathfrak{a}}_0$. Similarly, for a standard Levi M , restriction by $M \subset G$ and $A_G \subset A_M$ gives canonical direct sum decompositions $\mathfrak{a}_M = \mathfrak{a}_G \oplus \mathfrak{a}_M^G$ and $\check{\mathfrak{a}}_M = \check{\mathfrak{a}}_G \oplus \check{\mathfrak{a}}_M^G$. When $M = M_0$ we write $\check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_G \oplus \check{\mathfrak{a}}_0^G$.

The roots of A_0 in G form a root system Φ_0 in $\check{\mathfrak{a}}_0$; Φ_0 spans $\check{\mathfrak{a}}_0^G$. Fix a minimal parabolic subgroup $P_0 \supset M_0$. This fixes positive roots $\Phi_0^+ \subset \Phi_0$, a system of simple roots $\Delta_0 \subset \Phi_0^+$, a Weyl chamber $\check{\mathfrak{a}}_0^{G+}$ and its closure $\overline{\check{\mathfrak{a}}_0^{G+}}$, and a positive cone ${}^+\check{\mathfrak{a}}_0^G$ and its closure $\overline{{}^+\check{\mathfrak{a}}_0^G}$, and we have $\check{\mathfrak{a}}_0^{G+} \subset {}^+\check{\mathfrak{a}}_0^G$. We will also use the notation $\overline{\check{\mathfrak{a}}_0^+} = \check{\mathfrak{a}}_G + \overline{\check{\mathfrak{a}}_0^{G+}}$ and $\overline{{}^+\check{\mathfrak{a}}_0} = \check{\mathfrak{a}}_G + \overline{{}^+\check{\mathfrak{a}}_0^G}$.

The half-sum of roots in Φ_0 is denoted ρ_0 ; it belongs to $\check{\mathfrak{a}}_0^{G+}$. For any standard P , the half-sum of roots of A_M appearing in the nilradical of $\text{Lie } P(\mathbb{R})$ is denoted $\rho_P \in \check{\mathfrak{a}}_M^G$. An element $\nu \in \overline{\check{\mathfrak{a}}_0^{G+}}$ determines a standard parabolic $P(\nu)$ and standard Levi $M(\nu)$: The root group of $\alpha \in \Phi_0$ is contained in $P(\nu)$ if and only if $\langle \alpha, \nu \rangle \geq 0$. Then $\nu \in \check{\mathfrak{a}}_{M(\nu)}$.

For a standard Levi M let $H_M : A_M(\mathbb{R})^0 \rightarrow \mathfrak{a}_M$ be the logarithm map. Since $M(\mathbb{A}) = M(\mathbb{A})^1 \times A_M(\mathbb{R})^0$ (where $M(\mathbb{A})^1$ is the subgroup of g with $|\chi(g)|_{\mathbb{A}} = 1$ for all $\chi \in X^*(M)$) we get a map $H_M : M(\mathbb{A}) \rightarrow \mathfrak{a}_M$ by composing with the projection. (When $M = M_0$ we write H_0 .) Let $K_{\infty}^M = K_{\infty} \cap M(\mathbb{R})$. Then $\lambda \in (\check{\mathfrak{a}}_M)_{\mathbb{C}}$ defines a one-dimensional $(\mathfrak{m}, K_{\infty}^M) \times M(\mathbb{A}_f)$ -module \mathbb{C}_{λ}^M by taking the multiples of the function $m \mapsto e^{\langle \lambda, H_M(m) \rangle}$ with the action of $M(\mathbb{A})$ by right translation.

We will use an induction functor Ind_P^G from $(\mathfrak{p}, K_{\infty}^M) \times P(\mathbb{A}_f)$ -modules to $(\mathfrak{g}, K_{\infty}) \times G(\mathbb{A}_f)$ -modules. This is the functor used in [F1, §4] or [W, 3.3]; in particular it is normalized. (Our main concern will be its effect at the finite places, where it is the usual normalized induction of smooth representations.)

1.4. Spaces of functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})$. For $t \in \mathbb{R}$ let

$$A_0(t) := \{a \in A_0(\mathbb{R})^0 : \langle \alpha, H_0(a) \rangle > t \text{ for all } \alpha \in \Delta_0\}.$$

For a compact subset $\omega \subset P_0(\mathbb{A})$ and $t \in \mathbb{R}$ define the adelic Siegel set

$$\mathfrak{S} = \mathfrak{S}(t, \omega) = \{pak : p \in \omega, a \in A_0(t), k \in \mathbb{K}\}.$$

We take t small enough and ω large enough so that by reduction theory we have (1) $G(\mathbb{A}) = G(\mathbb{Q})\mathfrak{S}$ and (2) $\{\gamma \in G(\mathbb{Q}) : \gamma\mathfrak{S} \cap \mathfrak{S} \neq \emptyset\}$ is finite. Thus \mathfrak{S} is a coarse fundamental domain for $G(\mathbb{Q})$ in $G(\mathbb{A})$. We use this to define certain $(\mathfrak{g}, K_{\infty}) \times G(\mathbb{A}_f)$ -modules of functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ via growth conditions (following [F1, §2.1] or [W, §1]). Let $\|\cdot\|$ be a norm on \mathfrak{a}_0 .

Let $S(G)$ be the space of smooth K_{∞} -finite functions of uniform moderate growth on $G(\mathbb{Q}) \backslash G(\mathbb{A})$. Recall that this means there exists $N \in \mathbb{N}$ such that for all $X \in U(\mathfrak{g})$, there exists C such that for $g = pak \in \mathfrak{S}$,

$$|(Xf)(pak)| \leq Ce^{N\|H_0(a)\|}.$$

Let $R(G)$ be the subspace of $S(G)$ of functions satisfying the following condition: For all $X \in U(\mathfrak{g})$, there exists C such that for any $N \in \mathbb{Z}$ and $pak \in \mathfrak{S}$,

$$|(Xf)(pak)| \leq Ce^{(2\rho_0, H_0(a))} (1 + \|H_0(a)\|)^N.$$

Let $S_{\log}(G)$ be the space of functions in $S(G)$ satisfying the following: There exists an $N \in \mathbb{N}$ such that for all $X \in U(\mathfrak{g})$, there exists C such that for $pak \in \mathfrak{S}$,

$$|(Xf)(pak)| \leq Ce^{\langle \rho_0, H_0(a) \rangle} (1 + \|H_0(a)\|)^N.$$

Finally let $B(G)$ be the space of functions in $S(G)$ satisfying: $\exists N \in \mathbb{N}$ such that $\forall X \in U(\mathfrak{g}) \exists C$ such that for $pak \in \mathfrak{S}$,

$$|(Xf)(pak)| \leq C(1 + \|H_0(a)\|)^N.$$

In words, these are the functions which are bounded up to logarithmic factors, and have similarly bounded $U(\mathfrak{g})$ -derivatives. The constants belong to $B(G)$.

There are $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ -module inclusions

$$\mathbb{C} \subset B(G) \subset S_{\log}(G) \subset R(G) \subset S(G).$$

We will also consider the chain of inclusions

$$\mathbb{C} \subset B(G)_1 \subset S_{\log}(G)_1 \subset R(G)_1 \subset S(G)_1$$

of subspaces of functions which are A_∞ -invariant, i.e. descend to $G(\mathbb{Q})A_\infty \backslash G(\mathbb{A})$.

Remark 1.4.1. The relation to the spaces defined in [F1] is as follows:

$S(G)_1$ is the space denoted $S_\infty(G(\mathbb{Q})A_\infty \backslash G(\mathbb{A}))$ in [F1];

$R(G)_1$ is the space denoted $S_{\rho-\tau-\log}(G(\mathbb{Q})A_\infty \backslash G(\mathbb{A}))$ in [F1] for $\tau = \rho_0$;

$S_{\log}(G)_1$ is the space denoted $S_{\log}(G(\mathbb{Q})A_\infty \backslash G(\mathbb{A}))$ in [F1];

$B(G)_1$ is the space denoted $S_{\rho-\tau+\log}(G(\mathbb{Q})A_\infty \backslash G(\mathbb{A}))$ in [F1] for $\tau = -\rho_0$.

Franke's spaces are defined using weighted L^2 conditions whereas we have used weighted boundedness conditions (as in [W], where $S(G)$ and $S_{\log}(G)$ are defined). The equivalence between the two types of conditions follows from the Sobolev-type estimate of Proposition 2 on p. 198 of [F1].

1.5. Automorphic forms. Fix a Cartan subalgebra \mathfrak{h} of $\mathfrak{g}_\mathbb{C}$ containing \mathfrak{a}_0 (hence contained in $(\mathfrak{m}_0)_\mathbb{C}$). This gives a root system $\Phi = \Phi(\mathfrak{h}, \mathfrak{g}_\mathbb{C})$ and Weyl group $W = W(\mathfrak{h}, \mathfrak{g}_\mathbb{C})$. Fix a system of positive roots $\Phi^+ \subset \Phi = \Phi(\mathfrak{h}, \mathfrak{g}_\mathbb{C})$ compatible with Φ_0^+ (i.e. if $\beta \in \Phi^+$ then $\beta|_{\mathfrak{a}_0} \in \Phi_0^+ \cup \{0\}$). The half-sum of roots in Φ^+ will be denoted $\rho_{\mathfrak{h}}$; so $\rho_{\mathfrak{h}}|_{\mathfrak{a}_0} = \rho_0$.

Recall that the Harish-Chandra isomorphism $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$ identifies infinitesimal characters with the W -orbits in $\check{\mathfrak{h}}$ and ideals of finite codimension in $Z(\mathfrak{g})$ with finite W -invariant sets in $\check{\mathfrak{h}}$. For a finite W -invariant set $\Theta \subset \check{\mathfrak{h}}$ with corresponding ideal I_Θ and a (\mathfrak{g}, K_∞) -module V let

$$\mathfrak{F}in_\Theta V := \{v \in V : \exists n \text{ such that } I_\Theta^n v = 0\}.$$

Thus, for example, $\mathfrak{F}in_\Theta S(G)$ and $\mathfrak{F}in_\Theta R(G)$ are spaces of automorphic forms for G . The direct sum of $\mathfrak{F}in_\Theta S(G)$ as Θ runs over W -orbits in $\check{\mathfrak{h}}$ is the space of all automorphic forms.

For a set of finite primes S containing all but finitely many finite primes let $\mathbb{K}_S = \prod_{p \in S} \mathbb{K}_p$. In the spherical Hecke algebra $\mathcal{H}_S = \otimes_{p \in S} \mathcal{H}_p$, where $\mathcal{H}_p = \mathcal{H}(G(\mathbb{Q}_p) // \mathbb{K}_p)$ is the convolution algebra of \mathbb{K}_p -biinvariant smooth functions on $G(\mathbb{Q}_p)$, we have the maximal ideal \mathcal{J}_S annihilating the trivial representation. For an admissible $G(\mathbb{A}_f)$ -module V one considers the space of \mathbb{K}_S -spherical vectors killed by some power of \mathcal{J}_S :

$$V_{\mathcal{J}_S}^{\mathbb{K}_S} := \{v \in V^{\mathbb{K}_S} : \exists n \text{ s.t. } \mathcal{J}_S^n v = 0\}$$

and the direct limit over all such S

$$V_{\mathcal{J}} := \varinjlim_S V_{\mathcal{J}_S}^{\mathbb{K}_S}$$

is a direct summand of V . (For any $K^S \subset G(\mathbb{A}^S)$, $V^{\mathbb{K}_S K^S}$ has a decomposition according to maximal ideals of \mathcal{H}_S . Taking a direct limit over K^S and then over S gives a decomposition of V in which $V_{\mathcal{J}}$ is one summand.) For example, for the spaces of automorphic forms considered above (which are $G(\mathbb{A}_f)$ -admissible by the classical theorem of Harish-Chandra), we have

$$\mathfrak{F}in_\Theta R(G)_{1, \mathcal{J}_S}^{\mathbb{K}_S} = \{f \in \mathfrak{F}in_\Theta R(G)_1^{\mathbb{K}_S} : \exists n \text{ s.t. } \mathcal{J}_S^n f = 0\}$$

and

$$\mathfrak{Fin}_\Theta R(G)_{1,J} := \varinjlim_S \mathfrak{Fin}_\Theta R(G)_{1,J_S}^{\mathbb{K}_S}.$$

This is a direct summand of $\mathfrak{Fin}_\Theta R(G)_1$.

Henceforth fix the W -orbit in $\check{\mathfrak{h}}$:

$$\Theta := W \cdot \rho_{\check{\mathfrak{h}}}.$$

This orbit corresponds to the trivial character of $Z(\mathfrak{g})$, so that $\mathfrak{Fin}_\Theta R(G)_{1,J}$ contains the constant functions.

Theorem 1.5.1. $\mathfrak{Fin}_\Theta R(G)_{1,J}$ is the space of constant functions.

Proof. We will use one of Franke's filtrations on the space $\mathfrak{Fin}_\Theta S(G)_1$, which we recall following [F1, §6] or [W, 4.7, 6.4].

