

AUTOMORPHIC LEFSCHETZ PROPERTIES FOR NONCOMPACT ARITHMETIC MANIFOLDS

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ABSTRACT. We prove the injectivity of Oda-type restriction maps for the cohomology of non-compact congruence locally symmetric spaces. This includes results for restriction between (1) congruence real hyperbolic manifolds, (2) congruence complex hyperbolic manifolds, and (3) orthogonal Shimura varieties. These results generalize results for compact congruence manifolds by Bergeron-Clozel and Venkataramana. The proofs combine techniques of mixed Hodge theory and methods involving automorphic forms.

0.1. Main results. Let G be a connected semisimple algebraic group over \mathbb{Q} and X the symmetric space of $G(\mathbb{R})$. We write $G(\mathbb{R})^{nc}$ for the product of the noncompact factors of $G(\mathbb{R})$ and d_G for the (real) dimension of X .

For congruence subgroups $\Gamma \subset G(\mathbb{Q})$ we consider the quotients $M_\Gamma = \Gamma \backslash X$ and their cohomology groups $H^i(M_\Gamma)$ with complex coefficients. The direct limit

$$H^i(\mathcal{M}_G) := \operatorname{colim}_\Gamma H^i(M_\Gamma)$$

is a $G(\mathbb{Q})$ -module using pullback by the isomorphisms $M_{g\Gamma g^{-1}} \rightarrow M_\Gamma$ induced by pullback by g^{-1} on X .

For a semisimple subgroup $H \subset G$ let $\Gamma_H = \Gamma \cap H(\mathbb{Q})$ and $M_{H,\Gamma_H} = \Gamma_H \backslash X_H$. The totally geodesic embedding $X_H \subset X$ induces a proper map $M_{H,\Gamma_H} := \Gamma_H \backslash X_H \rightarrow M_\Gamma$. Pullback in cohomology defines an $H(\mathbb{Q})$ -equivariant map $\iota^* : H^*(\mathcal{M}_G) \rightarrow H^*(\mathcal{M}_H)$ and composing with the action of $G(\mathbb{Q})$ gives a map

$$\operatorname{Res} : H^*(\mathcal{M}_G) \longrightarrow I_H^G H^*(\mathcal{M}_H).$$

The target of Res (defined in 1.4 below) is a certain induced module contained in the product $\prod_{g \in G(\mathbb{Q})} H^*(\mathcal{M}_H)$, so that concretely we have that $\operatorname{Res}(\alpha) \neq 0$ if and only if $\iota^*(g^{-1} \cdot \alpha) \neq 0$ for some $g \in G(\mathbb{Q})$, i.e. some Hecke translate of α restricts nontrivially to \mathcal{M}_H .

Theorem 1. *Suppose that $H \subset G$ are semisimple groups of the same \mathbb{Q} -rank and that $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is one of the embeddings*

- (1) $SO(1, c) \subset SO(1, d)$ (the real hyperbolic case), with neither H nor G a triality form,
- (2) $SU(1, m) \subset SU(1, n)$ (the complex hyperbolic or ball quotient case), or
- (3) $SO(2, m) \subset SO(2, n)$ (the orthogonal Shimura variety case).

Then the map $\operatorname{Res} : H^(\mathcal{M}_G) \longrightarrow I_H^G H^*(\mathcal{M}_H)$ is injective in degrees $< d_H/2$ (and also in degree $i = d_H/2$ in the $SO(1, c) \subset SO(1, d)$ case).*

This *automorphic Lefschetz property* is well known if G is anisotropic (equivalently, M_Γ is compact): The injectivity in case (1) was proved in [BeC13] and in cases (2) and (3) it was proved in [Ven01] in degrees $i \leq d_H/2$.

In the noncompact situation, case (1) can be proved by adapting [Ber03, BeC13] with some care, and case (2) was proved in [Nai17b] (and [BeC17]), so that the most interesting new case is (3). This includes, for example, the most basic orthogonal Shimura varieties arising from quadratic forms over \mathbb{Q} of signature $(2, n)$ over \mathbb{R} with $n \geq 4$. The treatment of the ‘missing’ degree $i = d_H/2$ in the noncompact $SU(1, n)$ and $SO(2, n)$ cases requires arithmetic information and we leave it for another occasion.

For the inclusion $SO(1, n) \subset SU(1, n)$ we have the following, which is [BeC13, Theorem 1.7] in the compact case. For $i < n$, the group $H^i(M_\Gamma)$ carries a pure Hodge structure of weight i and $H^{i,0}(\mathcal{M}_G) := \operatorname{colim}_\Gamma H^{i,0}(M_\Gamma)$.

Theorem 2. *Suppose that $H \subset G$ are of the same \mathbb{Q} -rank, $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SO(1, n) \subset SU(1, n)$, and H is not a triality form. Then $\operatorname{Res} : H^{i,0}(\mathcal{M}_G) \rightarrow I_H^G H^*(\mathcal{M}_H)$ is injective in degrees $i \leq n/2$.*

We will discuss the proofs in 0.3 below.

0.2. Some history. Restriction maps between congruence quotients have been studied by numerous authors for almost forty years, starting with the pioneering work of Oda [Oda81]. We refer the reader to the surveys [Ven10, Ber18] for discussion of this work and restrict ourselves here to a brief review of the history of immediate relevance to us.

The first result of this type was proved by Oda [Oda81], who introduced the restriction map Res and proved Theorem 1 for $SU(1, m) \subset SU(1, n)$ in degree $i = 1$. Weissauer [Wei88] then proved the Lefschetz property for $SO(2, 2) \subset SO(2, 3)$ in degree $i = 2$. Arthur [Art89, §9] raised the question of whether the nonprimitive cohomology of Shimura varieties can be related to smaller Shimura varieties. Harris and Li [HL98] applied the Burger-Sarnak [BS91] method to prove the Lefschetz property in degree $i = 2$ in the (compact) complex hyperbolic and orthogonal Shimura variety cases, i.e. cases (2) and (3). They also conjectured injectivity in degrees $\leq d_H/2$ in these cases and showed that in case (2) it would follow from Arthur's conjectures [Art89] on the discrete spectrum. They also asked (when $d_G = d_H + 2$ in cases (2) and (3)) whether a linear combination of Hecke translates of the class of the divisor $M_{H,\Gamma}$ is the class of an ample divisor. Venkataramana [Ven01] showed that this is true in cohomology rather than on the level of cycles, i.e. a linear combination of translates of the cycle class in $H^2(M_\Gamma)$ is the hyperplane cohomology class in the Baily-Borel projective embedding, and used this to prove the conjecture of [HL98], i.e. Theorem 1 in compact cases (2) and (3).

The automorphic approach of [HL98] was taken up by Bergeron and Clozel [Ber03, BeC05, Ber06, BeC13], who made the remarkable discovery that Lefschetz properties hold for congruence hyperbolic manifolds (i.e. case (1) of Theorem 1) even though there is no complex structure available. This allows for a common approach to Lefschetz properties in different contexts, by using the Burger-Sarnak method to reduce them to uniform (in the level Γ) bounds for the nonzero eigenvalues of the Laplacian on forms on the smaller locally symmetric space. This eigenvalue bound was then deduced in case (1) in [BeC13] from Arthur's endoscopic classification [Art13] of automorphic forms on orthogonal groups, completing the proof of Theorems 1 and 2 in the compact case.

In the noncompact case it is less clear what should be true, although the analogues for singular varieties of the Lefschetz theorems in [GM88] are suggestive. Moreover, since the cohomology of noncompact quotients is influenced by the behaviour of L -functions (for example, through Eisenstein series constructions of cohomology), one expects that the question is more subtle, and this is reflected by the omission of $i = d_H/2$ in cases (2) and (3) for now. The complex hyperbolic case of Theorem 1 was proved in [Nai17a, Nai17b, BeC17] and here we will prove the rest of the theorem, with the orthogonal Shimura variety case being the main new result. In fact, our proof shows that the Lefschetz property for congruence real hyperbolic groups arises as a sort of *local* Lefschetz property at infinity for the noncompact orthogonal Shimura variety case. We will comment on this further below.

0.3. On the proofs. We sketch the proofs of the main theorems and some intermediate results proved along the way. There are, roughly speaking, three types of arguments involved:

- (a) automorphic arguments (mainly the Burger-Sarnak method as in [HL98, Ber03, BeC13], but also rank one residual Eisenstein cohomology)
- (b) geometric arguments (using cycle classes as in [Ven01] and mixed Hodge theory and compactifications as in [Nai17a, Nai17b])
- (c) elementary arguments with Lie algebra cohomology (as in [Nai17b]).

The proof of Theorem 1 in the different cases uses these ingredients differently: Case (1) uses (a) and (c), case (2) uses (b) and (c), while case (3) uses (b) and (c) explicitly, but also (a) through the use of case (1). The proof of Theorem 2 uses mainly (a) and (c), with some mild input from (b).

A basic role is played by the *minimal compactification* $M_\Gamma \hookrightarrow M_\Gamma^*$, which is the cusp compactification of the (real or complex) hyperbolic manifold in cases (1) and (2) and the Satake-Baily-Borel compactification in cases (2) and (3). This gives the basic exact sequence

$$0 \longrightarrow H_1^k(\mathcal{M}_G) \longrightarrow H^k(\mathcal{M}_G) \longrightarrow H^k(i^*j_*\mathbb{C}) \quad (0.1)$$

where the interior cohomology $H_1^k(\mathcal{M}_G)$ is, by definition, the image of $H_c^i(\mathcal{M}_G) := \operatorname{colim}_\Gamma H_c^i(M_\Gamma)$ in $H^k(\mathcal{M}_G)$ and the third term is the boundary cohomology. The sequence (0.1) is functorial for the inclusions $H \subset G$ considered in Theorems 1 and 2 because $M_{H,\Gamma_H} \rightarrow M_\Gamma$ extends to a morphism $M_{H,\Gamma_H}^* \rightarrow M_\Gamma^*$ of minimal compactifications. The obvious approach is to treat the interior cohomology and the contribution from the boundary separately, and this is what we do.

The Lefschetz property for interior cohomology is the following:

Theorem 3. (*Theorem 4.1, Corollary 2.2*) *The map Res is injective on $H_1^k(\mathcal{M}_G)$ for $k \leq d_H/2$ in cases (1) and (2) and for $k < d_H/2$ in case (3).*

The proof of this is different in the various cases. In the real hyperbolic case (1) we adapt the Burger-Sarnak approach of [HL98, Ber03, BeC13] to the noncompact case. This is a more-or-less straightforward matter of combining the method with well-known results about residual Eisenstein cohomology, but since the literature on this is less than satisfactory we treat it in some detail. In cases (2) and (3) when there is a complex structure available we adopt a different approach based on some mixed Hodge theory. (The complex hyperbolic case was treated in [Nai17a, Nai17b], but the approach here is slightly different.) Theorem 3 is then a corollary of the following:

Theorem 4. (*Theorem 2.1*) *The map Res is injective on $\operatorname{Gr}_k^W H^k(\mathcal{M}_G^*)$ for $k \leq d_H/2$.*

This result is deduced as a corollary of a general nonvanishing criterion (Theorem 2.11) for the map $\operatorname{Res} : \operatorname{Gr}_i^W H^i(\mathcal{M}_G^*) \rightarrow I_H^G \operatorname{Gr}_i^W H^i(\mathcal{M}_H^*)$ for a morphism between Shimura varieties, given in terms of the compact dual. This generalizes the criterion of [Ven01] in the compact case and has other applications (see Remark 2.14). The spirit of the proof of Theorem 2.11 is that given a functorial cohomology group, some Poincaré duality, and semisimplicity, the averaging argument can be used to show that a linear combination of $G(\mathbb{Q})$ -translates of the cycle class of the subvariety gives the class of the compact dual of H . The necessary ingredients are available thanks to some results in mixed Hodge theory (consequences of the weights and purity package of [BBD, Sai90], reviewed in 2.1) and the theory of Chern classes of automorphic vector bundles (results from [Mum77, GP02], reviewed in 2.2 and Appendix C). We remark that the purely automorphic (i.e. Burger-Sarnak) method cannot be made to work easily for interior cohomology in case (3) (see Remark 4.4 for details).

Having treated the interior cohomology we deal with the cohomology at infinity. In cases (1) and (2) this is straightforward: Given the sequence (0.1) and the identification of the boundary cohomology in terms of Lie algebra cohomology it reduces to an elementary computation with Kostant's theorem (as was already done in case (2) in [Nai17b]). The argument in case (3) of orthogonal Shimura varieties is more delicate: The boundary of M_Γ^* is more complicated, containing modular curves as well as cusps, and it is no longer true that the restriction is injective on the entire boundary cohomology. Instead, the argument is in two steps. First, one extends injectivity from interior cohomology $H_1^i(\mathcal{M}_G)$ to an intermediate subspace

$$H_1^i(\mathcal{M}_G) \subset \operatorname{Gr}_i^W \mathbb{H}_c^i(\mathcal{M}_G^1, j_*^1\mathbb{C}) \subset H^i(\mathcal{M}_G)$$

which takes into account some contributions from the one-dimensional boundary strata (see 6.1 for the notation). This is an elementary argument using Kostant's theorem as in the rank one cases. Next, one extends injectivity to all of $H^i(\mathcal{M}_G)$ by taking into account contributions from

the cusps. This reduces, using some arguments in which weights, purity, and the description of the restriction to strata of the direct image sheaves in M_Γ^* play a crucial role, to the Lefschetz property for real hyperbolic manifolds for the subgroup $SO(1, n-1)$ which appears in the Levi of $SO(2, n)$ with respect to the corresponding subgroup $SO(1, m-1)$ of $SO(2, m)$, i.e. the result from case (1), although with nontrivial coefficients. This completes the proof of Theorem 1.

0.4. Further remarks. The appearance of the Lefschetz property in the real hyperbolic case as a local Lefschetz property at the cusp singularities for the orthogonal Shimura variety suggests trying to reverse the logic and *deduce* the Lefschetz property in the real hyperbolic case from purely geometric facts. It seems likely that this would follow from showing that a linear combination of Hecke translates of the image of $M_{H, \Gamma_H}^* \rightarrow M_\Gamma^*$ is ample, i.e. resolving the question raised in [HL98]. Perhaps [Bor95] can be used profitably here.

Nonvanishing results for cup products in cohomology, which amount to injectivity of Res for the diagonal embedding $G \subset G \times G$, are known in the compact cases (see [Ven01, BeC13]) and for noncompact complex hyperbolic cases (see [Nai17b]). The nonvanishing of cup products in $H_1^*(\mathcal{M}_G)$ follows from the criterion of Theorem 2.11. The extension to $H^*(\mathcal{M}_G)$ should be possible using the arguments outlined here.

Finally, injectivity in degree $i = d_H/2$ in the complex hyperbolic and orthogonal Shimura variety cases remain to be resolved. The two cases are slightly different, since in the first we have injectivity on $H_1^{d_H/2}(\mathcal{M}_G)$ but not on the boundary cohomology, while in the second case we do not know injectivity on $H_1^{d_H/2}(\mathcal{M}_G)$. (For example, our result does not recover Weissauer's result [Wei88] in degree 2 for $SO(2, 2) \subset SO(2, 3)$.) In both cases, the classes potentially in the kernel of Res are constructed by residues of Eisenstein series, and their existence is caused by the nonvanishing of an L -value, while their survival under Res is also related to an L -value. We will consider this question in a sequel.

0.5. Contents. We end the introduction with a brief discussion of the contents of the paper.

Section 1 introduces the congruence quotients of interest and their minimal compactifications, recalls some well-known results on their local geometry and cohomology at infinity, and introduces the restriction maps above in detail.

Section 2 discusses restriction between Shimura varieties. We show using some standard mixed Hodge theory that there is a simple criterion for the injectivity of Res on the top weight quotient of $H^*(\mathcal{M}_G^*)$, and apply it to $SU(1, m) \subset SU(1, n)$ and $SO(2, m) \subset SO(2, n)$ to prove injectivity on the top weight quotient and on interior cohomology in these cases.

Section 3 contains Lie algebra cohomology computations using Kostant's theorem which are necessary to treat boundary contributions in the various cases. These are explicit elementary calculations with roots and weights.

Section 4 considers the congruence real hyperbolic case and contains the proof of case (1) of Theorem 1.

Section 5 considers the congruence complex hyperbolic case and contains the proofs of case (2) of Theorem 1 and of Theorem 2.

Section 6 considers the case of orthogonal Shimura varieties. The results of Sections 2, 3, and 4 are combined to prove the remaining case (3) of Theorem 1.

The three appendices contain some facts which are presumably well known but for which we could not find appropriate references in the literature. Appendix A contains some facts about L^2 cohomology used in Sections 4 and 5. In fact, we only need a very special case of what is proven (Proposition A.1 in the case $SO(1, d)$ for d odd), but the facts recorded will be useful elsewhere. Appendix B records some well-known facts about the construction of cohomology classes via residual Eisenstein series, for use in Sections 4 and 5. Appendix C discusses Chern classes of automorphic vector bundles, which are used in Section 2.

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1. PRELIMINARIES

1.1. Congruence arithmetic quotients. The general setup we work in is as follows. Let G be a semisimple algebraic group over \mathbb{Q} , K the maximal compact subgroup of $G(\mathbb{R})$, and $X = G(\mathbb{R})/K$ the symmetric space. For a congruence subgroup $\Gamma \subset G(\mathbb{Q})$ the quotient

$$M_\Gamma = \Gamma \backslash X$$

is noncompact when G is \mathbb{Q} -isotropic. The following three cases will be the main ones of interest to us:

- (i) $G(\mathbb{R})^{nc} = SO(1, d)$ for $d \geq 2$, so that X is real hyperbolic d -space and M_Γ is a congruence hyperbolic manifold
- (ii) $G(\mathbb{R})^{nc} = SU(1, n)$ for $n \geq 2$, so that X is the complex unit n -ball and M_Γ is a congruence ball quotient (or congruence complex hyperbolic manifold)
- (iii) $G(\mathbb{R})^{nc} = SO(2, n)$ for $n \geq 3$, so that

$$X = SO(2, n)/S(O(2) \times O(n)) = SO_0(2, n)/SO(2) \times SO(n)$$

and M_Γ is a Hermitian locally symmetric space which we will refer to, by an abuse of terminology, as an orthogonal Shimura variety.

In cases (ii) and (iii) the symmetric space X has a Hermitian structure, so that M_Γ is a smooth complex manifold if Γ is small enough. We will also be interested in the general case when X has a Hermitian structure; by an abuse of terminology we will then refer to M_Γ as a Shimura variety.

Example 1.1. The standard examples of congruence quotients of types (i)–(iii) are given by quadratic or Hermitian spaces over \mathbb{Q} or number fields. For example, if (V, q) is a quadratic space over \mathbb{Q} and $q_{\mathbb{R}}$ has signature $(1, d)$ then $G = SO(q)$ gives an example of (i), while if the signature is $(2, n)$ then it gives an example of (iii). If the number of variables is at least ≥ 5 and $q_{\mathbb{R}}$ is indefinite then $G = SO(q)$ is necessarily \mathbb{Q} -isotropic and M_Γ is noncompact. More generally, if $n + 2$ is odd and ≥ 5 then the only examples of type (iii) are the obvious ones, i.e. they come from quadratic forms. For $n + 2$ even there are more complicated examples, e.g. for $d + 1 = 8$ there are triality forms of type (i).

Let $G(\mathbb{R})^c$ be the compact real form of $G(\mathbb{R})$. The compact symmetric space dual to X is

$$X^c = G(\mathbb{R})^c/K.$$

In the three cases above X^c is (i) the d -sphere, (ii) complex projective n -space \mathbb{P}^n , and (iii) a quadric in \mathbb{P}^{n+1} . In all three cases there is a natural embedding $X \subset X^c$ and the action of $G(\mathbb{R})$ extends to the closure of X in X^c : In cases (ii) and (iii) it is the familiar $G(\mathbb{R})$ -equivariant Borel embedding of X in the flag variety and in case (i) it is clear e.g. from the upper halfspace model of hyperbolic space.

1.2. Minimal compactification. In all three cases above there is a canonical open immersion

$$j : M_\Gamma \hookrightarrow M_\Gamma^*$$

into a compact space which we will call the minimal compactification. For cases (i) and (ii) it is the obvious cusp compactification and also coincides with the reductive Borel-Serre compactification. For cases (ii) and (iii) and more generally, for any arithmetic quotient of a Hermitian symmetric domain, it is the Satake-Baily-Borel compactification of M_Γ as a projective variety, and we will describe it in some more detail in this generality.

The closure of X in X^c decomposes as a disjoint union of boundary components, which are (by definition) the maximal connected complex submanifolds of the closure. The stabilizer of a proper (i.e. $\neq X$) boundary component is a product of maximal parabolic subgroups of the simple factors of $G(\mathbb{R})$, and the boundary component is called rational if the stabilizer is defined over \mathbb{Q} , in which case it is a maximal \mathbb{Q} -parabolic of G . As a topological space, $M_\Gamma^* = \Gamma \backslash X^*$ where

$$X^* = \bigsqcup_{F \text{ rational}} F \subset X^c \tag{1.1}$$

is the union of all rational boundary components of X , equipped with the Satake topology. The action of $G(\mathbb{Q})$ on X extends to a continuous action on X^* ; the stabilizer of a rational boundary component F is a maximal \mathbb{Q} -parabolic subgroup (in which case F is proper, i.e. $F \subset X^* - X$) or G itself (the case $F = X$). The Baily-Borel theory [BB66] puts an analytic structure on M_Γ^* inducing the given holomorphic structure on each stratum, and this structure is unique. Moreover, M_Γ^* has a unique structure of projective algebraic variety compatible with this analytic structure, and this gives a canonical quasiprojective structure on M_Γ . The decomposition (1.1) induces an algebraic stratification of M_Γ^* .

Example 1.2. If G is isotropic and $G(\mathbb{R})^{nc}$ is isogenous to $SU(1, n)$ the boundary components are points and M_Γ^* is the cusp compactification of the complex hyperbolic manifold.

Example 1.3. If $G(\mathbb{R})^{nc}$ is isogenous to $SO(2, n)$ the boundary components have complex dimension one (i.e. they are upper half-planes) or zero (points). The natural filtration of X^* induces a filtration by Zariski open subsets

$$M_\Gamma \subset M_\Gamma^1 \subset M_\Gamma^*,$$

with $Z_\Gamma^1 = M_\Gamma^1 - M_\Gamma$ a disjoint union of curves and $Z_\Gamma^0 = M_\Gamma^* - M_\Gamma^1$ a finite set cusps.

The local geometry of the stratification of M_Γ^* is closely tied to the structure of parabolic subgroups, as we now review, see e.g. [BB66, §3], [AMRT, III.4.1–III.4.2], [LR91, 6.1], or [GP02, 7.1–7.3]; we will assume that the adjoint group G^{ad} is \mathbb{Q} -simple. Let P be a maximal rational parabolic subgroup. The unipotent radical W is an extension $1 \rightarrow U \rightarrow W \rightarrow V \rightarrow 1$ where U is the centre of W and V is abelian. The action of A on the Lie algebra $\mathfrak{u} = \text{Lie } U(\mathbb{R})$ is by the square of the positive (with respect to P) generator χ of $X^*(A)$, and the action on the Lie algebra $\mathfrak{v} = \text{Lie } V(\mathbb{R})$ is by χ (if $\mathfrak{v} \neq 0$). The Levi quotient $M = P/W$ has a decomposition $M = M_\ell M_h A$ where $A \cong \mathbb{G}_m$ is the maximal \mathbb{Q} -split central torus in M , M_ℓ and M_h commute, (any lift of) M_h centralizes U , M_h contains no nontrivial connected \mathbb{Q} -anisotropic subgroup, and $M_h(\mathbb{R})$ gives a Hermitian symmetric space, which is the rational boundary component corresponding to P . (The relation with the “five-factor decomposition” of [AMRT, §4.1] is the following: If P is the stabilizer of F and $P = G_h(F)G_\ell(F)M(F)V(F)U(F)$ as in [AMRT, §4.1], then $W = W(F)$, $U = U(F)$, $V \cong V(F)$, $M_\ell \cong G_\ell(F)M(F)$, and $M_h A \cong G_h(F)$.) Note that if G is simply connected the same is true of the derived group of the Levi, so $M^{der} = M_\ell^{der} \times M_h$, and hence M_h is also simply connected.

