MIXED STRUCTURES IN SHIMURA VARIETIES AND AUTOMORPHIC FORMS

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Abstract. We study the mixed structures appearing in the cohomology of noncompact Shimura varieties using a new spectral sequence which arises naturally from Morel's theory of weight truncations [M08]. The E_1 term of this spectral sequence is a direct sum of pure structures which are related to intersection cohomology (with coefficients) of strata of the minimal (=Baily-Borel) compactification. We use it to show that the weight-graded pieces of the cohomology of a noncompact Shimura variety are subquotients of intersection cohomology (with twisted coefficients) of minimal compactifications of smaller Shimura varieties. The main result is that this spectral sequence is automorphic, i.e. given by an explicit filtration of the space of automorphic forms (which computes the cohomology by Franke [F98]) by conditions on exponents. The main theorem is an application of a technical result of independent interest relating the weighted complexes on minimal compactifications defined by Morel with the weighted complexes of Goresky, Harder, and MacPherson [GHM94] defined on the reductive Borel-Serre compactification, and also uses results of [N99, F98].

1. Introduction

Let M be a connected component of a Shimura variety at finite level, i.e. $M = \Gamma\backslash G(\mathbb{R})/K$ for a semisimple $\mathbb{Q}$-algebraic group G, a maximal compact subgroup $K \subset G(\mathbb{R})$, and a congruence arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$. Let $\mathbb{E}$ be the local system on M given by a $\mathbb{Q}$-rational irreducible representation E of G. We assume that M is noncompact. The cohomology $H^*(M, \mathbb{E})$ has a mixed structure – a mixed Hodge structure or a mixed Hodge-de Rham structure over the number field of definition of M, or a mixed Galois module structure (if $\mathbb{E}$ is of geometric origin), in short a "mixed motive". The relation of this mixed structure (particularly its weight filtration) to automorphic forms is subtle.

In [M08] S. Morel introduced a general method of weight truncations using novel t-structures in derived categories of mixed sheaves. Let $j : M \hookrightarrow M^*$ be the inclusion of M in its minimal (i.e. Baily-Borel-Satake) compactification. The formalism of [M08] gives a natural spectral sequence converging to $H^*(M, \mathbb{E})$ in which the E_1 terms are related to intersection cohomology groups (with coefficients) of minimal compactifications of strata of M^*. In terms of Morel's weight truncation functors $w_{\leqslant a}$ (cf. [M08, §3] or 2.1) the E_1 term has the form

$$E_1^{p,q} = H^{p+q}(M^*, w_{\leqslant \dim M + w - p} w_{\geqslant \dim M + w - p} Rj_* \mathbb{E}^H) \quad \Rightarrow \quad H^{p+q}(M, \mathbb{E}^H) \qquad (1)$$

where $p \leq 0$. This is a second quadrant spectral sequence with edge terms given by $E_1^{0,q} = IH^q(M^*, \mathbb{E})$. The superscript H indicates that we are in Saito's derived category of mixed Hodge modules [S90], w is the weight of the natural variation of Hodge structure supported by $\mathbb{E}$, and $\mathbb{E}^H$ is the associated Hodge module; this is then a spectral sequence of mixed Hodge structures. In fact one can work in any theory of mixed sheaves in which $\mathbb{E}$ has a lift,

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e.g. mixed Hodge modules with de Rham rational structure [S06] or, if $\mathbb{E}$ is of geometric origin, mixed l-adic complexes [M11]. The analogue of (1) is then a spectral sequence in the appropriate category, e.g. mixed Hodge-de Rham structures over the field of definition, or Galois representations. In any of these settings, the E_1 term is a direct sum of *pure* structures (typically, of different weights), which are intersection cohomology groups (with coefficients) of closures of strata in M^*.

As an illustration of the utility of (1), we use it to give a quick proof of the following:

Theorem 1.1. *(cf. Theorem 3.10.1) The pure Hodge-de Rham structures appearing as weight-graded pieces of the cohomology $H^*(M, \mathbb{E})$ of a Shimura variety M are subquotients of the intersection cohomology (with coefficients) of minimal compactifications of the Shimura varieties which appear as strata of the minimal compactification M^*.*

By Theorem 1.1 the cohomology of noncompact Shimura varieties gives no new pure motives (i.e. besides those appearing in the intersection cohomology of minimal compactifications). ([1]) So the genuine interest in $H^*(M, \mathbb{E})$ lies in the *mixed* motives appearing in it. There is plenty of evidence (especially in the work of Harder [H93, H13]) that this is an interesting and difficult question, and (1) seems to provide the general setting in which to approach it.

The main result of this paper is that the spectral sequence (1) also has an *automorphic* description. Recall that there is an analytic Hodge theory available for $H^*(M, \mathbb{E})$. Let $\mathscr{A} = \mathscr{A}(\Gamma\backslash G(\mathbb{R}))$ be the space of automorphic forms, i.e. the space of C^∞ complex-valued functions on $\Gamma\backslash G(\mathbb{R})$ which are finite under K and under the centre of the universal enveloping algebra of $\mathfrak{g} = \operatorname{Lie} G(\mathbb{R})$ and satisfy a uniform moderate growth condition. By the fundamental theorem of Franke [F98],

$$H^*(M, \mathbb{E}) \otimes \mathbb{C} = H^*(\mathfrak{g}, K; \mathscr{A} \otimes E). \tag{2}$$

We prove (cf. Theorem 5.2.1(ii)):

Theorem 1.2. *There is an explicit decreasing filtration of the space of automorphic forms*

$$\cdots \supset F^p\mathscr{A} \supset F^{p+1}\mathscr{A} \supset \cdots$$

given by imposing conditions on the exponents (i.e. conditions on the Fourier expansions of the constant terms along split central tori of Levi subgroups, see 5.3 below for the precise definition) such that the associated spectral sequence with

$$E_1^{p,q} = H^{p+q}(\mathfrak{g}, K; F^p\mathscr{A}/F^{p+1}\mathscr{A} \otimes E) \quad \Rightarrow \quad H^*(M, \mathbb{E}) \otimes \mathbb{C} \tag{3}$$

is isomorphic to the spectral sequence (1) (tensored with $\mathbb{C}$) from the E_1 term onwards.

For the indexing to agree with that of (1) one has that the nonzero graded quotients of F are for $p \le 0$ and $F^0\mathscr{A}$ is the space of automorphic forms which are square-integrable up to some logarithmic factors, so that $E_1^{0,*}$ is the cohomology of the L^2 discrete spectrum.

This theorem allows us to apply analytic methods to (1). In [F98, §6] Franke introduced a family of similarly-defined filtrations and used Langlands theory of Eisenstein series to

[1] The same approach should work in the context of Galois representations, but we do not prove it here. Since one expects that motives appearing in intersection cohomology are polarizable (and this can be proved in many cases), the theorem suggests that there are no non-self-dual motives appearing in the cohomology of Shimura varieties. The introduction of [HLTT] mentions that this was known to Harris and to Clozel. (I thank Clozel for suggesting the inclusion of Theorem 1.1 to me after a discussion in December 2011.)

describe their graded pieces in terms of square-integrable automorphic forms on Levi subgroups. The filtration in the theorem is in fact *not* one of Franke's, but rather a degenerate limiting version of these. Franke's methods in [F98, §6] can be adapted to the filtration in the theorem to give an automorphic description of the E_1 term ($\otimes\mathbb{C}$), similar to that in [F98, §7] for the spectral sequences associated to Franke's filtrations. Thus the description of the discrete L^2 spectrum conjectured by Langlands and Arthur [Art89] should give a precise description of E_1 in terms of parameters, in a way that the mixed structures can be read off from the description, and the differentials can be related to Eisenstein series. ([2])

Since the detailed analysis of the spectral sequence requires mainly analytic techniques, while the arguments of this paper are mainly geometric (or topological), the discussion of the previous paragraph (and the consideration of examples) is postponed to a sequel. For the moment we use Theorem 1.2 and a result of Franke to show:

Theorem 1.3. *(cf. Theorem 5.2.1 (iii)) If the representation E has regular highest weight then the spectral sequence (1) degenerates at E_1.*

Note that for such coefficient systems $H^i(M, \mathbb{E})$ vanishes for $i < \dim M$ due to a result of Li-Schwermer [LS04] and Saper [Sap05a].

The general analysis of the cohomology of noncompact locally symmetric spaces, starting with the work of Borel and Harder in the 1970s, has traditionally used the nerve spectral sequence for the cohomology of the Borel-Serre boundary $\partial\overline{M}^{BS}$ and the theory of Eisenstein cohomology to lift classes from the boundary. In the Shimura variety case the boundary cohomology has a natural mixed structure such that the long exact sequence

$$\cdots \longrightarrow H^i_c(M, \mathbb{E}) \longrightarrow H^i(M, \mathbb{E}) \longrightarrow H^i(\partial\overline{M}^{BS}, \mathbb{E}|_{\partial\overline{M}^{BS}}) \longrightarrow \cdots$$

is one of mixed structures. In the sequence of papers [HZ94a, HZ94, HZ01], Harris and Zucker proved the remarkable fact that the nerve spectral sequence for the covering of $\partial\overline{M}^{BS}$ by its closed faces is a spectral sequence of mixed Hodge-de Rham structures.

More recently, the work of Franke [F98] provided a new general framework for the study of cohomology via Eisenstein series, one in which Franke's filtrations on the space of automorphic forms play an essential role. We refer to [F98, FS98, LS04, F08] etc. for some notable applications of these methods. Theorem 1.2 can then be seen as providing, in the Shimura variety case, a mixed Hodge-de Rham theory for this approach. ([3]) The theorem can also be seen as singling out, in the case of Shimura varieties, a particular filtration on the space of automorphic forms for which the spectral sequence has good motivic properties.

There is no simple relation between the nerve spectral sequence and (1) if G has $\mathbb{Q}$-rank > 1. Indeed, the E_1 term in the nerve spectral sequence is (in general) genuinely mixed, and the relation of its limit filtration to automorphic forms is complicated (cf. [OS90, HZ94]). So (1) seems (to me) rather better suited to the problem of understanding the mixed motives in Shimura varieties.

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[2] For purely geometric reasons (see the proof of Theorem 5.2.1 (i)), the E_1 term of (1) is a sum of intersection cohomology with coefficients of Shimura varieties appearing in the minimal boundary; the point here is that Theorem 1.2 describes the coefficient systems appearing in terms of the discrete spectrum of the linear subgroups of Levi subgroups (cf. 3.2).

[3] As mentioned above, (1) is related to a degenerate version of Franke's filtrations. The spectral sequences associated with all of Franke's filtrations are spectral sequences of mixed HdR structures (as can be proved using the methods of this paper, see Remark 5.3.1(ii)), but they do not have the purity property of (1) so we do not study them here.

Theorem 1.2 is an application of a technical result of independent interest. This is the comparison of two constructions of weighted complexes in the literature, namely Morel's weighted complexes, defined on the minimal compactification M^* of a locally symmetric variety using the weight truncations of [M08], and the weighted cohomology complexes of Goresky, Harder, and MacPherson [GHM94], defined on the reductive Borel-Serre (RBS) compactification $\overline{M}$ of an arbitrary locally symmetric space. (Morel's construction is meant to be an algebraic analogue of [GHM94], so that the existence of such a relation is not entirely surprising.)

Recall that given a weight, i.e. a quasicharacter $\nu \in X^*(A_0)_{\mathbb{Q}}$ on a maximal $\mathbb{Q}$-split torus $A_0 \subset G$, the construction of [GHM94] defines a complex of sheaves $W^{\geqslant\nu}C(\mathbb{E})$ on $\overline{M}$. There is a proper continuous map $p : \overline{M} \to M^*$ extending the identity on M, giving a constructible complex $Rp_*W^{\geqslant\nu}C(\mathbb{E})$ on M^*. On the other hand, for a $\mathbb{Q}$-valued function $\underline{a}$ on the set of strata of M^*, the construction of [M08] defines objects $w_{\leqslant\underline{a}}Rj_*\mathbb{E}^H$ and $w_{>\underline{a}}Rj_*\mathbb{E}^H$ in the derived category of mixed Hodge modules on M^*. The underlying $\mathbb{Q}$-complexes $rat(w_{\leqslant\underline{a}}Rj_*\mathbb{E}^H)$ and $rat(w_{\leqslant\underline{a}}Rj_*\mathbb{E}^H)$ are objects in the derived category $D^b_c(\mathbb{Q}_{M^*})$ of constructible complexes on M^*. In 4.3 we associate (in an explicit way) with $\nu \in X^*(A_0)_{\mathbb{Q}}$ a function $\underline{a}$ such that, denoting by $\underline{a}+w$ the sum of $\underline{a}$ with the constant function w, we have:

Theorem 1.4. *(cf. Theorem 4.3.1) For $\underline{a}$ associated with ν there are natural and Hecke-equivariant isomorphisms in $D^b_c(\mathbb{Q}_{M^*})$:*

$$rat(w_{\leqslant\underline{a}+w}Rj_*\mathbb{E}^H) = Rp_*W^{\geqslant\nu}C(\mathbb{E}) \tag{a}$$

$$rat(w_{>\underline{a}+w}Rj_*\mathbb{E}^H_M) = Rp_*\big(W^{\geqslant-\infty}C(\mathbb{E}_M)/W^{\geqslant\nu}C(\mathbb{E}_M)\big), \tag{b}$$

where $-\infty$ stands for any sufficiently negative weight (so that $Rp_*W^{\geqslant-\infty}C(\mathbb{E}_M) = Rj_*\mathbb{E}_M$).

Analogous statements hold in other categories of mixed sheaves (see Remark 4.3.7). There are some interesting special cases of (a):

When $\nu = -\rho$ (the half-sum of negative roots) then $\underline{a}$ is the constant function $\dim M$ and one gets the intersection complex $(j_{!*}\mathbb{E}_M[\dim M])[-\dim M]$ on both sides by [GHM94, Theorem 23.2] and [M08, Theorem 3.1.4].