First we need an elementary construction. For the finite set $\Theta \subset \check{\mathfrak{h}}$ define another finite set $\Theta_+ \subset \overline{\check{\mathfrak{a}}_0^+}$ as follows: For $\theta \in \Theta$ and a standard Levi M let $\theta_M \in (\check{\mathfrak{a}}_M)_\mathbb{C}$ be the restriction to $(\mathfrak{a}_M)_\mathbb{C}$. Considering $\text{Re}(\theta_M) \in \check{\mathfrak{a}}_M$ as an element of $\check{\mathfrak{a}}_0$, let $\text{Re}(\theta_M)_+ \in \overline{\check{\mathfrak{a}}_0^+}$ be the closest point to $\text{Re}(\theta_M)$ in $\overline{\check{\mathfrak{a}}_0^+}$. (Here we have fixed an inner product on $\check{\mathfrak{a}}_0$ invariant under the Weyl group of Φ_0 . The elements $\theta_M \in \check{\mathfrak{a}}_M$ belong to $\check{\mathfrak{a}}_M^G$, so we could as well work in $\check{\mathfrak{a}}_0^G$ here.) Taking the union over M and $\theta \in \Theta$ gives Θ_+ , i.e.:

$$\Theta_+ := \bigcup_M \{ \text{Re}(\theta_M)_+ : \theta \in \Theta \}.$$

Note that $\Theta_+ \subset \overline{\check{\mathfrak{a}}_0^{G+}}$. There is a natural way to filter Θ_+ . Let Θ_+^0 be the set of maximal elements in the standard ordering (viz. $\lambda \leq \mu \iff \mu - \lambda \in \overline{\check{\mathfrak{a}}_0^+}$). For $p > 0$ define Θ_+^p inductively to be the set of maximal elements of $\Theta_+ - \Theta_+^{\leq p-1}$ and set $\Theta_+^{\leq p} = \Theta_+^{\leq p-1} \cup \Theta_+^p$.

Let $f \in \mathfrak{Fin}_\Theta S(G)_1$. For each standard parabolic P the constant term of f along P admits a Fourier expansion in terms of characters of \mathfrak{a}_M (cf. [F1, §6]); the characters appearing form a finite set

$$\text{Exp}_P(f) \subset (\check{\mathfrak{a}}_M)_\mathbb{C} \subset (\check{\mathfrak{a}}_0)_\mathbb{C},$$

the P -exponents of f . (Since f is A_∞ -invariant these actually lie in $(\check{\mathfrak{a}}_M^G)_\mathbb{C} \subset (\check{\mathfrak{a}}_0^G)_\mathbb{C}$.) They are related to the infinitesimal character Θ by:

$$f \in \mathfrak{Fin}_\Theta S(G)_1 \text{ and } \lambda \in \text{Exp}_P(f) \implies \text{Re}(\lambda)_+ \in \Theta_+. \quad (1.5.1)$$

Define a finite decreasing filtration

$$\mathfrak{Fin}_\Theta S(G)_1 = F_S^0 \supseteq F_S^1 \supseteq F_S^2 \supseteq \cdots$$

by the following condition on exponents:

$$f \in F_S^p \iff \text{for all } P \text{ and } \lambda \in \text{Exp}_P(f) \text{ we have } \text{Re}(\lambda)_+ \in \Theta_+^{\leq p}.$$

Franke ([F1, Theorem 14] or [W, 4.7]) describes the graded quotients of this filtration: The graded piece F_S^p/F_S^{p+1} is isomorphic to a sum of induced modules

$$\text{Ind}_{P(\nu)}^G (\mathbb{C}_\nu^{M(\nu)} \otimes \pi) \quad (1.5.2)$$

where $\nu \in \Theta_+^p$ and π is an automorphic representation on $M(\nu)$ with unitary central character appearing in L^2 . (The notation $P(\nu), M(\nu)$ is as in 1.3. Here [F1, Thm 14] is applied with $\tau = \infty$, i.e. $\tau \in \check{\mathfrak{a}}_0^{G+}$ sufficiently large. The colimit in loc. cit. is replaced with a direct sum because Θ consists of regular elements of $\check{\mathfrak{h}}$ (see the proof of [F1, Thm 19.I]). Franke gives a more detailed description of π which appear, but we will not need it here.)

The following lemma says that $\mathfrak{Fin}_\Theta R(G)_1$ as a subspace of $\mathfrak{Fin}_\Theta S(G)_1$ is defined by a condition on exponents:

Lemma 1.5.2. *A function $f \in \mathfrak{Fin}_\Theta S(G)_1$ belongs to $\mathfrak{Fin}_\Theta R(G)_1$ if, and only if, for all standard P ,*

$$\lambda \in \text{Exp}_P(f) \implies \text{Re}(\lambda) \in \rho_0 - {}^+\check{\mathfrak{a}}_0^G. \quad (1.5.3)$$

Proof. This follows easily from (a variant of) Theorem 15 of [F1], which is a refinement of the square-integrability criterion of [L, Lemma 5.1]. Recall that in the notation of [F1], $R(G)_1$ is the space $S_{\rho-\tau-\log}(G(\mathbb{Q})A_\infty \backslash G(\mathbb{A}))$ for $\tau = \rho_0$. Theorem 15 of [F1] says that if $\tau \in \check{\mathfrak{a}}_0^{G+}$ then $f \in \mathfrak{Fin}_\Theta S(G)_1$ belongs to $\mathfrak{Fin}_\Theta S_{\rho-\tau+\log}$ if and only if $\text{Re}(\lambda) \in \tau - \overline{{}^+\check{\mathfrak{a}}_0^G}$ for all $\lambda \in \text{Exp}_P(f)$. The variant with “ $-\log$ ” in place of “ $+\log$ ” is the following: If $\tau \in \check{\mathfrak{a}}_0^{G+} \cap {}^+\check{\mathfrak{a}}_0^G$ then $f \in \mathfrak{Fin}_\Theta S(G)_1$ belongs to $\mathfrak{Fin}_\Theta S_{\rho-\tau-\log}$ if and only if any exponent $\lambda \in \text{Exp}_P(f)$ satisfies $\text{Re}(\lambda) \in (\tau - \varepsilon) - \overline{{}^+\check{\mathfrak{a}}_0^G}$ for some $\varepsilon \in \check{\mathfrak{a}}_0^{G+}$. (See the remarks after the proof of Theorem 15 on p. 242 of [F1].) Since $\rho_0 \in \check{\mathfrak{a}}_0^{G+} \subset {}^+\check{\mathfrak{a}}_0^G$ this can be applied to $R(G)_1$. The condition that $\text{Re}(\lambda) \in (\rho_0 - \varepsilon) - \overline{{}^+\check{\mathfrak{a}}_0^G}$ for some $\varepsilon \in \check{\mathfrak{a}}_0^{G+}$ is clearly the same as (1.5.3). \square

Now define a filtration $(F^p)_{p \in \mathbb{N}}$ of $\mathfrak{Fin}_\Theta R(G)_1$ by

$$F^p := F_S^p \cap \mathfrak{Fin}_\Theta R(G)_1.$$

The condition (1.5.3) is obviously compatible with the way the filtrations are defined, so that the graded quotient F^p/F^{p+1} is a sum of induced modules like (1.5.2) over those elements $\nu \in \Theta_+^p$ which satisfy

$$\nu \in \rho_0 - {}^+\check{\mathfrak{a}}_0^G \quad (1.5.4)$$

(cf. p. 242 of loc. cit.). Let F^N be the last nonzero step of the filtration, corresponding to $\Theta_+^N = \{0\}$, which is $F^N = \mathfrak{Fin}_\Theta S_{\log}(G)_1$. (By the geometrical lemma of Langlands (see e.g. Lemma 1 on p. 232 of [F1]), $\text{Re}(\lambda)_+ = 0$ implies $\text{Re}(\lambda) \in -\overline{{}^+\check{\mathfrak{a}}_0^G}$ (the negative of the closed positive cone). So F^N consists of automorphic forms all of whose exponents lie in $-\overline{{}^+\check{\mathfrak{a}}_0^G}$. This is precisely $\mathfrak{Fin}_\Theta S_{\log}(G)_1$ by [F1, Theorem 15] applied with $\tau = 0$.)

To prove the theorem it suffices to prove:

- (1) $(F^p/F^{p+1})_j = F_j^p/F_j^{p+1} = \{0\}$ for $p < N$
- (2) F_j^N is the space of constant functions.

We will use the following lemma, which will be proved later:

Lemma 1.5.3. *Let P be a parabolic subgroup with Levi factor M , τ an irreducible unitary $(\mathfrak{m}, K_\infty^M) \times M(\mathbb{A}_f)$ -module and $\nu \in (\check{\mathfrak{a}}_M^G)_\mathbb{C}$ with $\text{Re}(\nu) \in \check{\mathfrak{a}}_M^{G+}$. If the induced representation $\Pi = \text{Ind}_P^G(\mathbb{C}_\nu^M \otimes \tau)$ has a constituent which is trivial at all but finitely many finite primes, then $\text{Re}(\nu) = \rho_P$.*

To prove (1), note that each summand of F^p/F^{p+1} is an induced module as in the lemma for some $\nu \in \Theta_+^p$. By the lemma it makes no contribution to F_j^p/F_j^{p+1} unless $\nu = \rho_P$ for some P . But this possibility is excluded by (1.5.4) since $\rho_0 - \rho_P \in \overline{{}^+\check{\mathfrak{a}}_0^G} - {}^+\check{\mathfrak{a}}_0^G$.

To prove (2) we use Franke’s description of $F^N = \mathfrak{Fin}_\Theta S_{\log}(G)_1$ (see [F1, Theorem 13]). The precise result does not matter for us, it is enough to note that this space is a direct sum of modules of the form

$$\Pi = \left(\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} D_F \otimes \pi \right)^{W(M)}$$

with notation as follows: P is a parabolic subgroup with Levi M , $W(M)$ is a finite group (acting via intertwining operators), π is an automorphic representation of M with unitary central character appearing in the L^2 discrete spectrum, $F \subset i\check{\mathfrak{a}}_M$ is a finite set (depending on Θ), and $D_F = \bigoplus_{\lambda \in F} D_\lambda$ is the space of distributions supported on F . For $\lambda \in (\check{\mathfrak{a}}_M)_\mathbb{C}$ the space D_λ of distributions supported on λ is a $(\mathfrak{m}, K_\infty^M) \times M(\mathbb{A}_f)$ -module in such a way that it has a filtration (not of finite length) with graded quotients the modules \mathbb{C}_λ^M defined above in 1.3. If $P \neq G$ the previous lemma implies that $\Pi_{j_S}^{\mathbb{K}^S} = \{0\}$ for any S . If $P = G$ then $\Pi = \pi$ is a discrete L^2 automorphic representation of G appearing in $L_{dis}^2(G(\mathbb{Q})A_\infty \backslash G(\mathbb{A}))$ and hence $\Pi_{j_S}^{\mathbb{K}^S} = \{0\}$ unless Π is the space of constant functions. This completes the proof of (2) and of the theorem. \square

Proof of Lemma 1.5.3. Recall (cf. [L2, Lemma 1]) that the constituents of $\Pi = \text{Ind}_P^G(\mathbb{C}_\nu^M \otimes \tau)$ are of the form $\pi = \otimes_p \pi_p$ where for each p , π_p is a constituent of $\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \nu \otimes \tau_p$, and for almost all p where τ_p is spherical, π_p is the unique \mathbb{K}_p -spherical constituent of $\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \nu \otimes \tau_p$. (The component at p of \mathbb{C}_ν^M is the unramified character of $A_M(\mathbb{Q}_p)$ given by $\nu \in (\check{\mathfrak{a}}_M)_\mathbb{C}$, which we continue to denote by ν .) Let p be such that τ_p is spherical. Then π_p is the spherical subquotient of $\text{Ind}_{M_{0,p}(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)} \chi$ where $M_{0,p} \subset M_0$ is a minimal Levi of G/\mathbb{Q}_p and χ is an unramified character of $M_{0,p}$, unitary on $A_M(\mathbb{Q}_p)$, and we may assume that χ is dominant ([C]). Thus π_p is a constituent of

$$\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \text{Ind}_{M_{0,p}(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)} \nu \chi = \text{Ind}_{M_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \nu \chi.$$

We use the following fact: The unramified principal series representation $\text{Ind}_{M_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \lambda$ has no trivial constituent unless $\lambda \in W_0 \rho_{0,p}$, in particular for dominant λ we must have $\lambda = \rho_{0,p}$. (Here W_0 is the relative Weyl group. This fact follows easily from the structure of the Jacquet module of the unramified principal series [C].) Thus $\nu \chi = \rho_{0,p}$ and hence $\nu \chi|_{A_M} = \rho_P$. Since χ is unitary on $A_M(\mathbb{Q}_p)$ we have $\text{Re}(\nu) = \rho_P$. \square

1.6. Cohomology of the RBS compactification. To define the RBS compactification of the quotient

$$M_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty^0 A_\infty K = \text{Stab}_{G(\mathbb{Q})}(D) \backslash (D \times G(\mathbb{A}_f) / K)$$

we can either compactify each locally symmetric component (as in [Z1] or [GHM]) or else define

$$\bar{M}_K = \text{Stab}_{G(\mathbb{Q})}(D) \backslash (\bar{D} \times G(\mathbb{A}_f) / K)$$

where \bar{D} is the RBS bordification of D (see e.g. [GT, 1.3]). This is evidently compatible with change-of-level and the direct limit

$$H^*(\bar{\mathcal{M}}) = \varinjlim_K H^*(\bar{M}_K).$$

has an action of $G(\mathbb{A}_f)$ where $g \in G(\mathbb{A}_f)$ acts by pullback by the isomorphism $\bar{M}_{gKg^{-1}} \rightarrow \bar{M}_K$ induced by right translation by g on $G(\mathbb{A}_f)$. The Poincaré dual theory to the cohomology of \bar{M}_K is the weighted cohomology group $W^d H^*(\bar{M}_K)$ of [GHM] corresponding to the *dualizing profile* [GHM, §9]. This is defined via a complex of sheaves $W^d C(\mathbb{C})$ which is a version of the dualizing complex on \bar{M}_K [GHM, §19], thus $W^d H^*(\bar{M}_K) \cong H_{\dim M - *}(\bar{M}_K)$. The C^∞ de Rham model of [N1] for these groups referred to earlier is:

Proposition 1.6.1. *For each $K \subset G(\mathbb{A}_f)$ there are natural isomorphisms*

$$\begin{aligned} \mathrm{H}^*(\overline{M}_K) &= \mathrm{H}^*(\mathfrak{g}, K_\infty^0 A_\infty, B(G)_1^K) \\ \mathrm{W}^d \mathrm{H}^*(\overline{M}_K) &= \mathrm{H}^*(\mathfrak{g}, K_\infty^0 A_\infty, R(G)_1^K) \end{aligned}$$

such that for $g \in G(\mathbb{A}_f)$ the pullback by $\overline{M}_{gKg^{-1}} \rightarrow \overline{M}_K$ is the map induced by right translation by g on $G(\mathbb{A})$ on function spaces.