Example 1.4. Let $G = SO(q)$ for a quadratic form over \mathbb{Q} of signature $(2, n)$. Assume that G has \mathbb{Q} -rank two (this is automatic for $n \geq 6$). The maximal proper \mathbb{Q} -parabolics of G are the stabilizers of isotropic subspaces in V , which are of dimension one or two. We have:

- (1) If P is the stabilizer of an isotropic plane $I \subset V$ the unipotent radical is a nontrivial extension $1 \rightarrow \mathbb{G}_a \rightarrow W \rightarrow \mathbb{G}_a^{2(n-2)} \rightarrow 1$ and the Levi M is $GL(2) \times SO(I^\perp/I)$. Here $M_h A = GL(2)$ and $M_\ell = SO(I^\perp/I) \cong SO(n-2)$ is anisotropic over \mathbb{R} .
- (2) If P is the stabilizer of an isotropic line $I \subset V$ the unipotent radical is abelian $W \cong \mathbb{G}_a^{n-2}$ and the Levi is $M \cong \mathbb{G}_m \times SO(I^\perp/I)$. Here M_h is trivial and $M_\ell \cong SO(I^\perp/I)$ has \mathbb{Q} -rank one and $M_\ell(\mathbb{R}) = SO(1, n-1)$.

The corresponding strata of M_Γ^* are modular curves in case (1) and cusps in case (2).

Example 1.5. It can happen that $Z_\Gamma^1 = \emptyset$, e.g. for the \mathbb{Q} -rank one inner form of $Sp(4) = Spin(2, 3)$ associated with an indefinite quaternion algebra D over \mathbb{Q} and a rank two Hermitian space over D with respect to the involution of D extending the nontrivial Galois action on the maximal subfield of D (which is real quadratic). (This is the form denoted $C_{2,1}^{(2)}$ in [Tit65, p. 57].) In this case the stabilizer of a cusp is the inner form D^\times of $GL(2)$ and the boundary of M_Γ^* is a disjoint union of Shimura curves.

Example 1.6. Another example with $Z_\Gamma^1 = \emptyset$ is that of Hilbert modular surfaces, which are forms of $Spin(2, 2) \cong SL(2) \times SL(2)$ with \mathbb{Q} -rank one.

1.3. Direct limits. Let G be a semisimple \mathbb{Q} -algebraic group and $X = G(\mathbb{R})/K$ the symmetric space and $M_\Gamma = \Gamma \backslash X$ for congruence Γ . For $\Gamma' \subset \Gamma$ the covering map $M_{\Gamma'} \rightarrow M_\Gamma$ gives pullback maps in cohomology and compactly supported cohomology, so taking colimits over all congruence subgroups we define

$$\begin{aligned} \mathrm{H}^i(\mathcal{M}_G) &:= \operatorname{colim}_\Gamma \mathrm{H}^i(M_\Gamma) \\ \mathrm{H}_c^i(\mathcal{M}_G) &:= \operatorname{colim}_\Gamma \mathrm{H}_c^i(M_\Gamma) \\ \mathrm{H}_!^i(\mathcal{M}_G) &:= \operatorname{colim}_\Gamma \mathrm{H}_!^i(M_\Gamma) \end{aligned} \tag{1.2}$$

where, as usual, $\mathrm{H}_!^i(M_\Gamma) = \operatorname{im}(\mathrm{H}_c^i(M_\Gamma) \rightarrow \mathrm{H}^i(M_\Gamma))$ is the interior cohomology. All these are smooth $G(\mathbb{Q})$ -modules, in the sense that the stabilizer of a vector is a congruence subgroup. The action of $g \in G(\mathbb{Q})$ on $\mathrm{H}^i(M_\Gamma) \subset \mathrm{H}^i(\mathcal{M}_G)$ is given by pullback $\mathrm{H}^i(M_\Gamma) \rightarrow \mathrm{H}^i(M_{g\Gamma g^{-1}})$ by the isomorphism $M_{g\Gamma g^{-1}} \rightarrow M_\Gamma$ induced by left translation by g^{-1} on the universal cover X . The transition maps in the colimits are injective, and $\mathrm{H}^*(M_\Gamma)$ can be recovered as the Γ -invariants in $\mathrm{H}^*(\mathcal{M}_G)$. The same remarks apply to $\mathrm{H}_c^*(\mathcal{M}_G)$ and $\mathrm{H}_!^*(\mathcal{M}_G)$.

When the symmetric space X is Hermitian or $G(\mathbb{R})^{nc}$ is isogenous to $SO(1, d)$, we also have the minimal compactification M_Γ^* as in 1.2, and we can define

$$\mathrm{H}^i(\mathcal{M}_G^*) := \operatorname{colim}_\Gamma \mathrm{H}^i(M_\Gamma^*). \tag{1.3}$$

This is a smooth $G(\mathbb{Q})$ -module, and in the Hermitian case it carries a mixed (ind-)Hodge structure. In particular, it has a weight filtration with weights $\leq i$ in degree i and the graded pieces are

$$\operatorname{Gr}_j^W \mathrm{H}^i(\mathcal{M}_G^*) = \operatorname{colim}_\Gamma \operatorname{Gr}_j^W \mathrm{H}^i(M_\Gamma^*)$$

by strictness of the weight filtration.

The inductive setup requires the use of nontrivial coefficients (at the boundary) to treat the case of trivial coefficients. A finite-dimensional algebraic representation E of $G(\mathbb{C})$ gives a local system on M_Γ which, for simplicity we continue to denote E , and we can consider $\mathrm{H}^*(M_\Gamma, E)$, the colimit

$$\mathrm{H}^*(\mathcal{M}_G, E) = \operatorname{colim}_\Gamma \mathrm{H}^*(M_\Gamma, E)$$

and similarly $\mathrm{H}_c^*(\mathcal{M}_G, E)$ and $\mathrm{H}_!^*(\mathcal{M}_G, E)$. For minimal compactifications $j_\Gamma : M_\Gamma \hookrightarrow M_\Gamma^*$, we take the sheaf $\mathrm{H}^0(j_{\Gamma*} E)$ (this is the ordinary, i.e. underived pushforward) and let

$$\mathrm{H}^*(\mathcal{M}_G^*, E) := \operatorname{colim}_\Gamma \mathrm{H}^*(M_\Gamma^*, \mathrm{H}^0(j_{\Gamma*} E)).$$

1.4. Restriction maps. Now suppose that $H \subset G$ is an injective homomorphism of semisimple \mathbb{Q} -groups. Choosing (as we may) a maximal compact K in $G(\mathbb{R})$ such that $K_H = K \cap G(\mathbb{R})$ is maximal compact, and letting $\Gamma_H = \Gamma \cap H(\mathbb{Q})$ and $M_{H, \Gamma_H} = \Gamma_H \backslash H(\mathbb{R})/K_H$, we get a map $\iota : M_{H, \Gamma_H} \rightarrow M_\Gamma$ which is well known to be proper. Thus there are induced pullback maps $\mathrm{H}^i(M_\Gamma) \rightarrow \mathrm{H}^i(M_{H, \Gamma_H})$ and $\mathrm{H}_c^i(M_\Gamma) \rightarrow \mathrm{H}_c^i(M_{H, \Gamma_H})$ (the latter because $M_{H, \Gamma_H} \rightarrow M_\Gamma$ is proper). These are compatible under the natural maps $\mathrm{H}_c^*(\cdot) \rightarrow \mathrm{H}^*(\cdot)$ forgetting supports and hence induce $\mathrm{H}_!^i(M_\Gamma) \rightarrow \mathrm{H}_!^i(M_{H, \Gamma_H})$. In the limit over Γ we have $H(\mathbb{Q})$ -equivariant maps

$$\iota^* : \mathrm{H}_{(c)}^i(\mathcal{M}_G) \rightarrow \mathrm{H}_{(c)}^i(\mathcal{M}_H)$$

in cohomology and compactly-supported cohomology. There are induced homomorphisms of smooth $G(\mathbb{Q})$ -modules

$$\operatorname{Res} : \mathrm{H}_{(c)}^i(\mathcal{M}_G) \rightarrow I_H^G \mathrm{H}_{(c)}^i(\mathcal{M}_H)$$

where I_H^G is an induction functor such that for a smooth $H(\mathbb{Q})$ -module U , $I_H^G U$ consists of functions $f : G(\mathbb{Q}) \rightarrow U$ such that (1) $f(gh) = h^{-1} \cdot f(g)$ and (2) f is left-invariant by a congruence subgroup of $G(\mathbb{Q})$, and the action of $g \in G(\mathbb{Q})$ is by $(g \cdot f)(x) = f(g^{-1}x)$. Then I_H^G is exact, takes smooth modules to smooth modules and is right adjoint to restriction (these facts are completely elementary, see [Nai17a, 3.1]). Explicitly, Res is given by

$$\operatorname{Res}(\alpha)(g) = \iota^*(g^{-1} \cdot \alpha) = \iota^*((g \cdot)^* \alpha).$$

Note that Res restricts to a map $\operatorname{Res} : \mathrm{H}_!^i(\mathcal{M}_G) \rightarrow I_H^G \mathrm{H}_!^i(\mathcal{M}_H)$ on interior cohomology.

When both H and G give Hermitian symmetric spaces and K is chosen (as it may be) so that the map $H(\mathbb{R})/K_H \rightarrow G(\mathbb{R})/K$ is holomorphic, the map $M_{H,\Gamma_H} \rightarrow M_\Gamma$ extends to a morphism $M_{H,\Gamma_H}^* \rightarrow M_\Gamma^*$ of varieties of minimal compactifications. This well-known general fact (cf. [Sat80] or [Har89, 3.3]) is easily seen in our primary cases of interest using the description of M_Γ^* given in 1.2. Pullback induces an $H(\mathbb{Q})$ -equivariant map $H^i(\mathcal{M}_G^*) \rightarrow H^i(\mathcal{M}_H^*)$, which gives a homomorphism of mixed Hodge structures

$$\text{Res} : H^i(\mathcal{M}_G^*) \longrightarrow I_H^G H^i(\mathcal{M}_H^*)$$

by adjunction. There is a similar mapping in the real hyperbolic cases where $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SO(1, c) \subset SO(1, d)$ (up to isogeny) and in the ‘mixed’ case $SO(1, n) \subset SU(1, n)$, coming from the obvious extension of $M_{H,\Gamma_H} \rightarrow M_\Gamma$ to minimal compactifications.

Now assume that $H \subset G$ is such that the restriction of finite-dimensional representations from G to H is multiplicity-free. The situations we will treat are well known to be of this type, by classical branching laws e.g. [GW09, 8.1.1]). Choose Borel subgroups $B_H \subset B$ and maximal tori $T_H \subset B_H$ and $T \subset B$ of $H(\mathbb{C})$ and $G(\mathbb{C})$, and for E with highest weight $\lambda \in X^*(T)$ let E_H be the unique summand of $E|_{H(\mathbb{C})}$ with highest weight $\lambda|_{T_H}$, where $\lambda \in X^*(T)$ is the highest weight of E . The composition $H^*(\mathcal{M}_G, E) \rightarrow H^*(\mathcal{M}_H, E|_H) \rightarrow H^*(\mathcal{M}_H, E_H)$ is $H(\mathbb{Q})$ -invariant and induces a restriction map

$$\text{Res} : H^*(\mathcal{M}_G, E) \longrightarrow H^*(\mathcal{M}_H, E_H)$$

with coefficients. There are similar maps for $H_c^*(\mathcal{M}_G, E)$, $H_!^*(\mathcal{M}_G, E)$, and $H^*(\mathcal{M}_G^*, E)$.

The use of Res for several different maps should cause no confusion as we will always specify the domain when discussing injectivity results. We will also frequently write $H^i(\mathcal{M})$, $H^i(\mathcal{M}^)$ etc. for $H^i(\mathcal{M}_G)$, $H^i(\mathcal{M}_G^*)$ etc., i.e. drop the subscript G when it is clear from context.*

1.5. Higher direct images in the minimal compactification. We will use a well-known description of the restriction of $j_*\mathbb{Q}_{M_\Gamma}$ to a stratum of M_Γ^* in the case of Shimura varieties. (Here by j_* we mean the pushforward on the level of derived categories, i.e. Rj_* in the old-fashioned notation.)

To fix notation, let $i_S : S \hookrightarrow M_\Gamma^*$ be a stratum of the minimal compactification. Choose a rational boundary component $F \twoheadrightarrow S$ and let $P = MW$ be the stabilizer of F and $M = M_h M_l A$ as in 1.2. For the congruence subgroup Γ , let $\Gamma_W = \Gamma \cap W(\mathbb{Q})$, $\Gamma_P = \Gamma \cap P(\mathbb{Q})$, $\Gamma_M = \Gamma_P/\Gamma_W$, $\Gamma_{M_\ell} = \Gamma_M \cap M_\ell(\mathbb{Q})$, and $\Gamma_{M_h} = \Gamma_M/\Gamma_{M_\ell}$. These are all neat arithmetic subgroups when Γ is neat.

Proposition 1.7. *For a stratum $i_S : S \hookrightarrow M_\Gamma^*$ we have:*

(1) *There is a natural isomorphism in the derived category*

$$i_S^* j_* \mathbb{Q}_{M_\Gamma} = \bigoplus_k H^k(i_S^* j_* \mathbb{Q}_{M_\Gamma})[-k] \quad (1.4)$$

of sheaves on $S = \Gamma_{M_h} \backslash F$. The object $H^k(i_S^ j_* \mathbb{Q}_{M_\Gamma})$ is the local system on S associated with the representation of M_h on*

$$H^k(i_S^* j_* \mathbb{Q}_{M_\Gamma})_s \cong \bigoplus_{r+s=k} H^r(\Gamma_{M_\ell}, H^s(\mathfrak{w}, \mathbb{Q})) \quad (1.5)$$

for $s \in S$.

(2) *The weight filtration on $H^k(i_S^* j_* \mathbb{Q}_{M_\Gamma})_s$ is split by the action of A on $H^*(\mathfrak{w}, \mathbb{Q})$, i.e. $\text{Gr}_i^W H^k(i_S^* j_* \mathbb{Q}_{M_\Gamma})_s$ is identified with the subspace on which A acts by χ^{-i} (where $\chi \in X^*(A)$ is such that A acts on the centre \mathfrak{u} of \mathfrak{w} by χ^2 and on $\mathfrak{v} = \mathfrak{w}/\mathfrak{u}$ by χ , cf. 1.2).*

The description (1.5) of cohomology sheaves can be found e.g. in [LR91, 5.6] or [GHM94, 22.8]. The weight filtration on $H^k(i_S^* j_* \mathbb{Q}_{M_\Gamma})_s$ comes from the theory of mixed Hodge modules and the assertion in (2) is due to Looijenga and Rapoport [LR91, Proposition 5.6]. (The analogue in the l -adic setting is in [Pin92].) The existence of the decomposition (1.4) in the derived category can in fact be deduced from this, but instead one can use Theorem 2.9 of Burgos-Wildeshaus [BuW04], which proves the direct sum decomposition (1.4) in the derived

category of mixed Hodge modules and the identification of the graded in the context of Shimura varieties. The isomorphism (1.4) and (1.5) are equivariant for the actions of $(M_\ell AW)(\mathbb{Q})$ on both sides (which factor through $(M_\ell A)(\mathbb{Q})$).

Remark 1.8. The real hyperbolic $SO(1, d)$ case fits notationally into the above setup by taking $M_h = \{e\}$, $M_\ell = SO(d - 1)$, and then (1) remains true.

2. RESTRICTION BETWEEN SHIMURA VARIETIES

In this section we assume that H and G both give rise to Shimura varieties, and consider the restriction $\text{Res} : \text{Gr}_i^W \mathbb{H}^i(\mathcal{M}_G^*) \rightarrow I_H^G \text{Gr}_i^W \mathbb{H}^i(\mathcal{M}_H^*)$ on the top weight quotient. We prove a criterion (Theorem 2.11 below) for the nonvanishing of this restriction involving the compact dual symmetric space, which is the analogue of the criterion of [Ven01] in this situation. It implies the following:

Theorem 2.1. *If $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SU(1, m) \subset SU(1, n)$ or $SO(2, m) \subset SO(2, n)$ then Res is injective on $\text{Gr}_i^W \mathbb{H}^i(\mathcal{M}_G^*)$ in degrees $\leq m$.*

The unitary case of this is contained in [Nai17a, Theorem 3.17], although the proof here is slightly different (and more direct). As a corollary we get the following injectivity statements for interior cohomology, of which the first was proved earlier in [Nai17a] (see also [BeC17]):

Corollary 2.2. *If $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SU(1, m) \subset SU(1, n)$ then Res is injective on $\mathbb{H}_i^i(\mathcal{M}_G)$ in degrees $\leq m$.*

If $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SO(2, m) \subset SO(2, n)$ then Res is injective on $\mathbb{H}_i^i(\mathcal{M}_G)$ in degrees $\leq m$ if $\text{rk}_{\mathbb{Q}}(H) \leq 1$ and in degrees $\leq m - 1$ if $\text{rk}_{\mathbb{Q}}(H) = 2$.

2.1. Some cohomological facts. We will use some facts about the cohomology of (possibly) singular varieties, summarized in Proposition 2.3 and Lemma 2.5 below.

Recall that by [Del74] the rational cohomology $\mathbb{H}^*(X) = \mathbb{H}^*(X, \mathbb{Q})$ and homology $\mathbb{H}_*(X) = \mathbb{H}_*(X, \mathbb{Q})$ of a complex algebraic variety X carry rational mixed Hodge structures, in particular they have weight filtrations. The theory of mixed Hodge modules ([Sai90], see especially §4 of loc. cit.) gives a relative version of mixed Hodge structures and allows for sheaf-theoretic arguments, mirroring the situation in l -adic cohomology over finite fields [BBD]. Let X be an irreducible complex variety of dimension d . Let \mathbb{Q}_X^H be the canonical lift of \mathbb{Q}_X to an object in the derived category of mixed Hodge modules on X (i.e. $\mathbb{Q}_X^H = a_X^* \mathbb{Q}^H$ where $a_X : X \rightarrow \text{Spec}(\mathbb{C})$ and \mathbb{Q}^H is the trivial Hodge structure). The rational cohomology, compactly supported cohomology, homology, and Borel-Moore homology groups of X acquire mixed Hodge structures via

$$\begin{aligned} \mathbb{H}^i(X) &= \mathbb{H}^i(X, \mathbb{Q}_X^H), & \mathbb{H}_c^i(X) &= \mathbb{H}_c^i(X, \mathbb{Q}_X^H), \\ \mathbb{H}_i(X) &= \mathbb{H}_c^{-i}(X, \mathbb{D}\mathbb{Q}_X^H), & \mathbb{H}_i^{BM}(X) &= \mathbb{H}^{-i}(X, \mathbb{D}\mathbb{Q}_X^H) \end{aligned} \tag{2.1}$$

where \mathbb{D} is the Verdier duality functor, normalized so that $\mathbb{D}\mathbb{Q}_X^H = \mathbb{Q}_X^H[2d](d)$ if X is smooth. When X is proper, which is the main case of interest to us, we have $\mathbb{H}^i(X) = \mathbb{H}_c^i(X)$ and $\mathbb{H}_i(X) = \mathbb{H}_i^{BM}(X)$. The weights are determined by the fact that \mathbb{Q}_X^H has weights ≤ 0 , so e.g. $\mathbb{H}_c^i(X)$ has weights $\leq i$ and $\mathbb{H}_i^{BM}(X)$ has weights $\geq -i$.

We will also use the intersection complex $IC_X^H = (j_{!*} \mathbb{Q}_U^H[d])[-d]$ where $j : U \hookrightarrow X$ is the inclusion of an open dense smooth subset; this lifts the intersection complex $IC_X = (j_{!*} \mathbb{Q}_U[d])[-d]$ of X , and it is pure of weight 0. (Our notation is slightly different from [Sai90], where $IC_X(\mathbb{Q}^H)$ is used for $j_{!*} \mathbb{Q}_U[d]$.) Replacing \mathbb{Q}_X^H by IC_X^H in (2.1) defines rational mixed Hodge structures

$$\mathbb{I}\mathbb{H}^i(X), \quad \mathbb{I}\mathbb{H}_c^i(X), \quad \mathbb{I}\mathbb{H}_i(X), \quad \mathbb{I}\mathbb{H}_i^{BM}(X)$$

on the intersection cohomology, intersection cohomology with compact support, intersection homology, and Borel-Moore intersection homology. When X is proper these are all pure and the isomorphism $\mathbb{D}IC_X^H = IC_X^H[2d](d)$ extending $\mathbb{D}\mathbb{Q}_U^H = \mathbb{Q}_U^H[2d](d)$ on any smooth open subset $U \subset X$ induces a duality isomorphism $\mathbb{I}\mathbb{H}^i(X) \cong \mathbb{I}\mathbb{H}_{2d-i}(X)(-d)$.

Proposition 2.3. *If $f : X \rightarrow Y$ is a morphism of varieties there are maps*

$$\begin{aligned} f^* &: \mathrm{Gr}_i^W \mathbb{H}^i(Y) \rightarrow \mathrm{Gr}_i^W \mathbb{H}^i(X) \\ f_* &: W_{-i} \mathbb{H}_i(X) \rightarrow W_{-i} \mathbb{H}_i(Y) \end{aligned} \quad (2.2)$$

for each i satisfying

$$f_*(f^*(\alpha) \cap \beta) = \alpha \cap f_*(\beta) \quad \text{for } \alpha \in \mathrm{Gr}_i^W \mathbb{H}^i(Y), \beta \in W_{-j} \mathbb{H}_j(X). \quad (2.3)$$

If X is an irreducible proper variety of dimension d then

(1) $\mathbb{H}^i(X)$ has weights $\leq i$, $\mathbb{H}_i(X)$ has weights $\geq -i$, and the extreme weights are given by

$$\begin{aligned} \mathrm{Gr}_i^W \mathbb{H}^i(X) &= \mathrm{im}(\mathbb{H}^i(X) \rightarrow \mathrm{IH}^i(X)) \\ W_{-j} \mathbb{H}_j(X) &= \mathrm{im}(\mathrm{IH}_j(X) \rightarrow \mathbb{H}_j(X)). \end{aligned} \quad (2.4)$$

for all i, j .

(2) If $[X] \in \mathbb{H}_{2d}(X)(-d)$ is the fundamental class of X then

$$\cap[X] : \mathrm{Gr}_i^W \mathbb{H}^i(X) \longrightarrow W_{-(2d-i)} \mathbb{H}_{2d-i}(X)(-d) \quad (2.5)$$

is an isomorphism for all i .

(3) If $i : Z \hookrightarrow X$ is an irreducible closed subvariety of codimension c the cycle class

$$\xi_{X,Z} := (\cap[X])^{-1}(i_*[Z]) \in \mathrm{Gr}_{2c}^W \mathbb{H}^{2c}(X)(c)$$

has the property that if $\alpha \in \mathbb{H}^{2 \dim Z}(X)$ with $i^*(\alpha) = \xi_{Z,pt}$ then $\alpha \cdot \xi_{X,Z} = \xi_{X,pt}$.

Proof. The first statement is simply the functoriality of the weight filtration and the fact that when homology is considered as a module over the cohomology ring using cap product, push-forward in homology is a module over pullback in cohomology. (1) and (2) are contained in §4.5 of [Sai90], but for the reader's convenience we outline the arguments.