When $\nu = 0$ one has for $\underline{a}$ the dimension function $\underline{dim}(S) = \dim S$ and for E trivial one gets the identity

$$Rp_*\mathbb{Q}_{\overline{M}} = rat(w_{\leqslant\underline{dim}}Rj_*\mathbb{Q}^H). \tag{4}$$

This shows that the cohomology of the RBS compactification carries a natural mixed structure. In fact, according to Ayoub and Zucker [AZ12], the RBS compactification is even motivic (in the sense of Voevodsky). In [V12, V13] Vaish gives an alternate construction of the motive of Ayoub and Zucker along the lines suggested by (4); together with the l-adic analogue of (4) this recovers the main results of [AZ12] in a different way and in slightly strengthened form. Another application of (4) and related results is in [N10] where we use it (and results from [NV12]) to prove several new results about the cohomology of the RBS compactification by a mixture of analytic and geometric methods. (See 4.6 for some further discussion of these results related to $\overline{M}$.)

When $\underline{a}$ has constant value $a \geqslant \dim M$ the resulting complexes are relevant to (1) and its limit filtration.

As a corollary of Theorem 1.4, we see that the weighted cohomology groups of a locally symmetric variety carry natural mixed Hodge-de Rham structures and also Galois representations. In some cases (including those just mentioned) these provide mixed structures which can be related to automorphic forms. For example, when $\underline{a}$ has constant value $a \geqslant \dim M$ the cohomology $\mathbb{H}^*(M^*, w_{\leqslant a+w} Rj_* \mathbb{E}^H_M)$ is a mixed structure which can be computed in terms of automorphic forms (by the theorem and results of [N99, F98] recalled in 5.3). It surjects onto a mixed substructure of $H^*(M, \mathbb{E})$ (the corresponding step of the limit filtration of (1)). This suggests that some of these mixed structures should provide a simpler setting (i.e. simpler than $H^*(M, \mathbb{E})$) in which to study the general relation between automorphic forms and mixed motives. Thanks to Morel's formalism we can prove some properties of these mixed structures (e.g. determine the highest weight quotient or lowest weight subspace) in some cases. The results are summarized in Theorem 4.5.3.

We make some comments on the proofs of Theorems 1.1–1.4.

The proof of Theorem 1.1 uses (1), some simple properties of Morel's truncation functors, and a result from [HZ01] or [BW04].

The proof of Theorem 1.4 uses the ubiquitous local Hecke operator introduced by Looijenga [L88] and the relation between local Hecke weights and weights in the sense of mixed Hodge theory or Frobenius established by Looijenga and Rapoport [LR91]. We remark that some difficulties are caused by the fact that we are comparing the functorial image (under rat) of an object defined using a t-structure (viz. $w_{\leqslant \underline{a}+w} Rj_* \mathbb{E}^H$) with an object defined via an explicit complex (viz. $Rp_* W^{\geqslant \nu} C(\mathbb{E})$), and the comparison takes place in the category $D^b_c(\mathbb{Q}_X)$, where the t-structure is not available. A key input is a splitting property of the weighted complexes (of either type) in the derived category with respect to the action of a local Hecke operator. For the construction of [GHM94] this splitting property was already used for the proof of the main theorem of loc. cit.; we recall it as Lemma 4.2.2. For the construction of [M08] this is proved in Lemma 4.3.4 using the relation between local Hecke weights and mixed Hodge weights from [LR91]. The proof of Theorem 1.4 can be adapted to give versions of the theorem in some other categories of mixed sheaves, in particular mixed l-adic complexes (cf. Remark 4.3.7).

Theorem 1.2 is deduced from Theorem 1.4 using the main results of [N99] and [F98] as follows: Theorem 1.4 allows us to work with a particular complex representing weighted cohomology, namely the one defined in [GHM94, II] using special differential forms. In [N99] this complex was compared with certain complexes of differential forms with square-integrability conditions coming from the work of Franke [F98]. Specific properties of the $\mathbb{Q}$-root system of groups giving rise to Shimura varieties (verified in an appendix) then come into play to allow a further reduction to spaces of automorphic forms, using results of [F98]. ([4])

Theorem 1.3 follows easily from Theorem 1.2 and a result [F98, Theorem 19] about the degeneration of Franke's spectral sequences.

We briefly mention some applications of these results which will be taken up elsewhere. In [F08] Franke showed that the submodule of Hecke-invariant classes is a direct summand in $H^*(M, \mathbb{Q})$ and has a particularly simple topological model in terms of an open subset of

[4]Due to a technical difficulty to do with cones in the derived category we do not prove here that the isomorphism in Theorem 1.2 is Hecke-equivariant. This does not affect the applications mentioned here; we hope to deal with this elsewhere.

the compact dual. It should be possible to describe this summand more completely as a mixed structure (it is of mixed Tate type) using the results here. Secondly, in the regular coefficients case the degeneration at E_1 of the spectral sequence (1) should allow us to give a more detailed description of the mixed structure $H^*(M, \mathbb{E})$. Finally, the vanishing theorems of Borel [B87] and Saper [Sap05] for the cohomology of linear locally symmetric spaces give vanishing results for summands of the E_1 term; in some examples these should give interesting constraints on the weight filtration of $H^*(M, \mathbb{E})$.

The arguments in this paper are written in the context of mixed Hodge modules, but many of them work, with minor changes, in other settings of mixed sheaves. Rather than work in a general setting we make remarks about the necessary changes for this from time to time. We have also avoided the formalism of Shimura varieties for simplicity, although this would have been more elegant at some places (and would give results over the reflex field).

The contents of the various sections are as follows:

In section 2 we recall Morel's theory of weight truncations, as transposed to the setting of mixed Hodge modules (or, more generally, any theory of mixed sheaves for varieties over a subfield of $\mathbb{C}$) and prove some additional properties of the weight truncation functors.

In section 3 we recall background material on locally symmetric varieties and the minimal and reductive Borel-Serre compactifications and discuss the pure structures appearing as the weight-graded pieces of $H^*(M, \mathbb{E})$ (proving Theorem 1.1).

In section 4 we prove the basic relation between the weighted complexes and the weight truncations (i.e. Theorem 1.4). We also prove some basic properties of the resulting mixed structures on weighted cohomology.

In section 5 we discuss the spectral sequence (1) and its relation with automorphic forms and prove Theorems 1.2 and 1.3.

The appendix summarizes facts about the relative root system required in sections 4 and 5.

2. Weight truncation of mixed sheaves

2.1. **Weight truncation of mixed Hodge modules.** We first recall some facts from Saito's theory of mixed Hodge modules (cf. [S88, S90], especially §4 of the latter). For any complex algebraic variety X, Saito defines the abelian category $\mathrm{MHM}(X)$ of (algebraic) mixed Hodge modules on X. This is the analogue in Hodge theory of the category of mixed perverse complexes of [BBD82]. When X is a point $\mathrm{MHM}(X)$ is equivalent to the category of rational mixed Hodge structures with polarizable pure subquotients. When X is smooth $\mathrm{MHM}(X)$ contains polarizable variations of Hodge structure on X and, more generally, admissible variations of mixed Hodge structure with polarizable pure subquotients. In general, there is a faithful and exact functor rat from $\mathrm{MHM}(X)$ to the category of perverse complexes of $\mathbb{Q}_X$-sheaves constructible for an algebraic stratification. rat gives a functor on bounded derived categories:

$$rat : D^b\mathrm{MHM}(X) \to D^b_c(\mathbb{Q}_X).$$

For algebraic maps $f : X \to Y$ there are functors $Rf_*, f^*, Rf_!, f^!$ and $\otimes$ and $R\mathrm{Hom}$ between the appropriate derived categories of mixed Hodge modules, and these are compatible under *rat* with the corresponding functors on derived categories of complexes of $\mathbb{Q}$-sheaves ([5]). The cohomology functor $H^0 : D^b\mathrm{MHM}(X) \to \mathrm{MHM}(X)$ goes to the perverse cohomology functor ${}^pH^0$ on $D^b_c(\mathbb{Q}_X)$ under rat, i.e. $rat \circ H^0 = {}^pH^0 \circ rat$.

Mixed Hodge modules have weight filtrations and for $K \in \mathrm{MHM}(X)$ and $a \in \mathbb{Z}$ there is a canonical short exact sequence in $\mathrm{MHM}(X)$

$$0 \longrightarrow w_{\leqslant a}K \longrightarrow K \longrightarrow w_{>a}K \longrightarrow 0$$

where $w_{\leqslant a}K$ is the maximal subobject of K with weights $\leqslant a$ and $w_{>a}K$ is the maximal quotient object of K with weights $> a$ (equivalently, weights $\geqslant a+1$). This is, up to unique isomorphism, the unique short exact sequence $0 \to K' \to K \to K'' \to 0$ with K' having weights $\leqslant a$ and K'' having weights $> a$. By strictness of morphisms in $\mathrm{MHM}(X)$ with respect to weight filtrations, this implies that $K \mapsto w_{\leqslant a}K$ and $K \mapsto w_{>a}K = w_{\geqslant a+1}K$ are functors. Represent a complex K of $D^b\mathrm{MHM}(X)$ by a bounded complex $(C^i)_{i\in\mathbb{Z}}$ of objects in $\mathrm{MHM}(X)$. Setting $(w_{\leqslant a}K)^i := w_{\leqslant a}C^i$ and $(w_{>a}K)^i := w_{>a}C^i$ can be shown to define functors $w_{\leqslant a}, w_{>a} : D^b\mathrm{MHM}(X) \to D^b\mathrm{MHM}(X)$.

Morel's original definition of these functors (rather, the analogues in the ℓ-adic context) is via a t-structure. Following [M08, §3], define ${}^wD^{\leqslant a} = {}^wD^{\leqslant a}\mathrm{MHM}(X)$ (resp. ${}^wD^{>a}$) to be the full subcategory of $D^b\mathrm{MHM}(X)$ with objects

$$Ob\ {}^wD^{\leqslant a} = \{K \in D^b\mathrm{MHM}(X) : H^i(K) \text{ has weights } \leqslant a \text{ for all } i\}.$$

(resp. with objects

$$Ob\ {}^wD^{>a} = \{K \in D^b\mathrm{MHM}(X) : H^i(K) \text{ has weights } > a \text{ for all } i\}).$$

Then $({}^wD^{\leqslant a}, {}^wD^{>a})$ is a t-structure on $D^b\mathrm{MHM}(X)$. (The proof of this from [M08] carries over, the main point being that $\mathrm{Hom}^i(K, L) = 0$ if K has weights $\leqslant a$ and L has weights $> a + i$ [S90, 4.5.3]. Recall that a complex $K \in D^b\mathrm{MHM}(X)$ is said to have weights $\leqslant a$ (resp. $\geqslant a$) if $H^i(K)$ is of weights $\leqslant i + a$ (resp. $\geqslant i + a$) for all i ([S90, 4.5]).) The corresponding truncation functors are

$$w_{\leqslant a}, w_{>a} : D^b\mathrm{MHM}(X) \to D^b\mathrm{MHM}(X)$$

(the right adjoint to ${}^wD^{\leqslant a} \subset D^b\mathrm{MHM}(X)$ and the left adjoint to ${}^wD^{>a} \subset D^b\mathrm{MHM}(X)$, respectively). This t-structure has some unusual properties: ${}^wD^{\leqslant a}$ and ${}^wD^{>a}$ are full triangulated subcategories of $D^b\mathrm{MHM}(X)$. The functors $w_{\leqslant a}$ and $w_{>a}$ commute with shifts, take distinguished triangles to distinguished triangles, satisfy $w_{\leqslant a}(K(1)) = (w_{\leqslant a+2}K)(1)$ and $w_{>a}(K(1)) = (w_{>a+2}K)(1)$, and satisfy $\mathbb{D} \circ w_{\leqslant a} = w_{\geqslant -a} \circ \mathbb{D}$ where $\mathbb{D}$ is the Verdier duality functor. Any object in the heart ${}^wD^{\leqslant a} \cap {}^wD^{>a}$ of the t-structure is isomorphic to zero. The cohomology sequences (for the usual cohomology functor H^0) associated with the distinguished triangle $w_{\leqslant a}K \longrightarrow K \longrightarrow w_{>a}K \xrightarrow{+1}$ are short exact and $H^i(w_{\leqslant a}K) = w_{\leqslant a}H^i(K)$ and $H^i(w_{>a}K) = w_{>a}H^i(K)$. (The proofs of these facts in [M08] carry over, mutatis mutandis, to our setting.)

A key result is the following:

Theorem 2.1.1. *(Morel* [M08, Thm 3.1.4]*) If $j : U \hookrightarrow X$ is a nonempty open subset and K is a pure polarizable Hodge module of weight a on U then $w_{\geqslant a}j_!K = j_{!*}K = w_{\leqslant a}Rj_*K$.*

[5]We follow Saito's notation in [S88, S90], except that we use the slightly misleading notation $Rf_*, Rf_!$ instead of $f_*, f_!$ for direct images in the categories of mixed Hodge modules to agree with the notation for complexes of sheaves.

Suppose now that we are given a stratification of X, i.e. a partition $X = \coprod_{i=0}^{r} S_i$ into locally closed subvarieties with each S_d open in $X - \sqcup_{0\leqslant j<d} S_j$. Let $i_{S_d} : S_d \hookrightarrow X$ be the inclusion. Let $\underline{a} = (a_0, \ldots, a_r) \in (\mathbb{Z} \cup \{\pm\infty\})^{r+1}$. As in [BBD82, 1.4], one can glue the t-structures $({}^wD^{\leqslant a_i}(S_i), {}^wD^{>a_i}(S_i))$ on $D^b\mathrm{MHM}(S_i)$ to get a new t-structure $({}^wD^{\leqslant \underline{a}}, {}^wD^{>\underline{a}})$ on $D^b\mathrm{MHM}(X)$. Thus ${}^wD^{\leqslant \underline{a}}$ (resp. ${}^wD^{>\underline{a}}$) is the full subcategory of $D^b\mathrm{MHM}(X)$ consisting of complexes K such that $i_{S_d}^* K \in {}^wD^{\leqslant a_d}(S_d)$ for $d \in \{0, \ldots, r\}$ (resp. $i_{S_d}^! K \in {}^wD^{>a_d}(S_d)$ for $d \in \{0, \ldots, r\}$). The corresponding truncation functors, denoted

$$w_{\leqslant \underline{a}}, w_{>\underline{a}} : D^b\mathrm{MHM}(X) \to D^b\mathrm{MHM}(X)$$

take distinguished triangles to distinguished triangles, commute with the shift, and satisfy $w_{\leqslant (a_0,\ldots,a_r)}\big(K(1)\big) = \big(w_{\leqslant (a_0+2,\ldots,a_r+2)} K\big)(1)$ and $w_{>(a_0,\ldots,a_r)}\big(K(1)\big) = \big(w_{>(a_0+2,\ldots,a_r+2)} K\big)(1)$ ([M08, Prop. 3.4.1]). When $\underline{a} = (a, \ldots, a)$ we recover the previous construction, i.e. $w_{\leqslant (a,\ldots,a)} = w_{\leqslant a}$ ([M08, Lemme 3.3.3]).