Proof. For the first assertion take $\lambda = 0$ in Theorem 3.8 of [N1] and note that the weighted complex $W^0 C(\mathbb{C})$ of [GHM] is quasiisomorphic to the constant sheaf $\mathbb{C}_{\overline{M}_K}$ (cf. [GHM, §19]). The second assertion is the case $\lambda = 2\rho_0$ in [N1, Theorem 3.8]. \square

It follows from this description that one can also consider the direct limit

$$\mathrm{W}^d \mathrm{H}^*(\overline{\mathcal{M}}) = \varinjlim_K \mathrm{W}^d \mathrm{H}^*(\overline{M}_K)$$

which is a $G(\mathbb{A}_f)$ -module. In the direct limits $\mathrm{H}^*(\overline{\mathcal{M}})$ and $\mathrm{W}^d \mathrm{H}^*(\overline{\mathcal{M}})$ the K -invariants give back the groups $\mathrm{H}^*(\overline{M}_K)$ and $\mathrm{W}^d \mathrm{H}^*(\overline{M}_K)$, so that these direct limits are admissible. The induced action of the Hecke algebra of K -biinvariant functions on $\mathrm{H}^*(\overline{M}_K)$ and $\mathrm{W}^d \mathrm{H}^*(\overline{M}_K)$ is the familiar geometric action in terms of correspondences.

The inclusions $\mathbb{C} \subset B(G)_1 \subset R(G)_1$ give rise to $G(\mathbb{A}_f)$ -homomorphisms

$$\mathrm{H}^*(\check{D}) \rightarrow \mathrm{H}^*(\overline{\mathcal{M}}) \rightarrow \mathrm{W}^d \mathrm{H}^*(\overline{\mathcal{M}})$$

in cohomology. (At a finite level K these translate, on relative Lie algebra complexes, to the inclusion of $G(\mathbb{R})$ -invariant forms on $G(\mathbb{R})/K_\infty^0 A_\infty$ in spaces of differential forms on M_K .)

Theorem 1.6.2. *The homomorphism $\theta : \mathrm{H}^*(\check{D}) \rightarrow \mathrm{H}^*(\overline{\mathcal{M}})$ induces isomorphisms*

$$\mathrm{H}^*(\check{D}) \cong \mathrm{H}^*(\overline{\mathcal{M}})^{G(\mathbb{A}_f)} \cong \mathrm{H}^*(\overline{\mathcal{M}})_{G(\mathbb{A}_f)}.$$

The homomorphism $\mathrm{H}^(\check{D}) \rightarrow \mathrm{W}^d \mathrm{H}^*(\overline{\mathcal{M}})$ induces isomorphisms*

$$\mathrm{H}^*(\check{D}) \cong \mathrm{W}^d \mathrm{H}^*(\overline{\mathcal{M}})^{G(\mathbb{A}_f)} \cong \mathrm{W}^d \mathrm{H}^*(\overline{\mathcal{M}})_{G(\mathbb{A}_f)}.$$

Proof. It follows from results of Franke that the inclusion

$$\mathfrak{F} \mathrm{in}_\Theta R(G)_1 \subset R(G)_1$$

induces an isomorphism in $(\mathfrak{g}, K_\infty^0 A_\infty)$ -cohomology: Indeed, Theorem 16 on p. 246 of [F1] applies because $R(G)_1$ is $S_{\rho_{-\tau} - \log}(G(\mathbb{Q})A_\infty \backslash G(\mathbb{A}))$ for $\tau = \rho_0$ and $\rho_0 \in {}^+ \check{\mathfrak{a}}_0^G \cap \overline{\check{\mathfrak{a}}_0^{G+}}$, so that the derived functors $\mathfrak{F} \mathrm{in}_\Theta^i$ vanish on $R(G)_1$. By (3) of Theorem 7 on p. 208 of [F1] we deduce that $\mathfrak{F} \mathrm{in}_\Theta R(G)_1 \subset R(G)_1$ induces an isomorphism in cohomology.

For a set S of finite primes containing almost all primes we have (using Theorem 1.5.1):

$$\begin{aligned} \mathrm{W}^d \mathrm{H}^*(\overline{\mathcal{M}})_{j_S}^{\mathbb{K}_S} &= \mathrm{H}^*(\mathfrak{g}, K_\infty^0 A_\infty, \mathfrak{F} \mathrm{in}_\Theta R(G)_{j_S}^{\mathbb{K}_S}) \\ &= \mathrm{H}^*(\mathfrak{g}, K_\infty^0 A_\infty, \mathbb{C}) \\ &= \mathrm{H}^*(\check{D}). \end{aligned}$$

Thus $W^d H^*(\overline{\mathcal{M}})_j = H^*(\check{D})$ is a direct summand of $W^d H^*(\overline{\mathcal{M}})$ and the complement contains no almost-everywhere-trivial constituents, in particular neither invariants nor coinvariants. This proves the assertions about $W^d H^*(\overline{\mathcal{M}})$.

For $H^*(\overline{\mathcal{M}})$ we argue by duality: By the previous paragraph choosing a generator for $H^n(\check{D})$ gives a $G(\mathbb{A}_f)$ -invariant element in $W^d H^n(\overline{\mathcal{M}})$ (where $n = \dim M_K = \dim \check{D}$) which restricts to a generator of $W^d H^n(-)$ on each connected component of \overline{M}_K at each level K . This gives a linear form $W^d H^n(\overline{M}_K) \rightarrow \mathbb{C}$ by summing the contributions from each component of \overline{M}_K , and these give a $G(\mathbb{A}_f)$ -invariant form $W^d H^n(\overline{\mathcal{M}}) \rightarrow \mathbb{C}$. Composing with the cup product $H^i(\overline{\mathcal{M}}) \times W^d H^{n-i}(\overline{\mathcal{M}}) \rightarrow W^d H^n(\overline{\mathcal{M}})$ gives a $G(\mathbb{A}_f)$ -invariant pairing identifying $H^i(\overline{\mathcal{M}})$ with the contragredient of $W^d H^{n-i}(\overline{\mathcal{M}})$. The theorem follows. \square

Remark 1.6.3. In [F2, Cor. 3.5] Franke computes the invariants in $H_c^*(\mathcal{M}) = \varinjlim_K H_c^*(M_K)$ and the coinvariants in $H^*(\mathcal{M}) = \varinjlim_K H^*(M_K)$ in terms of a certain open subset of the compact dual. These are related to our spaces by maps

$$H_c^*(\mathcal{M})^{G(\mathbb{A}_f)} \rightarrow H^*(\overline{\mathcal{M}})^{G(\mathbb{A}_f)} = H^*(\check{D}) = W^d H^*(\overline{\mathcal{M}})_{G(\mathbb{A}_f)} \rightarrow H^*(\mathcal{M})_{G(\mathbb{A}_f)}.$$

The computation of [F2] is based on the observation that the Eisenstein series which contribute to the summand $H^*(\mathcal{M})_j$ of $H^*(\mathcal{M})$ are those starting from the constant function on a standard Levi (including G itself here) and evaluated at the half-sum of positive roots. The fact that $W^d H^*(\overline{\mathcal{M}}) \rightarrow H^*(\mathcal{M})$ corresponds to the inclusion $R(G)_1 \subset S(G)_1$ and that this submodule is given, after passing to subspaces of automorphic forms, by the condition (1.5.3) on exponents means that these Eisenstein series do not influence the summand $W^d H^*(\overline{\mathcal{M}})_j$ except for G itself. Thus we are left with the constants, which contribute $H^*(\check{D})$.

Remark 1.6.4. When M_K has a complex structure there is an analogue of the theorem for toroidal compactifications, cf. 4.2. The analogue of the theorem fails in an interesting way for the minimal compactification, cf. 4.3.

2. MIXED REALIZATIONS IN THE RBS COMPACTIFICATION

In this section we review the mixed realizations in the cohomology of \overline{M}_K when M_K is a Shimura variety. Two approaches are possible. The first, contained in [N2, NV] and outlined in 2.2–2.4, uses Morel’s weight truncations [Mo] in categories of mixed sheaves (in the sense of Saito [S2, S3]) built out of mixed realizations. The second, due to Ayoub and Zucker [AZ] and outlined in 2.5, depends on Ayoub’s theory of motivic sheaves and their realizations. Because some compatibilities between various realizations remain unchecked this second approach does not quite, at present, give a mixed realization, see 2.5 below. Moreover, since we are only interested in realizations (and not motivic results), the first approach is preferable: though elementary, it gives finer results.

2.1. Shimura varieties and compactifications. Let $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ and let $w : \mathbb{G}_{m,\mathbb{R}} \rightarrow \mathbb{S}$ be the canonical homomorphism. Let (G, X) be a motivic Shimura datum, i.e. a pair consisting of a connected reductive \mathbb{Q} -group G and a $G(\mathbb{R})$ -conjugacy class of homomorphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ satisfying

- (S1) The Hodge structure on \mathfrak{g} given by $\text{ad} \circ h$ is of type $(-1, 1) + (0, 0) + (1, -1)$.
- (S2) The automorphism $\text{Ad}(h(\sqrt{-1}))$ induces a Cartan involution on $G_0 = G^{\text{der}}(\mathbb{R})^0$ and G_0 has no compact factors defined over \mathbb{Q} .
- (S3) The weight homomorphism $h \circ w : \mathbb{G}_{m,\mathbb{R}} \rightarrow G_{\mathbb{R}}$ is defined over \mathbb{Q} .

We will further assume that

(S4) The maximal \mathbb{Q} -split torus A_G of the centre of G is maximally \mathbb{R} -split.

Thus G satisfies the conditions of §1 and we will use the notation fixed in 1.1. Under these conditions the stabilizer in $G(\mathbb{R})$ of a point is of the form $K_\infty^0 A_\infty$ where (as in 1.1) K_∞ is a maximal compact of $G(\mathbb{R})$ and $A_\infty = A_G(\mathbb{R})^0$. The quotient

$$M_K = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty^0 K_f$$

is the (complex points of) the Shimura variety at level K . By the theory of canonical models of Shimura, Deligne, Borovoi, Milne, the varieties M_K have models over a number field $E = E(G, X) \subset \mathbb{C}$ independent of K (the reflex field), and the morphisms $M_{K'} \rightarrow M_K$ for $K' \subset K$ are defined over E , so that the scheme

$$\varprojlim_K M_K$$

has a model over E . The action of $G(\mathbb{A}_f)$ on the scheme $\varprojlim_K M_K$ by isomorphisms $M_K \rightarrow M_{gKg^{-1}}$ is E -rational. (We will not distinguish notationally between the Shimura variety over E and its complex points.)

We assume that G^{der} is isotropic, so that M_K is noncompact. The compactifications of interest to us are:

- (1) The *minimal* (or *minimal Satake* or *Baily-Borel*) compactification:

$$M_K^* = G(\mathbb{Q}) \backslash X^* \times G(\mathbb{A}_f) / K$$

where $X^* = \bigsqcup_{D \subset X} D^*$ for D^* the union of the rational boundary components of D with the Satake topology. This is a normal projective variety with a model over E (the unique extension of the canonical model of M_K , see [P, 12.3]). It has a natural stratification defined over E in which the boundary strata are (up to finite covers) themselves Shimura varieties ([P, 12.3]).

- (2) The *RBS* compactification \bar{M}_K (defined earlier in 1.6). This is the union of the RBS compactifications of each component of M_K and it is easy to describe its strata. For a rational parabolic subgroup P let $L_P = P/R_u P$ be the Levi quotient and $M_P = \bigcap_{\chi: L_P \rightarrow \mathbb{G}_m} \chi^2$; then $M_P(\mathbb{R})$ contains any compact or arithmetic subgroup of $L_P(\mathbb{R})$. Write $\bar{M}_K = \bigsqcup_i \Gamma_i \backslash \bar{D}$ for congruence subgroups $\Gamma_i \subset G(\mathbb{Q})$. Then

$$\Gamma_i \backslash \bar{D} = \bigsqcup_P \Gamma_{i, M_P} \backslash M_P(\mathbb{R}) / K_P$$

where the union is over rational parabolic subgroups modulo Γ_i -conjugacy, and for a rational parabolic subgroup P , Γ_{i, M_P} is the projection of $\Gamma_i \cap P(\mathbb{R})$ to $L_P(\mathbb{R})$ and K_P is the projection of $K \cap P(\mathbb{R})$ to $L_P(\mathbb{R})$.

There is a proper continuous map

$$p : \bar{M}_K \rightarrow M_K^*$$

extending the identity. (See [GHM, §22] for a description of p .)