For the standard cohomology functor \mathbb{H}^0 on mixed Hodge modules (which corresponds to the perverse cohomology functor ${}^p\mathbb{H}^0$ on sheaves), we have dual natural isomorphisms

$$\begin{aligned} \mathrm{Gr}_d^W \mathbb{H}^d(\mathbb{Q}_X^H) &= IC_X^H[d] \\ W_{-d} \mathbb{H}^{-d}(\mathbb{D}\mathbb{Q}_X^H) &= IC_X^Hd \end{aligned} \quad (2.6)$$

(see [Sai90, §4.5] for details). If X is proper then the dual statements (2.6) and the hypercohomology spectral sequence imply that

$$\begin{aligned} \mathrm{Gr}_i^W \mathbb{H}^i(X) &= \mathrm{im}(\mathbb{H}^i(X) \rightarrow \mathrm{IH}^i(X)) \\ W_{-j} \mathbb{H}_j(X) &= \mathrm{im}(\mathrm{IH}_j(X) \rightarrow \mathbb{H}_j(X)) \end{aligned} \quad (2.7)$$

as claimed in (1).

An irreducible variety X has a fundamental class in Borel-Moore homology

$$[X] \in \mathbb{H}_{2d}^{BM}(X)(-d) = \mathbb{H}^0(X, (\mathbb{D}\mathbb{Q}_X^H)[-2d](-d)) = \mathrm{Hom}(\mathbb{Q}_X^H, (\mathbb{D}\mathbb{Q}_X^H)[-2d](-d))$$

giving the fundamental class homomorphism $\mathbb{Q}_X^H \rightarrow (\mathbb{D}\mathbb{Q}_X^H)[-2d](-d)$ which is an isomorphism if X is smooth. By the identities (2.6) and standard facts about the t -structure and weights, it factors as

$$\mathbb{Q}_X^H \longrightarrow IC_X^H \longrightarrow (\mathbb{D}IC_X^H)[-2d](-d) \longrightarrow (\mathbb{D}\mathbb{Q}_X^H)[-2d](-d) \quad (2.8)$$

where the first arrow is the unique extension of the identity morphism $\mathbb{Q}_U \rightarrow \mathbb{Q}_U$ on U and the third arrow is its dual (up to a twist). The second is the Verdier duality isomorphism extending $\mathbb{Q}_U^H = (\mathbb{D}\mathbb{Q}_U^H)[-2d](-d)$ and induces Poincaré duality isomorphisms

$$\mathrm{IH}_c^i(X) = \mathbb{H}_c^i(X, IC_X^H) \cong \mathbb{H}_c^{i-2d}(X, \mathbb{D}IC_X^H)(-d) = \mathrm{IH}_{2d-i}(X)(-d) = \mathrm{IH}^{2d-i}(X)^*(-d)$$

and

$$\mathrm{IH}^i(X) = \mathbb{H}^i(X, IC_X^H) \cong \mathbb{H}^{i-2d}(X, \mathbb{D}IC_X^H)(-d) = \mathrm{IH}_{2d-i}^{BM}(X)(-d),$$

and hence a nondegenerate pairing $\mathrm{IH}^i(X) \times \mathrm{IH}_c^{2d-i}(X) \rightarrow \mathbb{Q}(-d)$. The fundamental class homomorphism induces the identity isomorphism $\mathrm{Gr}_d^W \mathbb{H}^d(\mathbb{Q}_X^H) = W_{-d} \mathbb{H}^{-d}(\mathbb{D}\mathbb{Q}_X^H)(-d)$ from

(2.6) (as indeed it must since it is the unique extension of $\mathbb{Q}_U^H[d] = \mathbb{D}\mathbb{Q}_U^Hd$ on any smooth open $U \subset X$).

Now assume that X is proper. Then the duality of the first and third arrows in (2.8) implies that cap product with the fundamental class $[X]$ induces an isomorphism

$$\cap[X] : \mathrm{Gr}_i^W \mathrm{H}^i(X) \xrightarrow{\sim} W_{2d-i} \mathrm{H}_{2d-i}(X)(-d). \quad (2.9)$$

This proves (2). For (3) note that if $i^*(\alpha) = \xi_{Z,pt}$ then $[Z] \cap i^*(\alpha) = 1$ so that

$$1 = i_*(i^*(\alpha) \cap [Z]) = \alpha \cap i_*[Z] = [X] \cap (\alpha \cdot (\cap[X])^{-1}(i_*[Z])) = [X] \cap \alpha \cdot \xi_{X,Z} \quad (2.10)$$

so that $\alpha \cdot \xi_{X,Z} = (\cap[X])^{-1}(1) = \xi_{X,pt}$. \square

Remark 2.4. The assertion in (2) is related to some facts in classical mixed Hodge theory which we will also use. If X is irreducible and proper and $Y \rightarrow X$ is a resolution of singularities then

$$\mathrm{Gr}_i^W \mathrm{H}^i(X) = \mathrm{im}(\mathrm{H}^i(X) \rightarrow \mathrm{H}^i(Y))$$

for all i by [Del74, Prop. 8.2.5]. This equivalent to (2.4) because the pullback factors as $\mathrm{H}^i(X) \rightarrow \mathrm{IH}^i(X) \hookrightarrow \mathrm{H}^i(Y)$ for any inclusion $\mathrm{IH}^*(X) \subset \mathrm{H}^*(Y)$ coming from the decomposition theorem [BBD, Sai90]. Since one also has

$$\mathrm{Gr}_i^W \mathrm{H}_c^i(U) = \mathrm{im}(\mathrm{H}_c^i(U) \rightarrow \mathrm{IH}^i(X)) = \mathrm{im}(\mathrm{H}_c^i(U) \rightarrow \mathrm{H}^i(Y)),$$

one sees that $\mathrm{Gr}_i^W \mathrm{H}_c^i(U) \subset \mathrm{Gr}_i^W \mathrm{H}^i(X)$ for all i .

The following purity lemma will be used later in Section 7:

Lemma 2.5. *Let X be a normal complex variety of dimension d with $U \subset X^1 \subset X$ a filtration by open subsets such that U is smooth and open dense in X^1 , $Z^1 = X^1 - U$ is smooth of dimension one and, $Z^0 = X - X^1$ is smooth of dimension zero. Let $j : U \hookrightarrow X$ and $i : Z^0 \hookrightarrow X$ be the inclusions. Then $\mathrm{H}^i(i^{0*} j_* \mathbb{Q}_U^H)$ has weights $\leq i$ for $i \leq d - 2$. If $Z^1 = \emptyset$ then $\mathrm{H}^i(i^{0*} j_* \mathbb{Q}_U^H)$ has weights $\leq i$ for $i \leq d - 1$.*

Proof. Write $j = j^0 \circ j^1$ for $j^1 : U \hookrightarrow X^1$ and $j^0 : X^1 \hookrightarrow X$. Since $IC_X = \tau_{<d} j_*^0 \tau_{<d-1} j_*^1 \mathbb{C}_U$, it follows easily that $IC_X \rightarrow j_* \mathbb{C}_U$ induces an isomorphism on cohomology sheaves in degrees $\leq d - 2$. The same then holds for $IC_X^H \rightarrow j_* \mathbb{Q}_U^H$. On the other hand, by pointwise purity of the intersection complex, $\mathrm{H}^i(IC_X^H)_x$ has weights $\leq i$ in all degrees [BBD, Sai90]. This proves the first assertion of the lemma. If $Z^1 = \emptyset$ we have that $IC_X \rightarrow j_* \mathbb{C}_U$ induces isomorphisms on cohomology sheaves in degrees $\leq d - 1$, and purity proves the second assertion. \square

2.2. Invariants and the compact dual. We will assume from now on that G is semisimple and $X = G(\mathbb{R})/K$ is a Hermitian symmetric domain. In addition, we assume in this subsection that G is simply connected. We return to the use of cohomology with complex coefficients and ignore Tate twists.

We will consider the $G(\mathbb{Q})$ -module $\mathrm{IH}^i(\mathcal{M}^*) := \mathrm{colim}_\Gamma \mathrm{IH}^i(M_\Gamma^*)$ which is smooth and admissible. Note that (2.4) gives an inclusion

$$\mathrm{Gr}_*^W \mathrm{H}^*(\mathcal{M}^*) := \bigoplus_i \mathrm{Gr}_i^W \mathrm{H}^i(\mathcal{M}^*) \subset \mathrm{IH}^*(\mathcal{M}^*)$$

of $G(\mathbb{Q})$ -modules.

Proposition 2.6. *The $G(\mathbb{Q})$ -modules $\mathrm{Gr}_*^W \mathrm{H}^*(\mathcal{M}^*)$ and $\mathrm{IH}^*(\mathcal{M}^*)$ are semisimple and the summand of invariants is given by*

$$\mathrm{Gr}_*^W \mathrm{H}^*(\mathcal{M}^*)^{G(\mathbb{Q})} = \mathrm{IH}^*(\mathcal{M}^*)^{G(\mathbb{Q})} = \mathrm{H}^*(X^c).$$

The embedding of $\mathrm{H}^(X^c)$ in $\mathrm{Gr}_*^W \mathrm{H}^*(\mathcal{M}^*)$ is functorial: If $H \subset G$ gives a complex submanifold $X_H \subset X$ then the obvious diagram coming from $X_H^c \subset X^c$ and $M_{H,\Gamma_H}^* \rightarrow M_\Gamma^*$ commutes.*

Proof. There is a natural isomorphism

$$\mathrm{IH}^*(\mathcal{M}^*) = \mathrm{H}^*(\mathfrak{g}, K, L_{dis}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))) \quad (2.11)$$

thanks to [Loo88, SS90] and [BoC83]. Here L_{dis}^2 is the L^2 discrete spectrum; see [Nai17a, 3.8] for a detailed discussion of (2.11). This proves the semisimplicity statements. The inclusion of the constants in L^2 functions induces an embedding of $\mathrm{H}^*(X^c) = \mathrm{H}^*(\mathfrak{g}, K, \mathbb{C})$ in $\mathrm{IH}^*(\mathcal{M}^*)$. It follows from (2.11) using strong approximation and the density of $G(\mathbb{Q})$ in $G(\mathbb{R})$ (weak approximation) (see e.g. the proof of Proposition 3.8 in [Nai17a], which works verbatim here), that these are all the invariants.

To show that the invariants are actually in $\mathrm{Gr}_*^W \mathrm{H}^*(\mathcal{M}^*)$ we will use Chern classes of automorphic vector bundles [Mum77, GP02]. A finite-dimensional representation V of K gives a homogenous bundle \mathcal{V}^c on $X^c = G(\mathbb{R})^c/K$. Restricting by the Borel embedding $X \subset X^c$ (see 1.1) and dividing by Γ gives a bundle \mathcal{V}_Γ on $M_\Gamma = \Gamma \backslash X$ for any Γ . The bundle \mathcal{V}_Γ does not, in general, extend to a vector bundle on M_Γ^* (see Example 2.7 below for an important exception), but Goresky and Pardon [GP02] defined classes $c_k^*(\mathcal{V}_\Gamma) \in \mathrm{H}^{2k}(M_\Gamma^*)$ which behave like the Chern classes of a putative extension \mathcal{V}_Γ^* to M_Γ^* . The main property is that for $\pi : M_\Gamma^\Sigma \rightarrow M_\Gamma^*$ a smooth toroidal desingularization [AMRT], the pullback $\pi^*(c_k^*(\mathcal{V}_\Gamma)) = c_k(\mathcal{V}_\Gamma^\Sigma)$ is the Chern class of Mumford's canonical extension $\mathcal{V}_\Gamma^\Sigma$ [Mum77, Har89]. It is a well-known consequence of Mumford's generalization of Hirzebruch proportionality that the classes $c_k(\mathcal{V}_\Gamma^\Sigma)$ generate a copy of $\mathrm{H}^*(X^c)$ in $\mathrm{H}^*(M_\Gamma^\Sigma)$. More precisely, there is an injective homomorphism $\theta : \mathrm{H}^*(X^c) \rightarrow \mathrm{H}^*(M_\Gamma^\Sigma)$ such that $\theta(c_k(\mathcal{V}^c)) = (-1)^k c_k(\mathcal{V}_\Gamma^\Sigma)$ for all k, \mathcal{V} (see Lemma C.1 in Appendix C for a proof, following [Nai14, Lemma 3.7.2]). They are contained in $\mathrm{Gr}_*^W \mathrm{H}^*(M_\Gamma^*) = \mathrm{im}(\mathrm{H}^*(M_\Gamma^*) \rightarrow \mathrm{H}^*(M_\Gamma^\Sigma))$ since $\pi^*(c_k^*(\mathcal{V})) = c_k(\mathcal{V}_\Gamma^\Sigma)$ by [GP02]. Moreover, the compatibility of the construction for different Σ (see e.g. [Har89, 4.3.1]) shows that we have a well-defined embedding $\theta : \mathrm{H}^*(X^c) \rightarrow \mathrm{Gr}_*^W \mathrm{H}^*(M_\Gamma^*)$.

It remains to show that the classes are $G(\mathbb{Q})$ -invariant and the embedding is functorial. The direct limit $\mathrm{colim}_{\Sigma, \Gamma} \mathrm{H}^*(M_\Gamma^\Sigma)$ over all pairs (Σ, Γ) where Σ is admissible for Γ is a $G(\mathbb{Q})$ -module, and contains $\mathrm{Gr}_*^W \mathrm{H}^*(\mathcal{M}^*)$ as a $G(\mathbb{Q})$ -submodule. Standard properties of the canonical extensions listed in [Har89, 4.3] show that the Chern classes are $G(\mathbb{Q})$ -invariants in $\mathrm{colim}_{\Sigma, \Gamma} \mathrm{H}^*(M_\Gamma^\Sigma)$, and hence in $\mathrm{Gr}_*^W \mathrm{H}^*(\mathcal{M}^*)$. Finally, functoriality follows from [Har89, 4.3.4] and the definition of the map θ given in Lemma C.1. \square

Example 2.7. The representation of K on the top exterior power of \mathfrak{p} , where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition given by K , gives a special automorphic bundle called the Baily-Borel bundle. This extends as a line bundle \mathcal{L}^{bb} over M_Γ^* and some power of \mathcal{L}^{bb} is the $\mathcal{O}(1)$ in the Baily-Borel projective embedding (see [Mum77, Prop. 3.4(b)]). So \mathcal{L}^{bb} is ample and the Chern class $c_1^{bb} := c_1^*(\mathcal{L}) = c_1(\mathcal{L}^{bb})$ fixes a generator

$$(c_1^{bb})^n \in \mathrm{Gr}_{2n}^W \mathrm{H}^{2n}(\mathcal{M}^*) = \mathrm{IH}^{2n}(\mathcal{M}^*) \quad (2.12)$$

in top degree.

Example 2.8. If $G(\mathbb{R})^{nc} = SU(1, n)$ then $X^c = SU(1+n)/S(U(1) \times U(n)) \cong \mathbb{P}^n$. So the invariant part of $\mathrm{Gr}_*^W \mathrm{H}^*(\mathcal{M}^*)$ is $\mathbb{C}[c_1^{bb}]/((c_1^{bb})^{n+1})$. (See e.g. [Nai17a, 1.2] for an intrinsic description of \mathcal{L}^{bb} .)

Example 2.9. If $G(\mathbb{R})^{nc} = Spin(2, n)$, the compact symmetric space dual to X is

$$X^c = Spin(2+n)/Spin(2) \times_{\{\pm 1\}} Spin(n) = SO(2+n)/SO(2) \times SO(n)$$

which is a quadric in \mathbb{P}^{n+1} . The complex cohomology ring of quadrics is well known. Let E_2 and E_n be the vector bundles on X^c corresponding to the natural representations of $SO(2) \times SO(n)$ of dimension 2 and n , respectively. When n is odd the complex cohomology is generated by the Euler class (or first Chern class) $c_1 = c_1(E_2) \in \mathrm{H}^2(X^c)$, i.e. $\mathrm{H}^*(X^c) = \mathbb{C}[c_1]/(c_1^{n+1})$. When $n = 2d$ is even the complex cohomology of X^c is generated as a ring by c_1 and the Euler class $c_d = c_d(E_n) \in \mathrm{H}^{n=2d}(X^c)$, with the relations $c_1^{n+1} = 0$, $c_d^2 = (-1)^d c_1^{2d}$ and $c_1 c_d = 0$.

For later use we remark that if $n = 2d \geq 4$ and $X_a^c \subset X^c$ is the inclusion of quadrics coming from $SO(2, a) \subset SO(2, n)$ for $a < n$, then $c_d|_{X_a^c} = 0$. (Indeed, $0 = (c_1 c_d)|_{X_{n-1}^c} = c_1|_{X_{n-1}^c} \cdot c_d|_{X_{n-1}^c} = c_1 \cdot (c_d|_{X_{n-1}^c})$. Since $c_1 \cdot$ is injective on $H^*(X_{n-1}^c) = \mathbb{C}[c_1]/(c_1^n)$ in degrees $< 2(n-1)$ we must have $c_d|_{X_{n-1}^c} = 0$ if $d \geq 2$.)

2.3. Cycle classes and an injectivity criterion. We will prove a criterion for the nonvanishing of Res between Shimura varieties and apply it to prove Theorem 2.1 and Corollary 2.2. Since we are only interested in cohomology with complex coefficients we will ignore Tate twists henceforth and write $H^*(X)$ for $H^*(X, \mathbb{C})$.

Suppose now that $\iota : M_{H, \Gamma_H}^* \rightarrow M_\Gamma^*$ is the extension to minimal compactifications of a morphism of Shimura varieties (cf. 1.4), and let $n = \dim M_\Gamma, m = \dim M_{H, \Gamma_H}$. Let

$$\xi_\Gamma := (\cap[M_\Gamma^*])^{-1}(\iota_*[M_{H, \Gamma_H}^*]) \in \mathrm{Gr}_{2(n-m)}^W H^{2(n-m)}(M_\Gamma^*)$$

be the cycle class defined earlier in Proposition 2.3, ignoring Tate twists and simplifying the notation (in the notation of loc. cit. this would be $\xi_{M_\Gamma^*, M_{H, \Gamma_H}^*}$). It is easily checked that if $\Gamma' \subset \Gamma$ is normal then $\xi_\Gamma = |\Gamma/\Gamma'|^{-1} \sum_{\gamma \in \Gamma/\Gamma'} \gamma \cdot \xi_{\Gamma'}$, where $\xi_{\Gamma'} = (\cap[M_\Gamma^*])^{-1}(\iota'_*([M_{H, \Gamma'_H}^*]))$ for $\iota' : M_{H, \Gamma'_H}^* \rightarrow M_{\Gamma'}^*$ at level Γ' .

We will also consider the closed immersion $\iota^c : X_H^c \rightarrow X^c$, which gives the cycle class

$$\xi_{X_H^c} := (\cap[X^c])^{-1}(\iota_*^c[X_H^c])$$

which is nonzero since X_H^c is a subvariety of the algebraic variety X^c .

Proposition 2.10. *The $G(\mathbb{Q})$ -submodule of $\mathrm{IH}^*(\mathcal{M}^*)$ generated by ξ_Γ contains the cycle class $\xi_{X_H^c} \in H^{2(n-m)}(X^c)$ of X_H^c in X^c .*

Proof. The $G(\mathbb{Q})$ -submodule $V \subset \mathrm{Gr}_{2(n-m)}^W H^{2(n-m)}(\mathcal{M}^*) \subset \mathrm{IH}^{2(n-m)}(\mathcal{M}^*)$ generated by ξ_Γ admits a decomposition $V = V^0 \oplus V^1$ where V^1 has no invariants or coinvariants and V^0 is contained in the summand of invariants $H^{2(n-m)}(X^c) \subset \mathrm{Gr}_{2m}^W H^{2m}(M_\Gamma^*)$. Write $\xi_\Gamma = \xi_\Gamma^0 + \xi_\Gamma^1$ with $\xi_\Gamma^i \in V^i$. Since V^1 has no coinvariants, $\xi_\Gamma^0 \cdot \alpha = \xi_\Gamma \cdot \alpha$ for any $\alpha \in H^{2m}(X^c)$, so that

$$\begin{aligned} [M_\Gamma^*] \cap (\xi_\Gamma^0 \cdot \alpha) &= [M_\Gamma^*] \cap (\xi_\Gamma \cdot \alpha) \\ &= [M_\Gamma^*] \cap \xi_\Gamma \cap \alpha \\ &= \iota_*[M_{H, \Gamma_H}^*] \cap \alpha \\ &= \iota_*([M_{H, \Gamma_H}^*] \cap \iota^*(\alpha)) \end{aligned} \tag{2.13}$$

On the other hand,

$$\begin{aligned} [X^c] \cap (\xi_{X_H^c} \cdot \alpha) &= [X^c] \cap \xi_{X_H^c} \cap \alpha \\ &= \iota_*^c[X_H^c] \cap \alpha \\ &= \iota_*^c([X_H^c] \cap \iota^{c*}(\alpha)). \end{aligned} \tag{2.14}$$

Now the pullbacks ι^{c*} and ι^* are compatible, while under the isomorphism $H^{2n}(X^c) \cong \mathrm{Gr}_{2n}^W H^{2n}(M_\Gamma^*)$ the two nonzero linear forms $[M_\Gamma^*] \cap : \mathrm{Gr}_{2n}^W H^{2n}(M_\Gamma^*) \rightarrow H_0(M_\Gamma^*) = \mathbb{C}$ and $[X^c] \cap : H^{2n}(X^c) \rightarrow H_0(X^c) = \mathbb{C}$ are necessarily proportional, and the same holds for M_{H, Γ_H}^* and X_H^c . It follows from (2.13) and (2.14) that for $\alpha \in H^{2m}(X^c)$, we have

$$[X^c] \cap (\xi_\Gamma^0 \cdot \alpha) \sim [X^c] \cap (\xi_{X_H^c} \cdot \alpha)$$

where \sim means up to a fixed nonzero constant independent of α . Thus $\xi_\Gamma^0 \cdot \alpha \sim \xi_{X_H^c} \cdot \alpha$ for any α and so $\xi_\Gamma^0 \sim \xi_{X_H^c}$ by Poincaré duality for X^c . \square

Theorem 2.11. *If $\alpha \in \mathrm{Gr}_i^W H^i(\mathcal{M}^*)$ and $\mathrm{Res}(\alpha) = 0$ then $\alpha \cdot \xi_{X_H^c} = 0$.*

Proof. Suppose that $\alpha \in \mathrm{Gr}_i^W H^i(M_\Gamma^*) \subset \mathrm{Gr}_i^W H^i(\mathcal{M}^*)$ is such that $\mathrm{Res}(\alpha) = 0$. Let $g \in G(\mathbb{Q})$ and choose Γ' normal in Γ with $\Gamma' \subset \Gamma \cap g^{-1}\Gamma g$. Let $\gamma_1, \dots, \gamma_r$ be the representatives for cosets

of Γ' in Γ , let $p : M_{\Gamma'}^* \rightarrow M_{\Gamma}^*$, and let $\iota' : M_{H, \Gamma'_H}^* \rightarrow M_{\Gamma'}^*$ be the natural map at level Γ' . Then we have

$$p^{-1}(\iota(M_{H, \Gamma_H}^*)) = \bigcup_i \gamma_i \cdot \iota'(M_{H, \Gamma'_H}^*).$$

If $\text{Res}(\alpha) = 0$ for $\alpha \in \text{Gr}_i^W \text{H}^i(M_{\Gamma}^*)$ then $(g\gamma_i^{-1})^* \alpha = (\gamma_i^{-1})^* g^* \alpha$ restricts to zero on $\iota'(M_{H, \Gamma'_H}^*)$ for each i , i.e. $\iota'^*((\gamma_i^{-1})^* g^* \alpha) = 0$ for each i . Using (2.3) we have

$$\begin{aligned} 0 &= \iota'_* \left(\iota'^*((\gamma_i^{-1})^* g^* \alpha) \cap [M_{H, \Gamma'_H}^*] \right) \\ &= (\gamma_i^{-1})^* g^* \alpha \cap \iota'_*([M_{H, \Gamma'_H}^*]) \\ &= [M_{\Gamma'}^*] \cap ((\gamma_i^{-1})^* g^* \alpha \cdot \xi_{\Gamma'}) \\ &= [M_{\Gamma'}^*] \cap (g^* \alpha \cdot \gamma_i^* \xi_{\Gamma'}). \end{aligned} \tag{2.15}$$

By (2) of Proposition 2.3 we have that $g^* \alpha \cdot \gamma_i^* \xi_{\Gamma'} = 0$. Summing over Γ/Γ' gives that

$$0 = g^* \alpha \cdot \xi_{\Gamma} = \alpha \cdot (g^{-1})^* \xi_{\Gamma}.$$

Since this holds for all $g \in G(\mathbb{Q})$, Proposition 2.10 implies that $\alpha \cdot \xi_{X_H^c} = 0$. \square

Proof of Theorem 2.1. In the $SU(1, m) \subset SU(1, n)$ case $X^c = \mathbb{P}^n$, so that $\xi_{X_H^c} \sim (c_1^{bb})^{n-m}$, where c_1^{bb} is the first Chern class of the ample Baily-Borel line bundle in (2.12). So $\cdot c_1^{bb}$ is injective in degrees $< n$ on $\bigoplus_i \text{Gr}_i^W \text{H}^i(M_{\Gamma}^*) \subset \text{IH}^*(M_{\Gamma}^*)$ because of the hard Lefschetz property for intersection cohomology [BBD, Sai90]), and hence $\cdot \xi_{X_H^c}$ is injective in degrees $i \leq m$. Theorem 2.11 implies the injectivity of Res in degrees $\leq m$.