The following lemma is stronger than the Hom-vanishing required to prove that $({}^wD^{\leqslant \underline{a}}, {}^wD^{>\underline{a}})$ is a t-structure (which follows by taking H^0). The proof in [M08] works mutatis mutandis.

Lemma 2.1.2. *(*[M08, Prop. 3.4.1]*) If $K \in {}^wD^{\leqslant \underline{a}}$ and $L \in {}^wD^{>\underline{a}}$ then $R\mathrm{Hom}(K, L) = 0$.*

For $k \in \{0, \ldots, r\}$ and $a \in \mathbb{Z} \cup \{\pm\infty\}$ define $w^k_{\leqslant a} := w_{\leqslant (+\infty,\ldots,+\infty,a,+\infty,\ldots,+\infty)}$ where a appears in the kth place.

Lemma 2.1.3. *(*[M08, Prop. 3.3.4]*) (i) $w_{\leqslant (a_0,\ldots,a_r)} = w^r_{\leqslant a_r} \circ w^{r-1}_{\leqslant a_{r-1}} \circ \cdots \circ w^0_{\leqslant a_0}$.*

(ii) For $K \in D^b\mathrm{MHM}(X)$ and $k \in \{0, \ldots, r\}$ there is a distinguished triangle $w^k_{\leqslant a} K \longrightarrow K \longrightarrow Ri_{S_k *} w_{>a} i^*_{S_k} K \xrightarrow{+1}$.

Lemma 2.1.4. *(*[M08, Prop. 3.4.2]*) If $K \in {}^wD^{\leqslant a_0}(S_0) \cap {}^wD^{\geqslant a_0}(S_0)$ then*

$$w_{\leqslant (a_0,a_1,\ldots,a_r)} Rj_* K = w_{\geqslant (a_0,a_1+1,\ldots,a_r+1)} j_! K$$

and this is the unique extension of K in ${}^wD^{\leqslant (a_0,\ldots,a_r)} \cap {}^wD^{\geqslant (a_0,a_1+1,\ldots,a_r+1)}$.

2.2. **Some further properties of weight truncation.** Fix a stratification $X = \coprod_{i=0}^{r} S_i$ of the variety X. For $d = 0, \ldots, r$ let $U_d := \coprod_{i=0}^{d} S_d$ and let

$$U_{d-1} \xhookrightarrow{j_d} U_d \xhookleftarrow{i_d} S_d \tag{2.2.1}$$

be the inclusions.

Lemma 2.2.1. *There are natural isomorphisms of functors*

$$\begin{aligned} w_{\leqslant (a_0,\ldots,a_r)} R(j_r \cdots j_1)_* &= w^r_{\leqslant a_r} Rj_{r*} \circ w^{r-1}_{\leqslant a_{r-1}} Rj_{r-1*} \circ \cdots \circ w^1_{\leqslant a_1} Rj_{1*} \circ w_{\leqslant a_0}. \\ &= w^r_{\leqslant a_r} Rj_{r*} \circ w_{\leqslant (a_0,\ldots,a_{r-1})} R(j_{r-1} \cdots j_1)_*. \end{aligned}$$

Proof. The lemma follows from Lemma 2.1.3 (i) and the observation that for $k < d$ we have the identity $w^k_{\leqslant a} Rj_{d*} = Rj_{d*} w^k_{\leqslant a}$. To prove this identity, it suffices to show that the two obvious natural transformations

$$w^k_{\leqslant a} Rj_{d*} \leftarrow w^k_{\leqslant a} Rj_{d*} w^k_{\leqslant a} \to Rj_{d*} w^k_{\leqslant a}$$

are isomorphisms. For the second transformation, note that Rj_{d*} takes ${}^wD^{\leqslant (+\infty,\ldots,a,\ldots,+\infty)}(U_{d-1})$ to ${}^wD^{\leqslant (+\infty,\ldots,a,\ldots,+\infty,+\infty)}(U_d)$ (where the a is in the kth place in each case), so that $w^k_{\leqslant a}$ is the identity on any object like $Rj_{d*}w^k_{\leqslant a}K$. For the first, note that for K on U_{d-1}, applying $w^k_{\leqslant a} \circ Rj_{d*}$ to the t-structure distinguished triangle for $({}^wD^{\leqslant(+\infty,\dots,a,\dots,+\infty)}(U_{d-1}), {}^wD^{>(+\infty,\dots,a,\dots,+\infty)}(U_{d-1}))$ gives a distinguished triangle

$$w^k_{\leqslant a}Rj_{d*}w^k_{\leqslant a}K \longrightarrow w^k_{\leqslant a}Rj_{d*}K \longrightarrow w^k_{\leqslant a}Rj_{d*}w_{>(+\infty,\dots,a,\dots,+\infty)}K \xrightarrow{+1} .$$

The third term is zero since Rj_{d*} takes ${}^wD^{>(+\infty,\dots,a,\dots,+\infty)}(U_{d-1})$ into ${}^wD^{>(+\infty,\dots,a,\dots,+\infty,+\infty)}(U_d)$ (because $i^!_dRj_{d*} = 0$) and ${}^wD^{\leqslant \underline{a}} \cap {}^wD^{>\underline{a}} = \{0\}$ for any $\underline{a}$, in particular for $\underline{a} = (+\infty, \dots, a, \dots, +\infty, +\infty)$. □

Lemma 2.2.2. *(i) If $\underline{a} \leqslant \underline{b}$ (i.e. $a_d \leqslant b_d$ for all d) then $w_{\leqslant \underline{a}}w_{\leqslant \underline{b}} = w_{\leqslant \underline{a}}$ and $w_{>\underline{a}}w_{\leqslant \underline{b}} = w_{\leqslant \underline{b}}w_{>\underline{a}}$.*

(ii) If $(a_0, \cdots, a_r)$ is nonincreasing then $w_{\leqslant \underline{a}}$ is right t-exact, i.e. $w_{\leqslant \underline{a}}(D^{\leqslant 0}) \subset D^{\leqslant 0}$.

Proof. (i) follows from the inclusion ${}^wD^{\leqslant \underline{a}} \subset {}^wD^{\leqslant \underline{b}}$.

(ii) If $a_d \geqslant a_{d+1}$ for all d, then

$$\begin{aligned} w_{\leqslant(a_0,\dots,a_r)} &= w_{\leqslant(a_0,\dots,a_r)} \circ \cdots \circ w_{\leqslant(a_0,a_1,\dots,a_1)} \circ w_{\leqslant(a_0,\dots,a_0)} \\ &= w_{\leqslant(+\infty,\dots,+\infty,a_r)} \circ \cdots \circ w_{\leqslant(+\infty,a_1,\dots,a_1)} \circ w_{\leqslant(a_0,\dots,a_0)} \end{aligned}$$

Now $w_{\leqslant(+\infty,\dots,+\infty,a_d,\dots,a_d)}$ is the functor $w_{\leqslant(+\infty,a_d)}$ for the coarser stratification of X with two strata $S'_0 = \sqcup_{k\leqslant d-1}S_k$ and $S'_1 = \sqcup_{k\geq d}S_k$. In such a situation any $w_{\leqslant(+\infty,b)}$ preserves $D^{\leqslant 0}$. Indeed, if $K \in D^{\leqslant 0}$, then $i^*_{S'_0}w_{\leqslant(+\infty,b)}K = i^*_{S'_0}K \in D^{\leqslant 0}$ by definition and $i^*_{S'_1}w_{\leqslant(+\infty,b)}K = w_{\leqslant b}i^*_{S'_1}K \in D^{\leqslant 0}$ since $i^*_{S'_1}K \in D^{\leqslant 0}$ and $w_{\leqslant b}$ is t-exact. □

For later use we note some properties of the "diagonal" functors $w_{\leqslant a} = w_{\leqslant(a,\dots,a)}$.

Lemma 2.2.3. *We have*

(i) $w_{\geqslant a}w_{\leqslant b} = w_{\leqslant b}w_{\geqslant a}$

(ii) *For $K \in {}^wD^{\leqslant a} \cap {}^wD^{\geqslant a}$ there is a canonical isomorphism $K = \oplus_k H^k(K)[-k]$. Each cohomology object $H^k(K)$ is pure (of weight a) and so is a sum of simple pure Hodge modules supported on irreducible closed subvarieties.*

(iii) *For $K, L \in {}^wD^{\leqslant a} \cap {}^wD^{\geqslant a}$,* $\mathrm{Hom}(K, L) = \oplus_k\mathrm{Hom}(H^k(K), H^k(L))$.

Proof. (i) is a special case of Lemma 2.2.2(i).

(ii) This is proved by induction on the cardinality of $\{k \in \mathbb{Z} : H^k(K) \neq 0\}$, the case of cardinality one being obvious. Let $K \in {}^wD^{\leqslant a} \cap {}^wD^{\geqslant a}$ and let k_0 be the minimal element of $\{k \in \mathbb{Z} : H^k(K) \neq 0\}$), so that $\tau_{\leqslant k_0}K = H^{k_0}(K)[-k_0]$. In the distinguished triangle

$$H^{k_0}(K)[-k_0] \longrightarrow K \longrightarrow \tau_{\geqslant k_0+1}K \xrightarrow{+1}$$

the first two terms belong to ${}^wD^{\leqslant a} \cap {}^wD^{\geqslant a}$ and hence $\tau_{\geqslant k_0+1}K$ belongs to ${}^wD^{\leqslant a} \cap {}^wD^{\geqslant a}$. By the induction hypothesis there is a canonical isomorphism $\tau_{\geqslant k_0+1}K = \oplus_{k\geqslant k_0+1}H^k(K)[-k]$. Now $H^k(K)[-k]$ is pure of weight $a - k$, so $\tau_{\geqslant k_0+1}K$ is of weights $\leqslant a - k_0 - 1$, while $H^{k_0}(K)[-k_0 + 1]$ is pure of weight $a - k_0 + 1$. Thus in the distinguished triangle above the $+1$ morphism must be zero, i.e. the triangle must split. The splitting is unique: Two splittings differ by a morphism $\tau_{\geqslant k_0+1}K \to H^{k_0}(K)[-k_0]$, which must be zero for weight reasons. So there is a canonical isomorphism $K = H^{k_0}(K)[-k_0] \oplus \tau_{\geqslant k_0+1}K$, completing the inductive step.

(iii) In the canonical isomorphisms given by (ii), an element of $\mathrm{Hom}(K, L)$ is a sum of its components in $\mathrm{Hom}(H^k(K)[-k], H^j(L)[-j])$ for various k, j. If $k < j$ then

$$\mathrm{Hom}(H^k(K)[-k], H^j(L)[-j]) = 0$$

by the standard t-structure vanishing. If $k > j$ then

$$\mathrm{Hom}(H^k(K)[-k], H^j(L)[-j]) = 0$$

since $H^k(K)[-k]$ has weight $a-k$ and $H^j(L)[-j]$ has weight $a-j > a-k$. □

There is a pointwise criterion for the weights of a complex $K \in D^b\mathrm{MHM}(X)$: K is of weights $\leqslant a$ if and only if for any point $i_x : \{x\} \hookrightarrow X$, the mixed Hodge structure $H^i(i_x^*K)$ has weights $\leqslant i+a$ ([S90, 4.6.1]). There is a t-structure $({}^cD^{\leqslant 0}, {}^cD^{\geqslant 0})$ on $D^b\mathrm{MHM}(X)$ related to the classical t-structure on $D^b_c(\mathbb{Q}_X)$ (see [S90, 4.6]). It is defined by $K \in {}^cD^{\leqslant 0}$ if and only if $rat(K) \in D^b_c(\mathbb{Q}_X)^{\leqslant 0}$. Let ${}^cH^0$ be the corresponding cohomology functor, so that $rat \circ {}^cH^0 = H^0 \circ rat$.

For a smooth variety X we say that a mixed Hodge module $K \in \mathrm{MHM}(X)$ is *smooth* if $rat(K)[-\dim X]$ is a local system. ([6]) A complex $K \in D^b\mathrm{MHM}(X)$ is smooth if $H^i(K)$ is smooth for all i.

Lemma 2.2.4. *Suppose that X is smooth and $K \in D^b\mathrm{MHM}(X)$ is smooth. Then*

(i) $H^i(K) = {}^cH^{i-\dim X}(K)[\dim X]$ *for all* i

(ii) $K \in {}^wD^{\leqslant a}$ *if, and only if, for each* $x \in X$, $H^i(i_x^*K)$ *has weights* $\leqslant a - \dim X$ *for all* i.

Proof. (i) follows from the identity ${}^pH^i(rat(K)) = H^{i-\dim X}(rat(K))[\dim X]$ in $D^b_c(\mathbb{Q}_X)$.

(ii) Since K is smooth, $H^i(i_x^*K) = {}^cH^i(i_x^*K) = i_x^*{}^cH^i(K) = i_x^*H^{i+\dim X}(K)[-\dim X]$. By definition, $K \in {}^wD^{\leqslant a}$ if $H^i(K)$ has weights $\leqslant a$ for all i, which translates to the condition that $H^i(i_x^*K)$ has weights $\leqslant a - \dim X$ for all i. □

2.3. **Mixed sheaves.** Everything so far works in any theory of A-mixed sheaves in the sense of Saito [S06] (A a field of characteristic zero) for varieties over a fixed field $k \subset \mathbb{C}$. Recall that such a theory gives an A-linear abelian category $\mathscr{M}(X)$ for every variety X/k with an A-linear forgetful functor For to perverse A-sheaves on $X(\mathbb{C})$, satisfying a number of axioms similar to the properties of mixed Hodge modules [S88, S90]. In particular, objects have weight filtrations, associated with a morphism of varieties are four functors satisfying conditions with respect to weights, and there is a (Verdier) duality functor. The basic example is the theory of $\mathbb{Q}$-mixed sheaves given by $\mathscr{M}(X) = \mathrm{MHM}(X(\mathbb{C}))$ with $For = rat$.