- (3) The *toroidal* compactifications M_K^Σ (cf. [AMRT, H2, P]): For suitable data Σ this is a smooth projective variety over E with an open immersion $M_K \hookrightarrow M_K^\Sigma$ and the complement of M_K is a normal crossings divisor with smooth components. There is a proper morphism

$$\pi : M_K^\Sigma \rightarrow M_K^*$$

over E extending the identity.

We will return to these compactifications in 2.4.

2.2. Mixed sheaves. Saito has defined and studied the notion of a theory of A -mixed sheaves (A a field of characteristic zero) on varieties over a subfield $k \subset \mathbb{C}$ ([S2], see [S3, §1] for a summary). Briefly, such a theory gives an A -linear abelian category $\mathcal{M}(X)$ for every variety X/k with an A -linear faithful exact functor For to perverse A -sheaves on $X(\mathbb{C})$, satisfying a certain set of axioms listed in [S2, 1.1–1.6]. These include the existence of weight filtrations for objects of $\mathcal{M}(X)$ and the semisimplicity of graded pieces, a contravariant duality functor $\mathbb{D} : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ with $\mathbb{D}^2 = id$, the existence of pullbacks by open immersions and pushforward by affine morphisms, and the existence of a constant object $A^{\mathcal{M}} \in \mathcal{M}(\text{Spec}(k))$. Given this setup, Saito shows in [S2] (cf. [S3, Theorem 1.2] and the remarks after it) that for a morphism $f : X \rightarrow Y$ there are four functors $f^*, f_*, f^!, f_!$ between derived categories $D^b\mathcal{M}(X)$ and $D^b\mathcal{M}(Y)$ having the appropriate adjointness properties and compatible with the usual functors between $D_c^b(\mathbb{Q}_X)$ and $D_c^b(A_Y)$ under the functor $For : D^b\mathcal{M}(X) \rightarrow D_c^b(A_X)$ induced by For . The duality functor extends to $D^b\mathcal{M}(X)$ and the expected properties, e.g. $\mathbb{D} \circ f^* = f^! \circ \mathbb{D}$ etc. hold. The notion of weights extends to the objects of $D^b\mathcal{M}(X)$ (see [S2, §6]) and the functors $f_*, f_!, f^*, f^!$ have the correct behaviour with respect to weights [S2, 6.7], so that there is a decomposition theorem [S2, 6.10]. For each variety X there is an object $A_X^{\mathcal{M}}$ in $D^b\mathcal{M}(X)$ with $For(A_X^{\mathcal{M}}) = A_X$; in fact $A_X^{\mathcal{M}} = a_X^* A^{\mathcal{M}}$ for $a_X : X \rightarrow \text{Spec}(k)$ the structure morphism.

The basic example is the theory of mixed Hodge modules [S1], which is a theory of \mathbb{Q} -mixed sheaves on varieties over \mathbb{C} with $\mathcal{M}(X) = \text{MHM}(X(\mathbb{C}))$ and $For = rat$ (i.e. taking the underlying perverse sheaf of a mixed Hodge module).

The theory of \mathbb{Q} -mixed sheaves we will use here is an enrichment of mixed Hodge modules described in [S2, 1.8(iv-v)] or [S3, 1.1] for any subfield $k \subset \mathbb{C}$. An object of $\mathcal{M}(X)$ is a triple $((M_k, F, W), (K, W), (K_l, W)_l)$ where

- (M_k, F) is a filtered regular holonomic D -module on X/k with a finite increasing filtration W on $M_k \otimes \mathbb{C}$
- (K, W) is a perverse \mathbb{Q} -complex on $X(\mathbb{C})^{an}$ with an increasing filtration W
- for each prime l , (K_l, W) is a filtered perverse étale \mathbb{Q}_l -complex on $X \times_k \bar{k}$ with a continuous $\text{Gal}(\bar{k}/k)$ -action compatible with the action on $X \times_k \bar{k}$

such that the pair $((M_k \otimes \mathbb{C}, F, W), (K, W))$ is a mixed Hodge module on $X(\mathbb{C})$, and we are also given comparison isomorphisms relating $(K, W) \otimes \mathbb{Q}_l$ and (K_l, W_l) (see [S3, 1.1] or [S2, 1.8] for the precise description; note that the comparison isomorphisms are part of the data of the object). The category $\mathcal{M}(\text{Spec}(k))$ is essentially the category of mixed realizations over k (cf. e.g. [D, J, Hu1]), except that we use the fixed embedding $k \subset \mathbb{C}$ rather than all embeddings of k in \mathbb{C} . Thus for a variety X/k and $K \in D^b\mathcal{M}(X)$, the object

$$H^i(a_{X*}K) \in \mathcal{M}(\text{Spec}(k))$$

($a_X : X \rightarrow \text{Spec}(k)$ is the structure morphism) is a triple consisting of a \mathbb{Q} -mixed Hodge structure, a filtered vector space over k , and a family of l -adic $\text{Gal}(\bar{k}/k)$ -representations, all related by comparison isomorphisms. In particular, for a variety X and $K = \mathbb{Q}_X^{\mathcal{M}}$ this gives the usual mixed realization on the cohomology:

$$\begin{aligned} H^*(X) &:= (H_B^*(X), H_{dR}^*(X/k), (H_l^*(X))_l) \\ &= ((H^*(X(\mathbb{C})^{an}, \mathbb{Q}), W_{\bullet}, F^{\bullet}), (H_{dR}^*(X/k), F^{\bullet}), (H_{et}^*(X \times_k \bar{k}, \mathbb{Q}_l)_l)). \end{aligned}$$

For X irreducible of dimension d , the intersection complex $K = IC_X(\mathbb{Q}^{\mathcal{M}})[d] = j_{!*}\mathbb{Q}_U^{\mathcal{M}}[d]$ where $j : U \hookrightarrow X$ is a smooth open subset belongs to $\mathcal{M}(X)$ (cf. e.g. [S2, §6]), so this gives

a mixed realization on intersection cohomology:

$$\mathrm{IH}^*(X) := \left((\mathrm{IH}^*(X(\mathbb{C})^{an}, \mathbb{Q}), W_\bullet, F^\bullet), (\mathrm{IH}_{dR}^*(X/k), F^\bullet), (\mathrm{IH}_{et}^*(X \times_k \bar{k}, \mathbb{Q}_l)_l) \right).$$

When X is proper this is pure, i.e. $\mathrm{Gr}_j^W \mathrm{IH}^i(X) = 0$ for $j \neq i$.

Henceforth, $\mathcal{M}(\cdot)$ will stand for the theory of mixed sheaves fixed here.

2.3. Truncation by weights. Morel [Mo] found new t -structures in categories of mixed sheaves. (The arguments in [Mo] are given for l -adic sheaves over a finite field but they work in any theory of mixed sheaves with trivial changes.) Briefly, Morel shows that for any $a \in \mathbb{Z} \cup \{\pm\infty\}$ the pair $({}^w D^{\leq a} \mathcal{M}(X), {}^w D^{> a} \mathcal{M}(X))$ of full (and, in fact, triangulated) subcategories defined by $K \in {}^w D^{\leq a} \mathcal{M}(X) \iff \mathrm{H}^i(K) \in \mathcal{M}(X)$ has weights $\leq a$ (respectively, $K \in {}^w D^{> a} \mathcal{M}(X) \iff \mathrm{H}^i(K)$ has weights $> a$) defines a t -structure on $D^b \mathcal{M}(X)$. The associated truncation functors are denoted $w_{\leq a}, w_{> a}$.

By a stratification of X we mean a partition $X = \bigsqcup_{i=0}^r X_i$ into locally closed subvarieties with each X_d open in $X - \bigsqcup_{i < d} X_j$. Given a stratification of X by equidimensional subvarieties, and any function \underline{a} from the set of strata to $\mathbb{Z} \cup \{\pm\infty\}$, gluing the t -structures above along the strata using the theory of [BBD, §1.4] produces a new t -structure $({}^w D^{\leq \underline{a}} \mathcal{M}(X), {}^w D^{> \underline{a}} \mathcal{M}(X))$ on $D^b \mathcal{M}(X)$ (cf. [Mo, §3]). Thus $K \in {}^w D^{\leq \underline{a}} \mathcal{M}(X)$ (respectively, $K \in {}^w D^{> \underline{a}} \mathcal{M}(X)$) iff $i_{X_k}^* K \in {}^w D^{\leq \underline{a}(X_k)} \mathcal{M}(X_k)$ (respectively, $i_{X_k}^! K \in {}^w D^{\leq \underline{a}(X_k)} \mathcal{M}(X_k)$), where $i_{X_k} : X_k \hookrightarrow X$. We will be interested in the particular function \underline{dim} defined (for any stratification) by:

$$\underline{dim}(S) := \dim S.$$

The truncation functors of the glued t -structure are denoted $w_{\leq \underline{dim}}$ and $w_{> \underline{dim}}$.

Proposition 2.3.1. *Suppose that U is smooth of dimension n , $j : U \hookrightarrow X$ is an open immersion as a Zariski-dense subset of an irreducible variety, and $\pi : Y \rightarrow X$ is a proper morphism from Y smooth such that $\pi|_{\pi^{-1}(U)}$ is an isomorphism. Assume X is given a stratification such that $j_! \mathbb{Q}_U^{\mathcal{M}}[n]$, $j_* \mathbb{Q}_U^{\mathcal{M}}[n]$, and $\pi_* \mathbb{Q}_Y^{\mathcal{M}}[n]$ are constructible. Then*

$$w_{\leq \underline{dim}} j_* \mathbb{Q}_U^{\mathcal{M}}[n] = w_{\leq \underline{dim}} j_! \mathbb{Q}_U^{\mathcal{M}}[n] = w_{\leq \underline{dim}} \pi_* \mathbb{Q}_Y^{\mathcal{M}}[n].$$

Proof. The equality of extreme terms is [NV, Proposition 4.1.2]. The first equality (also noted in [NV]) follows by applying $w_{\leq \underline{dim}}$ to Morel's formula $j_! \mathbb{Q}_U^{\mathcal{M}}[n] = w_{\leq n} j_* \mathbb{Q}_U^{\mathcal{M}}[n]$ ([Mo, Theorem 3.1.4]). \square

2.4. The RBS compactification as a weight truncation. Let $j : M_K \hookrightarrow M_K^*$ be the inclusion. We work in the theory $\mathcal{M}(-)$ on varieties over the reflex field E outlined above and with the canonical stratification of M_K^* . Recall the compactifications in 2.1.

Proposition 2.4.1. *There is a natural isomorphism*

$$p_* \mathbb{Q}_{\bar{M}_K} = \mathrm{For}(w_{\leq \underline{dim}} j_* \mathbb{Q}_{M_K}^{\mathcal{M}})$$

in the derived category of constructible sheaves on M_K^ .*

Proof. This is a special case of [N2, Theorem 4.3.1] (see [N2, 4.6] for a discussion). [N2] is written in the context of mixed Hodge modules, but as remarked there (Remark 4.3.7 of loc. cit.) the proof works in any theory for which For factors through $\mathrm{rat} : D^b \mathrm{MHM}(X) \rightarrow D_c^b(\mathbb{Q}_X)$, in particular for $\mathcal{M}(-)$. \square

Thus the cohomology of \overline{M}_K carries a mixed realization

$$\mathbf{H}^*(\overline{M}_K) = (\mathbf{H}_B^*(\overline{M}_K), \mathbf{H}_{dR}^*(\overline{M}_K), (\mathbf{H}_l^*(\overline{M}_K))_l) := \mathbf{H}^*(a_{M_K^*} w_{\leq \dim j_*} \mathbb{Q}_{M_K}^{\mathcal{M}}) \in \mathcal{M}(\mathrm{Spec}(E)).$$

Combining Propositions 2.4.1 and 2.3.1 gives identities

$$p_* \mathbb{Q}_{\overline{M}_K} = \mathrm{For}(w_{\leq \dim j_*} \mathbb{Q}_{M_K}^{\mathcal{M}}) = \mathrm{For}(w_{\leq \dim j_*} \mathbb{Q}_{M_K}^{\mathcal{M}}[n][-n]) = \mathrm{For}(w_{\leq \dim} \pi_* \mathbb{Q}_{M_K^\Sigma}^{\mathcal{M}}).$$

The consequences of interest to us are summarized in the following commutative diagram in $\mathcal{M}(\mathrm{Spec}(E))$:

$$\begin{array}{ccc} & & \mathbf{H}^*(M_K^\Sigma) \\ & \nearrow \gamma^* & \\ \mathbf{H}^*(\overline{M}_K) & \xrightarrow{\rho} & \mathbf{IH}^*(M_K^*) \\ & \nwarrow p^* & \uparrow \iota \\ & & \mathbf{H}^*(M_K^*) \\ & & \nearrow \pi^* \end{array} \quad (2.4.1)$$

Here $\iota : \mathbf{IH}^*(M_K^*) \hookrightarrow \mathbf{H}^*(M_K^\Sigma)$ is given by any homomorphism $j_{!*} \mathbb{Q}_{M_K}^{\mathcal{M}}[n] \rightarrow \pi_* \mathbb{Q}_{M_K^\Sigma}^{\mathcal{M}}[n]$ coming from the decomposition theorem ([BBD], [S1], [S2, 6.10]). The morphisms other than ι are canonical and Hecke-equivariant in the appropriate sense (e.g. for π^* , γ^* one keeps in mind that a Hecke operator goes from $\mathbf{H}^*(M_K^\Sigma) \rightarrow \mathbf{H}^*(M_K^{\Sigma'})$ for some Σ').