In the $SO(2, m) \subset SO(2, n)$ case the previous argument can be applied to the simply connected covers $\tilde{H} \subset \tilde{G}$. Now we claim $\xi_{X_H^c} \sim (c_1^{bb})^{n-m}$. If $n \neq 2m$ this is clear since $\text{Gr}_{2(n-m)}^W \text{H}^{2(n-m)}(M_{\Gamma}^*) = \mathbb{C} (c_1^{bb})^{n-m}$, while if $n = 2m$ then it holds because $c_m|_{X_H^c} = 0$ (see Example 2.9). It follows that $\cdot \xi_{X_H^c}$ is injective on $\bigoplus_i \text{Gr}_i^W \text{H}^i(M_{\Gamma}^*)$ in degrees $\leq m$ because of the hard Lefschetz property of c_1^{bb} on $\text{IH}^*(M_{\Gamma}^*)$. Theorem 2.11 implies the injectivity of Res in degrees $\leq m$. \square

Proof of Corollary 2.2. The complex hyperbolic case follows easily from the observation that

$$\text{H}_1^k(M_{\Gamma}) = \text{Gr}_k^W \text{H}_c^k(M_{\Gamma}) \quad \text{for } k \leq n$$

(see the proof of [Nai17b, Proposition 1.6]) and the fact that $\text{Gr}_k^W \text{H}_c^k(M_{\Gamma}) \subset \text{Gr}_k^W \text{H}^k(M_{\Gamma}^*)$ for all k (Remark 2.4).

Now consider the orthogonal case. First note that

$$\text{H}_1^i(M_{\Gamma}) = \text{im}(\text{H}_c^i(M_{\Gamma}) \rightarrow \text{IH}^i(M_{\Gamma}^*)) = \text{Gr}_i^W \text{H}_c^i(M_{\Gamma}) \quad \text{for } i \leq n-1. \tag{2.16}$$

The first equality holds because $\text{IH}^i(M_{\Gamma}^*) \rightarrow \text{H}^i(M_{\Gamma})$ is injective for $i \leq n-1$ (and an isomorphism for $i \leq n-2$) because the boundary has dimension one, and the second holds because $\text{Gr}_i^W \text{H}_c^i(M_{\Gamma}) \hookrightarrow \text{IH}^i(M_{\Gamma}^*)$ for all i (see Remark 2.4). On the other hand,

$$\text{Gr}_i^W \text{H}_c^i(M_{\Gamma}) \subset \text{Gr}_i^W \text{H}^i(M_{\Gamma}^*)$$

for all i (by Remark 2.4 again). Thus $\text{H}_1^i(\mathcal{M}) \subset \text{Gr}_i^W \text{H}^i(\mathcal{M}^*)$ for $i \leq n-1$ and similarly for \mathcal{M}_H . So the corollary follows from Theorem 2.1 in degrees $\leq m-1$ in the case $rk_{\mathbb{Q}}(G) = 2$. If $rk_{\mathbb{Q}}(H) = 1$ then this can be improved slightly because the singularities of M_{H, Γ_H}^* are isolated and so $\text{H}_1^i(\mathcal{M}) \subset \text{Gr}_i^W \text{H}^i(\mathcal{M}^*)$ holds for $i = m$ also. \square

Remark 2.12. The use of the embedding $\text{H}^*(X^c) \subset \text{Gr}_*^W \text{H}^*(\mathcal{M}^*)$ (of Proposition 2.6) in the proof of Theorem 2.1 can be avoided in the unitary case and also in the orthogonal case (except possibly when $n = 2m$). As noted in Example 2.7, the first Chern class of the Baily-Borel bundle can be made sense of in $\text{Gr}_*^W \text{H}^*(\mathcal{M}^*)$ and the invariants in $\text{IH}^{2(n-m)}(M_{\Gamma}^*)$ are reduced to $\mathbb{C} (c_1^{bb})^{n-m}$. So the argument can be run with $\text{H}^*(X^c)$ replaced by the subring $\bigoplus_{0 \leq i \leq n} \mathbb{C} (c_1^{bb})^i$.

Remark 2.13. The arguments above can be modified to treat nontrivial coefficients, using the fact that the local system E on M_Γ underlies a pure polarizable variation of Hodge structure (see [LR91, §4] or for a more canonical approach in the context of Shimura varieties see [BuW04]). This allows us to use mixed Hodge modules and arguments with weights.

Remark 2.14. The criterion of [Ven01] in the compact case has been used in [Ber09] to prove a number of other results about restriction using computations in the compact dual. The analogues for the top weight quotient of $H^*(\mathcal{M}^*)$ in general then follow immediately using the criterion of Theorem 2.11 instead. It seems likely that (suitably formulated) they should extend to $H^*(\mathcal{M})$ using the methods of later sections.

The following example shows that these bounds can sometimes be improved on:

Example 2.15. Consider the case of $SO(2, 2) \subset SO(2, n)$ for $n \geq 3$, and assume $rk_{\mathbb{Q}}(H) = 2$, so that $H = SL(2) \times SL(2)$. Theorem 2.1 gives the injectivity of $\mathrm{Gr}_2^W H^2(\mathcal{M}^*) \rightarrow I_H^G \mathrm{Gr}_2^W H^2(\mathcal{M}_H^*)$. Since $H_1^2(M_\Gamma) = \mathrm{IH}^2(M_\Gamma^*) = \mathrm{Gr}_2^W H_c^2(M_\Gamma)$, the map $\mathrm{Gr}_2^W H^2(M_\Gamma^*) \rightarrow \mathrm{Gr}_2^W (M_{H, \Gamma_H}^*)$ factors as

$$\mathrm{Gr}_2^W H_c^2(M_\Gamma) \rightarrow \mathrm{Gr}_2^W H_c^2(M_{H, \Gamma_H}) \rightarrow \mathrm{Gr}_2^W H^2(M_{H, \Gamma_H}^*).$$

Now $M_{H, \Gamma_H} = X_1 \times X_2$ is a product of two modular curves, so $\mathrm{Gr}_2^W H_c^2(X_1 \times X_2) = \mathrm{Gr}_1^W H_c^1(X_1) \otimes \mathrm{Gr}_1^W H_c^1(X_2)$ injects into $\mathrm{Gr}_2^W H^2(X_1^* \times X_2^*)$ and so the second map is injective. It follows that Res is always injective on $H_1^2(\mathcal{M})$, improving Corollary 2.2 slightly.

Remark 2.16. In fact, when $m = 2$ and $n = 3$, Res is injective on all of $H^2(\mathcal{M})$ by a result of Weissauer [Wei88]. This is not covered by our results since $H_1^2(\mathcal{M})$ is a proper subspace of $H^2(\mathcal{M})$.

Remark 2.17. The basic idea of this section is that in the presence of some functoriality, semisimplicity, and duality, one can use the cycle class argument. This can also be applied to the reductive Borel-Serre (RBS) compactification, to get a slight generalization of Theorems 2.11 and 2.1.

The RBS compactification is a (nonalgebraic) compactification of M_Γ dominating M_Γ^* , i.e. the identity of M_Γ extends to $M_\Gamma^{rbs} \rightarrow M_\Gamma^*$. The cohomology $H^i(M_\Gamma^{rbs})$ carries a mixed Hodge structure like that of a proper variety, i.e. with weights $\leq i$ in degree i , and the top weight quotient $\mathrm{Gr}_i^W H^i(M_\Gamma^{rbs})$ is the image of a natural map $H^i(M_\Gamma^{rbs}) \rightarrow \mathrm{IH}^i(M_\Gamma^*)$. For $\iota : M_{H, \Gamma_H} \rightarrow M_\Gamma$ there is no continuous map $M_{H, \Gamma_H}^{rbs} \rightarrow M_\Gamma^{rbs}$ extending $M_{H, \Gamma_H} \rightarrow M_\Gamma$, but nevertheless there is a natural pullback map $H^*(M_\Gamma^{rbs}) \rightarrow H^*(M_{H, \Gamma_H}^{rbs})$, which is a homomorphism of mixed Hodge structures. (See e.g. the survey [Nai17c] where these results are discussed.) Theorem 2.1 can be improved to injectivity of the induced map Res on $\mathrm{Gr}_i^W H^i(\mathcal{M}^{rbs}) = \mathrm{colim}_\Gamma \mathrm{Gr}_i^W H^i(M_\Gamma^{rbs})$ for $i \leq m$. Since the canonical mapping $H^i(M_\Gamma^*) \rightarrow \mathrm{IH}^i(M_\Gamma^*)$ factors through $H^*(M_\Gamma^{rbs})$, it follows that $\mathrm{Gr}_i^W H^i(\mathcal{M}^*) \subset \mathrm{Gr}_i^W H^i(\mathcal{M}^{rbs})$. (However, in general $H^i(\mathcal{M}^*) \rightarrow H^*(\mathcal{M}^{rbs})$ is not injective.) It can be shown using methods of Eisenstein series that in the case at hand this inclusion is proper, so we would have an improvement of Theorem 2.1.

3. LIE ALGEBRA COHOMOLOGY COMPUTATIONS

3.1. Kostant's theorem. We recall results of [Kos61]. Fix a complex semisimple Lie group G , a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$, and let $\Phi = \Phi(T, G)$ be the root system, Φ^+ the positive roots determined by B , $\Phi^- = -\Phi^+$ the negative roots, $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ the half-sum of positive roots, and $W = W(T, G)$ the Weyl group of T in G . Let P be a standard parabolic subgroup of G , N its unipotent radical and \mathfrak{n} its Lie algebra. The Weyl group of the Levi $L = P/N$ is a subgroup $W_L \subset W$, and we let W^P be the set of minimal length coset representatives of $W_L \backslash W$. For each $w \in W^P$ the associated set of positive roots

$$\Phi(w) = \{\alpha \in \Phi(T, G) : \alpha > 0, w^{-1}\alpha < 0\} = \Phi^+ \cap w\Phi^-$$

which has cardinality $\ell(w)$. For a dominant $\lambda \in X^*(T)$ the weights $w(\lambda + \rho) - \rho$ for $w \in W^P$ are dominant for L and distinct. The mapping $w \mapsto \Phi(w)$ sets up a bijection between W^P and

the subsets S of $\Phi(\mathfrak{n}) = \{\alpha \in \Phi(T, G) : \mathfrak{g}^\alpha \subset \mathfrak{n}\}$ for which both S and $\Phi^+ - S$ are closed under \dagger ([Kos61, 5.10]; recall that $\alpha \dagger \beta$ is $\alpha + \beta$ when this is a root and empty otherwise).

Let E_λ be the irreducible finite-dimensional G -representation with highest weight $\lambda \in X^*(T)$ with respect to B . The Lie algebra cohomology $H^*(\mathfrak{n}, E_\lambda)$ is the cohomology of $\wedge^* \mathfrak{n}^* \otimes E_\lambda$ with the Lie algebra differential. The natural P -module structure on $\wedge^* \mathfrak{n}^* \otimes E_\lambda$ descends to an $L = P/N$ -module structure in cohomology. For an L -dominant weight $\mu \in X^*(T)$ let E_μ^L denote the irreducible finite-dimensional algebraic representation of L with highest weight μ . Then by [Kos61, Theorem 5.14] there is a multiplicity-free decomposition of L -modules

$$H^k(\mathfrak{n}, E_\lambda) = \bigoplus_{w \in W^P, \ell(w)=k} E_{w(\lambda+\rho)-\rho}^L. \quad (3.1)$$

Kostant also identified a highest weight vector in each summand above. Let \mathfrak{n}^- be the nilradical of the Lie algebra of the parabolic subgroup opposite to P . The Killing form gives isomorphisms $\mathfrak{n}^- \cong \mathfrak{n}^*$ and $\wedge^i \mathfrak{n}^- \cong \wedge^i \mathfrak{n}^*$. Choose a nonzero vector e_α in the root space \mathfrak{g}^α for each $\alpha \in \Phi(T, G_\mathbb{C})$, and for $w \in W^P$ let $e_w := \bigwedge_{\alpha \in \Phi(w)} e_{-\alpha} \in \wedge^{\ell(w)} \mathfrak{n}^-$. Let $v_{w\lambda} \in E_\lambda$ be a weight vector for the extremal weight $w\lambda$. Then under the identification of \mathfrak{n}^* with \mathfrak{n}^- , the element

$$e_w \otimes v_{w\lambda} \in \wedge^{\ell(w)} \mathfrak{n}^- \otimes E_\lambda \quad (3.2)$$

is closed in $\wedge^* \mathfrak{n}^* \otimes E_\lambda$ and its cohomology class is a highest weight vector for the summand $E_{w(\lambda+\rho)-\rho}^L$ in (3.1) (see [Kos61, Theorem 5.14]); it is a harmonic representative for a natural Laplacian. A lowest weight vector is given by $\bigwedge_{\alpha \in \Phi(w)} e_{-w_0^L(\alpha)} \otimes v_{w_0^L w\lambda}$, where w_0^L is the longest element of $W_L \subset W$. (See [Kos61, Remark 8.2].) In fact, taking the sum of the L -submodules of $\wedge^* \mathfrak{n}^- \otimes E_\lambda$ generated by the $e_w \otimes v_{w\lambda}$ as w runs over W^P gives (using the identification $\wedge^* \mathfrak{n}^- \cong \wedge^* \mathfrak{n}^*$) a canonical L -equivariant inclusion $H^*(\mathfrak{n}, E_\lambda) \subset \wedge^* \mathfrak{n}^* \otimes E_\lambda$ inducing the identity in cohomology and compatible with products (see [Kos61, Theorem 5.7]).

3.2. Restriction maps in \mathfrak{n} -cohomology. Now consider the situation where $\iota : H \rightarrow G$ is a homomorphism of real semisimple groups with finite kernel. Then for a parabolic P of G with Levi $L = P/N$, we have the parabolic $P_H = \iota^{-1}(P)$ of H with unipotent radical $N_H = \iota^{-1}(N)$ and Levi $L_H = P_H/N_H$. Let $\mathfrak{n} = \text{Lie } N(\mathbb{R})$ and $\mathfrak{n}_H = \text{Lie } N_H(\mathbb{R})$. For a finite-dimensional $G(\mathbb{C})$ -representation E and E_H a summand of $E|_{H(\mathbb{C})}$, the restriction map

$$H^*(\mathfrak{n}_\mathbb{C}, E) \longrightarrow H^*(\mathfrak{n}_{H,\mathbb{C}}, E) \longrightarrow H^*(\mathfrak{n}_{H,\mathbb{C}}, E_H)$$

is $L_H(\mathbb{C})$ -equivariant. Consider the map

$$Res_{\mathfrak{n}} : H^*(\mathfrak{n}_\mathbb{C}, E) \longrightarrow \prod_{m \in L(\mathbb{C})} H^*(\mathfrak{n}_{H,\mathbb{C}}, E_H)$$

with coordinate indexed by $m \in L(\mathbb{C})$ given by precomposing the previous map with the adjoint action of m . Note that the kernel of $Res_{\mathfrak{n}}$ is an $L(\mathbb{C})$ -module; in particular, for each irreducible summand $E_{w(\lambda+\rho)-\rho}^L$, we have that $Res_{\mathfrak{n}}$ is injective on $E_{w(\lambda+\rho)-\rho}^L \iff Res_{\mathfrak{n}}$ is nonzero on $E_{w(\lambda+\rho)-\rho}^L \iff Res_{\mathfrak{n}}(e_w \otimes v_{w\lambda}) \neq 0$.

Let us assume that the restriction of finite-dimensional irreducible representations by $H(\mathbb{C}) \subset G(\mathbb{C})$ is multiplicity-free. Choose maximal tori $T_H \subset T$ and Borel subgroups $B_H \subset B$, and for an irreducible representation E with highest weight $\lambda \in X^*(T)$ let E_H be the summand of $E|_{H(\mathbb{C})}$ with highest weight $\lambda|_{T_H} \in X^*(T_H)$. The following propositions are proved by explicit elementary calculations with roots and weights using Kostant's theorem and take up the rest of this section. In each case the restriction from $G(\mathbb{C})$ to $H(\mathbb{C})$ is multiplicity-free by classical results ([GW09, 8.1.1]).

Proposition 3.1. *Let $G = SO(d, 1)$ and $H = SO(c, 1)$ for $2 \leq c < d$ embedded in the standard way in G . Let $P = LN$ be a proper parabolic subgroup of G . Then $Res_{\mathfrak{n}}$ is injective in degrees $i \leq c/2$ except in the case $(d, c, i) = (2k + 1, 2k, k)$. In this case $H^k(\mathfrak{n}, E)$ has two L -irreducible summands and $Res_{\mathfrak{n}}$ is injective on either one.*

Proposition 3.2. *Let $G = SU(n, 1)$ and $H = SU(m, 1)$ for $2 \leq m < n$ embedded in the standard way in G . Let $P = MW$ be a proper parabolic subgroup of G . Then $Res_{\mathfrak{w}}$ is injective on $H^i(\mathfrak{w}, E)$ in degrees $i < m$.*

Proposition 3.3. *Let $G = SO(2, n)$ and $H = SO(2, m)$, embedded in G in the standard way for $2 \leq m < n$, and $E = \mathbb{C}$. If $P = MW$ is the stabilizer of an isotropic plane then $Res_{\mathfrak{w}}$ is injective on $H^i(\mathfrak{w}, \mathbb{C})$ for $i \leq m - 2$.*

To treat $SO(1, n) \subset SU(1, n)$ we will need:

Proposition 3.4. *Let $G = SU(n, 1)$ and $H = SO(n, 1)$ embedded in the standard way in G with $n \geq 2$. Let $P = MW$ be a proper parabolic subgroup of G . Then $Res_{\mathfrak{w}}$ is injective on $H^{i,0}(\mathfrak{w}, \mathbb{C})$ in degrees $i < n$.*

The bigrading in Proposition 3.4 refers to the Hodge structure coming from the identification of $H^i(\mathfrak{w}, \mathbb{C})$ with the link cohomology $H^i(i_{\{x\}}^* j_* \mathbb{C})$, where $j : M_\Gamma \hookrightarrow M_\Gamma^*$ and $i_{\{x\}} : \{x\} \hookrightarrow M_\Gamma^*$ is the inclusion of the cusp corresponding to P (see Proposition 1.7 or [Nai17b, Lemma 1.2].) It is also seen from the decomposition (3.1) (see [Nai17b, Remark 1.11]).

The rest of this section will be taken up with the proofs of these propositions.

3.3. Proof of Proposition 3.3. To make computations we will fix some notation for roots. We may assume $\mathfrak{g} = \mathfrak{so}(2, n) = \mathfrak{so}(J)$ where

$$J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & I_{n-2} & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

Fix a Cartan subalgebra \mathfrak{s} of $\mathfrak{so}(n-2)_\mathbb{C}$ and let $\mathfrak{t} \subset \mathfrak{g}_\mathbb{C}$ be the Cartan subalgebra defined by

$$\mathfrak{t} := \{diag(a, d, C, -d, -a) : a, d \in \mathbb{C}, C \in \mathfrak{s}\}.$$

Then \mathfrak{t} is defined and maximally split over \mathbb{R} and the subspace $\mathfrak{a}_\mathbb{C} \subset \mathfrak{t}$ given by $C = 0$ is the complexification of the Lie algebra $\mathfrak{a} \subset \mathfrak{g}$ of a maximal \mathbb{R} -split Cartan in \mathfrak{g} . Let $\alpha_1, \alpha_2 \in \mathfrak{t}^*$ be defined by

$$\begin{aligned} \alpha_1(diag(a, d, C, -d, -a)) &= a \\ \alpha_2(diag(a, d, C, -d, -a)) &= d \end{aligned} \tag{3.3}$$

The relative roots are $\Phi(\mathfrak{a}_\mathbb{C}, \mathfrak{g}_\mathbb{C}) = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 - \alpha_2), \pm(\alpha_1 + \alpha_2)\}$.

Now choose for \mathfrak{s} the Cartan subalgebra of block-diagonal matrices

$$\mathfrak{s} = \left\{diag \left(\begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_k \\ -b_k & 0 \end{pmatrix} \right) : b_1, \dots, b_k \in \mathbb{C} \right\} \tag{3.4}$$

when $n - 2 = 2k$ is even, and

$$\mathfrak{s} = \left\{diag \left(\begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_k \\ -b_k & 0 \end{pmatrix}, 0 \right) : b_1, \dots, b_k \in \mathbb{C} \right\} \tag{3.5}$$

when $n - 2 = 2k + 1$ is odd. Let $\eta_1, \dots, \eta_k \in \mathfrak{s}^*$ be defined by

$$\eta_i \left(diag \left(\begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_k \\ -b_k & 0 \end{pmatrix}, (0) \right) \right) = \sqrt{-1} b_i \tag{3.6}$$

where (0) means the entry is omitted when $n - 2$ is even and the entry is zero when $n - 2$ is odd. Fix the positive system $\Phi^+(\mathfrak{s}, \mathfrak{so}(n-2)_\mathbb{C})$ with simple roots $\{\eta_i - \eta_{i+1} : 1 \leq i < k\} \sqcup \{\eta_k\}$ for $n - 2 = 2k + 1$ odd and $\{\eta_i - \eta_{i+1} : 1 \leq i < k\} \sqcup \{\eta_k + \eta_{k-1}\}$ for $n - 2 = 2k$ even, and take the positive system in $\Phi(\mathfrak{t}, \mathfrak{g}_\mathbb{C})$ containing it and compatible with $\Phi^+(\mathfrak{a}_\mathbb{C}, \mathfrak{g}_\mathbb{C}) = \{\alpha_1, \alpha_2, \alpha_1 - \alpha_2, \alpha_1 + \alpha_2\}$. An explicit computation shows that the positive roots are

$$\Phi^+(\mathfrak{t}, \mathfrak{g}_\mathbb{C}) = \Phi^+(\mathfrak{s}, \mathfrak{so}(n-2)_\mathbb{C}) \sqcup \{\alpha_1 \pm \eta_i, \alpha_2 \pm \eta_i : 1 \leq i \leq k\} \sqcup \{\alpha_1 - \alpha_2, \alpha_1 + \alpha_2\}$$

for $n - 2 = 2k$ even, and

$$\Phi^+(\mathfrak{t}, \mathfrak{g}_\mathbb{C}) = \Phi^+(\mathfrak{s}, \mathfrak{so}(n-2)_\mathbb{C}) \sqcup \{\alpha_1 \pm \eta_i, \alpha_2 \pm \eta_i : 1 \leq i \leq k\} \sqcup \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 - \alpha_2\}$$

for $n - 2 = 2k + 1$ odd.