A second example is the theory of $\mathbb{Q}$-mixed sheaves defined in [S06, 1.8(ii)], which we will denote $\mathrm{MHM}(X/k)$ and refer to as mixed Hodge modules with de Rham structure. In essence, for X/k, $\mathrm{MHM}(X/k)$ is the category of mixed Hodge modules on $X(\mathbb{C})$ such that the underlying bifiltered D-module on $X(\mathbb{C})$ comes (by extension of scalars) from a bifiltered D-module on X/k. (For $X = Spec(k)$ this gives the category of mixed Hodge-de Rham structures over k with polarizable graded pieces.) Working in this theory of mixed sheaves allows us to keep track of the de Rham rational structure in various cohomology groups, i.e. to work with Hodge-de Rham structures everywhere.

[6]By Theorem 3.27 of [S90] (cf. also the remark after the proof of the theorem regarding the algebraic case) a smooth Hodge module on a smooth variety X is the same thing as an admissible variation of mixed Hodge structure with polarizable pure subquotients.

In [M11] Morel constructs a theory of mixed l-adic complexes on varieties over a number field $k \subset \mathbb{C}$ (l is any fixed prime), using earlier work of Huber [Hu97]. Denote the category of l-adic mixed perverse complexes having a weight filtration by $\mathscr{M}_l(X)$; this has a forgetful functor to the category of perverse $\mathbb{Q}_l$-complexes in the classical topology on $X(\mathbb{C})$ which factors through the perverse l-adic complexes in the étale topology. Morel shows that given a morphism $f : X \to Y$ the four functors $f^*, f^!, Rf_*, Rf_!$ on l-adic complexes can be lifted to functors between the derived categories $D^b\mathscr{M}_l(X)$ and $D^b\mathscr{M}_l(Y)$ (cf. [M11, Thm 2.1] and §5 of loc. cit.). Working in this theory will allow us to conclude that some statements hold in the context of $\mathrm{Gal}(\bar{\mathbb{Q}}/k)$-modules. (The theory $\mathscr{M}_l(\cdot)$ is not quite a theory of mixed $\mathbb{Q}_l$-sheaves in the sense of Saito as one does not have semisimplicity of pure objects in $\mathscr{M}_l(\cdot)$. Nevertheless, the preceding discussion goes through in this theory (cf. [M11, §6]), with the following caveats about Lemma 2.2.3: The isomorphism $K = \oplus_k H^k(K)[-k]$ in Lemma 2.2.3 (ii) is not canonical. Each cohomology object $H^k(K)$ in (ii) breaks up canonically "by support", i.e. is canonically isomorphic to a direct sum of intersection complexes with coefficient local systems which are pure but not necessarily semisimple. Lemma 2.2.3 (iii) fails.)

Remark 2.3.1 (Effectivity). Recall that a Hodge structure H is *effective* if $H^{p,q} \neq 0$ only if $p, q \geq 0$. A mixed Hodge structure is effective if its weight graded pieces are effective. (In particular, its weights are ≥ 0.) For example, if X is a complex algebraic variety the mixed Hodge structures on $H^*(X, \mathbb{Q})$ and $H^*_c(X, \mathbb{Q})$ given by Deligne's Hodge theory [D74] are effective. More generally, if X is smooth and $\mathbb{V}$ is an effective variation of Hodge structure on X then $H^*(X, \mathbb{V})$ and $H^*_c(X, \mathbb{V})$ are effective (cf. [S90, 3.10]).

For a complex $K \in D^b\mathrm{MHM}(X)$ we will say K is *pointwise effective* if the mixed Hodge structure $H^k(i_x^* K)$ is effective for all $i_x : \{x\} \hookrightarrow X$ and $k \in \mathbb{Z}$. Pointwise effective complexes form a triangulated subcategory of $D^b\mathrm{MHM}(X)$. The following can be proved by standard arguments: (1) Pointwise effectivity is preserved by the functors $Rf_*, Rf_!, f^*$ for an arbitrary morphism and by $f^!$ for f a locally closed immersion. (2) For a Whitney stratification, the functors $w_{\leqslant \underline{a}}$ and $w_{>\underline{a}}$ applied to objects constructible for the stratification preserve pointwise effectivity. (3) If K in $D^b\mathrm{MHM}(X)$ is pointwise effective then $\mathbb{H}^*_c(X, K)$ and $\mathbb{H}^*(X, K)$ are effective mixed Hodge structures. Thus with the notation of 2.2 objects like $w_{\leqslant \underline{a}} Rj_* \mathbb{Q}^H_U$ (for a Whitney stratification) are pointwise effective and their cohomology groups are effective. (We will not use these statements below so we do not give complete proofs here.)

2.4. **Properties of some weight-truncated cohomology groups.** In this subsection we prove some results about the hypercohomology groups of some special weight truncations which will used later (in 4.5) for locally symmetric varieties. Thus we fix the following notation: $j : M \hookrightarrow M^*$ is an open immersion of a smooth variety M as a Zariski-dense subset of a complete variety M^*, $\mathbb{E}^H_M[\dim M]$ is the pure Hodge module of weight $w+\dim M$ coming from a polarizable variation on M, and $M = \coprod_i M_i$ is a stratification by equidimensional smooth varieties for which $Rj_*\mathbb{E}^H_M$ and $j_!\mathbb{E}^H_M$ are constructible. (In the application in 4.5 M will be a locally symmetric variety and M^* its minimal compactification with its canonical stratification.)

Lemma 2.4.1. *(i) Suppose that* $(a_1, \ldots, a_r)$ *is nonincreasing,* $a_i < \dim M$ *for* $i \geqslant 1$*, and* $a_0 \geqslant \dim M$*. Then*

$$rat(w_{\leqslant (a_0+w, \ldots, a_r+w)} Rj_* \mathbb{E}^H_M)[\dim M] \in {}^pD^{\leqslant 0}(\mathbb{Q}_{M^*}) \tag{2.4.1}$$

and

$$w_{\geqslant \dim M+w} w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M[\dim M] = j_{!*}(\mathbb{E}^H_M[\dim M]). \tag{2.4.2}$$

The mixed structure on $\mathbb{H}^i(M^*, rat(w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M))$ *has weights* $\leqslant i+w$.

(ii) Suppose that $(a_1,\dots,a_r)$ *is nondecreasing,* $a_i \geqslant \dim M$ *for* $i \geqslant 1$ *and* $a_0 \geqslant \dim M$. *Then*

$$rat(w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M)[\dim M] \in {}^pD^{\geqslant 0}(\mathbb{Q}_{M^*}) \tag{2.4.3}$$

and

$$w_{\leqslant \dim M+w} w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M[\dim M] = j_{!*}(\mathbb{E}^H_M[\dim M]). \tag{2.4.4}$$

The mixed structure on $\mathbb{H}^i(M^*, rat(w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M))$ *has weights* $\geqslant i+w$.

Proof. Write $d_0 := \dim M$ to simplify the notation.

(i) If $(a_1,\dots,a_r)$ is nonincreasing, $a_i \leqslant d_0$ for all i and $a_0 = d_0$ then by Lemma 2.2.2(i)

$$w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M[d_0] = w_{\leqslant(a_0+w,\dots,a_r+w)} w_{\leqslant d_0+w} Rj_*\mathbb{E}^H_M[d_0] \tag{2.4.5}$$

and this lies in $D^{\leqslant 0}$ by Lemma 2.2.2 since $w_{\leqslant d_0+w} Rj_*\mathbb{E}^H_M[d_0] = j_{!*}\mathbb{E}^H_M[d_0] \in D^{\leqslant 0}$. This proves (2.4.1) if $(a_1,\dots,a_r)$ is nonincreasing and $a_0 \geqslant d_0$ since we can always replace a_0 by d_0 without changing $w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M[d_0]$.

For $a_0 = d_0$, we have:

$$\begin{aligned} w_{\geqslant d_0+w} w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M[d_0] &= w_{\geqslant d_0+w} w_{\geqslant(a_0+w,a_1+1+w,\dots,a_r+1+w)} j_!\mathbb{E}^H_M[d_0] \\ &= w_{\geqslant d_0+w} j_!\mathbb{E}^H_M[d_0] = j_{!*}\mathbb{E}^H_M[d_0] \end{aligned}$$

by Prop. 3.4.2 of [M08] (i.e. Lemma 2.1.4), Lemma 2.2.2(i), and Theorem 2.1.1. This proves (2.4.2) if $a_0 = d_0$; the general case follows since we can replace $a_0 = d_0$ by any $a_0 \geqslant d_0$ without changing $w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M$.

For the assertion about weights it is enough to prove that $w_{\leqslant(a_0+w,\dots,a_r+w)} Rj_*\mathbb{E}^H_M[d_0]$ has weights $\leqslant d_0+w$. Suppose $X = S_0 \sqcup S_1$ with S_1 closed. If K on X is of weights $\leqslant a$ then $i_1^* w_{\leqslant(+\infty,b)} K = w_{\leqslant b} i_1^* K$ and $H^k(i_1^* w_{\leqslant(+\infty,b)} K) = w_{\leqslant b} H^k(i_1^* K)$ has weights $\leqslant \min(b, k+a) \leqslant k+a$. So $i_1^* w_{\leqslant(+\infty,b)} K$ has weights $\leqslant a$. Since $i_0^* w_{(\leqslant(+\infty,b)} K = i_0^* K$ has weights $\leqslant a$, we conclude that $w_{\leqslant(+\infty,b)} K$ has weights $\leqslant a$. Thus $w_{\leqslant(+\infty,b)}$ preserves the subcategory of complexes with weights $\leqslant a$. Now for $(a_0,\dots,a_r)$ nonincreasing we have

$$w_{\leqslant(a_0,\dots,a_r)} = w_{\leqslant(+\infty,\dots,+\infty,a_r)} \circ \dots \circ w_{\leqslant(+\infty,a_1,\dots,a_1)} \circ w_{\leqslant(a_0,\dots,a_0)}.$$

Each functor here is of the form $w_{\leqslant(+\infty,b)}$ for a suitable restratification of M^*. So we conclude that if K is of weights $\leqslant a$ and $(a_0,\dots,a_r)$ is nonincreasing, then $w_{\leqslant(a_0,\dots,a_r)} K$ is of weights $\leqslant a$. Combining this with (2.4.5) gives the required assertion in the case $a_0 = d_0$ and the general case then follows.

(ii) will follow from (i) by duality. Suppose that $(a_0,\dots,a_r)$ satisfies the conditions of (ii). Define $(b_0,\dots,b_r)$ by $b_0 = 2d_0 - a_0$ and $b_d = 2d_0 - a_d - 1$ for $d \geqslant 1$. Then $(b_0,\dots,b_r)$ satisfies the conditions of (i) and so (2.4.1) holds, i.e. $w_{\leqslant(b_0+w,\dots,b_r+w)} Rj_*\mathbb{E}^H_M[d_0] \in D^{\leqslant 0}$. Dualizing (and replacing $\check{\mathbb{E}}^H_M$ by $\mathbb{E}^H_M$) gives

$$w_{\geqslant(-b_0-w,\dots,-b_r-w)}\big(j_!\mathbb{E}^H_M[d_0](d_0+w)\big) \in D^{\geqslant 0}.$$

Using the identities

$$\begin{aligned} w_{\geqslant(-b_0-w,\dots,-b_r-w)}\big(j_!\mathbb{E}^H_M[d_0](d_0+w)\big) &= w_{\geqslant(2d_0-b_0+w,\dots,2d_0-b_r+w)}\, j_!\mathbb{E}^H_M[d_0] \\ &= w_{\geqslant(2d_0-b_0+w,2d_0-b_1+w-1,\dots,2d_0-b_r+w-1)}\, j_!\mathbb{E}^H_M[d_0] \\ &= w_{\leqslant(2d_0-b_0+w,2d_0-b_1+w-1,\dots,2d_0-b_r+w-1)}\, Rj_*\mathbb{E}^H_M[d_0] \\ &= w_{\leqslant(a_0+w,a_1+w,\dots,a_r+w)}\, Rj_*\mathbb{E}^H_M[d_0] \end{aligned}$$

(using Lemma 2.1.4 for the third equality) gives (2.4.3). Similarly, dualizing (2.4.2) gives

$$\begin{aligned} j_{!*}\mathbb{E}^H_M[d_0](d_0+w) &= w_{\leqslant-d_0-w}\, w_{\geqslant(-b_0-w,\dots,-b_r-w)}\big(j_!\mathbb{E}^H_M[d_0](d_0+w)\big) \\ &= w_{\leqslant-d_0-w}\,\big(w_{\geqslant(2d_0-b_0-w,\dots,2d_0-b_r-w)}\, j_!\mathbb{E}^H_M[d_0](d_0+w)\big) \\ &= w_{\leqslant d_0-w}\, w_{\geqslant(2d_0-b_0+w,\dots,2d_0-b_r+w)}\, j_!\mathbb{E}^H_M[d_0] \\ &= w_{\leqslant d_0+w}\, w_{\leqslant(2d_0-b_0+w,2d_0-b_1+w-1\dots,2d_0-b_r+w-1)}\, Rj_*\mathbb{E}^H_M[d_0] \\ &= w_{\leqslant d_0+w}\, w_{\leqslant(a_0+w,a_1+w\dots,a_r+w)}\, Rj_*\mathbb{E}^H_M[d_0] \end{aligned}$$

(using Lemma 2.1.4 for the fourth equality) gives (2.4.4). □

In the case of "diagonal" truncations $w_{\leqslant a}$ and the special truncation $w_{\leqslant(\dim M_0,\dots,\dim M_r)}$ we have information about the weights in (global) hypercohomology:

Proposition 2.4.2. *(i) For $a \leqslant \dim M$ the weights of*

$$\mathbb{H}^i(M^*, w_{\geqslant a+w} j_!\mathbb{E}^H_M[\dim M]) = \mathbb{H}^i(M^*, w_{\leqslant(\dim M+w,a+w-1,\dots,a+w-1)} Rj_*\mathbb{E}^H_M[\dim M])$$

are $\leqslant i+\dim M+w$. The top weight quotient is the image in $IH^{i+\dim M}(M^*,\mathbb{E})$ under the map induced by (2.4.2).