Dualizing gives mixed realizations on the groups $\mathbf{W}^d \mathbf{H}^*(\overline{M}_K)$. The ring structure of $\mathbf{H}^*(\overline{M}_K)$, the natural map $\mathbf{H}^*(\overline{M}_K) \rightarrow \mathbf{W}^d \mathbf{H}^*(\overline{M}_K)$, and the action of $\mathbf{H}^*(\overline{M}_K)$ on $\mathbf{W}^d \mathbf{H}^*(\overline{M}_K)$ all respect mixed structures.

We will not use the following fact, but it is useful to keep it in mind:

Lemma 2.4.2. *The weights of $\mathbf{H}^i(\overline{M}_K)$ are $\leq i$. The top weight quotient is*

$$\mathrm{Gr}_i^{\mathbf{W}} \mathbf{H}^i(\overline{M}_K) = \mathrm{im}(\mathbf{H}^i(\overline{M}_K) \rightarrow \mathbf{IH}^*(M_K^*)) = \mathrm{im}(\mathbf{H}^i(\overline{M}_K) \xrightarrow{\gamma^*} \mathbf{H}^i(M_K^\Sigma)).$$

Dually, the weights of $\mathbf{W}^d \mathbf{H}^i(\overline{M}_K)$ are $\geq i$ and the bottom weight piece is the image of the map $\mathbf{IH}^i(M_K^*) \rightarrow \mathbf{W}^d \mathbf{H}^i(\overline{M}_K)$ Poincaré dual to $\mathbf{H}^*(\overline{M}_K) \rightarrow \mathbf{IH}^*(M_K^*)$.

Proof. The constraint on weights and the equality of the extreme terms is proved in [N2, Proposition 2.4.2(ii)] or [NV, Lemma 4.3.1]. Since ι is injective the second equality holds. \square

Remark 2.4.3. Goresky and Tai [GT] proved the existence of a map $\mathbf{H}^*(\overline{M}_K, \mathbb{Z}) \rightarrow \mathbf{H}^*(M_K^\Sigma, \mathbb{Z})$ factoring $\pi^* : \mathbf{H}^*(M_K^*, \mathbb{Z}) \rightarrow \mathbf{H}^*(M_K^\Sigma, \mathbb{Z})$. This agrees $(\otimes \mathbb{Q})$ with γ^* .

2.5. Motivic approach. This approach uses the theory of motivic sheaves (see [A] and the references there), unlike that of [N2, NV], which works with the more classical objects in 2.2. It does not give the whole diagram (2.4.1) but gives the outer triangle, i.e. the factorization $\pi^* = \gamma^* \circ p^*$, modulo some compatibilities which have not been checked in the literature. This factorization is enough for the proof of Theorem 2 in §3.

Theorem 4.1 of [AZ] gives an object $\mathbb{E}_{M_K^*}$ in Ayoub's triangulated category $\mathbf{DA}(M_K^*, \mathbb{Q})$ of (étale) motivic sheaves on M_K^* which realizes (in Ayoub's Betti realization) to $p_* \mathbb{Q}_{\overline{M}_K}$. Ayoub's theory of motivic sheaves has a formalism of six functors compatible with the Betti realization, and over $\mathrm{Spec}(E)$ it gives Voevodsky's triangulated category of motives over E (cf. [A] for a survey of the theory and specific references). So pushing

$\mathbb{E}_{M_K^*}$ forward by $M_{K^*} \rightarrow \text{Spec}(E)$ gives a Voevodsky motive over E with Betti realization $R\Gamma(\overline{M}_K, \mathbb{Q}) \cong \bigoplus_i H^i(\overline{M}_K, \mathbb{Q})[-i]$. Huber [Hu1, Hu2, Hu3] has constructed realization functors from Voevodsky's triangulated category to the derived category of mixed realizations, so that one also gets a mixed realization in this way. This would give a mixed realization with Betti part $H^*(\overline{M}_K, \mathbb{Q})$ and the factorization $\pi^* = \gamma^* \circ p^*$, except that the compatibility of Huber's Betti realization with Ayoub's has not been checked (see Remarks 4.4 and 4.9 of [AZ].)

Remark 2.5.1. The mixed realizations coming from the two approaches described here could, in principle, be different. Vaish [V] shows that given suitable realization functors (to derived categories of mixed sheaves) on Ayoub's category of motivic sheaves the two approaches yield the same objects. Such realizations have recently been constructed by Ivorra [I]. (The agreement of the Betti realization in [I] with the one used in [AZ] has not been checked, so that one cannot combine [AZ] and [I] to get a mixed realization with Betti part $H^*(\overline{M}_K, \mathbb{Q})$.)

3. CHERN CLASSES OF AUTOMORPHIC VECTOR BUNDLES

We continue in the setting of Shimura varieties of §2, i.e. (G, X) satisfies conditions (S1)–(S4) of 2.1. We make the additional assumption that

$$G^{\text{der}} \text{ is simply connected.}$$

Thus $G_0 = G^{\text{der}}(\mathbb{R})$, and the compact form G_0^c , being a maximal compact subgroup of $G^{\text{der}}(\mathbb{C})$, is simply connected.

If H is a compact group (or an algebraic group over a subfield of \mathbb{C}) the category of finite-dimensional continuous (or rational) complex representations is denoted $\text{Rep}(H)$.

3.1. Chern classes. For the reader's convenience we summarize the properties of Chern classes that we will use. For a clear discussion see [DMOS, I.1, pp. 19–22]. Let $k \subset \mathbb{C}$.

Let $\mathbb{Q}_B(1)$ be the Tate Hodge structure, i.e. \mathbb{C} with \mathbb{Q} -structure $(2\pi\sqrt{-1})\mathbb{Q}$ and Hodge filtration $F^{-1} = \mathbb{C}, F^0 = 0$. Let $\mathbb{Q}_l(1)$ be \mathbb{Q}_l with the action of $\text{Gal}(\overline{k}/k)$ by the l -adic cyclotomic character $\chi_l : \text{Gal}(\overline{k}/k) \rightarrow \mathbb{Q}_l^*$ for any prime l , and $\mathbb{Q}_{dR}(1) = k$ with the filtration $F^{-1} = k, F^0 = \{0\}$. Thus $(\mathbb{Q}_B(1), \mathbb{Q}_{dR}(1), (\mathbb{Q}_l(1))_l)$ is the Tate object in mixed realizations, i.e. the mixed realization $H_2(\mathbb{P}^1)$. For each $i \in \mathbb{Z}$ the comparison isomorphisms $\mathbb{Q}_B(i) \otimes \mathbb{Q}_l = \mathbb{Q}_l(i)$ (over \overline{k}) and $\mathbb{Q}_B(i) \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{Q}_{dR}(i) \otimes_k \mathbb{C}$ give comparison isomorphisms

$$\begin{aligned} \sigma_l : H_B^*(X)(i) \otimes \mathbb{Q}_l &\rightarrow H_l^*(X)(i) \\ \sigma_{dR} : H_B^*(X)(i) \otimes_{\mathbb{Q}} \mathbb{C} &\rightarrow H_{dR}^*(X)(i) \otimes_k \mathbb{C} \end{aligned}$$

for any smooth variety X over k , where, as in 2.2, we have

$$(H_B^*(X), H_{dR}^*(X/k), H_l^*(X)) = (H^*(X(\mathbb{C})^{\text{an}}, \mathbb{Q}), H_{dR}^*(X/k), H_{\text{et}}^*(X \times_k \overline{k}, \mathbb{Q}_l))$$

and $H_l^*(X)(i) = H_l^*(X) \otimes \mathbb{Q}_l(1)^{\otimes i}$ for $? = B, dR, l$.

If X/k is smooth and complete and \mathcal{F} is a vector bundle (i.e. locally free sheaf of \mathcal{O}_X -modules) on X/k , there are classes:

$$c_i^B(\mathcal{F}) \in H_B^{2i}(X)(i), \quad c_i^{dR}(\mathcal{F}) \in H_{dR}^{2i}(X/k)(i), \quad c_i^l(\mathcal{F}) \in H_l^{2i}(X)(i)$$

and these are related under comparison isomorphisms, i.e. $\sigma_{dR}(c_i^B(\mathcal{F})) = c_i^{dR}(\mathcal{F})$ and $\sigma_l(c_i^B(\mathcal{F})) = c_i^l(\mathcal{F})$. The class $c_i^B(\mathcal{F})$ is of weight zero, $c_i^{dR}(\mathcal{F})$ belongs to F^0 , and $c_i^l(\mathcal{F})$ is Galois-invariant.

For $X/k, \mathcal{F}$ as above there is also an underlying topological complex vector bundle F (with sheaf of holomorphic sections \mathcal{F}^{an}) on the space $X(\mathbb{C})^{an}$. This gives topological Chern classes

$$c_i(F) \in H^{2i}(X(\mathbb{C})^{an}, \mathbb{Q}) = H_B^{2i}(X)$$

(see [MS, §14], we ignore the integral theory) where one chooses $\sqrt{-1} \in \mathbb{C}$ to orient F . This choice also fixes an isomorphism $\mathbb{Q}_B \rightarrow \mathbb{Q}_B(1)$, hence isomorphisms $\mathbb{Q}_l \rightarrow \mathbb{Q}_l(1)$ (over \bar{k}) and $\mathbb{C} \rightarrow \mathbb{Q}_{dR}(1) \otimes \mathbb{C}$. These induce isomorphisms $H_B^*(X) \rightarrow H_B^*(X)(i)$, $H_l^*(X)(i) \cong H_l^*(X)$ (over \bar{k} , i.e. not Galois-equivariantly) and $H_{dR}^*(X/k) \otimes \mathbb{C} = H_{dR}^*(X/k)(i) \otimes \mathbb{C}$. The isomorphism $H_B^*(X) \rightarrow H_B^*(X)(i)$ maps $c_i(F)$ to $c_i^B(\mathcal{F})$, so that one has:

$$\sigma_{dR}((2\pi\sqrt{-1})^i c_i(F)) = c_i^{dR}(\mathcal{F}) \quad (3.1.1)$$

$$\text{Gal}(\bar{k}/k) \text{ acts on } \sigma_l(c_i(\mathcal{F})) \text{ by } \chi_l^{-i}. \quad (3.1.2)$$

The first point is easily checked (see the diagram on p. 20 of [DMOS]). The second holds because $H_l^*(X)(i) = H_l^*(X) \otimes \chi_l^i$ as $\text{Gal}(\bar{k}/k)$ -modules and $c_i^l(\mathcal{F})$ is Galois invariant.

In any context, the Chern classes of a vector bundle depend only on its class in the Grothendieck group of vector bundles, so that if two vector bundles have filtrations with isomorphic gradeds then they have the same Chern classes.

The following remark will not be used below but it is useful to keep it in mind: If X/k is an algebraic variety which is embeddable in a smooth variety, there is (by [Har] and [Gr]) a good theory of Chern classes in algebraic de Rham cohomology, i.e. for a vector bundle \mathcal{F} on X/k there are classes $c_i^{dR}(\mathcal{F}) \in H_{dR}^{2i}(X/k)$ which are related to the Chern classes of the underlying topological vector bundle on $X(\mathbb{C})^{an}$ as in (3.1.1).

In the sequel we use the same notation for algebraic vector bundles and their underlying topological vector bundles and suppress comparison isomorphisms from the notation (as in the introduction).

3.2. Automorphic vector bundles. We summarize some standard facts (see e.g. [H1, §3] or [Mi, Ch. III]) about the compact dual and automorphic vector bundles.

Fix a point $h \in X$. The stabilizer of h in $G(\mathbb{R})$ is of the form $K_\infty^0 A_\infty$ where $K_\infty \subset G(\mathbb{R})$ is maximal compact. There is a unique maximal parabolic P_h of $G(\mathbb{C})$ such that

- (i) $P_h \cap G_0 = K_{0,\infty} = K_\infty^0 \cap G_0$ and
- (ii) the induced map $G_0/K_{0,\infty} \rightarrow G(\mathbb{C})/P_h$ is an open holomorphic immersion.

The inclusion $G_0^c \subset G(\mathbb{C})$ induces an identification

$$\check{D} = G_0^c/K_{0,\infty} = G(\mathbb{C})/P_h$$

so that the compact dual is a $G(\mathbb{C})$ -homogeneous space. The open immersion $D \subset \check{D}$ in (ii) extends to give the $G(\mathbb{R})$ -equivariant *Borel embedding*:

$$\beta : X \hookrightarrow \check{D}.$$

The functor $V \mapsto \check{\mathcal{V}}$, where

$$\check{\mathcal{V}} = G(\mathbb{C}) \times_{P_h} V = G(\mathbb{C}) \times V/(g, v) \sim (gp, p^{-1} \cdot v),$$

gives an equivalence of $\text{Rep}(P_h)$ with the category of $G(\mathbb{C})$ -homogeneous vector bundles on \check{D} . A $G(\mathbb{C})$ -homogeneous vector bundle $\check{\mathcal{V}}$ on \check{D} gives an *automorphic vector bundle* $(\mathcal{V}_K)_K$ on the Shimura variety where

$$\mathcal{V}_K = G(\mathbb{Q}) \backslash (\beta^* \check{\mathcal{V}} \times G(\mathbb{A}_f)/K).$$

Then

$$\check{\mathcal{V}} \mapsto (\mathcal{V}_K)_K \quad (3.2.1)$$

defines an exact tensor functor from $G(\mathbb{C})$ -homogeneous vector bundles on \check{D} to $G(\mathbb{A}_f)$ -equivariant algebraic vector bundles on $\varprojlim M_K$.