The parabolic $P = MW$ in the proposition is the stabilizer of an isotropic plane, which may be assumed to be the obvious plane $\mathbb{R}e_1 + \mathbb{R}e_2$ in \mathbb{R}^{n+2} . We will need the set $\Phi(\mathfrak{w}) = \{\alpha \in \Phi(\mathfrak{t}, \mathfrak{g}_{\mathbb{C}}) : \mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{w}_{\mathbb{C}}\}$. Using the description above and explicit matrix descriptions we have

$$\Phi(\mathfrak{w}) = \{\alpha_1 \pm \eta_i\}_{1 \leq i \leq k} \sqcup \{\alpha_2 \pm \eta_i\}_{1 \leq i \leq k} \sqcup \{\alpha_1 + \alpha_2\}$$

if $n - 2 = 2k$ is even and

$$\Phi(\mathfrak{w}) = \{\alpha_1 \pm \eta_i\}_{1 \leq i \leq k} \sqcup \{\alpha_2 \pm \eta_i\}_{1 \leq i \leq k} \sqcup \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$$

if $n - 2 = 2k + 1$ is odd.

Now let us prove Proposition 3.3. We may assume that $\mathfrak{h} \subset \mathfrak{g}$ is given by the subspace $\mathbb{R}^{m+2} = (\mathbb{R}e_{m+1} + \cdots + \mathbb{R}e_n)^{\perp} \subset \mathbb{R}^{n+2}$, i.e. that $\mathfrak{h} = \mathfrak{so}(2, m)$ is embedded in $\mathfrak{g} = \mathfrak{so}(2, n)$ in a way that the $m + 1, m + 2, \dots, n$ rows and columns are zero. We will consider the cases n even and n odd separately.

First assume $n - 2 = 2k$ is even. Then we have

$$\Phi(\mathfrak{w}) = \{\alpha_1 \pm \eta_i\}_{1 \leq i \leq k} \sqcup \{\alpha_2 \pm \eta_i\}_{1 \leq i \leq k} \sqcup \{\alpha_1 + \alpha_2\}.$$

For $w \in W^P$ of length $\leq m - 2 = 2k - (n - m)$ the set of roots $\Phi(w) \subset \Phi(\mathfrak{w})$ has cardinality $\leq 2k - (n - m)$, so that $\Phi^+ - \Phi(w)$ contains at least $n - m$ elements which belong to $\{\alpha_2 \pm \eta_i\}_{1 \leq i \leq k}$, which has cardinality $2k$. Since $\Phi^+ - \Phi(w)$ is closed under \dagger and $\alpha_1 - \alpha_2 \in \Phi^+ - \Phi(w)$, we get that we may choose sets I_+ and I_- in $\{1, \dots, k\}$ such that

- (1) I_+ and I_- are disjoint and $|I_+ \sqcup I_-| = n - m$
- (2) for $i \in I_+$ we have $\{\alpha_1 + \eta_i, \alpha_2 + \eta_i\} \subset \Phi^+ - \Phi(w)$
- (3) for $i \in I_-$ we have $\{\alpha_1 - \eta_i, \alpha_2 - \eta_i\} \subset \Phi^+ - \Phi(w)$.

(These sets are not unique, but any choice suffices for our purposes.) Let \mathfrak{h}' be the copy of $\mathfrak{so}(2, m)$ given by the embedding of the subspace

$$\left(\bigoplus_{i \in I_+} \mathbb{R}e_{2i+2} \oplus \bigoplus_{i \in I_-} \mathbb{R}e_{2i+1} \right)^{\perp} \subset \mathbb{R}^{n+2}$$

i.e. the $2i + 2$ nd row and column are zero for $i \in I_+$ and the $2i + 1$ st row and column are zero for $i \in I_-$. Then the restriction of the harmonic representative $e_w = \wedge_{\alpha \in \Phi(w)} e_{-\alpha}$ in (3.2) to $\mathfrak{n}_{H'}$ is nonzero, and equals (up to a nonzero scalar) the harmonic representative of a similar class in $\mathfrak{n}_{H'}$. Since the subspace above is conjugate to the subspace $(\mathbb{R}e_{m+1} + \cdots + \mathbb{R}e_n)^{\perp}$ by an element $m \in M(\mathbb{C}) = Spin(n - 2, \mathbb{C})$, we see that \mathfrak{h}' is conjugate to \mathfrak{h} by $m \in M(\mathbb{C})$, and the highest weight vector e_w restricts nontrivially to $Ad(m^{-1})(\mathfrak{h})$. This proves the proposition in this case.

Next assume $n - 2 = 2k + 1$ is odd. Then

$$\Phi(\mathfrak{w}) = \{\alpha_1 \pm \eta_i\}_{1 \leq i \leq k} \sqcup \{\alpha_2 \pm \eta_i\}_{1 \leq i \leq k} \sqcup \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}.$$

For $w \in W^P$ of length $\leq m - 2 = 2k + 1 - (n - m)$ the set of roots $\Phi(w) \subset \Phi(\mathfrak{w})$ has cardinality $\leq 2k + 1 - (n - m)$, so that $\Phi^+ - \Phi(w)$ contains at least $n - m$ elements from the set $\{\alpha_2 \pm \eta_i : 1 \leq i \leq k\} \sqcup \{\alpha_2\}$ of cardinality $2k + 1$. As in the previous case $\alpha_1 - \alpha_2 \in \Phi^+ - \Phi(w)$, and $\Phi(w)$ is closed under \dagger , so we may choose (not necessarily unique) sets of indices I_+ and I_- such that

- (1) I_+ and I_- are disjoint and $|I_+ \sqcup I_-| = n - m - 1$ if $\alpha_2 \in \Phi^+ - \Phi(w)$ and $|I_+ \sqcup I_-| = n - m$ if $\alpha_2 \notin \Phi^+ - \Phi(w)$
- (2) for $i \in I_+$ we have $\{\alpha_1 + \eta_i, \alpha_2 + \eta_i\} \subset \Phi^+ - \Phi(w)$
- (3) for $i \in I_-$ we have $\{\alpha_1 - \eta_i, \alpha_2 - \eta_i\} \subset \Phi^+ - \Phi(w)$.

As before, if $\alpha_2 \notin \Phi^+ - \Phi(w)$ we can define the subspace

$$\left(\bigoplus_{i \in I_+} \mathbb{R}e_{2i+2} \oplus \bigoplus_{i \in I_-} \mathbb{R}e_{2i+1} \right)^{\perp} \subset \mathbb{R}^{n+2}$$

of dimension m and the harmonic representative e_w restricts nontrivially to the corresponding $\mathfrak{h}' = \mathfrak{so}(2, m)$ in \mathfrak{g} . If $\alpha_2 \in \Phi^+ - \Phi(w)$ then one adds on $\mathbb{R}e_n$ to the subspace above and e_2 restricts nontrivially to the corresponding $\mathfrak{h}' = \mathfrak{so}(2, m)$. In either case since \mathfrak{h}' is conjugate to \mathfrak{h} by $m \in M(\mathbb{C})$, we have proved the proposition. \square

3.4. Proof of Proposition 3.1. For $E = \mathbb{C}$ the injectivity in degrees $i \leq c/2$ except in the exceptional case is immediate from the fact that \mathfrak{n}^* is the natural representation of the factor $SO(d-1)$ of L and so $H^i(\mathfrak{n}, \mathbb{C}) = \wedge^i \mathfrak{n}^*$ are irreducible. This can be easily generalized to the case of general E , but we give a computational proof using Kostant's theorem as we will have to verify slightly more.

We may assume $\mathfrak{g} = \mathfrak{so}(1, d) = \mathfrak{so}(J)$ where

$$J = \begin{pmatrix} & & 1 \\ & I_{d-1} & \\ 1 & & \end{pmatrix}.$$

Fix a Cartan subalgebra \mathfrak{s} of $\mathfrak{so}(d-2)_{\mathbb{C}}$ and let $\mathfrak{t} \subset \mathfrak{g}_{\mathbb{C}}$ be the Cartan subalgebra defined by

$$\mathfrak{t} := \{\text{diag}(a, C, -a) : a, d \in \mathbb{C}, C \in \mathfrak{s}\}.$$

Then \mathfrak{t} is defined and maximally split over \mathbb{R} and the subspace $\mathfrak{a}_{\mathbb{C}} \subset \mathfrak{t}$ given by $C = 0$ is the complexification of the Lie algebra $\mathfrak{a} \subset \mathfrak{g}$ of a maximal \mathbb{R} -split subspace in \mathfrak{g} . Let $\alpha \in \mathfrak{t}^*$ be defined by

$$\alpha(\text{diag}(a, C, -a)) = a. \quad (3.7)$$

We choose for \mathfrak{s} the same Cartan subalgebra of block-diagonal matrices in $\mathfrak{so}(d-1)_{\mathbb{C}}$ specified earlier in (3.4) and (3.5) and use the same roots η_i and the same positive system used there. An explicit computation shows that the positive roots are

$$\Phi^+(\mathfrak{t}, \mathfrak{g}_{\mathbb{C}}) = \Phi^+(\mathfrak{s}, \mathfrak{so}(d-1)_{\mathbb{C}}) \sqcup \{\alpha \pm \eta_i : 1 \leq i \leq k\}$$

for $d-1 = 2k$ even, and

$$\Phi^+(\mathfrak{t}, \mathfrak{g}_{\mathbb{C}}) = \Phi^+(\mathfrak{s}, \mathfrak{so}(d-1)_{\mathbb{C}}) \sqcup \{\alpha \pm \eta_i : 1 \leq i \leq k\} \sqcup \{\alpha\}$$

for $d-1 = 2k+1$ odd. We also have

$$\Phi(\mathfrak{n}) = \{\alpha \pm \eta_i : 1 \leq i \leq k\}$$

in the case $d-1 = 2k$ even and

$$\Phi(\mathfrak{n}) = \{\alpha \pm \eta_i : 1 \leq i \leq k\} \sqcup \{\alpha\}$$

in the case $d-1 = 2k+1$ odd.

We will list the relevant $w \in W^P$ and the sets $\Phi(w)$. We will consider the even and odd cases separately.

First assume $d-1 = 2k$ is even. Let $\alpha_1, \alpha_2, \dots, \alpha_{k+1}$ be the set of simple roots of $\mathfrak{so}(1, d)_{\mathbb{C}}$ determined by the positive system fixed above, i.e. $\alpha_1 := \alpha - \eta_1$, $\alpha_i = \eta_{i-1} - \eta_i$ for $2 \leq i \leq k$ and $\alpha_{k+1} = \eta_{k-1} + \eta_k$. The minimal length representatives in W^P of length $\leq k$ are

$$\{s_0, s_1, \dots, s_k, t_k\}$$

where $s_0 := 1$, $s_j := s_{\alpha_1} \cdots s_{\alpha_j}$ for $1 \leq j \leq k$ has length j , and $t_k = s_{k-1} s_{\alpha_{k+1}}$ has length k (cf. [BW99, VI.3.1], as usual, s_{α_i} denotes the reflection in α_i). The set $\Phi(w)$ is easily computed for these representatives: $\Phi(1) = \emptyset$ and $\Phi(s_j) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_j\} = \{\alpha - \eta_1, \alpha_1 - \eta_2, \dots, \alpha - \eta_j\}$ for $j \leq k$, while $\Phi(t_k) = \{\alpha_1, \dots, \alpha_1 + \cdots + \alpha_{k-1}, \alpha_1 + \cdots + \alpha_{k-1} + \alpha_{k+1}\} = \{\alpha - \eta_1, \alpha_1 - \eta_2, \dots, \alpha - \eta_{k-1}, \alpha + \eta_k\}$.

Now suppose $d-1 = 2k+1$ is odd. Let $\alpha_1, \alpha_2, \dots, \alpha_{k+1}$ be the simple roots of $\mathfrak{so}(1, d)_{\mathbb{C}}$ determined by the positive system, i.e. $\alpha_1 := \alpha - \eta_1$, $\alpha_i = \eta_{i-1} - \eta_i$ for $2 \leq i \leq k$ and $\alpha_{k+1} = \eta_k$. The minimal length representatives in W^P of length $\leq k$ are

$$\{s_0, s_1, \dots, s_k\}$$

where $s_0 := 1$, $s_j := s_{\alpha_1} \cdots s_{\alpha_j}$ for $1 \leq j \leq k$ has length j (cf. [BW99, VI.4.4], where this set is denoted ${}^P W$). The set $\Phi(s_j)$ has the same description for $j \leq k$ as above.

Now consider the setup of the proposition. So $\mathfrak{h} = \mathfrak{so}(1, c)$ for $c \leq d - 1$, embedded in the standard way, i.e. using the subspace $\mathbb{R}^{c+1} \subset \mathbb{R}^{d+1}$ spanned by $e_1, e_2, \dots, e_c, e_{d+1}$. To show that $Res_{\mathfrak{n}}$ is injective in a given degree i it will suffice to show that the harmonic representative (i.e. L -highest weight vector) $e_w \otimes v_{w\lambda}$ restricts nontrivially in $H^i(\mathfrak{n}_H, E_H)$ for each $w \in W^P$ of length i . In the case at hand for $j < c/2$ there is a unique element in W^P of length j , namely s_j . The L -highest-weight vector $e_{s_j} \otimes v_{s_j\lambda} = \bigwedge_{1 \leq i \leq j} e_{-(\alpha - \eta_i)} \otimes v_{s_j\lambda}$ maps in $\wedge^j \mathfrak{n}_H^* \otimes E_H$ to a (nonzero multiple of) the harmonic representative $\bigwedge_{1 \leq i \leq j} e_{-(\alpha - \eta_i)}^H \otimes v_{s_j^H \lambda_H}^H$, where $v_{s_j^H \lambda_H}^H$ is the $s_j^H(\lambda_H)$ -weight vector of E_H , hence is nonvanishing in cohomology. This proves that $Res_{\mathfrak{n}}(e_w \otimes v_{w\lambda}) \neq 0$, and hence that $Res_{\mathfrak{n}}$ is injective for $j < c/2$. The same proof works if $j \leq c/2$ as long as we are not in the exceptional case $(d, c, i) = (2k + 1, 2k, k)$.

In the remaining case we are considering $Res_{\mathfrak{n}}$ on $H^k(\mathfrak{n}, E)$ for $d = 2k + 1, c = 2k, i = k$. In this case there are two L -irreducible summands with highest weight vectors $e_{s_k} \otimes v_{s_k\lambda}$ and $e_{t_k} \otimes v_{t_k\lambda}$ respectively. In the embedding $\mathfrak{so}(2k + 1, \mathbb{C}) \subset \mathfrak{so}(2k + 2, \mathbb{C})$ the weight space $\mathfrak{h}^{-\alpha_k}$ is embedded diagonally in the weight spaces $\mathfrak{g}^{-\alpha_k}$ and $\mathfrak{g}^{-\alpha_{k+1}}$. Under the restriction from T to T_H we have $\alpha_k|_{T_H} = \alpha_{k+1}|_{T_H} = \eta_{k-1}^H = \alpha_k^H$. Thus the vector $e_{s_k} \otimes v_{s_k\lambda} = \bigwedge_{1 \leq i \leq k} e_{-(\alpha - \eta_i)} \otimes v_{s_k\lambda}$ goes to (a nonzero multiple of) the vector $\bigwedge_{1 \leq i \leq k-1} e_{-(\alpha - \eta_i)}^H \wedge e_{-\eta_{k-1}}^H \otimes v_{s_k\lambda}^H$ which is nonzero since η_{k-1} is not one of $\alpha - \eta_i, i \leq k - 1$. A similar argument applies to $e_{t_k} \otimes v_{t_k\lambda}$. \square

3.5. Proof of Proposition 3.2. This was proved for $E = \mathbb{C}$ in [Nai17b, §1.6] and the elements of W^P were explicitly listed there. The proof extends to general coefficients exactly as in the previous proof. \square

3.6. Proof of Proposition 3.4. A tedious computational proof is possible, but we will argue differently. Recall the notation $P = MW$ for the parabolic in $SU(1, n)$ and $P_H = P \cap H = LN$ for the parabolic in $SO(1, n)$. For $k < n$ we have a diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{u}_{\mathbb{C}}^* \otimes \wedge^{k-2} \mathfrak{v}_{\mathbb{C}}^* & \longrightarrow & \wedge^k \mathfrak{v}_{\mathbb{C}}^* & \longrightarrow & H^k(\mathfrak{w}_{\mathbb{C}}, \mathbb{C}) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \wedge^k \mathfrak{n}_{\mathbb{C}}^* & = & H^k(\mathfrak{n}_{\mathbb{C}}, \mathbb{C}) \end{array} \quad (3.8)$$

The second vertical map is induced by $\mathfrak{n} \subset \mathfrak{w}$. The first vertical map is induced by the identification of \mathfrak{v} with the χ -eigenspace for A in \mathfrak{w} ; since $A \subset P_H$ and it acts by χ on \mathfrak{u} we have $\mathfrak{n} \subset \mathfrak{v}$ (cf. 1.2 for notation). The top row comes from the long exact sequence for the boundary divisor in the toroidal compactification, and is exact in degrees $k < n$ (see [Nai17b, Lemma 1.3]). It is also a sequence of Hodge structures (see loc. cit.). The bigrading on $\wedge^k \mathfrak{v}_{\mathbb{C}}^*$ comes from $\mathfrak{v}_{\mathbb{C}}^* = (\mathfrak{v}_{\mathbb{C}}^*)^{1,0} + (\mathfrak{v}_{\mathbb{C}}^*)^{0,1}$ given by the complex structure on $\mathfrak{v} = \mathfrak{w}/\mathfrak{u}$ (given by the central $U(1) \subset M(\mathbb{R})$) and the $(k, 0)$ -subspace $\wedge^k (\mathfrak{v}_{\mathbb{C}}^*)^{1,0}$ maps isomorphically onto $H^{k,0}(\mathfrak{w}, \mathbb{C})$. This is because the first term in the sequence has Hodge types $(k - 1, 1), \dots, (1, k - 1)$ because \mathfrak{u}^* amounts to a Tate twist (see the proof of [Nai17b, Lemma 1.3]). Now the composition $(\mathfrak{v}_{\mathbb{C}}^*)^{1,0} \hookrightarrow \mathfrak{v}_{\mathbb{C}}^* \rightarrow \mathfrak{n}_{\mathbb{C}}^*$ is an isomorphism since the kernel of $\mathfrak{v}_{\mathbb{C}}^* \rightarrow \mathfrak{n}_{\mathbb{C}}^*$ is the complexification of a real subspace, hence does not meet $(\mathfrak{v}_{\mathbb{C}}^*)^{1,0}$. Thus $\wedge^k (\mathfrak{v}_{\mathbb{C}}^*)^{1,0} \rightarrow \wedge^k \mathfrak{n}_{\mathbb{C}}^*$ is nonzero and since $H^{k,0}(\mathfrak{w}, \mathbb{C})$ is M -irreducible (see [Nai17b, Remark 1.11]), $Res_{\mathfrak{w}}$ is injective on $H^{k,0}(\mathfrak{w}_{\mathbb{C}}, \mathbb{C})$ for $k < n$. \square

4. CONGRUENCE REAL HYPERBOLIC MANIFOLDS

We restate the main result Theorem 1 for congruence hyperbolic manifolds:

Theorem 4.1. *Suppose that $H \subset G$ are semisimple groups of the same \mathbb{Q} -rank such that $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SO(1, c) \subset SO(1, d)$ for $2 \leq c < d$, and that neither H nor G is triality. Then $Res : H^i(\mathcal{M}_G, E) \rightarrow I_H^G H^i(\mathcal{M}_H, E_H)$ is injective for $i \leq c/2$.*

In the compact case (and $E = \mathbb{C}$) this is Theorem 1.5 of [BeC13] of Bergeron and Clozel, who, using the Burger-Sarnak method [BS91] following Harris and Li [HL98], deduced it from Arthur's endoscopic classification [Art13] of automorphic representations for orthogonal groups. Their arguments can be adapted to the noncompact case to prove injectivity on the interior cohomology $H_1^i(\mathcal{M}, E)$ for $i \leq c/2$. Since the state of the literature on this adaptation is less than satisfactory, we will sketch the argument in some detail below, although the ingredients are all well known. Combined with an elementary argument at infinity using Proposition 3.1 from the previous section, this proves the theorem in general. Before starting the proofs we will need to recall some general facts.

Recall the classification of unitarizable (\mathfrak{g}, K) -modules with cohomology with coefficients in E for $G(\mathbb{R}) = SO(d, 1)$ from [BW99, VI.4] or [RS87, 1.3]. (We refer to [BW99] for more details; this reference deals with $SO_0(d, 1)$ and $E = \mathbb{C}$ but it is easy to extend to the general case using e.g. translation functors as in [BW99, VI.0].) Let $0 \leq i_E \leq \lfloor d/2 \rfloor$ be the minimal degree for which there exists a unitary representation V with $H^*(\mathfrak{g}, K, V \otimes E) \neq \{0\}$ (so $i_{\mathbb{C}} = 0$ and $i_E = \lfloor d/2 \rfloor$ if λ is regular). For each degree $i_E \leq i \leq \lfloor d/2 \rfloor$ there is a unique unitary cohomological representation π_i with cohomology in degree i with respect to E , and it has cohomology in exactly degrees i and $d - i$ if d is odd or $i < d/2$ and if $d = 2k$ it has cohomology in degree k . In the case $d = 2k$ the representation π_k of $SO(1, d)$ is a discrete series representation and its restriction to $SO_0(1, d)$ is a sum of two discrete series representations. If $d = 2k + 1$ is odd the representation π_k is tempered. This completes the list of unitarizable (\mathfrak{g}, K) -modules with cohomology with coefficients in E . (All these depend on E , but to keep the notation simple we do not indicate this.) When we use these objects for $H(\mathbb{R})$ we will write π_i^H .

We recall some well-known facts about noncompact arithmetic quotients, for which we refer to [BoC83] or Appendix A. There is a decomposition $L^2(\Gamma \backslash G(\mathbb{R})) = L_{dis}^2(\Gamma \backslash G(\mathbb{R})) \oplus L_{cts}^2(\Gamma \backslash G(\mathbb{R}))$ into discrete and continuous spectrum and a further decomposition $L_{dis}^2(\Gamma \backslash G(\mathbb{R})) = L_{cusp}^2(\Gamma \backslash G(\mathbb{R})) \oplus L_{res}^2(\Gamma \backslash G(\mathbb{R}))$ into cuspidal and residual spectrum. For $? \in \{cusp, dis, cts\}$ let

$$H_?^*(\Gamma, E) = H^*(\mathfrak{g}, K, L_?^2(\Gamma \backslash G(\mathbb{R})) \otimes E)$$

where the (\mathfrak{g}, K) -cohomology of a unitary $G(\mathbb{R})$ -representation (π, V) is understood to be that of the space V^∞ of smooth vectors. It is well-known that the natural map

$$H^*(\mathfrak{g}, K, L^2(\Gamma \backslash G(\mathbb{R})) \otimes E) \longrightarrow H^*(\Gamma, E)$$

induced by $L^2(\Gamma \backslash G(\mathbb{R}))^\infty \subset C^\infty(\Gamma \backslash G(\mathbb{R}))$ is injective on $H_{cusp}^*(\Gamma, E)$ and zero on $H_{cts}^*(\Gamma, E)$ (see the Appendix A for a proof of the last fact).