(ii) For $b \geqslant \dim M$ the weights of

$$\mathbb{H}^i(M^*, w_{\leqslant b+w} Rj_*\mathbb{E}^H_M[\dim M])$$

are $\geqslant i+\dim M+w$. The lowest weight piece is the image of $IH^{i+\dim M}(M^*,\mathbb{E})$ under the map induced by (2.4.4).

(iii) The mixed structure on

$$\mathbb{H}^i(M^*, w_{\leqslant(\dim M+w,\dim M_1+w,\dots,\dim M_r+w)} Rj_*\mathbb{E}^H_M)$$

has weights $\leqslant i+w$. The top weight quotient is the image in $IH^{i+\dim M}(M^*,\mathbb{E})$ under the map induced by (2.4.2).

Proof. Let $d_0 := \dim M$. In each of (i)–(iii) the first assertion follows from the previous lemma, so we concentrate on the second assertions.

(i) From the long exact sequence in cohomology associated with the triangle

$$w_{<d_0+w}w_{\geqslant a+w}j_!\mathbb{E}^H_M[d_0] \to w_{\geqslant a+w}j_!\mathbb{E}^H_M[d_0] \to w_{\geqslant d_0+w}j_!\mathbb{E}^H_M[d_0] \xrightarrow{+1}$$

it suffices to show that for $a \leqslant d_0$, $w_{<d_0+w}w_{\geqslant a+w}j_!\mathbb{E}^H_M[d_0]$ has weights $< d_0+w$. Since

$$w_{<d_0+w}w_{\geqslant a+w}j_!\mathbb{E}^H_M[d_0] = w_{\geqslant a+w}w_{<d_0+w}j_!\mathbb{E}^H_M[d_0]$$

and $w_{\geqslant a+w}$ preserves the complexes of weight $< d_0+w$, it is enough to show that $w_{<d_0+w}j_!\mathbb{E}^H_M[d_0]$ has weights $< d_0+w$. The triangle $w_{<d_0+w}j_!\mathbb{E}^H_M[d_0] \to j_!\mathbb{E}^H_M[d_0] \to w_{\geqslant d_0+w}j_!\mathbb{E}^H_M[d_0] \xrightarrow{+1}$ shows that for any $d \geqslant 1$,

$$i_d^* w_{<d_0+w}j_!\mathbb{E}^H_M[d_0] = i_d^* w_{\geqslant d_0+w}j_!\mathbb{E}^H_M[d_0]\,[-1]$$

has weights $\leqslant d_0 + w - 1$ since $w_{\geqslant d_0+w} j_! \mathbb{E}_M^H[d_0] = j_{!*}\mathbb{E}_M^H[d_0]$ is of weights $\leqslant d_0 + w$. Since $i_0^* w_{<d_0+w} j_! \mathbb{E}_M^H[d_0] = 0$ we have that $w_{<d_0+w} j_! \mathbb{E}_M^H[d_0]$ has weights $\leqslant d_0 + w - 1$.

(ii) follows from (i) by duality.

(iii) Let $d_i = \dim M_i$. From the triangle

$$w_{\leqslant(d_0+w,\ldots,d_r+w)} Rj_*\mathbb{E}_M^H \to w_{\leqslant d_0+w} Rj_*\mathbb{E}_M^H \to w_{>(d_0+w,\ldots,d_r+w)} w_{\leqslant d_0+w} Rj_*\mathbb{E}_M^H \xrightarrow{+1}$$

we see that it suffices to show that $K := w_{>(d_0+w,\ldots,d_r+w)} w_{\leqslant d_0+w} Rj_*\mathbb{E}_M^H$ has weights $\leqslant w$. For this we will use the pointwise criterion (cf. 2.2): K has weights $\leqslant w$ if and only if for every point $i_x : \{x\} \hookrightarrow M^*$, $H^i(i_x^* K)$ has weights $\leqslant i + w$. If $x \in M_0$ this obviously holds, so assume $x \in M_k$ for $k \geqslant 1$. Then $i_k^* w_{\leqslant(d_0+w,\ldots,d_r+w)} Rj_*\mathbb{E}_M^H$ belongs to ${}^w D^{\leqslant d_k+w}$, i.e. $H^i(i_k^* w_{\leqslant(d_0+w,\ldots,d_r+w)} Rj_*\mathbb{E}_M^H)$ has weights $\leqslant d_k + w$ for all i. Since these cohomology objects are smooth, Lemma 2.2.4 implies that for any point $x \in M_k$, $H^i(i_x^* w_{\leqslant(d_0+w,\ldots,d_r+w)} Rj_*\mathbb{E}_M^H)$ has weights $\leqslant w + d_k - d_k = w$ for all i. In the long exact sequence

$$\to H^i(i_x^* w_{\leqslant d_0+w} Rj_*\mathbb{E}_M^H) \to H^i(i_x^* K) \to H^{i+1}(i_x^* w_{\leqslant(d_0+w,\ldots,d_r+w)} Rj_*\mathbb{E}_M^H) \to$$

the first group has weights $\leqslant i + w$ (because $w_{\leqslant d_0+w} Rj_*\mathbb{E}_M^H = (j_{!*}\mathbb{E}_M^H[d_0])[-d_0]$ has weight w) and the last group has weights $\leqslant w$. So $H^i(i_x^* K)$ has weights $\leqslant i + w$ for all i. □

Remarks 2.4.3. (i) Everything in this subsection holds in any theory of A-mixed sheaves in which $\mathbb{E}_M \otimes A$ underlies an object, in particular in the theories mentioned in 2.3 (and with the same proofs).

(ii) There are bounds on weights generalizing Prop. 2.4.2 (iii) in the noncomplete case, cf. [NV12, 4.3].

3. Locally symmetric varieties and their compactifications

3.1. **Locally symmetric varieties.** Let G be a semisimple simply-connected and almost $\mathbb{Q}$-simple $\mathbb{Q}$-algebraic group. We assume that the symmetric space D of maximal compact subgroups of $G(\mathbb{R})$ is a Hermitian symmetric domain. Let $\Gamma \subset G(\mathbb{Q})$ be a neat arithmetic subgroup. The quotient

$$M = \Gamma \backslash D$$

is a smooth quasiprojective complex algebraic variety. We assume that M is noncompact (equivalently, that G is $\mathbb{Q}$-isotropic).

3.2. **Rational parabolic subgroups.** (cf. [BB66, §3], [AMRT75, III.4.1–III.4.2], [LR91, 6.1], [GP02, 7.1–7.3]) Let P be a rational parabolic subgroup of G. The Levi quotient $M = P/R_uP$ has an almost-direct product decomposition

$$M = M_\ell \cdot A \cdot M_h$$

where A is the $\mathbb{Q}$-split part of the centre of M and such that

(i) M_h contains no nontrivial proper $\mathbb{Q}$-anisotropic connected subgroup

(ii) the symmetric space of $M_h(\mathbb{R})$ is of Hermitian type.

Since G is simply-connected, so is M^{der}, and therefore $M^{\mathrm{der}} = M_\ell^{\mathrm{der}} \times M_h$. Thus M_ℓ^{der} and M_h are simply-connected. Note that M_h satisfies similar assumptions as G in (3.1) (except that it may be $\mathbb{Q}$-anisotropic), namely it is semisimple and simply-connected and almost $\mathbb{Q}$-simple. We will refer to M_ℓ and M_h as the *linear* and *Hermitian* parts of the Levi quotient.

If P is a maximal proper rational parabolic subgroup, then in addition to (i) and (ii), we have

(iii) If U is the centre of $W := R_u P$ then the quotient $V := W/U$ is abelian. The adjoint action of $A = \mathbb{G}_m$ on $\operatorname{Lie} U(\mathbb{R})$ is by the square χ^2 of a generator of the character group. If $V = W/U$ is nontrivial then A acts by χ on $\operatorname{Lie} V(\mathbb{R})$.

(iv) M_h centralizes U

(v) $M_\ell(\mathbb{R})A(\mathbb{R})^0$ acts (via the adjoint action) properly and transitively on an open cone $C \subset \operatorname{Lie} U(\mathbb{R})$. The stabilizer of a point in C is maximal compact modulo the centre of M_ℓ and this identifies $C/A(\mathbb{R})^0$ with the symmetric space of $M_\ell^{\mathrm{der}}(\mathbb{R})$.

The projection from a parabolic P to its Levi quotient will be denoted ν_P.

For a parabolic P with Levi quotient $M = M_h A M_\ell$, we set

$$\begin{aligned} \Gamma_P &:= \Gamma \cap P(\mathbb{Q}), \\ \Gamma_M &:= \nu_P(\Gamma_P), \\ \Gamma_{M_\ell} &:= \Gamma_M \cap M_\ell(\mathbb{Q}), \\ \Gamma_{M_h} &:= \Gamma_M/\Gamma_{M_\ell}. \end{aligned} \tag{3.2.1}$$

These are neat arithmetic subgroups of their respective groups.

Fix a minimal parabolic P_0. The standard (with respect to P_0) maximal parabolic subgroups are partially ordered by

$$Q \prec Q' \iff U \subset U'.$$

Under our assumption that G is almost $\mathbb{Q}$-simple this is a total order. Define a map $P \mapsto P^+$ from standard rational parabolics to standard maximal rational parabolics as follows: If P is written as an intersection of maximal parabolics $P = Q_1 \cap \dots \cap Q_d$ with $Q_i \prec Q_{i+1}$, then $P^+ = Q_d$, i.e. P^+ is the last maximal parabolic (i.e. the one with largest U) containing P in the ordering.

Lemma 3.2.1. *The map $P \mapsto P^+$ can be extended to all rational parabolics compatibly with conjugation, i.e. with $(gPg^{-1})^+ = gP^+g^{-1}$. It then has the following properties:*

(i) *If $P \subset Q$ then $P^+ \succ Q^+$.*

(ii) *If $P^+ = Q$ and $Q/W = M_h A M_\ell$ then the parabolic $\nu_Q(P)$ of Q/W contains M_h. The projection $\nu_{\nu_Q(P)}$ identifies M_h with the Hermitian part of $P/R_u P = \nu_Q(P)/R_u\nu_Q(P)$.*

(iii) *If $P^+ = Q^+$ then P and Q are Γ-conjugate if and only if they are $\Gamma_{P^+=Q^+}$-conjugate.*

(iv) *If Q is maximal and $Q/W = M_\ell A M_h$, then there is a bijection between Γ-conjugacy classes in $\{P : P^+ \text{ is } \Gamma\text{-conjugate to } Q\}$ and Γ_{M_ℓ}-conjugacy classes of parabolics in M_ℓ. (The bijection is the following: If $\gamma P^+ \gamma^{-1} = Q$ then $P \mapsto \nu_Q(\gamma P \gamma^{-1}) \cap M_\ell$.)*

(v) *Two standard parabolics P_1 and P_2 are associates (i.e. have conjugate Levi subgroups) if and only if $P_1^+ = P_2^+$ (and hence $M_{1,h} = M_{2,h}$) and $M_{1,\ell}$ and $M_{2,\ell}$ are conjugate in $M_{1,\ell}^+ = M_{2,\ell}^+$.*

3.3. **Minimal (Baily-Borel-Satake) compactification.** This is a normal projective algebraic variety M^* with an open immersion

$$j : M \hookrightarrow M^*.$$

As a topological space, M^* is the quotient $M^* = \Gamma\backslash D^*$ where

$$D^* = \coprod\nolimits_{F \text{ rational}} F \tag{3.3.1}$$

is the union of all rational boundary components equipped with the Satake topology. The action of $G(\mathbb{Q})$ on D extends to a continuous action on D^*; the stabilizer of a rational boundary component F is a maximal $\mathbb{Q}$-parabolic subgroup (in which case F is proper, i.e. $F \subset D^* - D$) or G itself (the case $F = D$). Thus the proper rational boundary components are indexed bijectively by maximal $\mathbb{Q}$-parabolic subgroups, and M^* has a natural stratification induced by (3.3.1) in which M is the open and dense stratum and the boundary strata are indexed by Γ-conjugacy classes of such subgroups. If $F \subset D^*$ is a rational boundary component and Q the associated parabolic, then $(M_\ell A)(\mathbb{R})$ acts trivially on F while $M_h(\mathbb{R})$ acts properly transitively, identifying F with the symmetric space of $M_h(\mathbb{R})$. The stratum S of M^* covered by F is $S = \Gamma_{M_h}\backslash F$, where Γ_{M_h} is defined in (3.2.1). Thus M^* has a stratification in which each stratum is a smooth locally symmetric variety satisfying the same assumptions as M in 3.1. The closure relations between the strata are determined by those between rational boundary components in D^*, which in turn are determined by the associated parabolic subgroups: Let F, F' be rational boundary components with stabilizers Q, Q'. Then $F' \subset \overline{F} \iff U' \supset U$.

The Baily-Borel theory [BB66] puts an analytic structure on M^* inducing the given complex structure on each stratum, and this structure is unique. Moreover, M^* has a unique structure of projective algebraic variety compatible with this analytic structure. The stratification induced by (3.3.1) is an algebraic Whitney stratification by smooth varieties. The minimal compactification has a hereditary nature: The normalization of the closure of a boundary stratum in M^* is isomorphic to its minimal compactification.