The compact dual \check{D} has a natural E -structure (see [H1, 3.1] or [Mi, III.1]). By one of the main results of Harris's [H1] (see also [Mi, III.5]) the functor (3.2.1) is E -rational, i.e. takes a $G(\mathbb{C})$ -homogeneous vector bundle on \check{D} with L -rational structure for $L \supset E$ to an L -rational vector bundle on the canonical model of M_K .

3.3. For technical reasons we will need a slightly different construction of bundles on M_K . These come from the fact that $M_K = \bigsqcup_i \Gamma_i \backslash D$ where $\Gamma_i \subset G^{\text{der}}(\mathbb{Q})$ (for K small enough). Let $P_{0,h} := P_h \cap G^{\text{der}}$. This is a parabolic subgroup of G^{der} with $K_{0,\infty}^{\mathbb{C}}$ as a Levi subgroup, and

$$\check{D} = G(\mathbb{C})/P_h \cong G^{\text{der}}(\mathbb{C})/P_{0,h} \cong G_0^{\mathbb{C}}/K_{0,\infty}.$$

Given $E \in \text{Rep}(K_{0,\infty})$ one has the $G_0^{\mathbb{C}}$ -homogeneous bundle $\check{\mathcal{E}} = G_0^{\mathbb{C}} \times_{K_{0,\infty}} E$ on \check{D} . This has an algebraic structure by inflating E to a $P_{0,h}$ -module (i.e. by letting the unipotent radical of $P_{0,h}$ act trivially) and using the identification $\check{\mathcal{E}} = G^{\text{der}}(\mathbb{C}) \times_{P_{0,h}} E$ induced by the inclusion $G_0^{\mathbb{C}} \times E \subset G^{\text{der}}(\mathbb{C}) \times E$. Restricting $\check{\mathcal{E}}$ to D (via β) and dividing by $\Gamma_i \subset G^{\text{der}}(\mathbb{Q})$ for each i defines a vector bundle \mathcal{E}_K on M_K for K small enough, i.e. $\mathcal{E}_K|_{\Gamma_i \backslash D} = (\Gamma_i \backslash G_0) \times_{K_{0,\infty}} E$. This defines an exact tensor functor

$$E \mapsto \mathcal{E}_K \quad (3.3.1)$$

from $\text{Rep}(K_{0,\infty})$ to algebraic vector bundles on M_K for K small enough.

Remark 3.3.1. For $V \in \text{Rep}(P_h)$ the $G(\mathbb{C})$ -homogeneous vector bundle $\check{\mathcal{V}}$ on \check{D} as in 3.2 is C^∞ isomorphic to the bundle associated with $E = V|_{K_{0,\infty}}$ by (3.3.1) under the map induced by the inclusion $G_0^{\mathbb{C}} \times V \subset G(\mathbb{C}) \times V$. Similar remarks clearly apply to the bundles on M_K , so that an automorphic vector bundle on the Shimura variety (i.e. as in 3.2) is, at any finite level, C^∞ isomorphic to a bundle coming (by (3.3.1)) from a $K_{0,\infty}$ -representation. (Conversely, a bundle coming from (3.3.1) is C^∞ isomorphic to an automorphic vector bundle on M_K because the restriction functor $\text{Rep}(P_h) \rightarrow \text{Rep}(K_{0,\infty})$ is essentially surjective.)

Henceforth, to avoid confusion between the constructions of 3.2 and 3.3, we will usually use the letters $V, \check{\mathcal{V}}, \mathcal{V}_K$ etc. for the former and the letters $E, \check{\mathcal{E}}, \mathcal{E}_K$ etc. for the latter. We will also avoid using the term automorphic vector bundle for the construction of 3.3, reserving it for that of 3.2.

3.4. **Toroidal canonical extensions.** The vector bundles in 3.2, 3.3 admit canonical extensions to algebraic vector bundles toroidal compactifications.

Let M_K^Σ be a smooth projective toroidal compactification in which the complement of M_K is a simple normal crossings divisor. The vector bundle \mathcal{E}_K (as in 3.3) defined by $E \in \text{Rep}(K_{0,\infty})$ has a canonical extension to an algebraic vector bundle \mathcal{E}_K^Σ on M_K^Σ (Mumford's canonical extension [Mu]). Then

$$E \mapsto \mathcal{E}_K^\Sigma \quad (3.4.1)$$

defines an exact tensor functor from $\text{Rep}(K_{0,\infty})$ to algebraic vector bundles on M_K^Σ . This is compatible with changing the level K or refinement of Σ , i.e. if Σ' refines Σ then \mathcal{E}_K^Σ pulls back to $\mathcal{E}_K^{\Sigma'}$ under the morphism $M_K^{\Sigma'} \rightarrow M_K^\Sigma$ extending the identity of M_K , and if $K' \subset K$ then \mathcal{E}_K^Σ pulls back to $\mathcal{E}_{K'}^\Sigma$ under the covering $M_{K'} \rightarrow M_K$.

Let M_K^Σ be a smooth projective toroidal compactification in which the complement of M_K is a simple normal crossings divisor. An automorphic vector bundle \mathcal{V}_K defined as in 3.2 has an extension to a vector bundle \mathcal{V}_K^Σ on M_K^Σ ([H2]). This defines a functor

$$\check{\mathcal{V}} \mapsto \mathcal{V}_K^\Sigma \quad (3.4.2)$$

from $G(\mathbb{C})$ -homogeneous vector bundles on \check{D} to algebraic vector bundles on M_K^Σ . This is an exact tensor functor compatible with change of level and refinement of Σ . It is also compatible with the $G(\mathbb{A}_f)$ -action on the collection of all toroidal compactifications ([H2, 4.3]). When M_K^Σ is defined over E the functor (3.4.2) is E -rational by a result of Harris ([H2, Theorem 4.2]), i.e. if $\check{\mathcal{V}}$ is L -rational for $L \supset E$ then so is \mathcal{V}_K^Σ .

Lemma 3.4.1. *If $V \in \text{Rep}(P_h)$ and $E = V|_{K_{0,\infty}} \in \text{Rep}(K_{0,\infty})$ then the bundles \mathcal{V}_K^Σ and \mathcal{E}_K^Σ have the same Chern classes.*

Proof. Filter V by P_h -stable subspaces such that the unipotent radical of P_h acts trivially on the graded quotients. The sum of graded quotients carries a representation of the Levi $K_{0,\infty}^{\mathbb{C}} Z(G)$ (here $Z(G)$ is the centre of G) which restricts on $K_{0,\infty}^{\mathbb{C}}$ to E . This gives a filtration of \mathcal{V}_K by subbundles with the graded isomorphic to \mathcal{E}_K . Canonical extension is exact so we get a filtration of \mathcal{V}_K^Σ with graded isomorphic to \mathcal{E}_K^Σ . It follows that they have the same Chern classes. \square

3.5. RBS canonical extensions. The underlying topological bundles of the constructions in 3.2, 3.3 admit canonical extensions to topological vector bundles on the RBS compactification.

For a bundle \mathcal{E}_K on M_K coming from a $K_{0,\infty}$ -representation (as in 3.3) there is a natural topological vector bundle $\bar{\mathcal{E}}_K$ on the RBS compactification \bar{M}_K extending \mathcal{E}_K . This is described in [GT, 9.2] or [Z2, 1.10] and is easily described stratumwise. In the notation of 2.1(2), the restriction of $\bar{\mathcal{E}}_K$ to the P -boundary stratum of the component $\Gamma_i \backslash \bar{D}$ is the vector bundle

$$\mathcal{E}_{i,P} = (\Gamma_{i,P} \backslash M_P(\mathbb{R})) \times_{K_P} E,$$

i.e. the extension $\bar{\mathcal{E}}_K$ is obtained by gluing the various $\mathcal{E}_{i,P}$ together (see [GT, 9.2]). The construction gives an exact tensor functor from $\text{Rep}(K_{0,\infty})$ to topological vector bundles on \bar{M}_K .

For an automorphic vector bundle \mathcal{V}_K as in 3.2 defined by $V \in \text{Rep}(P_h)$ let $E = V|_{K_{0,\infty}}$ and *define*:

$$\bar{\mathcal{V}}_K = \bar{\mathcal{E}}_K. \quad (3.5.1)$$

Then $V \mapsto \bar{\mathcal{V}}_K$ defines an exact tensor functor from $\text{Rep}(P_h)$ to vector bundles on \bar{M}_K (which factors through the earlier functor).

Remark 3.5.1. The definition (3.5.1) ignores the algebraic structure of \mathcal{V}_K . Since \bar{M}_K is motivic, it is interesting to ask what sort of algebraic/motivic object on \bar{M}_K we should associate with $V \in \text{Rep}(P_h)$. As a first approximation, one can ask for Chern classes in the cycle groups of the motive of \bar{M}_K refining the $c_k(\bar{\mathcal{V}}_K)$.

3.6. **Chern classes and the homomorphism θ .** We now show that

$$\theta(c_i(\check{\mathcal{Y}})) = (-1)^i c_i(\overline{\mathcal{Y}}_K) \quad (\text{in } H^*(\overline{M}_K, \mathbb{Q})) \quad (3.6.1)$$

for a homogeneous bundle $\check{\mathcal{Y}}$. We will use the natural connections on homogeneous bundles, which descend to bundles on quotients.

Recall that the Chern-Weil theory (see [MS, Appendix C]) gives a C^∞ way of computing the Chern classes of a complex vector bundle F on a smooth manifold Y , at least in $H^{2i}(Y, \mathbb{C})$, as follows: Given a Hermitian connection ∇ in F , one has the curvature form Ω which is a smooth 2-form on Y with values in $\text{End}(F)$. The Chern forms $c_i(F, \nabla)$ are closed forms given by applying the natural conjugation-invariant polynomial functions $\text{End}(F) \rightarrow \mathbb{C}$. Thus $c_i(F, \nabla) = P_i(\Omega)$ where P_i is the i th symmetric function of the eigenvalues of an endomorphism. Then the image of $c_i(F) \in H^{2i}(Y, \mathbb{Q})$ in $H^*(Y, \mathbb{C})$ is the class of $\frac{1}{(2\pi\sqrt{-1})^i} c_i(F, \nabla)$.

Let $(E, \tau) \in \text{Rep}(K_{0,\infty})$. Then \mathfrak{g}_0 (the Lie algebra of G_0) and $\text{End}(E)$ are $\mathfrak{k}_{0,\infty}$ -modules under $\text{ad}|_{\mathfrak{k}_{0,\infty}}$ and $\text{ad} \circ d\tau$ respectively. The G_0 -invariant connections in the vector bundle $G_0 \times_{K_{0,\infty}} E$ are given by $\mathfrak{k}_{0,\infty}$ -module homomorphisms

$$\mathfrak{g}_0 \rightarrow \text{End}(E)$$

which extend $d\tau : \mathfrak{k}_{0,\infty} \rightarrow \text{End}(E)$. (See [GP, Proposition 5.3]: As in [GP, 5.2], we identify $\text{End}(E)$ -valued forms on D with $\text{End}(E)$ -valued forms on G_0 which are basic (i.e. $K_{0,\infty}$ -equivariant and $\mathfrak{k}_{0,\infty}$ -horizontal). A connection is determined by its $\text{End}(E)$ -valued 1-form, which in the case of a G_0 -invariant connection reduces to a mapping $\mathfrak{g}_0 \rightarrow \text{End}(E)$ with the properties above.) The Cartan decomposition of \mathfrak{g}_0 gives a projection $\theta_0 : \mathfrak{g}_0 \rightarrow \mathfrak{k}_{0,\infty}$ and the G_0 -invariant connection given by

$$d\tau \circ \theta_0 : \mathfrak{g}_0 \rightarrow \text{End}(E)$$

is called the *Nomizu connection* and denoted ∇^{Nom} . The curvature 2-form of ∇^{Nom} is the G_0 -invariant differential form given at the identity by $\Omega_0(X, Y) = -d\tau([(1-\theta_0)(X), (1-\theta_0)(Y)])$ where $X, Y \in \mathfrak{g}$. (See [GP, Proposition 5.3, Example 5.5.1]). The Chern forms of ∇^{Nom} are G_0 -invariant differential forms on D (see [GP, Proposition 5.3]). The Nomizu connection descends to one on \mathcal{E}_K which will also be denoted ∇^{Nom} .

Lemma 3.6.1. *Let $E \in \text{Rep}(K_{0,\infty})$, \mathcal{E}_K the associated vector bundle on M_K (as in 3.3), and $\overline{\mathcal{E}}_K$ the canonical extension to the RBS compactification (as in 3.5). In the isomorphism*

$$H^*(\overline{M}_K, \mathbb{C}) = H^*(\mathfrak{g}, K_\infty^0 A_\infty, B(G)_1^K)$$

(of Proposition 1.6.1) the invariant differential form $\frac{1}{(2\pi\sqrt{-1})^k} c_k(\mathcal{E}_K, \nabla^{\text{Nom}})$ represents the k th Chern class of $\overline{\mathcal{E}}_K$.

Proof. This is essentially proved in [Z2], but for the L^p cohomology model there. The same proof (minus the last step) works in our context; we sketch it for the reader's convenience.