Lemma 4.2. *For Γ arithmetic in $SO(1, d)$ we have (1) $H_{cusp}^*(\Gamma, E) = H_1^*(\Gamma, E)$ and (2) $H_{dis}^i(\Gamma, E) \rightarrow H^i(\Gamma, E)$ is injective in degrees $i \leq d/2$ and an isomorphism for $i \leq \lfloor d/2 \rfloor - 1$.*

Proof. This follows from methods of Harder [Har73] recalled in Appendix B and also Rohlfs-Speh [RS87]. According to Theorem 1.5.1 of [RS87] the cohomology in degrees $i < k = \lfloor d/2 \rfloor$ is all square-integrable, and by the results in 1.4 of loc. cit., the noncuspidal square-integrable classes are generated using residues of Eisenstein series. These classes restrict nontrivially to the boundary (see the proof of Proposition 1.4.4 of loc. cit. or Lemma B.1 in Appendix B), so they do not belong to interior cohomology. This proves (1) of the lemma in degrees $i < k = \lfloor d/2 \rfloor$. Moreover, the restriction to the boundary is injective on the residual classes (ibid.), proving (2) of the lemma in degrees $i < k = \lfloor d/2 \rfloor$. (1) follows in degrees $i > d - \lfloor d/2 \rfloor$ by duality. This leaves degree k when $d = 2k$ is even and degrees $k, k + 1$ when $d = 2k + 1$ is odd. In both cases the only contributions to $H_1^*(\Gamma, E)$ (or to $H_{dis}^i(\Gamma, E)$) in these degrees are from tempered representations (namely the discrete series when $d = 2k$ is even and the tempered representation π_k when $d = 2k + 1$ is odd), so that they are cuspidal by a well-known observation of Wallach [Wal84], and then (1) and (2) are clear for these contributions. \square

Proposition 4.3. *Assume that neither G nor H is triality. Then Res is injective on $H_1^i(\mathcal{M}, E)$ for $i \leq c/2$.*

Proof. We sketch how to adapt the argument of [HL98, Ber03, Ber06, BeC13] to the noncompact case. The main points are:

(1) For $i \leq c/2$ the abstract restriction of π_i to (\mathfrak{h}, K_H) contains π_i^H as a direct summand and multiplicity one holds, i.e. $\dim \text{Hom}_{(\mathfrak{h}, K_H)}(\pi_i|_H, \pi_i^H) = 1$. Moreover, the induced map

$$H^i(\mathfrak{g}, K, \pi_i \otimes E) \rightarrow H^i(\mathfrak{h}, K_H, \pi_i^H \otimes E_H) \quad (4.1)$$

is an isomorphism of one-dimensional spaces. This is proved in [Ber03, Theorem 3.4] (see also [HL98, §1, §6] and [Ber06, Théorème 5.3]). The references treat the case $E = \mathbb{C}$ but the proof works in general; alternately one can use translation functors as in [BW99, VI.0] to reduce the general case to this one.

(2) Let $R : C^\infty(\Gamma \backslash G(\mathbb{R})) \rightarrow C^\infty(\Gamma_H \backslash H(\mathbb{R}))$ denote restriction of functions and $R^* : H^*(\Gamma, E) \rightarrow H^*(\Gamma_H, E_H)$ the induced map in cohomology. Given an irreducible summand π of $L^2_{cusp}(\Gamma \backslash G(\mathbb{R}))$ with smooth vectors $\pi^\infty = \pi_i$, the image $R(\pi_i)$ consists of bounded functions, so we may consider its closure $R(\pi)$ in $L^2(\Gamma \backslash H(\mathbb{R}))$. Suppose that an irreducible summand σ of $L^2(\Gamma \backslash H(\mathbb{R}))$ with $\sigma^\infty = \pi_i^H$ appears as a direct summand of $R(\pi)$. We claim that $R^* : H^i(\Gamma, E) \rightarrow H^i(\Gamma_H, E_H)$ is injective on the summand $H^i(\mathfrak{g}, K, \pi \otimes E)$. To see this, note that $R(\pi) = R(\pi)_{dis} \oplus R(\pi)_{cts}$ where $R(\pi)_? = R(\pi) \cap L^2_?(\Gamma \backslash G(\mathbb{R}))$ for $? = dis, cts$, and the map R^* on $H^i(\mathfrak{g}, K, \pi \otimes E)$ is induced by the composition

$$\pi^\infty \rightarrow R(\pi)^\infty \rightarrow R(\pi)_{dis}^\infty \rightarrow C^\infty(\Gamma_H \backslash H(\mathbb{R})).$$

(We use that $H^*(\mathfrak{h}, K_H, R(\pi)_{cts} \otimes E_H) \rightarrow H^*(\Gamma_H, E_H)$ is zero since $R(\pi)_{cts}$ is a summand of $L^2_{cts}(\Gamma_H \backslash H(\mathbb{R}))$, see Appendix A for a proof of this fact.) The last map induces an injection in degree $i \leq c/2$ (by (2) of Lemma 4.2). The map $\pi^\infty \rightarrow R(\pi)_{dis}^\infty$ is nonzero in cohomology because the composition $\pi^\infty \rightarrow R(\pi)_{dis}^\infty \rightarrow \sigma^\infty$ is a nonzero multiple of the map $\pi_i|_H \rightarrow \pi_i^H$ in (1) (by multiplicity one), and hence induces the nontrivial map (4.1) in cohomology. This proves that R^* is nonzero on $H^*(\mathfrak{g}, K, \pi \otimes E)$, hence injective.

(3) The Burger-Sarnak argument shows that given π as in (2), and assuming a certain isolation hypothesis on π_i^H (recalled below), we can arrange for a summand σ as in (2) above, perhaps after replacing H by a conjugate, i.e. replacing the map R above by $R_g : C^\infty(\Gamma \backslash G(\mathbb{R})) \rightarrow C^\infty(g\Gamma g^{-1} \cap H \backslash H(\mathbb{R}))$ for some $g \in G(\mathbb{Q})$. This is [HL98, Proposition 3.1] and the refinement [Ber03, Proposition 3.2]. The observation of [BS91] is that matrix coefficients of the cuspidal representation on $\Gamma \backslash G(\mathbb{R})$ are, when restricted to $H(\mathbb{R})$, the limits, uniform on compacta, of finite sums of matrix coefficients of $H(\mathbb{R})$ appearing in the spaces $L^2(g^{-1}\Gamma g \cap H \backslash H(\mathbb{R}))$ for $g \in G(\mathbb{Q})$. A key point explained in [HL98] (see the remarks at the bottom of p. 93 of loc. cit.) is that this applies to cuspidal cohomology on a noncompact quotient because cuspidal functions are of rapid decrease, hence uniformly continuous on $\Gamma \backslash G(\mathbb{R})$, and this suffices for the argument in [BS91]. Thus under the isolation hypothesis, if π_i^H is weakly contained in $R(\pi)$, then there is a direct summand σ of $R_g(\pi)$ as in (2), and so by (2), R_g^* is injective on $H^i(\mathfrak{g}, K, \pi \otimes E)$.

(4) The isolation hypothesis required in (3) is that π_i^H is isolated in $\{\pi_i^H\} \cup \{(\rho, V_\rho) \in \widehat{H}_{Aut} : d \equiv 0 \text{ on } C^i(\mathfrak{g}, K, V_\rho \otimes E)\}$. (We refer to [HL98, Ber03] for the precise definitions of \widehat{H}_{Aut} and the relevance of this condition.) This was shown in [Ber03] to follow from a uniform (in Γ) lower bound for the first nonzero eigenvalue of the Laplacian on i -forms on M_Γ (Conjecture 2.3 of [Ber03]). This eigenvalue bound was later proved in [BeC13, Theorem 1.3] using Arthur's endoscopic classification [Art13] of representations. (At this point triality forms must be excluded, although they do not occur if the \mathbb{Q} -rank is one, see Remark 4.5 below.)

(5) Putting (1)–(4) together, we conclude that Res is injective on the cuspidal cohomology for $i \leq c/2$, and hence, by Lemma 4.2, on $H^i(\mathcal{M}, E)$ for $i \leq c/2$. \square

We will make some remarks below as to the necessity of the contortions in the previous proof after finishing the proof of the theorem.

Proof of Theorem 4.1. We will use the standard commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}_!^i(\mathcal{M}, E) & \longrightarrow & \mathrm{H}^i(\mathcal{M}, E) & \longrightarrow & I_P^G \mathrm{H}^i(\mathfrak{n}, E) \\
& & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res}_\infty \\
0 & \longrightarrow & I_H^G \mathrm{H}_!^i(\mathcal{M}_H, E) & \longrightarrow & I_H^G \mathrm{H}^i(\mathcal{M}_H, E) & \longrightarrow & I_H^G I_{P_H}^H \mathrm{H}^i(\mathfrak{n}_H, E_H).
\end{array} \tag{4.2}$$

This diagram comes from the properness of $M_{H, \Gamma_H} \rightarrow M_\Gamma$, or can be seen using the fact that the cusp (i.e. reductive Borel-Serre) compactification is functorial for $H \subset G$ in this case. The identification of the boundary cohomology with $I_P^G \mathrm{H}^i(\mathfrak{n}, E)$ is standard. It suffices to prove the injectivity of Res_∞ for $i \leq c/2$. By transitivity of induction $I_H^G I_{P_H}^H = I_P^G I_{P_H}^P$ so it suffices to prove the injectivity of $\mathrm{H}^i(\mathfrak{n}, E) \rightarrow I_{P_H}^P \mathrm{H}^i(\mathfrak{n}_H, E_H)$. Now the action of $N(\mathbb{Q})$ on $\mathrm{H}^i(\mathfrak{n}, E)$ is trivial, so this factors through $\left(I_{P_H}^P \mathrm{H}^i(\mathfrak{n}_H, E_H) \right)^{N(\mathbb{Q})} = I_{L_H}^L \mathrm{H}^i(\mathfrak{n}_H, E_H)$ and it is enough to prove the injectivity of $\mathrm{H}^i(\mathfrak{n}, E) \rightarrow I_{L_H}^L \mathrm{H}^i(\mathfrak{n}_H, E)$. This follows from the injectivity of $\mathrm{Res}_\mathfrak{n}$ in degrees $i \leq c$ proved in Proposition 3.1, except possibly in the case $(d, c, i) = (2k+1, 2k, k)$.

The remaining case is restriction from $SO(1, 2k+1)$ to $SO(1, 2k)$ in degree $i = k$. It suffices to prove Res_∞ is injective on the image of $\mathrm{H}^k(\mathcal{M}, E) \rightarrow I_P^G \mathrm{H}^k(\mathfrak{n}, E)$. This map is induced by the $P(\mathbb{Q})$ -map $\mathrm{H}^k(\mathcal{M}, E) \rightarrow \mathrm{H}^k(\mathfrak{n}, E)$ given by restriction to a deleted neighbourhood of the cusp given by $P = LN$ and the image of $\mathrm{H}^k(\mathcal{M}, E) \rightarrow I_P^G \mathrm{H}^k(\mathfrak{n}, E)$ is $I_P^G U$ where U is the image of $\mathrm{H}^k(\mathcal{M}) \rightarrow \mathrm{H}^k(\mathfrak{n}, E)$. There is a nondegenerate duality pairing on $\mathrm{H}^k(\mathfrak{n}, E)$ given by the cup product and the self-duality of E . Now U is a maximal isotropic subspace for the duality pairing on $\mathrm{H}^k(\mathfrak{n}, E)$, i.e. $U^\perp = U$, and it is also $L(\mathbb{C}) = \mathbb{C}^* \times SO(2k, \mathbb{C})$ -stable. Since $\mathrm{H}^k(\mathfrak{n}, E) = U_+ \oplus U_-$ is a sum of two inequivalent $SO(2k, \mathbb{C})$ -modules of the same dimension (they form a single $O(2k, \mathbb{C})$ -module), either $U = U_+$ or $U = U_-$, and the image $I_P^G U$ of $\mathrm{H}^k(\mathcal{M}) \rightarrow I_P^G \mathrm{H}^k(\mathfrak{n}, E)$ is either $I_P^G U_+$ or $I_P^G U_-$. In the notation of 3.4, U_+ and U_- are the $SO(2k, \mathbb{C})$ -modules with highest weight vectors $e_{s_k} \otimes v_{s_k \lambda}$ and $e_{t_k} \otimes v_{t_k \lambda}$ respectively. Now the restriction $\mathrm{Res}_\mathfrak{n} : \mathrm{H}^k(\mathfrak{n}, E) \rightarrow \prod_m \mathrm{H}^k(\mathfrak{n}_H, E_H)$ induced by $\mathfrak{n}_H \subset \mathfrak{n}$ is nonzero on either summand, because it is nonzero on these highest weight vectors, as was proved in Proposition 3.1. It follows that $I_P^G \mathrm{H}^k(\mathfrak{n}, E) \rightarrow I_P^G I_{P_H}^P \mathrm{H}^k(\mathfrak{n}_H, E_H)$ is injective on each of $I_P^G U_\pm$ individually, and hence on the image of $\mathrm{H}^k(\mathcal{M}, E) \rightarrow I_P^G \mathrm{H}^k(\mathfrak{n}, E)$, whichever of these modules it is. This completes the proof of the theorem. \square

Remark 4.4. (On the Burger-Sarnak method for noncompact quotients) In general, it is not clear to us that the argument of [HL98, Ber03, Ber06, BeC13] can be adapted to treat noncuspidal interior cohomology classes on a general arithmetic quotient without a better understanding of the latter, e.g. using Eisenstein series. Since the argument for injectivity in cohomology treats one summand π of L^2 at a time, one needs to know that π^∞ , or at least π , contains some uniformly continuous functions, the diagonal matrix coefficients of which can then be used in the Burger-Sarnak argument. If π is cuspidal then the functions in π^∞ are of rapid decrease, hence uniformly continuous, and this suffices. It is not clear to us that the automorphic representatives of noncuspidal interior cohomology classes are uniformly continuous on $\Gamma \backslash G(\mathbb{R})$ – they are not of rapid decrease as they would then be cuspidal – or, indeed, that the summand π^∞ contributing to such cohomology contains any uniformly continuous functions.

In the case at hand, Lemma 4.2 shows that there is no noncuspidal interior cohomology and so this problem does not occur. For the congruence ball quotients discussed in [Nai17b], [BeC17, §3] and in the next section one can show that the noncuspidal interior cohomology consists of nonprimitive classes (see the discussion in the next section), and the analogue of Proposition 4.3 for $SU(1, m) \subset SU(1, n)$ can be proved similarly. However, for $SO(2, n)$ the situation is more complicated and something more is required. In any case, we will not use automorphic arguments to treat interior cohomology in the $SU(1, n)$ and $SO(2, n)$ cases since the geometric arguments of Section 2 are available.

Remark 4.5. A triality form over a totally real field F which becomes $SO(1, 7)$ over \mathbb{R} for some real embedding of F is necessarily anisotropic over \mathbb{Q} . This follows by looking at Tits indices, see the table on p. 58 of [Tit65]. So in the \mathbb{Q} -rank one case we may ignore triality altogether.

5. CONGRUENCE COMPLEX HYPERBOLIC MANIFOLDS

The first main result for congruence complex hyperbolic manifolds is the following:

Theorem 5.1. *Suppose that $H \subset G$ are groups of the same \mathbb{Q} -rank and $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is the inclusion $SU(1, m) \subset SU(1, n)$ with $2 \leq m < n$. Then Res is injective on $H_1^i(\mathcal{M}_G)$ for $i \leq m$ and on $H^i(\mathcal{M}_G)$ for $i < m$.*

This follows immediately from Corollary 2.2 and Proposition 3.2. This is simpler than the proof in [Nai17b].

The rest of this section consists of the proof of the following, which is Theorem 2 of the introduction:

Theorem 5.2. *Suppose that $H \subset G$ are groups of the same \mathbb{Q} -rank and $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is the inclusion $SO(1, n) \subset SU(1, n)$ with $n > 2$. Then Res is injective on $H^{i,0}(\mathcal{M}_G)$ for $i \leq n/2$.*

Proof of Theorem 5.2. The proof is broadly the same as that of Theorem 4.1: Given the injectivity on $H_1^i(\mathcal{M})$ for $i \leq n/2$, the diagram (4.2), strictness of the Hodge filtration, and Proposition 3.4 combine to prove the theorem.

The proof of injectivity on $H_1^{i,0}(\mathcal{M})$ for $i \leq n/2$ follows the outline of the proof of Proposition 4.3, with step (1) there replaced by the following:

(1') For each $i < n$ there is a unique cohomological (\mathfrak{g}, K) -module $J_{i,0}$ with cohomology in bidegree $(i, 0)$. This is immediate from the classification of (\mathfrak{g}, K) -modules with cohomology for $SU(1, n)$ (see [BW99, VI.4.7-VI.4.12]). For $i \leq n/2$ the abstract restriction of $J_{i,0}$ to (\mathfrak{h}, K_H) contains π_i^H (the unique cohomological representation for $SO(1, n)$ with $H^i(\mathfrak{h}, K_H, \pi_i^H) \neq 0$, see Section 4) as a direct summand with multiplicity one, i.e. $\dim \text{Hom}_{(\mathfrak{h}, K_H)}(J_{i,0}|_H, \pi_i^H) = 1$, and the induced map

$$H^i(\mathfrak{g}, K, J_{i,0}) \longrightarrow H^i(\mathfrak{h}, K_H, \pi_i^H)$$

is an isomorphism of one-dimensional spaces. This is [Ber06, Théorème 5.6].

Given (1'), steps (2)–(5) in the proof of Proposition 4.3 work verbatim to prove that Res is injective on $H_{cusp}^{i,0}(\mathcal{M})$ for $i \leq n/2$. (We are using here the agreement of the two possible Hodge structures on $H_1^i(\mathcal{M})$, the first coming from geometry (hence having good properties for the boundary exact sequence and in diagram (4.2)), and the second from the inclusion $H_1^i(M_\Gamma) \subset \text{IH}^i(M_\Gamma^*)$ and the L^2 Hodge theory on the latter coming from (2.11) (hence agreeing with the Hodge types in (\mathfrak{g}, K) -cohomology). This is known by [Zuc87] because M_Γ^* has isolated singularities.

Lemma 5.3 below completes the proof of the theorem. □

It remains to prove:

Lemma 5.3. *For Γ arithmetic in $SU(1, n)$, we have $H_1^{*,0}(M_\Gamma) = H_{cusp}^{*,0}(M_\Gamma)$.*

Proof. The proof is similar to that of Lemma 4.2 for $SO(1, n)$. Suppose first that $i < n$. Since $\text{IH}^i(M_\Gamma^*) = H^i(M_\Gamma)$ for $i < n$ we are actually dealing with the cohomology

$$H^i(\mathfrak{g}, K, L_{dis}^2(\Gamma \backslash G(\mathbb{R}))) = \bigoplus_{\pi \subset L_{dis}^2} H^i(\mathfrak{g}, K, \pi)$$

where the sum is over irreducible closed summands, only finitely many of which contribute to the sum. For π to contribute to the $(i, 0)$ summand of cohomology we must have $\pi^\infty = J_{i,0}$. For such a summand π , the theory of Eisenstein series gives a mapping $I_{i,0} \rightarrow L_{dis}^2$ onto $\pi^\infty = J_{i,0}$. Here $I_{i,0}$ denotes the standard module of which $J_{i,0}$ is the Langlands quotient (see [BW99, VI.4.8]). Now the minimal degree in which $J_{i,0}$ has cohomology is i , so we are in the situation

of Appendix B and applying Lemma B.1 gives Lemma 5.3 for $i < n$. (Note that the assumption (*) required in Lemma B.1 holds since $I_{i,0}$ has cohomology in degrees $2n - i, 2n - i - 1$ and $I_{i,0} \rightarrow J_{i,0}$ induces an isomorphism in degree $2n - i$, see [BW99, p. 133].)

The statement in degrees $i > n$ follows by duality. Finally, the equality in degree n holds because the component at infinity of a class in $H_1^{n,0}(M_\Gamma)$ is of discrete series type, hence is already cuspidal by [Wal84]. \square

Remark 5.4. The proof of the lemma shows more generally that for $i + j \leq n$, we have

$$H_1^{i,j}(M_\Gamma)^{prim} = H_{cusp}^{i,j}(M_\Gamma)^{prim}.$$

where the primitive is taken with respect to the Lefschetz class. Thus the Burger-Sarnak method can be applied to prove injectivity on $H_1^{i,j}(M_\Gamma)^{prim}$ for the restriction by $SU(1, m) \subset SU(1, n)$. Using the action of the Lefschetz operator this can be used to give another proof of Theorem 5.1 for complex hyperbolic manifolds.

6. ORTHOGONAL SHIMURA VARIETIES

The main theorem in this case is the following:

Theorem 6.1. *Suppose that $H \subset G$ are of the same \mathbb{Q} -rank and that $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is the inclusion $SO(2, m) \subset SO(2, n)$ with $n > m \geq 2$. Then Res is injective on $H^i(\mathcal{M}_G, E)$ for $i \leq m - 1$.*

We will argue as if the \mathbb{Q} -rank of both G and H is two and indicate how the argument simplifies in the \mathbb{Q} -rank one case. The proof is by a kind of induction on the stratification of the minimal compactification, going from injectivity on interior cohomology $H_1^i(\mathcal{M}_G)$ (proved earlier as Corollary 2.2) to injectivity on a larger subspace $\text{Gr}_i^W \mathbb{H}_c^i(\mathcal{M}_G^1, j_*^1 \mathbb{C})$ of $H^i(\mathcal{M}_G)$ (defined below) which takes into account some contributions from the one-dimensional boundary strata, and then to the injectivity on all of $H^i(\mathcal{M}_G)$ by taking into account some contributions from the cusps (using the Lefschetz property for real hyperbolic manifolds from Section 4). A similar, but simpler, argument was used in [Nai17b] for ball quotients.

To simplify the notation somewhat we will write \mathcal{M} for \mathcal{M}_G . (We continue to write \mathcal{M}_H for the Shimura variety associated to H of course.)

6.1. We will introduce some notation which will be useful. Recall the stratification of M_Γ^* discussed in Section 1 and denote the inclusions by

$$M_\Gamma \xrightarrow{j_\Gamma^1} M_\Gamma^1 \xrightarrow{j_\Gamma^0} M_\Gamma^*$$

with $j_\Gamma^0 \circ j_\Gamma^1 = j_\Gamma$. This gives two cohomology long exact sequences:

(1) The distinguished triangle associated with $j_{\Gamma^*}^1 \mathbb{C}$ on M_Γ^1 and the open-closed decomposition

$$M_\Gamma \xrightarrow{j_\Gamma^1} M_\Gamma^1 \xleftarrow{i_\Gamma^1} Z_\Gamma^1 \text{ is } j_{\Gamma^*}^1 \mathbb{C} \longrightarrow j_{\Gamma^*}^1 \mathbb{C} \longrightarrow i_{\Gamma^*}^1 i_{\Gamma^*}^{1*} j_{\Gamma^*}^1 \mathbb{C} \xrightarrow{+1} \text{ and gives}$$

$$\cdots \longrightarrow \mathbb{H}_c^i(M_\Gamma) \longrightarrow \mathbb{H}_c^i(M_\Gamma^1, j_{\Gamma^*}^1 \mathbb{C}) \longrightarrow \mathbb{H}_c^i(Z_\Gamma^1, i_{\Gamma^*}^{1*} j_{\Gamma^*}^1 \mathbb{C}) \longrightarrow \cdots .$$

(2) The distinguished triangle associated with $j_{\Gamma^*} \mathbb{C}$ and $M_\Gamma^1 \xrightarrow{j_\Gamma^0} M_\Gamma^* \xleftarrow{i_\Gamma^0} Z_\Gamma^0$ is $j_{\Gamma^*}^0 j_{\Gamma^*}^1 \mathbb{C} \longrightarrow$

$$j_{\Gamma^*} \mathbb{C} \longrightarrow i_{\Gamma^*}^0 i_{\Gamma^*}^{0*} j_{\Gamma^*}^1 \mathbb{C} \xrightarrow{+1} \text{ and gives}$$

$$\cdots \longrightarrow \mathbb{H}_c^i(M_\Gamma^1, j_{\Gamma^*}^1 \mathbb{C}) \longrightarrow \mathbb{H}^i(M_\Gamma) \longrightarrow \mathbb{H}^i(Z_\Gamma^0, i_{\Gamma^*}^{0*} j_{\Gamma^*} \mathbb{C}) \longrightarrow \cdots .$$

Both are long exact sequences of mixed Hodge structures by [Sai90].