3.4. **Reductive Borel-Serre (RBS) compactification.** The RBS compactification $\overline{M}$, first constructed by Zucker in [Z83] by modifying the earlier construction of Borel and Serre, should perhaps be thought of as a desingularization of M^*, although it is in general neither algebraic nor smooth. The construction is similar to that of M^*: There is an extension $\overline{D}$ of D which is the union of locally closed strata indexed by $\mathbb{Q}$-parabolic subgroups of G:

$$\overline{D} = \coprod\nolimits_P D_P. \tag{3.4.1}$$

If $M = P/R_uP$ is the Levi quotient, $D_P = M(\mathbb{R})/A(\mathbb{R})^0 K_M$ where A is the $\mathbb{Q}$-split centre of M and K_M is the image of $K \cap P(\mathbb{R})$ in $M(\mathbb{R})$. ([7]) $\overline{D}$ carries a topology for which D_P is contained in the closure of D_Q if and only if $P \subset Q$. There is a continuous action of $G(\mathbb{Q})$ on $\overline{D}$ and the quotient

$$\overline{M} = \Gamma\backslash\overline{D}$$

is the RBS compactification of M. (3.4.1) induces a decomposition of $\overline{M}$ into locally closed strata indexed by Γ-conjugacy classes of $\mathbb{Q}$-parabolic subgroups of G:

$$\overline{M} = \coprod_{\Gamma\backslash\{P\}} M_P, \qquad \text{where } M_P = \Gamma_M\backslash D_P = \Gamma_M\backslash M(\mathbb{R})/A(\mathbb{R})^0 K_M. \tag{3.4.2}$$

The RBS construction can be made for spaces like M_P and the closure of a stratum M_P in $\overline{M}$ is naturally identified with its RBS compactification $\overline{M}_P$. Note that $\overline{M}$ can have strata of odd real dimension.

The identity mapping of M extends to a proper continuous map

$$p : \overline{M} \to M^*.$$

[7] In general, $M(\mathbb{R})/A(\mathbb{R})^0 K_M$ is the product of a symmetric space and a Euclidean space; we will refer to it as a symmetric space, and its quotient by an arithmetic group as a locally symmetric space in what follows. RBS compactifications can be constructed for arbitrary locally symmetric spaces in this sense.

This was proved in [Z83]; we describe p following [GHM94, §22]. For a rational parabolic P let $M = P/R_uP$ and $M = M_hAM_\ell$ as in (3.2). Then

$$D_P = M_\ell(\mathbb{R})/K_{M_\ell} \times M_h(\mathbb{R})/K_{M_h}$$

where K_{M_h} and K_{M_ℓ} are the respective intersections with $M_h(\mathbb{R})$ and $M_\ell(\mathbb{R})$ of K_M. The Hermitian part M_h is naturally identified with the Hermitian part of the Levi quotient of P^+ (Lemma 3.2.1(ii)), so the rational boundary component associated with P^+ is $F = M_h(\mathbb{R})/K_{M_h}$. Thus projection to the second factor defines a map $D_P \to F$. Varying P defines a Γ-equivariant continuous extension of the identity $\overline{D} \to D^*$ and the induced map on quotients is $p : \overline{M} \to M^*$ (cf. [GHM94, Cor. 22.5]). The next lemma describes the restriction of p to the closure of a stratum. (When P is maximal it describes the fibres of p over the P-stratum of M^* cf. [GHM94, Cor. 22.7].)

Lemma 3.4.1. *Let P be parabolic, $P/R_uP = M_hAM_\ell$ (as in 3.2), and let $S = \Gamma_{M_h}\backslash M_h(\mathbb{R})/K_{M_h}$ be the P^+-stratum of M^*. Then $p|_{\overline{M}_P}$ is given by*

$$\overline{M}_P \xrightarrow{p_{M_\ell}} \overline{S} \xrightarrow{p_{M_h}} S^* \xrightarrow{\hat{i}_S} M^*$$

where $\overline{S}$ (resp. S^*) is the RBS (resp. minimal) compactification of S. The map $p_{M_\ell} : \overline{M}_P \to \overline{S}$ is induced by $D_P \to F$ and is a fibre bundle with fibres isomorphic to $\overline{\Gamma_{M_\ell}\backslash M_\ell(\mathbb{R})/K_{M_\ell}}$, the map $p_{M_h} : \overline{S} \to S^*$ is the unique extension of the identity on S, and the map $\hat{i}_S : S^* \to M^*$ is the composite of normalization with the inclusion of the closure of S in M^*.

3.5. **Stratifications.** We will use coarser stratifications of M^* and $\overline{M}$ than those defined above. Recall that we fixed (in 3.2) an enumeration $Q_1, \dots, Q_r$ of the standard maximal parabolics with the property that $i < j \iff Q_i \prec Q_j$ ($\iff U_i \subset U_j \iff \overline{F_i} \supset F_j$). For $d = 0, 1, \dots, r$ define $D^*_{\leqslant d}$ to be the union of D with the rational boundary components corresponding to parabolic subgroups conjugate to Q_i for some $i \in \{1, \dots, d\}$. Let $M^*_{\leqslant d} := \Gamma\backslash D^*_{\leqslant d}$. This defines a filtration

$$M = M^*_{\leqslant 0} \subset M^*_{\leqslant 1} \subset \ldots \subset M^*_{\leqslant r} = M^* \tag{3.5.1}$$

by Zariski-open and dense sets with $M_d := M^*_{\leqslant d} - M^*_{\leqslant d-1}$ (and $M_0 = M$) a union of finitely many copies of the Hermitian locally symmetric space associated with the Hermitian part $M_{d,h}$ of the standard parabolic Q_d. $M^* = \coprod_{d=0,\dots,r} M_d$ is a stratification in the sense used in 2.1.

The filtration (3.5.1) of M^* defines one of $\overline{M}$ by setting $\overline{M}_{\leqslant d} := p^{-1}(M^*_d)$. This is a filtration by open sets

$$M = \overline{M}_{\leqslant 0} \subset \overline{M}_{\leqslant 1} \subset \ldots \subset \overline{M}_{\leqslant r} = \overline{M} \tag{3.5.2}$$

such that $p^{-1}(M_d) = \overline{M}_{\leqslant d} - \overline{M}_{\leqslant d-1}$ contains, as a dense open subset, finitely many copies of the RBS boundary stratum M_{Q_d} attached to Q_d. We fix notation for inclusions as follows:

$$\begin{array}{ccccc} \overline{M}_{\leqslant d-1} & \xhookrightarrow{\bar{j}_d} & \overline{M}_{\leqslant d} & \xhookleftarrow{\bar{i}_d} & \overline{M}_{\leqslant d} - \overline{M}_{\leqslant d-1} \\ \downarrow & & \downarrow p & & \downarrow \\ M^*_{\leqslant d-1} & \xhookrightarrow{j_d} & M^*_{\leqslant d} & \xhookleftarrow{i_d} & M_d \end{array} \tag{3.5.3}$$

3.6. **Correspondences.** For $g \in G(\mathbb{Q})$ and $\Gamma' \subset \Gamma \cap g^{-1}\Gamma g$ of finite index let $M' = \Gamma' \backslash D$. The pair of morphisms

$$(c_1, c_2) : M' \rightrightarrows M \qquad \text{by} \qquad \begin{array}{l} c_1(\Gamma' x) = \Gamma x \\ c_2(\Gamma' x) = \Gamma g x \end{array} \tag{3.6.1}$$

give a finite correspondence on M. When $\Gamma' = \Gamma \cap g^{-1}\Gamma g$ this is called the *Hecke correspondence* associated with g; in general (3.6.1) is a finite cover of the Hecke correspondence.

(3.6.1) extends to a finite correspondence

$$(c_1, c_2) : M'^* \rightrightarrows M^* \tag{3.6.2}$$

where $M'^* = \Gamma' \backslash D^*$.

(3.6.1) also extends to RBS compactifications: For $g \in G(\mathbb{Q})$ and $\Gamma' \subset \Gamma \cap g^{-1}\Gamma g$ of finite index, let $\overline{M'}$ be the RBS compactification of $M' = \Gamma' \backslash D$. Then there is a correspondence

$$(\bar{c}_1, \bar{c}_2) : \overline{M'} \rightrightarrows \overline{M} \tag{3.6.3}$$

which has finite fibres ([GM03, 6.3]). There is a commutative diagram

$$\begin{array}{ccccc} \overline{M} & \xleftarrow{\bar{c}_2} & \overline{M'} & \xrightarrow{\bar{c}_1} & \overline{M} \\ \downarrow p & & \downarrow p' & & \downarrow p \\ M^* & \xleftarrow{c_2} & M'^* & \xrightarrow{c_1} & M^*. \end{array} \tag{3.6.4}$$

The filtrations (3.5.1) and (3.5.2) are preserved by these correspondences, i.e. (c_1, c_2) as in (3.6.2) restricts to $(c_1, c_2) : M'^*_{\leqslant d} \rightrightarrows M^*_{\leqslant d}$ for each d and similarly for $(\bar{c}_1, \bar{c}_2)$.

3.7. **Coefficient systems.** A $\mathbb{Q}$-irreducible finite-dimensional rational representation of G gives a polarizable irreducible variation of rational Hodge structures of a certain weight w on $\mathbb{E}_M$. This is explained in [Z81] and also in [LR91, §4]. (A more elegant treatment is possible in the context of Shimura varieties, cf. [HZ01, BW04].) By [S90, Theorem 3.27] (cf. also the remark after 3.27 regarding the algebraic case) a polarizable variation of Hodge structure gives a smooth Hodge module

$$\mathbb{E}_M^H[\dim M] \in \mathrm{MHM}(M)$$

which is pure of weight $w + \dim M$, irreducible, and polarizable, and has $rat(\mathbb{E}_M^H(\dim M)) = \mathbb{E}_M[\dim M]$. Note that $\mathbb{E}_M^H$ is effective. For a correspondence (c_1, c_2) as in (3.6.1) there are isomorphisms $c_2^*\mathbb{E}_M^H = \mathbb{E}_{M'}^H = c_1^*\mathbb{E}_M^H = c_1^!\mathbb{E}_M^H$. If $\Gamma' = g\Gamma g^{-1}$ and $(g\cdot) : M' \to M$ is the isomorphism induced by $g\cdot : D \to D$ there is a natural isomorphism $(g\cdot)^*\mathbb{E}_M^H = \mathbb{E}_{M'}^H$.

Remark 3.7.1. It is well-known that M, being a connected component of a Shimura variety at finite level, has a model over a number field k contained in $\mathbb{C}$. (The theory of canonical models shows that k is, up to an abelian extension, independent of the congruence subgroup Γ.) Let $\mathscr{M}(\cdot)$ be a theory of A-mixed sheaves over k in the sense of Saito [S06] (cf. 2.3). If the rational local system $\mathbb{E}_M$ is of geometric origin (in the sense of [BBD82, S06]) then $\mathbb{E}_M \otimes A$ underlies an object $\mathbb{E}_M^{\mathscr{M}} \in D^b\mathscr{M}(M)$. (If $A = \mathbb{Q}$ and the forgetful functor to perverse sheaves on $X(\mathbb{C})$ factors through mixed Hodge modules then $\mathbb{E}_M^{\mathscr{M}}$ maps to $\mathbb{E}_M^H$.) The local systems $\mathbb{E}_M$ are known to be of geometric origin in many cases where M can be related to a moduli problem involving abelian varieties, cf. [P92, 5.6], but not in general.

For the particular theory of mixed sheaves $\mathrm{MHM}(\cdot/k)$ in [S06, 1.8(ii)] (mentioned earlier in 2.3), the local system $\mathbb{E}_M$ coming from a $\mathbb{Q}$-rational representation of G underlies an object in the category $\mathrm{MHM}(M/k)$ where k is the field of definition of M.

A well-known prescription (of Langlands, cf. [P92, 5.1]) makes $\mathbb{E}_M \otimes \mathbb{Q}_l$ into an l-adic sheaf. For Morel's theory of mixed l-adic complexes (cf. 2.3) $\mathbb{E}_M \otimes \mathbb{Q}_l$ underlies a mixed l-adic sheaf if $\mathbb{E}_M$ is of geometric origin (in the sense of [BBD82]).

3.8. **Correspondences and weight truncations.** In the next section we will consider weight truncations of $Rj_*\mathbb{E}_M^H$ with respect to the stratification of (3.5). We note some basic facts about the action of correspondences on them (cf. [M08, §5]).

Let $\underline{a} = (a_0, \ldots, a_r) \in (\mathbb{Z} \cup \{\pm\infty\})^{r+1}$. A correspondence $(c_1, c_2) : M'^* \rightrightarrows M^*$ as in (3.6.2) has a unique lift to $Rj_*\mathbb{E}_M^H$ extending the isomorphism $c_2^*\mathbb{E}_M^H = \mathbb{E}_{M'}^H = c_1^!\mathbb{E}_M^H$. It is given by

$$c_2^* Rj_*\mathbb{E}_M^H \xrightarrow{BC} Rj'_* c_2^*\mathbb{E}_M^H = Rj'_*\mathbb{E}_{M'}^H = Rj'_* c_1^!\mathbb{E}_M^H \xrightarrow{BC} c_1^! Rj_*\mathbb{E}_M^H \tag{3.8.1}$$

(Arrows marked BC are base change morphisms.)

Lemma 3.8.1. *Let $(c_1, c_2) : M'^* \rightrightarrows M^*$ be a finite correspondence on M^* as above.*

(i) There is a unique morphism $c_2^* w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H \longrightarrow c_1^! w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H$ such that

$$\begin{array}{ccc} c_2^* w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H & \longrightarrow & c_1^! w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H \\ \downarrow & & \downarrow \\ c_2^* Rj_*\mathbb{E}_M^H & \xrightarrow{(3.8.1)} & c_1^! Rj_*\mathbb{E}_M^H \end{array}$$

commutes.

(ii) The morphism in (i) factors as

$$c_2^* w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H \longrightarrow w_{\leqslant \underline{a}} Rj'_*\mathbb{E}_{M'}^H \longrightarrow c_1^! w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H. \tag{3.8.2}$$

Proof. (i) This is a case of [M08, Lemme 5.1.3] (and follows easily from Lemma 2.1.2).