In [Z2, §4], Zucker shows how to represent Chern classes of vector bundles on a stratified space like \overline{M}_K using a choice of control data, i.e. how to do Chern-Weil theory in a stratified setting. The class $(2\pi\sqrt{-1})^k c_k(\overline{\mathcal{E}}_K)$ is represented by the Chern form $c_k(\overline{\mathcal{E}}_K, \nabla^{\text{ctrl}})$ of a controlled connection ∇^{ctrl} , which is a controlled differential form on \overline{M}_K (Theorem 4.3.5 of loc. cit.). Lemma 5.4.2(i) of loc. cit. shows that such controlled differential forms are bounded, hence belong to the relative Lie algebra complex computing the cohomology of $B(G)_1^K$. On the other hand, Proposition 5.4.3 of loc. cit. shows that the difference

$\omega = c_k(\bar{\mathcal{E}}_K, \nabla^{\text{ctrl}}) - c_k(\mathcal{E}_K, \nabla^{\text{Nom}})$ is bounded, and by standard formulas (cf. (4.3.4.1) of loc. cit.), $\omega = d\eta_k$ for a bounded differential form η_k . Thus $c_k(\bar{\mathcal{E}}_K, \nabla^{\text{ctrl}})$ and $c_k(\mathcal{E}_K, \nabla^{\text{Nom}})$ represent the same class in the cohomology of $B(G)_1^K$. \square

Lemma 3.6.2. *For $E \in \text{Rep}(K_{0,\infty})$, $\theta(c_k(\check{\mathcal{E}})) = (-1)^k c_k(\bar{\mathcal{E}}_K)$.*

Proof. This is essentially Hirzebruch's original observation (see [Mu, p. 263]). The G_0^c -invariant connections in the bundle $\check{\mathcal{E}} = G_0^c \times_{K_{0,\infty}} E$ are given (as above, using [GP, 5.2, 5.3]) by $\mathfrak{k}_{0,\infty}$ -module homomorphisms $\mathfrak{g}_0^c \rightarrow \text{End}(E)$ extending $d\tau$. As before, $d\tau$ composed with the projection $\mathfrak{g}_0^c \rightarrow \mathfrak{k}_{0,\infty}$ gives a canonical connection $\check{\nabla}^{\text{Nom}}$. The curvature 2-forms $\check{\Omega}_0$ and Ω_0 of $\check{\nabla}^{\text{Nom}}$ and ∇^{Nom} satisfy $\beta^*(\check{\Omega}_0) = -\Omega_0$ where $\beta : D \hookrightarrow \check{D}$ and hence $c_k(\check{\mathcal{E}}, \check{\nabla}^{\text{Nom}}) = (-1)^k c_k(\mathcal{E}_K, \nabla^{\text{Nom}})$ as invariant differential forms. \square

This proves (3.6.1), since for a $G(\mathbb{C})$ -homogeneous bundle $\check{\mathcal{Y}}$ we have

$$c_k(\check{\mathcal{Y}}) = c_k(\check{\mathcal{E}}) = (-1)^k c_k(\bar{\mathcal{E}}_K) = (-1)^k c_k(\bar{\mathcal{Y}}_K).$$

by Remark 3.3.1 and the definition (3.5.1).

3.7. Proof of Proposition 1. Recall the map $\gamma^* : H^*(\bar{M}_K) \rightarrow H^*(M_K^\Sigma)$ from 2.4 and the diagram (2.4.1). We shall show that there is a commutative diagram

$$\begin{array}{ccc} & & H^*(M_K^\Sigma, \mathbb{Q}) \\ & \nearrow^{\theta^\Sigma} & \uparrow \gamma^* \\ H^*(\check{D}, \mathbb{Q}) & & H^*(\bar{M}_K, \mathbb{Q}) \\ & \searrow_{\theta} & \end{array} \quad (3.7.1)$$

with θ^Σ injective. This will prove Proposition 1.

Lemma 3.7.1. *For $E \in \text{Rep}(K_{0,\infty})$, $\gamma^*(c_k(\bar{\mathcal{E}}_K)) = c_k(\mathcal{E}_K^\Sigma)$.*

Proof. This could be proved using Lemma 3.6.1 and simple arguments from [Mu], but we will use [GP, Z2] instead. According to the main result of [GP], there are classes $c_k^{GP}(\mathcal{E}_K) \in H^{2k}(M_K^*, \mathbb{C})$ such that $\pi^*(c_k^{GP}(\mathcal{E}_K)) = c_k(\mathcal{E}_K^\Sigma)$. By [Z2], $p^*(c_k^{GP}(\mathcal{E}_K)) = c_k(\bar{\mathcal{E}}_K)$. Thus $\gamma^*(c_k(\bar{\mathcal{E}}_K)) = \pi^*(c_k^{GP}(\mathcal{E}_K)) = c_k(\mathcal{E}_K^\Sigma)$. \square

Lemma 3.7.2. *There is an injective ring homomorphism*

$$\theta^\Sigma : H^*(\check{D}, \mathbb{Q}) \longrightarrow H^*(M_K^\Sigma, \mathbb{Q})$$

with $\theta^\Sigma(c_k(\check{\mathcal{E}})) = (-1)^k c_k(\mathcal{E}_K^\Sigma)$ for $E \in \text{Rep}(K_{0,\infty})$.

Proof. Following a suggestion of N. Fakhruddin we will use K -theory to prove this. Let $K^0(-)$ denote the topological K -theory of a space and $\text{ch} : K^0(-) \rightarrow H^*(-, \mathbb{Q})$ the Chern character homomorphism. We write $R(H)$ for the representation ring of a compact group or an algebraic group over a subfield of \mathbb{C} , i.e. $R(H)$ is the Grothendieck group of $\text{Rep}(H)$.

We first show that the ring homomorphism

$$R(K_{0,\infty}) \longrightarrow H^*(M_K^\Sigma, \mathbb{Q}) \quad (3.7.2)$$

defined by $E \mapsto \text{ch}(\mathcal{E}_K^\Sigma)$ and extended \mathbb{Q} -linearly defines a ring homomorphism

$$\kappa : K^0(\check{D}) \otimes \mathbb{Q} \longrightarrow H^*(M_K^\Sigma, \mathbb{Q}). \quad (3.7.3)$$

(Here \mathcal{E}_K is the bundle as in 3.3 and \mathcal{E}_K^Σ is the construction in (3.4.1). That κ is a ring homomorphism follows from the compatibility of canonical extension with tensor product [H2, 4.2] and the fact that the Chern character is one.) Since $\check{D} = G_0^c/K_{0,\infty}$ with G_0^c simply connected and $K_{0,\infty} \subset G_0^c$ is a subgroup of maximal rank, the construction $E \mapsto \check{\mathcal{E}}$ gives an isomorphism

$$\mathrm{R}(K_{0,\infty}) \otimes_{\mathrm{R}(G_0^c)} \mathbb{Z} \longrightarrow \mathrm{K}^0(\check{D}) \quad (3.7.4)$$

where \mathbb{Z} is a $\mathrm{R}(G_0^c)$ -module via the dimension homomorphism (by [Pi, Theorem 3]). Since the left-hand side is the quotient of $\mathrm{R}(K_{0,\infty})$ by the ideal generated by $\ker(\dim : \mathrm{R}(G_0^c) \rightarrow \mathbb{Z})$ and ch is a ring homomorphism, it suffices to check that $\mathrm{ch}(\mathcal{E}_K^\Sigma) = \dim E$ if E is a G_0^c -representation. Since the degree zero term of the Chern character of a bundle is its rank, it suffices to check that $c_k(\mathcal{E}_K^\Sigma) = 0$ for $k > 0$ for such E . By Lemma 3.7.1, $c_k(\mathcal{E}_K^\Sigma) = \gamma^*(c_k(\overline{\mathcal{E}}_K))$, so it is enough to show that $c_k(\overline{\mathcal{E}}_K)$ vanishes in $\mathrm{H}^*(\overline{M}_K, \mathbb{C})$ for $k > 0$. By Lemma 3.6.1 $c_k(\overline{\mathcal{E}}_K)$ is represented (up to $(2\pi\sqrt{-1})^k$) by the k th Chern form of the Nomizu connection ∇^{Nom} . But if E is a G_0^c -representation the curvature 2-form of ∇^{Nom} vanishes identically (see e.g. [GP, Proposition 5.3]), hence so do its Chern forms for $k > 0$. Thus $c_k(\mathcal{E}_K^\Sigma) = 0$ for $k > 0$, $\mathrm{ch}(\mathcal{E}_K^\Sigma) = \dim E$, and we have κ as in (3.7.3).

Since \check{D} is a flag variety it has only even-degree cohomology so the Chern character gives an isomorphism $\mathrm{ch} : \mathrm{K}^0(\check{D}) \otimes \mathbb{Q} \rightarrow \mathrm{H}^*(\check{D}, \mathbb{Q})$ (cf. [AH, 2.4]). Now define

$$\theta^\Sigma := \kappa \circ \mathrm{ch}^{-1} \circ \sigma$$

where $\sigma : \mathrm{H}^*(\check{D}, \mathbb{Q}) \rightarrow \mathrm{H}^*(\check{D}, \mathbb{Q})$ is defined by $\sigma(\alpha) = (-1)^{\deg(\alpha)/2} \alpha$. (Since $\mathrm{H}^*(\check{D}, \mathbb{Q})$ is concentrated in even degrees this makes sense and σ is a ring homomorphism.) Note that $\theta^\Sigma(\sigma(\mathrm{ch}(\check{\mathcal{E}}))) = \mathrm{ch}(\mathcal{E}_K^\Sigma)$, from which it follows that $\theta^\Sigma(c_k(\check{\mathcal{E}})) = (-1)^k c_k(\mathcal{E}_K^\Sigma)$. This implies that θ^Σ is injective (i.e. nonzero) in top degree $2n = 2 \dim_{\mathbb{C}} D$: Choose a nonzero monomial $c_{k_1}(\check{\mathcal{E}}_1) \cdots c_{k_r}(\check{\mathcal{E}}_r)$ with $\sum_i k_i = n$; it spans $\mathrm{H}^{2n}(\check{D}, \mathbb{Q})$. Then

$$\int_{M_K^\Sigma} \theta^\Sigma(c_{k_1}(\check{\mathcal{E}}_1) \cdots c_{k_r}(\check{\mathcal{E}}_r)) = (-1)^n \cdot C \cdot \int_{\check{D}} c_{k_1}(\check{\mathcal{E}}_1) \cdots c_{k_r}(\check{\mathcal{E}}_r) \neq 0$$

where C is a nonzero constant. (This is Mumford's version of proportionality [Mu, Theorem 3.2]; it also follows from the equality of Chern forms in Lemma 3.6.2 and a computation on \overline{M}_K , since γ^* is an isomorphism in top degree. The constant C is the volume of M_K .) It follows that θ^Σ is injective: For nonzero $\alpha \in \mathrm{H}^i(\check{D})$ choose $\beta \in \mathrm{H}^{2n-i}(\check{D})$ such that $\alpha \cdot \beta \neq 0$. Then $0 \neq \theta^\Sigma(\alpha \cdot \beta) = \theta^\Sigma(\alpha) \cdot \theta^\Sigma(\beta)$, so that $\theta^\Sigma(\alpha) \neq 0$. \square

The commutativity of (3.7.1) follows from the previous two lemmas, Lemma 3.6.2, and the surjectivity of (3.7.4). Proposition 1 follows.

3.8. Proof of Theorem 2. We work in the category $\mathcal{M}(\mathrm{Spec}(E))$ of mixed realizations from 2.2. The action of the Hecke algebra of K -biinvariant functions on $G(\mathbb{A}_f)$ on $\mathrm{H}^*(\overline{\mathcal{M}})^K = \mathrm{H}^*(\overline{M}_K)$ respects all structures, i.e. is an action in $\mathcal{M}(\mathrm{Spec}(E))$. (The action on $\mathrm{H}^*(\overline{M}_K)$ in $\mathcal{M}(\mathrm{Spec}(E))$ is induced by an action by cohomological correspondences on the object $w_{\leq \dim} j_* \mathbb{Q}_{M_K}^{\mathcal{M}}$, see [Mo, 5.1]. By naturality of the isomorphism in Proposition 2.4.1, it agrees with the action by cohomological correspondences on $p_* \mathbb{Q}_{\overline{M}_K}$, which induces the action of the Hecke algebra on $\mathrm{H}^*(\overline{M}_K)$ discussed after Proposition 1.6.1.) The direct summand of invariants can be projected out using an element of the unramified Hecke algebra outside a finite set of finite places, so it is a direct summand in the category $\mathcal{M}(\mathrm{Spec}(E))$. For a $G(\mathbb{C})$ -homogeneous bundle $\check{\mathcal{Y}}$ given by $V \in \mathrm{Rep}(P_h)$ and the automorphic vector bundle

\mathcal{V}_K , let $\tilde{\mathcal{E}}$ and \mathcal{E}_K be the bundles on \check{D} and M_K associated with $E = V|_{K_{0,\infty}}$ (as in 3.3). Since $\overline{\mathcal{V}}_K = \overline{\mathcal{E}}_K$ (by definition, cf. 3.5), we have

$$\gamma^*(c_k(\overline{\mathcal{V}}_K)) = \gamma^*(c_k(\overline{\mathcal{E}}_K)) = c_k(\mathcal{E}_K^\Sigma) = c_k(\mathcal{V}_K^\Sigma)$$

by Lemmas 3.7.1 and 3.4.1. Since γ^* is injective on the direct summand of invariants, the class $c_k(\overline{\mathcal{V}}_K)$ has the same properties in realizations as its lift $c_k(\mathcal{V}_K^\Sigma)$. The properties (i)–(iii) in Theorem 2 then follow from general properties of Chern classes ((3.1.1) and (3.1.2)) and the rationality of the functor $\check{\mathcal{V}} \mapsto \mathcal{V}_K^\Sigma$ due to Harris noted after (3.4.2) of 3.4. \square

4. REMARKS ON OTHER COMPACTIFICATIONS

We make some remarks on analogues of the main results for other compactifications. For toroidal compactifications or for the intersection cohomology of the minimal compactification there are straightforward analogues of the main results, which we state first in 4.1, 4.2. For the cohomology of the minimal compactification the situation is rather more interesting, and we discuss what we expect to hold in 4.3. (Since our main purpose is to sketch a conjectural picture we will not give complete proofs in the discussion in 4.3.) The setting is that of §3, i.e. (G, X) satisfies (S1)–(S4) of 2.1 and G^{der} is assumed simply connected.