The exact sequences (1) and (2) are natural with respect to passing to subgroups of Γ of finite index, and this leads us to introduce the following suggestive notation:

$$\begin{aligned}\mathbb{H}_c^i(\mathcal{M}^1, j_*^1 \mathbb{C}) &:= \operatorname{colim}_\Gamma \mathbb{H}_c^i(M_\Gamma^1, j_{\Gamma*}^1 \mathbb{C}) \\ \mathbb{H}_c^i(\mathcal{Z}^1, i^{1*} j_*^1 \mathbb{C}) &:= \operatorname{colim}_\Gamma \mathbb{H}_c^i(Z_\Gamma^1, i_{\Gamma*}^{1*} j_{\Gamma*}^1 \mathbb{C}) \\ \mathbb{H}^i(\mathcal{Z}^0, i^{0*} j_* \mathbb{C}) &:= \operatorname{colim}_\Gamma \mathbb{H}^i(Z_\Gamma^0, i_{\Gamma*}^{0*} j_{\Gamma*} \mathbb{C})\end{aligned}\tag{6.1}$$

where all colimits are over congruence subgroups. These are smooth $G(\mathbb{Q})$ -modules and the map $\mathbb{H}_c^*(\mathcal{M}) \rightarrow \mathbb{H}^*(\mathcal{M})$ factors through $\mathbb{H}_c^i(\mathcal{M}^1, j_*^1 \mathbb{C}) \rightarrow \mathbb{H}^i(\mathcal{M})$. The exact sequences above give exact sequences

$$\cdots \rightarrow \mathbb{H}_c^i(\mathcal{M}) \rightarrow \mathbb{H}_c^i(\mathcal{M}^1, j_*^1 \mathbb{C}) \rightarrow \mathbb{H}_c^i(\mathcal{Z}^1, i^{1*} j_*^1 \mathbb{C}) \rightarrow \cdots\tag{6.2}$$

and

$$\cdots \rightarrow \mathbb{H}_c^i(\mathcal{M}^1, j_*^1 \mathbb{C}) \rightarrow \mathbb{H}^i(\mathcal{M}) \rightarrow \mathbb{H}^i(\mathcal{Z}^0, i^{0*} j_* \mathbb{C}) \rightarrow \cdots\tag{6.3}$$

which are exact sequences of (colimits of) mixed Hodge structures.

We also note the following useful consequence of the purity lemma (Lemma 2.5):

$$\mathbb{H}_!^i(\mathcal{M}) = \operatorname{im}(\operatorname{Gr}_i^W \mathbb{H}_c^i(\mathcal{M}) \rightarrow \operatorname{Gr}_i^W \mathbb{H}_c^i(\mathcal{M}^1, j_*^1 \mathbb{C})) \text{ for } i \leq n-1.\tag{6.4}$$

Indeed, Lemma 2.5 implies that $\mathbb{H}_!^i(M_\Gamma) = \operatorname{im}(\operatorname{Gr}_i^W \mathbb{H}_c^i(M_\Gamma) \rightarrow \operatorname{Gr}_i^W \mathbb{H}_c^i(M_\Gamma^1, j_{\Gamma*}^1 \mathbb{C}))$ for $i \leq n-1$ because $\operatorname{Gr}_i^W \mathbb{H}_c^i(M_\Gamma^1, j_{\Gamma*}^1 \mathbb{C}) \subset \operatorname{Gr}_i^W \mathbb{H}^i(M_\Gamma)$ by the second exact sequence above. Since Gr_i^W commutes with the colimits we get (6.4).

6.2. Proof of Theorem 6.1. Now consider the situation of $H \subset G$ and the morphism $M_{H, \Gamma_H}^* \rightarrow M_\Gamma^*$. The stratifications are compatible, in the sense that the stratification of M_{H, Γ_H}^* is the pullback of that of M_Γ^* , i.e. the relevant diagrams relating strata are Cartesian. It follows that both the exact sequences above are functorial, i.e. there are $H(\mathbb{Q})$ -module maps from each exact sequence for G to the corresponding one for H . Frobenius reciprocity gives a commutative diagram of $G(\mathbb{Q})$ -modules with exact rows

$$\begin{array}{ccccccc} \longrightarrow & \mathbb{H}_c^i(\mathcal{M}) & \longrightarrow & \mathbb{H}_c^i(\mathcal{M}^1, j_*^1 \mathbb{C}) & \longrightarrow & \mathbb{H}_c^i(\mathcal{Z}^1, i^{1*} j_*^1 \mathbb{C}) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & I_H^G \mathbb{H}_c^i(\mathcal{M}_H) & \longrightarrow & I_H^G \mathbb{H}_c^i(\mathcal{M}_H^1, j_{H*}^1 \mathbb{C}) & \longrightarrow & I_H^G \mathbb{H}_c^i(\mathcal{Z}_H^1, i_{H*}^{1*} j_{H*}^1 \mathbb{C}) & \longrightarrow \end{array}$$

from the first sequence and a similar diagram with exact rows

$$\begin{array}{ccccccc} \longrightarrow & \mathbb{H}_c^i(\mathcal{M}^1, j_*^1 \mathbb{C}) & \longrightarrow & \mathbb{H}^i(\mathcal{M}) & \longrightarrow & \mathbb{H}^i(\mathcal{Z}^0, i^{0*} j_* \mathbb{C}) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & I_H^G \mathbb{H}_c^i(\mathcal{M}_H^1, j_{H*}^1 \mathbb{C}) & \longrightarrow & I_H^G \mathbb{H}^i(\mathcal{M}_H) & \longrightarrow & I_H^G \mathbb{H}^i(\mathcal{Z}_H^0, i_{H*}^{0*} j_{H*} \mathbb{C}) & \longrightarrow \end{array}$$

from the second. Taking Gr_i^W and using (6.4) in the first diagram gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}_!^i(\mathcal{M}) & \longrightarrow & \operatorname{Gr}_i^W \mathbb{H}_c^i(\mathcal{M}^1, j_*^1 \mathbb{C}) & \longrightarrow & \operatorname{Gr}_i^W \mathbb{H}_c^i(\mathcal{Z}^1, i^{1*} j_*^1 \mathbb{C}) \\ & & \downarrow \operatorname{Res}_! & & \downarrow \operatorname{Res}^1 & & \downarrow \operatorname{Res}_\infty^1 \\ 0 & \longrightarrow & I_H^G \mathbb{H}_!^i(\mathcal{M}_H) & \longrightarrow & I_H^G \operatorname{Gr}_i^W \mathbb{H}_c^i(\mathcal{M}_H^1, j_{H*}^1 \mathbb{C}) & \longrightarrow & I_H^G \operatorname{Gr}_i^W \mathbb{H}_c^i(\mathcal{Z}_H^1, i_{H*}^{1*} j_{H*}^1 \mathbb{C}) \end{array}\tag{6.5}$$

in which the upper row is exact for $i \leq n-1$ and the lower row is exact for $i \leq m-1$. (We have used the purity of $\mathbb{H}_!^i(\mathcal{M})$.) Similarly, taking Gr_i^W and using (6.4) in the second diagram

gives a commutative diagram with exact rows for $i \leq m - 1$:

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathrm{Gr}_i^W \mathbb{H}_c^i(\mathcal{M}^1, j_*^1 \mathbb{C}) & \longrightarrow & \mathrm{H}^i(\mathcal{M}) & \longrightarrow & \mathrm{Gr}_i^W \mathbb{H}^i(\mathcal{Z}^0, i^{0*} j_* \mathbb{C}) & (6.6) \\
 & & \downarrow \mathrm{Res}^1 & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res}_\infty^0 \\
 0 & \longrightarrow & I_H^G \mathrm{Gr}_i^W \mathbb{H}_c^i(\mathcal{M}_H^1, j_{H*}^1 \mathbb{C}) & \longrightarrow & I_H^G \mathrm{Gr}_i^W \mathbb{H}^i(\mathcal{M}_H) & \longrightarrow & I_H^G \mathrm{Gr}_i^W \mathbb{H}^i(\mathcal{Z}_H^0, i_H^{0*} j_{H*} \mathbb{C})
 \end{array}$$

We have used the purity of $\mathrm{H}^i(\mathcal{M})$ in degrees $\leq m - 1$, which follows from the fact that $\mathrm{IH}^i(M_\Gamma^*) = \mathrm{H}^i(M_\Gamma)$ in degrees $i \leq n - 2$.

We see from these diagrams that Theorem 6.1(i), namely the injectivity of Res on $\mathrm{H}^i(\mathcal{M})$ in degrees $\leq m - 1$, follows from the conjunction of Corollary 2.2 (injectivity on interior cohomology in degrees $\leq m - 1$) and the following two statements:

Proposition 6.2. *The map Res_∞^1 in (6.5) is injective in degrees $i \leq m - 1$.*

Proposition 6.3. *The map Res_∞^0 in (6.6) is injective in degrees $i \leq m - 1$.*

The rest of this subsection will be taken up with the proofs of these two propositions. The first will use Proposition 3.3 while the second will use the Lefschetz property for real hyperbolic manifolds in Theorem 4.1.

Proof of Proposition 6.2. Recall that Proposition 6.2 asserts the injectivity of

$$\mathrm{Res}_\infty^1 : \mathrm{Gr}_i^W \mathbb{H}_c^i(\mathcal{Z}^1, i^{1*} j_*^1 \mathbb{C}) \longrightarrow I_H^G \mathrm{Gr}_i^W \mathbb{H}_c^i(\mathcal{Z}_H^1, i_H^{1*} j_{H*}^1 \mathbb{C})$$

in degrees $i \leq m - 1$. We will deduce from Proposition 3.3 the a priori stronger assertion that the map

$$\mathbb{H}_c^i(\mathcal{Z}^1, i^{1*} j_*^1 \mathbb{C}) \longrightarrow I_H^G \mathbb{H}_c^i(\mathcal{Z}_H^1, i_H^{1*} j_{H*}^1 \mathbb{C}) \quad (6.7)$$

is injective in this range; since this is a morphism of mixed Hodge structures the statement about the i th graded follows. By definition,

$$\mathbb{H}_c^i(\mathcal{Z}^1, i^{1*} j_*^1 \mathbb{C}) = \mathrm{colim}_\Gamma \mathbb{H}_c^i(Z_\Gamma^1, i_\Gamma^{1*} j_{\Gamma*}^1 \mathbb{C}).$$

Choose a rational boundary component F of dimension one and let $P = MW$ be its stabilizer, which is the maximal parabolic stabilizing an isotropic plane in V . The stratum of M_Γ^* given by F is $S_\Gamma := \Gamma_{M_h} \backslash F$ and it is a component of Z_Γ^1 ; let $i_{S_\Gamma} : S_\Gamma \hookrightarrow M_\Gamma^1$ be the inclusion. Then we have natural identifications

$$\begin{aligned}
 \mathbb{H}_c^i(\mathcal{Z}^1, i^{1*} j_*^1 \mathbb{C}) &= I_P^G \left(\mathrm{colim}_\Gamma \mathbb{H}_c^i(S_\Gamma, i_{S_\Gamma}^* j_{\Gamma*} \mathbb{C}_{M_\Gamma}) \right) \\
 &= \bigoplus_k I_P^G \left(\mathrm{colim}_\Gamma \mathbb{H}_c^{i-k}(S_\Gamma, \mathrm{H}^k(i_{S_\Gamma}^* j_{\Gamma*} \mathbb{C}_{M_\Gamma})) \right) \\
 &= \bigoplus_k I_P^G \left(\mathrm{colim}_\Gamma \mathbb{H}_c^{i-k}(S_\Gamma, \mathrm{H}^k(\mathfrak{w}, \mathbb{C})) \right) \\
 &= \bigoplus_k I_P^G \mathbb{H}_c^{i-k}(\mathcal{M}_M, \mathrm{H}^k(\mathfrak{w}, \mathbb{C})).
 \end{aligned} \quad (6.8)$$

Here the first equality is an elementary argument keeping track of the cusps (see [Nai17a, Lemma 3.3] for a similar argument for ball quotients), the second is given by Proposition 1.7. We have also used that $M_\ell(\mathbb{R})$ is compact, so that $\Gamma_{M_\ell A} = \{e\}$ for neat Γ . There is a similar expression for the $H(\mathbb{Q})$ -module $\mathbb{H}_c^i(\mathcal{Z}_H^1, i_H^{1*} j_{H*}^1 \mathbb{C})$ and hence for the target of (6.7), namely

$$I_H^G \mathbb{H}_c^i(\mathcal{Z}_H^1, i_H^{1*} j_{H*}^1 \mathbb{C}) = \bigoplus_k I_H^G I_{P_H}^H \left(\mathrm{colim}_{\Gamma_H} \mathbb{H}_c^{i-k}(S_{\Gamma_H}, \mathrm{H}^k(\mathfrak{w}_H, \mathbb{C})) \right).$$

By transitivity of induction $I_H^G I_{P_H}^H = I_P^G I_{P_H}^P$ we are reduced to showing that

$$\mathrm{colim}_\Gamma \mathbb{H}_c^{i-k}(S_\Gamma, \mathrm{H}^k(\mathfrak{n}, \mathbb{C})) \longrightarrow I_{P_H}^P \mathrm{colim}_{\Gamma_H} \mathbb{H}_c^{i-k}(S_{\Gamma_H}, \mathrm{H}^k(\mathfrak{n}_H, \mathbb{C}))$$

is injective for $i \leq m - 1$. Now the action of $W(\mathbb{Q})$ on the left-hand-side is trivial, and

$$\left(I_{P_H}^P \mathbb{H}_c^{i-k}(S_\Gamma, \mathrm{H}^k(\mathfrak{n}_H, \mathbb{C})) \right)^{W(\mathbb{Q})} = I_{M_H}^M \mathbb{H}_c^{i-k}(S_{\Gamma_H}, \mathrm{H}^k(\mathfrak{n}_H, \mathbb{C})),$$

so we must show

$$\operatorname{colim}_{\Gamma} \mathbb{H}_c^{i-k}(S_{\Gamma}, \mathbb{H}^k(\mathfrak{n}, \mathbb{C})) \longrightarrow I_{M_H}^M \operatorname{colim}_{\Gamma_H} \mathbb{H}_c^{i-k}(S_{\Gamma_H}, \mathbb{H}^k(\mathfrak{n}_H, \mathbb{C}))$$

is injective for $i \leq m-1$. But now note that $S_{\Gamma} = S_{\Gamma_H}$ for neat Γ (M and M_H differ only in the compact factor), and the case $i = k$ does not occur (because $\mathbb{H}_c^0(S_{\Gamma}, V) = 0$ for any V), so this follows from Proposition 3.3. \square

Proof of Proposition 6.3. Recall that the proposition asserts the injectivity of

$$\operatorname{Res}_{\infty}^0 : \operatorname{Gr}_i^W \mathbb{H}^i(\mathcal{Z}^0, i^{0*} j_* \mathbb{C}) \longrightarrow I_H^G \operatorname{Gr}_i^W \mathbb{H}^i(\mathcal{Z}_H^0, i_H^{0*} j_{H*} \mathbb{C})$$

in degrees $i \leq m-1$. We will reduce this to the Lefschetz property for congruence hyperbolic manifolds, i.e. Theorem 4.1.

Let $P = MW$ be the stabilizer of an isotropic line I in V , and F the associated rational boundary component. Let $i_{s_{\Gamma}} : \{s_{\Gamma}\} \hookrightarrow M_{\Gamma}^1$ be the stratum given by F in M_{Γ}^* . Then by Proposition 1.7 there is an isomorphism in the derived category

$$i_{s_{\Gamma}}^* j_{\Gamma*} \mathbb{C} = \bigoplus_k \mathbb{H}^k(i_{s_{\Gamma}}^* j_{\Gamma*} \mathbb{C})[-k]$$

and moreover

$$\mathbb{H}^i(i_{s_{\Gamma}}^* j_{\Gamma*} \mathbb{C}) = \bigoplus_k \mathbb{H}^{i-k}(\Gamma_M, \wedge^k \mathfrak{u}_{\mathbb{C}}^*)$$

where we have used that $\Gamma_M = \Gamma_{M_{\ell}}$ (assuming Γ is neat) is in $SO(1, n-1)$ and $\mathfrak{w} = \mathfrak{u}$ is abelian, so that $\mathbb{H}^*(\mathfrak{w}, \mathbb{C}) = \wedge^* \mathfrak{u}^*$. By Proposition 1.7, the k -summand is pure of weight $2k$, so that

$$\operatorname{Gr}_i^W \mathbb{H}^i(i_{s_{\Gamma}}^* j_{\Gamma*} \mathbb{C}) = \mathbb{H}^{i/2}(\Gamma_M, \wedge^{i/2} \mathfrak{u}_{\mathbb{C}}^*)$$

if i is even and zero if i is odd. We then have

$$\begin{aligned} \operatorname{Gr}_i^W \mathbb{H}^i(\mathcal{Z}^0, i^{0*} j_* \mathbb{C}) &= I_P^G \operatorname{colim} \mathbb{H}^{i/2}(\Gamma_M, \wedge^{i/2} \mathfrak{u}_{\mathbb{C}}^*) \\ &= I_P^G \mathbb{H}^{i/2}(\mathcal{M}_M, \wedge^{i/2} \mathfrak{u}_{\mathbb{C}}^*) \end{aligned}$$

for i even and zero for i odd. This discussion applies also to $\operatorname{Gr}_i^W \mathbb{H}^i(\mathcal{Z}_H^0, i_H^{0*} j_{H*} \mathbb{C})$ and gives

$$\begin{aligned} \operatorname{Gr}_i^W \mathbb{H}^i(\mathcal{Z}^0, i^{0*} j_* \mathbb{C}) &= I_{P_H}^H \operatorname{colim} \mathbb{H}^{i/2}(\Gamma_{M_H}, \wedge^{i/2} \mathfrak{u}_{H, \mathbb{C}}^*) \\ &= I_P^G \mathbb{H}^{i/2}(\mathcal{M}_{M_H}, \wedge^{i/2} \mathfrak{u}_{H, \mathbb{C}}^*) \end{aligned}$$

for i even and zero for i odd. By transitivity of induction, the injectivity of $\operatorname{Res}_{\infty}^0$ in degree i is reduced to that of

$$\mathbb{H}^{i/2}(\mathcal{M}_M, \wedge^{i/2} \mathfrak{u}_{\mathbb{C}}^*) \longrightarrow I_{P_H}^P \mathbb{H}^{i/2}(\mathcal{M}_{M_H}, \wedge^{i/2} \mathfrak{u}_{H, \mathbb{C}}^*).$$

The action of $W(\mathbb{Q}) = U(\mathbb{Q})$ on the source is trivial so this factors through the $U(\mathbb{Q})$ -invariants of the target, i.e.

$$\left(I_{P_H}^P \operatorname{colim} \mathbb{H}^{i/2}(\Gamma_{M_H}, \wedge^{i/2} \mathfrak{u}_{H, \mathbb{C}}^*) \right)^{U(\mathbb{Q})} = I_{M_H}^M \operatorname{colim} \mathbb{H}^{i/2}(\Gamma_{M_H}, \wedge^{i/2} \mathfrak{u}_{H, \mathbb{C}}^*).$$

We are thus reduced to the injectivity of

$$\operatorname{colim} \mathbb{H}^{i/2}(\Gamma_M, \wedge^{i/2} \mathfrak{u}_{\mathbb{C}}^*) \longrightarrow I_{M_H}^M \operatorname{colim} \mathbb{H}^{i/2}(\Gamma_{M_H}, \wedge^{i/2} \mathfrak{u}_{H, \mathbb{C}}^*).$$

Suppose $i \leq m-1$. Then $\wedge^{i/2} \mathfrak{u}_{\mathbb{C}}^*$ is irreducible and $\wedge^{i/2} \mathfrak{u}_{H, \mathbb{C}}^*$ is the M_H -summand containing the M -highest weight vector, so injectivity of the previous map follows from the Lefschetz property in Theorem 4.1. (The subgroup M_H is never triality.) This concludes the proof. \square

This concludes the proof of Theorem 6.1.

APPENDIX A. SOME FACTS ABOUT L^2 COHOMOLOGY

Let G be a semisimple algebraic group over \mathbb{Q} , K a maximal compact subgroup of $G(\mathbb{R})$, $X = G(\mathbb{R})/K$ and $\Gamma \subset G(\mathbb{Q})$ a congruence subgroup.

The L^2 cohomology of Γ with coefficients in a finite-dimensional algebraic representation E of $G(\mathbb{C})$ is

$$H_{(2)}^*(\Gamma, E) = H^*(\mathfrak{g}, K, L^2(\Gamma \backslash G(\mathbb{R})) \otimes E).$$

(The (\mathfrak{g}, K) -cohomology of a $G(\mathbb{R})$ -representation (π, V) is, by definition, that of the space V^∞ of smooth vectors.) This is not the usual definition, which requires looking at the complex of L^2 differential forms with L^2 weak differential (or smooth L^2 forms with L^2 differential) but they are known to agree [BoC83, Prop. 5.4]. The decompositions

$$\begin{aligned} L^2(\Gamma \backslash G(\mathbb{R})) &= L_{dis}^2(\Gamma \backslash G(\mathbb{R})) \oplus L_{cts}^2(\Gamma \backslash G(\mathbb{R})) \\ &= L_{cusp}^2(\Gamma \backslash G(\mathbb{R})) \oplus L_{res}^2(\Gamma \backslash G(\mathbb{R})) \oplus L_{cts}^2(\Gamma \backslash G(\mathbb{R})) \end{aligned}$$

induce decompositions

$$\begin{aligned} H_{(2)}^*(\Gamma, E) &= H_{dis}^*(\Gamma, E) \oplus H_{cts}^*(\Gamma, E) \\ &= H_{cusp}^*(\Gamma, E) \oplus H_{res}^*(\Gamma, E) \oplus H_{cts}^*(\Gamma, E) \end{aligned}$$

where $H_{\gamma}^*(\Gamma, E) = H^*(\mathfrak{g}, K, L_{\gamma}^2(\Gamma \backslash G(\mathbb{R})) \otimes E)$. The summand $H_{dis}^*(\Gamma, E)$ is identified with the (finite-dimensional) space of E -valued L^2 harmonic forms for the invariant metric (a result due to Borel and Garland, see [BoC83, Prop. 4.4(i)]), while the summand $H_{cts}^*(\Gamma, E)$ either vanishes (e.g. when G is equal-rank) or is infinite-dimensional (see [BoC83]). The inclusion $L^2(\Gamma \backslash G(\mathbb{R}))^\infty \subset C^\infty(\Gamma \backslash G(\mathbb{R}))$ induces a natural map

$$H_{(2)}^*(\Gamma, E) \longrightarrow H^*(\Gamma, E)$$

which is well known to be injective on cuspidal cohomology. Proposition A.1 below is well known to experts but for lack of a suitable reference we give a proof, which is a simple matter of applying results of [Fra98] and [BoC83]. It goes beyond the existing literature ([BoC83]) only in the cases where $G(\mathbb{R})$ does not have a discrete series, e.g. $GL(n)$, and we need to use it in the main body of the paper for the case $SO(1, d)$, d odd.

Proposition A.1. *The map $H_{(2)}^*(\Gamma, E) \rightarrow H^*(\Gamma, E)$ is zero on $H_{cts}^*(\Gamma, E)$.*

Proof. Following Franke [Fra98], let $S_1(\Gamma \backslash G(\mathbb{R})) \subset L^2(\Gamma \backslash G(\mathbb{R}))$ be the submodule of smooth functions which are uniformly in L^2 , i.e. smooth functions f such that $Df \in L^2$ for all $D \in U(\mathfrak{g})$. The inclusion $S_1 \subset L^2$ induce isomorphisms in cohomology (by [Fra98, Theorem 3]), so $S_1 \subset C^\infty$ induces the map in question. It factors as

$$S_1(\Gamma \backslash G(\mathbb{R})) \subset S_{\log}(\Gamma \backslash G(\mathbb{R})) \subset C^\infty(\Gamma \backslash G(\mathbb{R}))$$

where S_{\log} is the space of functions which are uniformly L^2 up to logarithmic terms (see [Fra98, §5] or [Wal97, 6.1]). It suffices to show that $H_{cts}^*(\Gamma, E)$, as a summand of $H^*(\mathfrak{g}, K, S_1 \otimes E)$, goes to zero in $H^*(\mathfrak{g}, K, S_{\log} \otimes E)$. We will do this using further reductions to the bounded spectra $S_{1,b} \subset S_1$ and $S_{\log,b} \subset S_{\log}$ with respect to the Casimir operator, a notion introduced in [Fra98, 5.1] (cf. also [Wal97, 6.3]). We will show that:

- (1) $S_{1,b} \subset L^2$ induces an isomorphism $H^*(\mathfrak{g}, K, S_{1,b} \otimes E) = H_{(2)}^*(\Gamma, E)$,
- (2) $S_{\log,b} \subset S_{\log}$ induces an isomorphism in cohomology, and
- (3) $S_{1,b} \subset S_{\log,b}$ induces zero on $H_{cts}^*(\Gamma, E)$.