(ii) Consider the composition $c_2^* w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H \xrightarrow{can} c_2^* Rj_*\mathbb{E}_M^H \xrightarrow{BC} Rj'_*\mathbb{E}_{M'}^H$. Since c_2 is finite, $c_2^* w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H \in {}^wD^{\leqslant \underline{a}}$ and so $BC \circ can$ factors through a unique morphism

$$c_2^* w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H \longrightarrow w_{\leqslant \underline{a}} Rj'_*\mathbb{E}_{M'}^H.$$

A dual argument (using Lemma 2.1.4) gives a morphism

$$w_{\leqslant \underline{a}} Rj'_*\mathbb{E}_{M'}^H \longrightarrow c_1^! w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H,$$

so that we can compose as in (3.8.2). The compatibility with (3.8.1) is clear. By uniqueness the composition is the unique lift. □

We will also use the following fact: If $g\Gamma g^{-1} = \Gamma'$ and $(g\cdot) : M'^* \to M^*$ is the isomorphism induced by $g\cdot : D^* \to D^*$ then $(g\cdot)^* w_{\leqslant \underline{a}} Rj_*\mathbb{E}_M^H = w_{\leqslant \underline{a}} Rj'_*\mathbb{E}_{M'}^H$.

3.9. **Automorphic complexes.** Following [LR91, §6], let $D^b\mathrm{MHM}_{\mathrm{aut}}(M^*)$ be the (full) triangulated subcategory of $D^b\mathrm{MHM}(M^*)$ consisting of *automorphic complexes*. Thus $K \in Ob\, D^b\mathrm{MHM}(M^*)$ is automorphic if for each stratum S (of the canonical stratification of M^*) and integer k the cohomology object $H^k(i_S^* K) \in \mathrm{MHM}(S)$ is smooth and satisfies

(1) its underlying local system is the local system associated with a finite-dimensional rational representation of M_h and
(2) the weight graded pieces $Gr_j^W H^k(i_S^* K)$ are pure Hodge modules associated with the locally homogeneous variation coming (as in 3.7) from the M_h-representation on $Gr_j^W H^k(i_S^* K)$.

The definition makes sense for any subvariety $X \subset M^*$ which is a union of strata and locally closed, for example $X = M^*_{\leqslant d}$ for $d \in \{0, \dots, r\}$ (as in (3.5.1)) or X the closure of a stratum.

Lemma 3.9.1. *(i) Let $X \subset Y$ be subvarieties of M^* which are unions of strata of the canonical stratification and let $i : X \hookrightarrow Y$ be the inclusion. Then the functors $Ri_*, Ri_!, i^*, i^!$ define functors between categories of automorphic complexes of mixed Hodge modules.*

(ii) The functors $w_{\leqslant \underline{a}}, w_{>\underline{a}}$ preserve categories of automorphic complexes of mixed Hodge modules.

Proof. (i) This is stated in [LR91, p. 256], but only the weight filtration is fully treated there. The results of Harris and Zucker [HZ01, III.4.3] or Burgos and Wildeshaus [BW04, Thm 2.6] show that Rj_* preserves automorphic complexes in the case where j is the inclusion of a locally symmetric variety in its minimal compactification. (We note that the result of [HZ01] is more than we need, and the simpler result of [BW04] suffices here.) The general case follows from this and from the standard triangles for open-closed pairs.

(ii) If X and K are smooth and K is automorphic then $H^i(w_{\leqslant a}K) \hookrightarrow H^i(K)$ for all i, so that $w_{\leqslant a}K$ is smooth and automorphic. It follows that $w_{>a}K$ is also smooth and automorphic. Now consider a general X. For an automorphic complex K and a stratum S_k consider the standard triangle

$$w^k_{\leqslant a}K \longrightarrow K \longrightarrow Ri_{S_k *} w_{>a} i^*_{S_k} K \xrightarrow{+1}$$

from Lemma 2.1.3 (ii). The third term is automorphic because $i^*_{S_k}K$ is smooth and Ri_{S_k*} is the composite of pushforward by an open immersion (which preserves automorphy by (i)) and pushforward by a closed immersion (which preserves automorphy trivially). Since K is automorphic, we conclude that $w^k_{\leqslant a}K$ is automorphic. Now Lemma 2.1.3 (i) and an induction prove the lemma. □

Remark 3.9.2. Let $\mathscr{M}(\,\cdot\,)$ be a theory of A-mixed sheaves. Suppose that for any G as in 3.1 and any finite-dimensional rational representation E of G, the local system $\mathbb{E}_M \otimes A$ underlies an object in $\mathscr{M}(M)$. (The theories $\mathrm{MHM}(\cdot)$ and $\mathrm{MHM}(\cdot/k)$ in 2.3 satisfy this condition; $\mathscr{M}_l(\cdot)$ satisfies this for $\mathbb{E}_M \otimes \mathbb{Q}_l$ of geometric origin.) Then the notion of automorphic complex makes sense in $D^b\mathscr{M}(X)$ where X is a union of strata of M^* as the conditions (1) and (2) of the previous paragraph do so. (If S is smooth an object $M \in \mathscr{M}(S)$ is called smooth if $For(M)$ is a shifted local system of A-vector spaces.) The proof of (i) of the lemma shows that if Rj_* preserves the automorphy condition, where j is the inclusion of a stratum in its closure (i.e. in its minimal compactification), then the other functors also do so, i.e. (i) of the lemma holds for $\mathscr{M}(\cdot)$. The proof of (ii) works in any category of mixed sheaves such that the direct image by an open immersion preserves automorphy.

3.10. **Pure and mixed.** We use the spectral sequence of the introduction to show that the pure structures appearing in the cohomology of noncompact Shimura varieties (i.e. things like $\mathrm{Gr}^W_i H^j(M, \mathbb{E})$) can already be found in the intersection cohomology (with homogeneous coefficients) of noncompact Shimura varieties of smaller dimension, at least in the setting of Hodge (or Hodge-de Rham) structures. (The existence of the spectral sequence is explained in 5.2 below.)

Theorem 3.10.1. *Assume that we are working in a theory of A-mixed sheaves $\mathscr{M}(\cdot)$ such that (a) the direct image preserves automorphy and (b) $\mathbb{E}_M \otimes A$ underlies an object in the theory.*

Then the weight-graded pieces $Gr_j^W H^i(M, \mathbb{E}_M^{\mathscr{M}})$ of the mixed structure on $H^i(M, \mathbb{E}_M^{\mathscr{M}})$ appear as subquotients of the intersection cohomology of (minimal compactifications of) boundary strata of M^* with homogeneous local systems as coefficients.

In particular this holds in the following situations: (i) $A = \mathbb{Q}$ and the theory of mixed Hodge modules (ii) $A = \mathbb{Q}$ and the theory of mixed Hodge modules with de Rham rational structure.

Proof. Assume that (a) and (b) hold for $\mathscr{M}(\cdot)$. There is a spectral sequence with

$$E_1^{p,q} = \mathbb{H}^{p+q}(M^*, w_{\leqslant \dim M + w - p} w_{\geqslant \dim M + w - p} Rj_* \mathbb{E}_M^{\mathscr{M}}) \quad \Rightarrow \quad H^{p+q}(M, \mathbb{E}_M^{\mathscr{M}}).$$

(This is explained in 5.2 below, specifically (5.2.3).) By Lemma 2.2.3 (ii) and the remark following it, the E_1 term breaks up as a sum of intersection complexes supported on the closures of boundary strata. The normalizations of these closures are minimal compactifications of the boundary locally symmetric varieties, and by Lemma 3.9.1 the intersection complexes in question are associated with homogeneous local systems and their canonical variations.

If $A = \mathbb{Q}$ and $\mathscr{M}(X) = \mathrm{MHM}(X)$ we have established (a) and (b) above. For $A = \mathbb{Q}$ and $\mathscr{M}(X) = \mathrm{MHM}(X/k)$ for k the field of definition of the canonical stratification of M^*, (b) holds and (a) also holds because $\mathrm{MHM}(X/k)$ is a subcategory of $\mathrm{MHM}(X)$. □

Remark 3.10.2. As remarked in 2.3, pure objects in Morel's theory $\mathscr{M}_l(\cdot)$ are not necessarily semisimple. However, it seems likely that (a) holds in $\mathscr{M}_l(\cdot)$ for $\mathbb{E}_M \otimes \mathbb{Q}_l$ of geometric origin and that the terms appearing in (5.2.3) should be (pure and) semisimple

4. Mixed structures in weighted cohomology

4.1. **Weighted complexes of** [GHM94]**.** Let P_0 be the fixed minimal parabolic and A_0 the split centre of the Levi quotient M_0. For a standard parabolic $P \supset P_0$ with Levi quotient MA (i.e. A is the split centre of the Levi and M is its natural complement), the inclusion of split radicals $R_d P \subset R_d P_0$ induces an inclusion $A \subset A_0$. Thus $\nu \in X^*(A_0)_{\mathbb{Q}}$ gives $\nu|_A \in X^*(A)_{\mathbb{Q}}$ by restriction. For nonstandard P an element of $X^*(A)_{\mathbb{Q}}$ is defined by conjugation. (Since $X^*(A)_{\mathbb{Q}} = X^*(P)_{\mathbb{Q}}$ one sees that this is well-defined.) Thus ν defines a (quasi)character on the split part of every Levi quotient.

Let $\mathbb{E}_M$ be the local system given by an irreducible rational representation of G (as in 3.7). For each $\nu \in X^*(A_0)_{\mathbb{Q}}$, the weighted cohomology complex

$$W^{\geqslant \nu} C(\mathbb{E}_M)$$

defined in [GHM94] is an explicit complex of $\mathbb{Q}$-sheaves on $\overline{M}$ with cohomology sheaves constructible with respect to natural stratification. In fact, two definitions are given, one a complex of $\mathbb{Q}$-sheaves [GHM94, IV] and the other a complex of $\mathbb{C}$-sheaves [GHM94, II] and their agreement (in the derived category) is proved [GHM94, §29]. [N99] gives another complex of $\mathbb{C}$-sheaves quasi-isomorphic to weighted cohomology using differential forms with square-integrability conditions from [F98]. We will not need to recall the actual definitions but only some properties of these complexes established in [GHM94].

The natural correspondences of 3.6 lift to weighted complexes: For $(\bar{c}_1, \bar{c}_2) : \overline{M'} \rightrightarrows \overline{M}$ as (3.6.1) there are natural isomorphisms

$$\bar{c}_2^* W^{\geqslant \nu} C(\mathbb{E}_M) = W^{\geqslant \nu} C(\mathbb{E}_{M'}) = \bar{c}_1^! W^{\geqslant \nu} C(\mathbb{E}_M)$$

(cf. [GM03, §13.3]). Applying Rp_* gives a lift of the correspondence $(c_1, c_2) : M'^* \rightrightarrows M^*$ to $Rp_* W^{\geqslant\nu} C(\mathbb{E}_M)$ by using the base change morphisms coming from (3.6.4):

$$\begin{array}{ccc} c_2^* Rp_* W^{\geqslant\nu} C(\mathbb{E}_M) & \xrightarrow{BC} & Rp'_* \bar{c}_2^* W^{\geqslant\nu} C(\mathbb{E}_M) \\ & & \parallel \\ & & Rp'_* W^{\geqslant\nu} C(\mathbb{E}_{M'}) \\ & & \parallel \\ & & Rp'_* \bar{c}_1^! W^{\geqslant\nu} C(\mathbb{E}_M) \xrightarrow{BC} c_1^! Rp_* W^{\geqslant\nu} C(\mathbb{E}_M). \end{array} \tag{4.1.1}$$

For $\Gamma' = g\Gamma g^{-1}$ and $(g\cdot) : M'^* \xrightarrow{\sim} M^*$ there is a natural isomorphism $(g\cdot)^* Rp_* W^{\geqslant\nu} C(\mathbb{E}_M) = Rp'_* W^{\geqslant\nu} C(\mathbb{E}_{M'})$ extending $(g\cdot)^* \mathbb{E}_M = \mathbb{E}_{M'}$.

4.2. **Local Hecke actions and the splitting property.** (cf. [L88, LR91, GHM94]) Let S be a stratum of M^*. Fix a rational boundary component F covering it and let Q be the maximal parabolic stabilizing it. Fix a lift $A \subset Q$ of the split centre of the Levi quotient. By [L88, 3.6] there exist elements $a \in A(\mathbb{Q})$ such that:

(i) $\chi(a) \in \mathbb{Z}_{>0}$
(ii) $a\Gamma_Q a^{-1} \subset \Gamma_Q$.

(Here $\chi \in X^*(A)$ is as in 3.2, i.e. χ^2 is the character in $\operatorname{Lie} U$.) Such elements are called *divisible.*

For a rational boundary component $F \subset D^*$ let $\operatorname{Star}(F)$ denote the union of all rational boundary components containing F in their closure. Similarly, for a stratum $S \subset M^*$ let $\operatorname{Star}(S)$ denote the union of all strata containing S in their closure. The following summarizes some well-known facts about Looijenga's local Hecke operators:

Lemma 4.2.1. *Let $a \in A(\mathbb{Q})$ be divisible. Let $W = R_u Q$.*

(i) There is a closed $W(\mathbb{R})$-invariant neighbourhood B_F of F in $\operatorname{Star}(F)$ such that

(a) *$\Gamma_Q \backslash \operatorname{Star}(F) \to \Gamma \backslash D^*$ identifies $\Gamma_Q \backslash B_F$ with a closed nbhd of S in $\operatorname{Star}(S)$*
(b) *the endomorphism of $\Gamma_Q \backslash \operatorname{Star}(F)$ defined by $\Phi(\Gamma_Q x) = \Gamma_Q a x$ restricts to an endomorphism of $\Gamma_Q \backslash B_F$ fixing S pointwise.*

(ii) There is a closed $W(\mathbb{R})$-invariant neighbourhood $\overline{V}_Q$ of $\overline{D}_Q$ in $\overline{D}$ such that

(a) *$\Gamma_Q \backslash \overline{V}_Q$ is identified with a closed nbhd of $\overline{M}_Q$ in $\overline{M}$*
(b) *$\bar{\Phi}(\Gamma_Q x) = \Gamma_Q a x$ defines an endomorphism of $\Gamma_Q \backslash \overline{V}_Q$ fixing $\overline{M}_Q$ pointwise.*
(c) *$\bar{\Phi}$ respects the stratification of $\Gamma_Q \backslash \overline{V}_Q$ induced by $\overline{M} = \coprod_P M_P$.*

The morphism $\bar{\Phi}$ is a component of the restriction to $\Gamma_Q \backslash \overline{V}_Q$ of the Hecke correspondence given by $\Gamma a \Gamma$.