4.1. Minimal compactification (intersection cohomology).

Proposition 4.1.1. *The invariants (and coinvariants) of the mixed realization $\text{IH}^*(M_K^*)$ are isomorphic to $\text{H}^*(\check{D})$, given by the composition*

$$\text{H}^*(\check{D}) \xrightarrow{\theta} \text{H}^*(\overline{M}_K) \xrightarrow{\rho} \text{IH}^*(M_K^*),$$

where $\rho : \text{H}^*(\overline{M}_K) \rightarrow \text{IH}^*(M_K^*)$ is as in the diagram (2.4.1).

Proof. A version of Matsushima’s formula for $\text{IH}^*(M_K^*, \mathbb{C})$ gives a natural isomorphism

$$\text{IH}^*(\mathcal{M}^*, \mathbb{C}) = \varinjlim_K \text{IH}^*(M_K^*) = \bigoplus_{\pi = \pi_f \otimes \pi_\infty} m_{\text{dis}}(\pi) \pi_f \otimes \text{H}^*(\mathfrak{g}, K_\infty^0 A_\infty, \pi_\infty)$$

where the (algebraic) direct sum is over π appearing in the L^2 discrete spectrum of G , and $m_{\text{dis}}(\pi)$ is the multiplicity. (See e.g. [Mo2, Theorems 2.1, 2.2].) It follows that invariants and coinvariants agree and are isomorphic to $\text{H}^*(\check{D}, \mathbb{C})$. The homomorphism $\rho : \text{H}^*(\overline{M}_K) \rightarrow \text{IH}^*(M_K^*)$ maps invariants into invariants. Since $\gamma^* = \iota \circ \rho$ is injective on the invariants, so is ρ . Hence $\rho \circ \theta$ is an isomorphism onto the invariants. The proposition then follows from Theorem 2. \square

Remark 4.1.2. This can be used, with the methods of [N2], to prove that $\text{H}_c^*(\mathcal{M})^{G(\mathbb{A}_f)}$ of $\text{H}_c^*(\mathcal{M})$, which is proved to be a direct summand in [F2], is a mixed Tate realization.

4.2. Toroidal compactifications. There is an analogue of Theorem 1 for toroidal compactifications, which we will state without proof. (The analogue of Theorem 2 is immediate from [H1, H2].) Let

$$\text{H}^*(\mathcal{M}^{\text{tor}}) := \varinjlim_K \varinjlim_\Sigma \text{H}^*(M_K^\Sigma).$$

Here the first (i.e. inner) limit is over all Σ adapted to K (see [H2, 2.5] or [P, §6]) and the second is over all compact open subgroups K . This is a smooth, but not usually admissible, module for $G(\mathbb{A}_f)$, and an (infinite-dimensional) mixed realization over E . Properties of canonical extensions verified in [H2, §4] and Lemma 3.7.2 give an injective homomorphism

$\theta^{tor} : H^*(\check{D}) \rightarrow H^*(\mathcal{M}^{tor})$. The $G(\mathbb{A}_f)$ -module $H^*(\mathcal{M}^{tor})$ is not semisimple, but nevertheless one has:

Theorem 4.2.1. *The homomorphism $\theta^{tor} : H^*(\check{D}) \rightarrow H^*(\mathcal{M}^{tor})$ induces isomorphisms*

$$H^*(\check{D}) \cong H^*(\mathcal{M}^{tor})^{G(\mathbb{A}_f)} \cong H^*(\mathcal{M}^{tor})_{G(\mathbb{A}_f)}.$$

The proof involves studying the graded pieces of the filtration of $H^*(\mathcal{M}^{tor})$ coming from the decomposition theorem ([BBD, S1] and [S2] for mixed realizations) and the perverse filtration of the direct image of the constant object by the morphisms $M_K^\Sigma \rightarrow M_K^*$. The (co)invariants come from the subquotient $\mathrm{IH}^*(\mathcal{M}^*) = \varinjlim_K \mathrm{IH}^*(M_K^*)$, which contributes $H^*(\check{D})$ (by 4.1), and the other constituents of $H^*(\mathcal{M}^{tor})$ make no contribution.

4.3. Minimal compactification (cohomology). We will consider the mixed realization

$$H^*(\mathcal{M}^*) := \varinjlim_K H^*(M_K^*) \in \mathcal{M}(\mathrm{Spec}(E))$$

on which $G(\mathbb{A}_f)$ acts. The classes $c_k^{GP}(\mathcal{V}_K)$ belong to the subring $H^*(\mathcal{M}^*)^{G(\mathbb{A}_f)}$ of invariants. For each K , Theorem 1, together with [GP], gives a surjection

$$H^*(M_K^*) \longrightarrow H^*(\check{D}) \tag{4.3.1}$$

in mixed realizations. (Indeed, the composition of $H^*(M_K^*) \rightarrow H^*(\overline{M}_K) \rightarrow H^*(\check{D})$ is surjective since it takes the class $c_k^{GP}(\mathcal{V}_K)$ to $(-1)^k c_k(\check{\mathcal{V}})$.)¹ This is compatible with limits, giving a surjection

$$H^*(\mathcal{M}^*) \longrightarrow H^*(\check{D}) \tag{4.3.2}$$

which is already surjective on the subring of invariants (which contains the classes $c_k^{GP}(\mathcal{V}_K)$). In fact $H^*(\check{D})$ is the top weight quotient of $H^*(\mathcal{M}^*)^{G(\mathbb{A}_f)}$ (e.g. by Lemma 2.4.2). Theorems 1 and 2 suggest the following question:

Does $H^(\mathcal{M}^*) \rightarrow H^*(\check{D})$ split naturally (in particular, $G(\mathbb{A}_f)$ -equivariantly) in the category of mixed realizations?*

The answer, perhaps surprisingly, is no. The first case showing this is the Siegel modular variety of degree 3.

Example 4.3.1. Let $G = Sp(6)$. The compact dual \check{D} is the space of totally isotropic 3-planes in \mathbb{Q}^6 and has even Betti numbers 1, 1, 1, 2, 1, 1, 1. A \mathbb{Q} -basis for $H^*(\check{D})$ is given by 1, c_1, c_1^2, \dots, c_1^6 and c_3 , where $c_i \in H^{2i}(\check{D})$ is the i th Chern class of the tautological bundle. Each basis element of degree $2k$ gives a summand $\mathbb{Q}(-k) \subset H^{2k}(\check{D})$.

By strong approximation, $M_K = \Gamma \backslash Sp(6, \mathbb{R})/U(3)$ where $\Gamma = K \cap Sp(6, \mathbb{Q})$. The boundary $M_K^* - M_K$ is a union of minimal compactifications of quotients like $\Gamma' \backslash Sp(4, \mathbb{R})/U(2)$, $\Gamma' \subset Sp(4, \mathbb{Q})$, with pairwise intersections certain (compactified) modular curves. It follows that $H^5(M_K^* - M_K) = 0$, and the long exact sequence

$$\cdots \rightarrow H_c^k(M_K) \rightarrow H^k(M_K^*) \rightarrow H^k(M_K^* - M_K) \rightarrow \cdots$$

¹This improves [GP, Theorem 16.4], where the authors prove that the subalgebra of $H^*(M_K^*)$ generated by the classes $c_k^{GP}(\mathcal{V}_K)$ surjects onto $H^*(\check{D})$ but excluded the cases where G_0 is isogenous to $SO(2, 2n)$. In fact, the use of Theorem 1 is unnecessary and an argument using L^2 /intersection cohomology as in 16.6 of loc. cit. is enough to prove (4.3.2).

implies that $H_c^6(M_K) \hookrightarrow H^6(M_K^*)$ for all K . Thus $H_c^6(\mathcal{M}) \hookrightarrow H^6(\mathcal{M}^*)$ and this gives an inclusion of mixed realizations

$$H_c^6(\mathcal{M})^{G(\mathbb{A}_f)} \subset H^6(\mathcal{M}^*)^{G(\mathbb{A}_f)}.$$

The mixed realization $H_c^6(\mathcal{M})^{G(\mathbb{A}_f)}$ has the following properties:

- (1) It is mixed Tate: There is a short exact sequence

$$0 \rightarrow \mathbb{Q}(0) \rightarrow H_c^6(\mathcal{M})^{G(\mathbb{A}_f)} \rightarrow \mathbb{Q}(-3) \rightarrow 0. \quad (4.3.3)$$

The top weight piece is the image of $H_c^6(\mathcal{M})^{G(\mathbb{A}_f)}$ in $\mathrm{IH}^6(\mathcal{M}^*)^{G(\mathbb{A}_f)}$ and under the identification $\mathrm{IH}^6(\mathcal{M}^*)^{G(\mathbb{A}_f)} = H^6(\check{D})$ it is the summand spanned by c_3 . Thus the map $H_c^6(\mathcal{M})^{G(\mathbb{A}_f)} \rightarrow \mathbb{Q}(-3)$ is the restriction of (4.3.2). The sequence comes with a canonical splitting $s_{\mathbb{C}}$ over \mathbb{C} .

- (2) The class of the extension $H_c^6(\mathcal{M})^{G(\mathbb{A}_f)}$ in rational mixed Hodge structures, i.e. in

$$\mathrm{Ext}_{\mathbb{Q}\text{-MHS}}^1(\mathbb{Q}(-3), \mathbb{Q}(0)) = \mathbb{C}/(2\pi\sqrt{-1})^3\mathbb{Q}.$$

is a nonzero real multiple of $\zeta(3)$, in particular it is nonzero.

Here (1) follows from results of [F2] and [N2]. The splitting in (1) comes from a particular Eisenstein series; the computation in (2) comes from an understanding of its residues at a particular point. (1) and (2) will be discussed in detail in a sequel to [N2] in preparation.

In particular, (4.3.3), and therefore $H^*(\mathcal{M}^*)^{G(\mathbb{A}_f)} \rightarrow H^*(\check{D})$, is nonsplit. Similar examples can be found in all $Sp(2g)$, $g \geq 3$, involving quantities like $\zeta(k)$ for k odd, and in other Shimura varieties.

In this example $H^*(M_K^*, \mathbb{C}) \rightarrow H^*(\check{D}, \mathbb{C})$ is canonically split. Working at full level $\mathbb{K} = Sp(6, \hat{\mathbb{Z}})$, we have that

$$M_{\mathbb{K}} = Sp(6, \mathbb{Z}) \backslash Sp(6, \mathbb{R}) / U(3) = A_3$$

is the moduli space of principally polarized abelian threefolds. Combining a result of Franke [F2, Cor. 3.5] with one of Hain [Ha, Theorem 1] we have $H_c^*(\mathcal{M})^{G(\mathbb{A}_f)} = H_c^*(A_3)$. Hain [Ha] also shows that $H^*(A_3^*)$ contains $\mathbb{Q}[\lambda]/(\lambda^7)$ where λ is first Chern class of the Hodge bundle on A_3 . A splitting $H^*(\check{D}, \mathbb{C}) \rightarrow H^*(A_3^*)$ is defined by $c_1 \mapsto \lambda$ and $c_3 \mapsto s_{\mathbb{C}}(c_3)$ where $s_{\mathbb{C}}$ is the splitting in (1) above. One can check that this defines a ring homomorphism $H^*(\check{D}, \mathbb{C}) \rightarrow H^*(A_3^*, \mathbb{C})$ and also that λ (respectively, $s_{\mathbb{C}}(c_3)$) is c_1^{GP} (respectively, c_3^{GP}) of the Hodge bundle. (The Hodge bundle is the automorphic vector bundle associated with the tautological bundle on \check{D} .)

Returning to the general situation one can formulate a conjectural picture:

- (1) The homomorphism $H^*(\mathcal{M}^*, \mathbb{C}) \rightarrow H^*(\check{D}, \mathbb{C})$ has a canonical splitting respecting the Hodge filtration. It is given by the Goresky-Pardon construction, i.e. by $c_k(\mathcal{E}) \mapsto (-1)^k c_k^{GP}(\mathcal{E}_K)$.
- (2) For $G = Sp(2g)$ the surjection of mixed realizations $H^*(\mathcal{M}^*)^{G(\mathbb{A}_f)} \rightarrow H^*(\check{D})$ is split over the summand $\mathbb{Q}(-k)$ of $H^{2k}(\check{D})$, $k \leq g$, given by the k th Chern class of the tautological bundle if and only if $k = 1$ or k is even.

I expect that the methods used in the example above (based on [F2, N2]) can be used to produce a splitting as in (1) and to verify the analogue of (2) in rational mixed Hodge structures.

As noted in 0.4, Scholze's p -adic Hodge-Tate map would give an approach to an analytic splitting, analogous to (1), but in the p -adic analytic world.

4.4. Satake compactifications. For a general locally symmetric space (i.e. not necessarily Hermitian), it seems reasonable to expect that the Hecke-invariants in the cohomology of *any* Satake compactification contains a copy of the cohomology of the compact dual, i.e. there should be well-defined Chern classes lifting to $c_k(\overline{\mathcal{Y}}_K)$ under pullback to the RBS compactification and these should generate a copy of $H^*(\overline{D}, \mathbb{C})$.

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