This will prove the proposition.

(1) To show that $S_{1,b} \subset L^2$ induces an isomorphism in cohomology we use Langlands spectral decomposition of L^2 . There are compatible direct sum decompositions

$$S_{1,b} = \bigoplus_{\{R\}} S_{1,b,\{R\}} \quad \subset \quad L^2 = \bigoplus_{\{R\}} L_{\{R\}}^2$$

indexed by associate classes of parabolic subgroups. The $R = G$ summands are $L_{dis}^{2,\infty}$ and L_{dis}^2 . (In the usual statement of $\{R\}$ -decompositions the $R = G$ summand is the cuspidal part; we are using the obvious rearrangement.) The inclusion $L_{dis}^{2,\infty} \subset L_{dis}^2$ induces an isomorphism by definition, so we must show the same for $S_{1,b,\{P\}} \subset L_{\{P\}}^2$ for proper P . For $P = MAN$, we have (cf. Theorem 11 of [Fra98]):

$$S_{1,b,\{P\}} = \left(\text{Ind}_P^G L_c^2(i\mathfrak{a}^*) \otimes A_2^M \right)^{W_M}$$

where A_2^M is the space of L^2 automorphic forms on $\Gamma_M \backslash M(\mathbb{R})$, and the Weyl group W_M of M acts by intertwining operators. The $\{P\}$ -summand of L^2 has the following description (see e.g. [BoC83, 4.3]). For each irreducible summand V of $L_{dis}^2(\Gamma_M \backslash M(\mathbb{R}))$ we have the direct integral

$$E_{P,V} = \int_{\mathfrak{a}^*}^{\oplus} \text{Ind}_P^G \mathbb{C}_{\rho+i\mu} \otimes V.$$

Then $L_{\{P\}}^2$ is the invariants under the action of W_M by intertwining operators on the (countable) Hilbert space direct sum of $E_{P,V}$ as V varies over all irreducible summands. One could, equivalently, restrict to a subset of V modulo W_M -equivalence and take the sum of direct integrals like the above over the positive Weyl chamber \mathfrak{a}^{*+} , which is the formulation in [BoC83, 4.3].

Now $H^*(\mathfrak{g}, K, E_{P,V} \otimes E) = \{0\}$ unless $E_{P,V}$ shares K -types with $\wedge^*(\mathfrak{g}/\mathfrak{k}) \otimes E^*$ and the Casimir acts by the correct scalar, so it follows that there is a finite set $\{V_i\}_{i \in I}$ of V such that the cohomology becomes a finite sum

$$H^*(\mathfrak{g}, K, L_{\{P\}}^2 \otimes E) = \left(\bigoplus_{i \in I} H^*(\mathfrak{g}, K, E_{P,V_i} \otimes E) \right)^{W_M}$$

(see [BoC83, Prop. 4.4(ii)]). The same applies to $S_{1,b,\{P\}}$, i.e. we may replace A_2^M by $\bigoplus_{i \in I} A_2^M \cap V_i^\infty$ and get the same cohomology. The computation of a single summand $H^*(\mathfrak{g}, K, E_{P,V_i} \otimes E)$ is contained in Theorem 3.4 of [BoC83] and is similar to the usual computation for induced representations in [BW99, III.3.3]. It gives that there is a unique $s \in W^P$ such that

$$H^*(\mathfrak{g}, K, E_{P,V_i} \otimes E) = H^{*-\ell(s)}(\mathfrak{m}, K_M, V_i \otimes E_{s(\lambda+\rho)-\rho}^M) \otimes H^* \left(\mathfrak{a}, \int_{\mathfrak{a}^*}^{\oplus} \mathbb{C}_{i\mu} d\mu \right)$$

where $E_{s(\lambda+\rho)-\rho}^M$ is the restriction to M of $E_{s(\lambda+\rho)-\rho}^{MA}$ (notation as in Kostant's theorem in 3.1). The parallel computation for $\text{Ind}_P^G L_c^2(i\mathfrak{a}^*) \otimes V_i$ (by the same arguments as in the proof of [BoC83, Theorem 3.4]) gives the same expression, with $\int_{\mathfrak{a}^*}^{\oplus} \mathbb{C}_{i\mu} d\mu$ replaced by $L_c^2(i\mathfrak{a}^*)$. The assertion that $S_{1,b,\{P\}} \subset L_{\{P\}}^2$ induces an isomorphism now boils down to the assertion that (for each P) the inclusion of $L_c^2(i\mathfrak{a}^*) = \text{colim}_{\Omega \subset \mathfrak{a}^*} \int_{\Omega}^{\oplus} \mathbb{C}_{i\mu} d\mu$ (the colimit taken over compact Ω) into the direct integral $\int_{\mathfrak{a}^*}^{\oplus} \mathbb{C}_{i\mu} d\mu$ is an isomorphism in \mathfrak{a} -cohomology. This elementary fact follows e.g. from Prop. 3.2 of [BoC83], which shows that this is already true of $\int_{\Omega}^{\oplus} \mathbb{C}_{i\mu} d\mu \subset \int_{\mathfrak{a}^*}^{\oplus} \mathbb{C}_{i\mu} d\mu$ if $0 \in \Omega$.

(2) To show that the inclusion $S_{\log,b} \subset S_{\log}$ induces an isomorphism in cohomology we simply combine [Fra98, Theorem 10] and the spectral sequence in (3) of [Fra98, Theorem 7]. (This may not be the simplest or most direct proof, but it is certainly the shortest to write down here!)

(3) We are reduced to considering the inclusion $S_{1,b} \subset S_{\log,b}$. Now $S_{\log,b}$ has a spectral decomposition analogous to that of $S_{1,b}$ in which the $\{P\}$ -summand for $P = MAN$ is

$$\left(\text{Ind}_P^G D'_c(i\mathfrak{a}^*) \otimes A_2^M \right)^{W_M}$$

where $D'_c(i\mathfrak{a}^*)$ is the space of compactly supported distributions (cf. [Fra98, Theorem 12]). The $P = G$ summand is still $L^{2,\infty}$. So the fact that $S_{1,b,\{P\}} \subset S_{\log,b,\{P\}}$ induces zero for P proper boils down to the fact that the inclusion of \mathfrak{a} -modules $L_c^2(i\mathfrak{a}^*) \subset D'_c(i\mathfrak{a}^*)$ induces zero in \mathfrak{a} -cohomology. This is immediate: The first has cohomology in degrees in $[1, \dim \mathfrak{a}]$ (by [BoC83, Prop. 3.2] as remarked above), while the latter has cohomology only in degree zero (e.g. by

[Fra98, Lemma 1]; this reduces to the fact that the complex $D'_c(\mathbb{R}) \xrightarrow{x} D'_c(\mathbb{R})$ has cohomology only in degree zero). This concludes the proof of the proposition. \square

The proof of the preceding proposition used three results (Theorems 10, 12, and 13 of [Fra98]), the proofs of which constitute the technical heart of Franke's work. It is possible that they can be avoided, but the method of proof gives rather more, as we now show. The results which follow are not used in the body of the paper, but will be useful elsewhere. The following is a corollary of the proof of the proposition:

Corollary A.2. *If E is rationally defined (i.e. has a rational structure preserved by $G(\mathbb{Q})$) then the square-integrable cohomology, which is (by definition) the image of $H^*_{(2)}(\Gamma, E) \rightarrow H^*(\Gamma, E)$, is a rational subspace.*

Proof. By [Nai99, Theorem A] the cohomology of S_{\log} is isomorphic to the lower middle weighted cohomology of [GHM94], and this has a rational structure ([GHM94, IV]) compatible with the map to $H^*(\Gamma, E)$. \square

In fact, we can refine this statement somewhat using the same methods. Recall that there is a subspace $S_{-\log}(\Gamma \backslash G(\mathbb{R})) \subset S_{\log}(\Gamma \backslash G(\mathbb{R}))$ defined by using a condition dual to the one defining S_{\log} (see [Fra98, §5] or [Wal97, 6.1]).

Proposition A.3. *The image of $H^*(\mathfrak{g}, K, S_{-\log} \otimes E) \rightarrow H^*(\mathfrak{g}, K, S_{\log} \otimes E)$ is identified with $H^*_{dis}(\Gamma, E)$, or, equivalently, with the space of E -valued L^2 harmonic forms.*

Proof. This was proved in [Nai99, Theorem B] under the assumption that G is equal-rank, in which case the map in the proposition is an isomorphism and both groups compute the L^2 cohomology. In general, we argue as follows. By Theorem 10 of [Fra98] and the spectral sequence of Theorem 7 of loc. cit., we know that $S_{\pm \log, b} \subset S_{\pm \log}$ induce isomorphisms in (\mathfrak{g}, K) -cohomology, so it is enough to consider the inclusion $S_{-\log, b} \subset S_{\log, b}$. For these spaces there are compatible decompositions

$$S_{-\log, b} = \bigoplus_{\{P\}} S_{-\log, b, \{P\}} \quad \subset \quad S_{\log, b} = \bigoplus_{\{P\}} S_{\log, b, \{P\}}$$

indexed by associate classes of parabolic subgroups. The $\{P\}$ -summand for $S_{-\log, P}$ is

$$\left(\text{Ind}_P^G C_c^\infty(\mathfrak{ia}^*) \otimes A_2^M \right)^{W_M}.$$

The $\{P\}$ -summand for $S_{\log, b}$ was recalled in the proof of the previous proposition and amounts to replacing $C_c^\infty(\mathfrak{ia}^*)$ by $D'_c(\mathfrak{ia}^*)$ in this expression; the map $S_{-\log, b, \{P\}} \subset S_{\log, b, \{P\}}$ is induced by $C_c^\infty(\mathfrak{ia}^*) \subset D'_c(\mathfrak{ia}^*)$. The map in \mathfrak{a} -cohomology induced by this inclusion is zero since $H^*(\mathfrak{a}, C_c^\infty(\mathfrak{ia}^*))$ is concentrated in degree $\dim \mathfrak{a}$ while $H^*(\mathfrak{a}, D'_c(\mathfrak{ia}^*))$ is concentrated in degree zero. By the standard computations of cohomology for induced representations (recalled earlier in the proof of the previous proposition) we see that $S_{-\log, b, \{P\}} \subset S_{\log, b, \{P\}}$ is zero in cohomology for $P \neq G$. Since the $P = G$ summands are both identified with L^2_{dis} , the first statement follows. \square

The previous corollary is refined by:

Corollary A.4. *If E is rationally defined then the space of square-integrable E -valued harmonic forms on $\Gamma \backslash X$ has a canonical (Betti) rational structure.*

Proof. By [Nai99, Theorem A] the groups in the proposition are the upper and lower middle weighted cohomology groups, which have natural \mathbb{Q} -structures ([GHM94, IV]). \square

Remark A.5. In contrast to the corollary, the space of cuspidal harmonic forms, which is simply the cuspidal cohomology, should not be expected to be Betti rational in general, e.g. for $Sp(4)$. Of course, it is well known to be so in the case of $GL(n)$, see [Fra98, 7.6], and some related cases, e.g. for $SO(1, d)$ it follows from Lemma 4.2.

Remark A.6. When $X = G(\mathbb{R})/K$ has a Hermitian structure something much stronger than the corollary is true thanks to (2.11), namely the space of L^2 harmonic forms is part of a mixed realization over the number field of definition of $\Gamma \backslash X$ (the reflex field if we work in the context of Shimura varieties), in particular it has both Betti and de Rham rational structures.

APPENDIX B. RESIDUAL EISENSTEIN COHOMOLOGY IN CORANK ONE

We summarize here some very well-known facts (essentially going back to [Har73]) on the construction of cohomology via residual Eisenstein series from cuspidal data on maximal parabolic subgroups. In the body of the paper these are applied to the rank one groups $SO(1, n)$ and $SU(1, n)$.

For a $G(\mathbb{R})$ -representation V the smooth vectors are denoted V^∞ and $H^*(\mathfrak{g}, K, V)$ is the (\mathfrak{g}, K) -cohomology of V^∞ . For a (\mathfrak{g}, K) -module or $G(\mathbb{R})$ -representation V let

$$d_{\min}(V) = \min\{i : H^i(\mathfrak{g}, K, V \otimes E) \neq 0\}, \quad d_{\max}(V) = \max\{i : H^i(\mathfrak{g}, K, V \otimes E) \neq 0\},$$

assuming these make sense and are finite.

Let $P = LN \subset G$ be a maximal parabolic subgroup and A the \mathbb{Q} -split part of the centre of L . Let $\sigma \in L_{\text{cusp}}^2(\Gamma_L A(\mathbb{R})^0 \backslash L(\mathbb{R}))$ be a cuspidal automorphic form on L . For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let $I_\lambda = \text{Ind}_P^G \sigma \otimes \mathbb{C}_\lambda$ be the (normalized) induced representation. The theory of Eisenstein series produces a (\mathfrak{g}, K) -homomorphism to the space of automorphic forms

$$E : I_\lambda^\infty \rightarrow \mathcal{A}(\Gamma \backslash G(\mathbb{R}))$$

which is meromorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. If λ is a pole of E (meaning that the Eisenstein series $E(\phi, \lambda)$ has a pole for generic ϕ in the space of σ), then P is self-associate and the pole is real and simple if $\text{Re}(\lambda) \in (\mathfrak{a}^*)^+$. Taking the residue at such a λ defines a residual Eisenstein operator

$$E^* : I_\lambda^\infty \rightarrow \mathcal{A}(\Gamma \backslash G(\mathbb{R})).$$

By [Fra98] the (\mathfrak{g}, K) -cohomology of $\mathcal{A}(\Gamma \backslash G(\mathbb{R})) \otimes E$ is $H^*(\Gamma, E)$.

Lemma B.1. *Suppose that E has a pole at λ and that the Langlands quotient J_λ of I_λ is cohomological. Suppose further that*

(*) $d_{\max}(I_\lambda) = d_{\max}(J_\lambda)$ and $H^{d_{\max}(I_\lambda)}(\mathfrak{g}, K, I_\lambda) \rightarrow H^{d_{\max}(J_\lambda)}(\mathfrak{g}, K, J_\lambda)$ is an isomorphism (see Remark B.2 below). Then the map in cohomology induced by $E^*(I_\lambda^\infty) \subset \mathcal{A}$ in degree $d_{\min}(J_\lambda)$ is injective and the classes in the subspace

$$H^{d_{\min}(J_\lambda)}(\mathfrak{g}, K, E^*(I_\lambda^\infty) \otimes E) \subset H^{d_{\min}(J_\lambda)}(\Gamma, E)$$

restrict nontrivially to the boundary, i.e. do not belong to $H_1^*(\Gamma, E)$.

Proof. We write I, J for I_λ, J_λ and ignore the coefficients E as they are not relevant. The residue of a cuspidal Eisenstein series at a point of the positive Weyl chamber is well-known to be square-integrable, so that $E^*(I^\infty) \subset \mathcal{A} \cap L_{\text{dis}}^2$ and as an abstract representation $E^*(I^\infty)$ is the Langlands quotient J^∞ . Taking the constant term of automorphic forms along P defines a mapping

$$I^\infty \xrightarrow{E^*} \mathcal{A} \longrightarrow I^{*, \infty}$$

where $I^* = \text{Ind}_P^G \sigma^* \otimes \mathbb{C}_{-\lambda}$ is the contragredient of I . (The usual expression for the constant term defines maps $I_\lambda \rightarrow \mathcal{A} \rightarrow I_\lambda \oplus I_\lambda^*$ for generic λ , but for the residual operator at a pole only the second term is nonvanishing.) The composite is a nonzero multiple of the standard intertwining operator $I \rightarrow I^*$, the image of which is the Langlands quotient J , and the factoring above is exactly $I^\infty \rightarrow J^\infty \subset I^{*, \infty}$.

Now by assumption $d_{\max}(I) = d_{\max}(J)$, and so by duality ([BW99, I.7.6] for the irreducible unitary module J and [BW99, III.3.3] combined with [BW99, I.7.6] for I), we have that

$$d_{\min}(J) = \dim X - d_{\max}(J) = \dim X - d_{\max}(I) = d_{\min}(I^*)$$

and $H^{d_{\min}(J)}(\mathfrak{g}, K, J) \cong H^{d_{\min}(J)}(\mathfrak{g}, K, I^*)$. Moreover, for a class in $H^{d_{\min}(J)}(\mathfrak{g}, K, E^*(I))$ in $H^*(\Gamma, \mathbb{C})$, the induced mapping

$$C^*(\mathfrak{g}, K, E^*(I^\infty)) \rightarrow C^*(\mathfrak{g}, K, I^{*,\infty})$$

gives, via the identification of $H^*(\mathfrak{g}, K, I^{*,\infty})$ with a summand of the cohomology of the P -boundary, the restriction of the class to the P -boundary. (This is contained in [Har73] in a differential-geometric language and [Sch83] in representation-theoretic terms.) The restriction is therefore nonzero (because $H^{d_{\min}(J)}(\mathfrak{g}, K, J) \cong H^{d_{\min}(J)}(\mathfrak{g}, K, I^*)$) and so $H^{d_{\min}(J)}(\mathfrak{g}, K, E^*(I)) \rightarrow H^*(\Gamma, \mathbb{C})$ is injective. The classes in this subspace survive on restriction to the boundary so they are not in interior cohomology. \square

Remark B.2. In fact (*) always holds for a unitary cohomological Langlands quotient, but rather than prove this general fact will verify it in the cases of interest.

APPENDIX C. CHERN CLASSES OF AUTOMORPHIC VECTOR BUNDLES

We are in the situation of 2.2: G is semisimple and simply-connected, $X = G(\mathbb{R})/K$ is Hermitian, and $M_\Gamma = \Gamma \backslash X$. Fix a smooth toroidal compactification $M_\Gamma \hookrightarrow M_\Gamma^\Sigma$ in which the boundary is a simple normal crossings divisor [AMRT]. Let $\text{Rep}(H)$ denote the category of finite-dimensional representations of a compact group H . With E in $\text{Rep}(K)$ are associated the homogeneous bundle \mathcal{E}^c on X^c , the automorphic vector bundle $\mathcal{E}_\Gamma^\Sigma$ on $M_\Gamma = \Gamma \backslash X$, and the canonical extension $\mathcal{E}_\Gamma^\Sigma$ of \mathcal{E}_Γ to M_Γ^Σ .

Lemma C.1. *There is an injective ring homomorphism $\theta : H^*(X^c, \mathbb{Q}) \rightarrow H^*(M_\Gamma^\Sigma, \mathbb{Q})$ with $\theta(c_k(\mathcal{E}^c)) = (-1)^k c_k(\mathcal{E}_\Gamma^\Sigma)$ for all $E \in \text{Rep}(K)$.*

Proof. Following a suggestion of N. Fakhruddin we will use K -theory to prove this. Let $K^0(-)$ denote the topological K -theory of a space and $ch : K^0(-) \rightarrow H^*(-, \mathbb{Q})$ the Chern character homomorphism. We write $R(H)$ for the representation ring of a compact group H , i.e. the Grothendieck group of the category $\text{Rep}(H)$

We first show that the ring homomorphism $R(K) \rightarrow H^*(M_\Gamma^\Sigma, \mathbb{Q})$ defined by $V \mapsto ch(\mathcal{V}_\Gamma^\Sigma)$ and extended \mathbb{Q} -linearly defines a ring homomorphism

$$\kappa : K^0(X^c) \otimes \mathbb{Q} \rightarrow H^*(M_\Gamma^\Sigma, \mathbb{Q}). \quad (\text{C.1})$$

Since $X^c = G(\mathbb{R})^c/K$ with $G(\mathbb{R})^c$ simply connected (it is the maximal compact of the simply-connected group $G(\mathbb{C})$) and $K \subset G(\mathbb{R})^c$ is a subgroup of maximal rank, the construction $V \mapsto \mathcal{V}^c$ gives an isomorphism

$$R(K) \otimes_{R(G(\mathbb{R})^c)} \mathbb{Z} \rightarrow K^0(X^c)$$

where \mathbb{Z} is an $R(G(\mathbb{R})^c)$ -module via the dimension homomorphism (by [Pit72, Theorem 3]). Since the left-hand side is the quotient of $R(K)$ by the ideal generated by $\ker(\dim : R(G(\mathbb{R})^c) \rightarrow \mathbb{Z})$ and ch is a ring homomorphism, to show that κ is well-defined it suffices to show that $ch(\mathcal{E}^\Sigma) = \dim E$ if $E \in \text{Rep}(G(\mathbb{R})^c)$. The degree zero term of the Chern character of a bundle is its rank, so it suffices to check that $c_k(\mathcal{E}_\Gamma^\Sigma) = 0$ for $k > 0$ for such E . Now Mumford showed that the k th Chern form of the invariant or Nomizu connection defines a current on M_Γ^Σ which represents (up to a factor of $(2\pi\sqrt{-1})^k$) the Chern class $c_k(\mathcal{E}_\Gamma^\Sigma)$ ([Mum77, Theorem 3.1 and Theorem 1.4]). But if E is a $G(\mathbb{R})^c$ -representation the curvature 2-form of the Nomizu connection vanishes identically (see e.g. [GP02, Proposition 5.3]), hence so do its Chern forms for $k > 0$. Thus $c_k(\mathcal{E}_\Gamma^\Sigma) = 0$ for $k > 0$ and $ch(\mathcal{E}_\Gamma^\Sigma) = \dim E$, and we have κ as in (C.1). It is a ring homomorphism because canonical extension is compatible with tensor product [Har89, 4.2] and the Chern character is multiplicative.

Since X^c is a flag variety it has only even-degree cohomology, so the Chern character gives an isomorphism $ch : K^0(X^c) \otimes \mathbb{Q} \rightarrow H^*(X^c, \mathbb{Q})$ (cf. [AH61, 2.4]). Now define

$$\theta := \kappa \circ ch^{-1} \circ \sigma$$

where $\sigma : H^*(X^c, \mathbb{Q}) \rightarrow H^*(X^c, \mathbb{Q})$ is defined by $\sigma(\alpha) = (-1)^{\deg(\alpha)/2}\alpha$. Since $H^*(X^c, \mathbb{Q})$ is concentrated in even degrees this makes sense and σ is a ring homomorphism. Note that $\theta(\sigma(ch(\mathcal{E}^c)) = ch(\mathcal{E}_\Gamma^\Sigma)$, from which it follows that $\theta(c_k(\mathcal{E}^c)) = (-1)^k c_k(\mathcal{E}_\Gamma^\Sigma)$. This implies that θ is injective (i.e. nonzero) in top degree $2n = 2\dim_{\mathbb{C}} X$: Choose a nonzero monomial $c_{k_1}(\mathcal{E}_1^c) \cdots c_{k_r}(\mathcal{E}_r^c)$ with $\sum_i k_i = n$; it spans $H^{2n}(X^c, \mathbb{Q})$. By Mumford's version of proportionality [Mum77, Theorem 3.2],

$$[M_\Gamma^\Sigma] \cap \theta(c_{k_1}(\mathcal{E}_1^c) \cdots c_{k_r}(\mathcal{E}_r^c)) = (-1)^n \cdot C \cdot [X^c] \cap c_{k_1}(\mathcal{E}_1^c) \cdots c_{k_r}(\mathcal{E}_r^c) \neq 0$$

where C is a nonzero constant independent of k_i, \mathcal{E}_i . It follows that θ is injective: For nonzero $\alpha \in H^i(X^c)$ choose $\beta \in H^{2n-i}(X^c)$ such that $\alpha \cdot \beta \neq 0$. Then $0 \neq \theta(\alpha \cdot \beta) = \theta(\alpha) \cdot \theta(\beta)$, so that $\theta(\alpha) \neq 0$. \square

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