(iii) Let $p : \overline{D} \to D^*$ be as in 3.4. If $\overline{V}_Q$ is as in (ii), let

$$\overline{B}_Q := \coprod_{P^+ = Q} \overline{V}_Q \cap M_P.$$

(This is preserved by $\bar{\Phi}$ by (ii)(c).) Then $B_F = p(\overline{B}_Q)$ satisfies (a) and (b) of (i), and $\bar{\Phi}$ and Φ are related by $p \circ \bar{\Phi} = \Phi \circ p$.

Proof. (i)(a) is well-known (cf. [L88]): For B_F one may take any closed $W(\mathbb{R})$-invariant nbhd of F in $\operatorname{Star}(F)$ which in Γ_Q-invariant and on which Γ-equivalence and Γ_Q-equivalence coincide. If, in addition, $B_F \cap D$ is invariant under the geodesic action of the semigroup $\{a \in A(\mathbb{R})^0 : \chi(a) \geqslant 1\}$ then (b) also holds.

For (ii) we recall some facts from [GM03] about how Hecke correspondences break up near the boundary:

(1) For $g \in G(\mathbb{Q})$ and a parabolic P, the double coset $\Gamma g \Gamma$ meets $P(\mathbb{Q})$ in finitely many double cosets for Γ_P:
$$\Gamma g \Gamma \cap P(\mathbb{Q}) = \coprod_i \Gamma_P g_i \Gamma_P \tag{4.2.1}$$
For a double coset $\Gamma_P g_i \Gamma_P$ here write $g = \gamma_1 g_i \gamma_2$. Let $\Gamma' = \Gamma \cap g^{-1}\Gamma g$. Then $P \mapsto \gamma_1 P \gamma_1^{-1} = g^{-1}\gamma_2^{-1} P \gamma_2 g$ gives a bijection between the double cosets on the right of (4.2.1) and the Γ'-conjugacy classes of parabolics Q with both Q and gQg^{-1} Γ-conjugate to P. (See [GM03, 7.3].)

(2) Consider the Hecke correspondence $(\bar{c}_1, \bar{c}_2)$ in (3.6.3) with $\Gamma' = \Gamma \cap g^{-1}\Gamma g$. The restriction to the P-boundary stratum M_P is the correspondence $(\bar{c}_1, \bar{c}_2) : \bar{c}_1^{-1}(M_P) \cap \bar{c}_2^{-1}(M_P) \rightrightarrows M_P$. This breaks up according to (4.2.1):
$$\bar{c}_1^{-1}(M_P) \cap \bar{c}_2^{-1}(M_P) = \coprod_i M'_{Q_i}$$
where Q_i corresponds to g_i under the bijection and $M'_Q = \Gamma' \backslash D_Q$ is the Q-stratum of $\overline{M'}$. The restriction of $(\bar{c}_1, \bar{c}_2)$ to M'_{Q_i} is isomorphic to the correspondence on M_P of the form (3.6.1) given by $\nu_P(g_i) \in (P/R_uP)(\mathbb{Q})$ for $\Gamma'_{P/R_uP} = \nu_P(\Gamma_P \cap g_i^{-1}\Gamma_P g_i)$ (cf. [GM03, 8.1, 8.6]).

(3) In fact this happens over a suitable neighbourhood of M_P in $\overline{M}$ ([GM03, Prop. 7.3]). Let V_P be a closed Γ_P-invariant and $(R_uP)(\mathbb{R})$-invariant neighbourhood of D_P in $\overline{D}$ on which Γ-equivalence and Γ_P-equivalence coincide. Assume further that V_P is invariant under the geodesic action of the semigroup $\{a \in A(\mathbb{R})^0 : \chi(a) \geqslant 1\}$. (Without the $(R_uP)(\mathbb{R})$-invariance condition this is called a Γ-parabolic neighbourhood of D_P in [GM03].) Then $\Gamma_P \backslash V_P$ is a neighbourhood of M_P in $\overline{M}$ and $\bar{c}_1^{-1}(\Gamma_P \backslash V_P) \cap \bar{c}_2^{-1}(\Gamma_P \backslash V_P)$ is a disjoint union of components indexed by (4.2.1). The g_i component is a neighbourhood of M'_{Q_i} in $\overline{M'}$ with properties similar to V_P.

Now if $a \in P(\mathbb{Q})$ there is a distinguished double coset in (4.2.1), namely $\Gamma_P a \Gamma_P$. Let $V'_P = V_P \cap a^{-1}V_P$. The corresponding component of the restriction of the Hecke correspondence is
$$(\bar{c}_1, \bar{c}_2) : \Gamma'_P \backslash V'_P \;\rightrightarrows\; \Gamma_P \backslash V_P \qquad \begin{aligned} c_1(\Gamma'_P x) &= \Gamma_P\, x \\ c_2(\Gamma'_P x) &= \Gamma_P\, ax \end{aligned} \tag{4.2.2}$$
where $\Gamma'_P = \Gamma_P \cap a^{-1}\Gamma_P a$. Suppose further that $a \in (R_dP)(\mathbb{Q})$ (the split radical). In this case (4.2.2) is a covering of the trivial correspondence when restricted to M_P.

Now let Q, A be as in the lemma. If $P \subset Q$ then $R_dQ \subset R_dP$, so it follows that $a \in A(\mathbb{Q}) \subset (R_dQ)(\mathbb{Q})$ induces local correspondences on neighbourhoods $\Gamma_P \backslash V_P$ for each $P \subset Q$. Letting $\overline{V}_Q := \cup_{P \subset Q} V_P$, $\overline{V}'_Q := \cup_{P \subset Q} V'_P$, and $\Gamma'_Q = \Gamma_Q \cap a^{-1}\Gamma_Q a$ we get a local correspondence
$$(\bar{c}_1, \bar{c}_2) : \Gamma'_Q \backslash \overline{V}'_Q \rightrightarrows \Gamma_Q \backslash \overline{V}_Q \tag{4.2.3}$$
on the closed neighbourhood $\Gamma_Q \backslash \overline{V}_Q$ of the closure $\overline{M}_Q$, and which is a covering of the trivial correspondence on each $M_P \subset \overline{M}_Q$.

Finally, suppose that a is divisible. Since $a^{-1}\Gamma_Q a \supset \Gamma_Q$ we have $a^{-1}\Gamma_P a \supset \Gamma_P$ for all $P \subset Q$ since $a \in P(\mathbb{Q})$. Thus $\Gamma'_P = \Gamma_P$ for all $P \subset Q$. Moreover, we can take $V'_P = V_P$ for all P since $aV_P \subset V_P$. Thus $\bar{\Phi} = \bar{c}_2$ defines an endomorphism of $\Gamma_Q \backslash \overline{V}_Q$ fixing each M_P for $P \subset Q$ pointwise.

(iii) Since $p \circ \bar{\Phi} = \Phi \circ p$ on $(\Gamma_Q \backslash \overline{B}_Q) \cap M$ they are the same. $\square$

For a maximal parabolic Q with associated boundary stratum $S \subset M^*$, restricting the lifts (4.1.1) to the relevant neighbourhoods as in Lemma 4.2.1 (and choosing the correct component of the correspondence) gives compatible lifts of the local Hecke operator $\bar{\Phi}$ to $W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{p^{-1}(S)}$ and of Φ to $Rp_*W^{\geqslant\nu}C(\mathbb{E}_M)\big|_S$. The basic splitting property of weighted complexes is the following (the notation $M^*_{\leqslant d}$ is as in 3.5):

Lemma 4.2.2. *(cf.* [GHM94, §16]*) Let $i_S : S \hookrightarrow M^*_{\leqslant d}$ be the inclusion of a connected component of M_d. Fix a maximal parabolic subgroup Q stabilizing a boundary component covering S, a lift $A \subset Q$ of the split centre, and let Φ be the local Hecke operator given by a divisible element $a \in A(\mathbb{Q})$. Let*

$$L_{d-1} := Rp_*W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{M^*_{\leqslant d-1}}.$$

(i) There is a (finite) Φ-invariant direct sum decomposition in the derived category

$$i_S^*Rj_{d*}L_{d-1} = \bigoplus_\lambda (i_S^*Rj_{d*}L_{d-1})_\lambda \tag{4.2.4}$$

such that Φ acts by the rational scalar λ on the λ-summand. The standard triangle

$$Rp_*W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{M^*_{\leqslant d}} \longrightarrow Rj_{d*}Rp_*\big(W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{M^*_{\leqslant d-1}}\big) \longrightarrow i_{d*}i_d^!\big(Rp_*W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{M^*_{\leqslant d}}\big)[1] \xrightarrow{+1} \tag{4.2.5}$$

(i.e. $L_d \to Rj_{d*}L_{d-1} \to i_{d*}i_d^!L_d[1] \xrightarrow{+1}$) on $M^*_{\leqslant d}$ becomes isomorphic to the split triangle

$$\bigoplus_{\lambda\leqslant\nu(a)^{-1}} (i_S^*Rj_{d*}L_{d-1})_\lambda \longrightarrow i_S^*Rj_{d*}L_{d-1} \longrightarrow \bigoplus_{\lambda>\nu(a)^{-1}} (i_S^*Rj_{d*}L_{d-1})_\lambda \xrightarrow{0}$$

coming from (4.2.4) when restricted to S, i.e. after applying i_S^*.

(ii) For any subset $Z \subset S$ the action of Φ in $\mathbb{H}^i(Z, Rj_{d*}L_{d-1})$ is semisimple over $\mathbb{Q}$ and (4.2.5) gives the short-exact sequence

$$0 \longrightarrow \mathbb{H}^i(Z, Rj_{d*}L_{d-1})_{\leqslant\nu(a^{-1})} \longrightarrow \mathbb{H}^i(Z, Rj_{d*}L_{d-1}) \longrightarrow \mathbb{H}^i(Z, Rj_{d*}L_{d-1})_{>\nu(a^{-1})} \longrightarrow 0 \tag{4.2.6}$$

given by decomposing with respect to weights of Φ.

Proof. Following the notation of [GHM94, §16], let

$$\mathbf{B}^\bullet := \bar{j}_{d*}\big(W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{\overline{M}_{\leqslant d-1}}\big)$$

This represents $R\bar{j}_{d*}\big(W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{\overline{M}_{\leqslant d-1}}\big)$ in the derived category and adjunction gives an inclusion $W^{\geqslant\nu}C(\mathbb{E}_M) \subset \mathbf{B}^\bullet$ representing $W^{\geqslant\nu}C(\mathbb{E}_M) \to R\bar{j}_{d*}\big(W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{\overline{M}_{\leqslant d-1}}\big)$. Lemma 16.9 of [GHM94] gives a quasi-isomorphic subcomplex $\mathbf{B}^\bullet_{\mathrm{sp}} \subset \mathbf{B}^\bullet$ containing $W^{\geqslant\nu}C(\mathbb{E}_M)$. Now $p^{-1}(S)$ is the closure in $\overline{M}_{\leqslant d}$ of the Q-stratum of $\overline{M}$, and [GHM94, 16.13] shows that the restriction of $\mathbf{B}^\bullet_{\mathrm{sp}}$ to such a closure admits a decomposition

$$\mathbf{B}^\bullet_{\mathrm{sp}}|_{p^{-1}(S)} = \bigoplus_\lambda \big(\mathbf{B}^\bullet_{\mathrm{sp}}|_{p^{-1}(S)}\big)_\lambda$$

such that $\bar{\Phi}$ acts by the scalar λ on the λ-summand. The scalars appearing here are the values of certain rational characters of A on the divisible element $a \in A(\mathbb{Q})$ and so lie in $\mathbb{Q}$. (The proof of) Proposition 16.4 of [GHM94] shows that the distinguished triangle on $\overline{M}_{\leqslant d}$

$$W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{\overline{M}_{\leqslant d}} \longrightarrow R\bar{j}_{d*}\big(W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{\overline{M}_{\leqslant d-1}}\big) \longrightarrow \bar{i}_{d*}\bar{i}_d^!\big(W^{\geqslant\nu}C(\mathbb{E}_M)\big|_{\overline{M}_{\leqslant d}}\big)[1] \xrightarrow{+1} \tag{4.2.7}$$

is, on restriction to $p^{-1}(S)$, isomorphic to the split triangle

$$\bigoplus_{\lambda\leqslant\nu(a)^{-1}} \big(\mathbf{B}^\bullet_{\mathrm{sp}}|_{p^{-1}(S)}\big)_\lambda \longrightarrow \mathbf{B}^\bullet_{\mathrm{sp}}|_{p^{-1}(S)} \longrightarrow \bigoplus_{\lambda>\nu(a)^{-1}} \big(\mathbf{B}^\bullet_{\mathrm{sp}}|_{p^{-1}(S)}\big)_\lambda \xrightarrow{0} .$$

Pushing forward by the proper map $p : \overline{M} \to M^*$ and using that $\Phi \circ p = p \circ \bar{\Phi}$ on neighbourhoods as in Lemma 4.2.1 gives the decomposition of

$$Rp_*\big(\mathbf{B}^\bullet_{\mathrm{sp}}|_{p^{-1}(S)}\big) = Rp_*\big(\mathbf{B}^\bullet|_{p^{-1}(S)}\big) = i_S^* Rj_{d*}L_{d-1}$$

and the splitting of the triangle (4.2.5) over S. This proves (i). ([8]) For (ii) note that the description of the restriction to $p^{-1}(S)$ of (4.2.7) above shows that for any subset $Y \subset \overline{M}_{\leqslant d} - \overline{M}_{\leqslant d-1}$ contained in a single connected component the resulting short-exact sequence in hypercohomology is the short-exact sequence

$$0 \longrightarrow \mathbb{H}^i(Y, \mathbf{B}^\bullet)_{\leqslant\nu(a^{-1})} \longrightarrow \mathbb{H}^i(Y, \mathbf{B}^\bullet) \longrightarrow \mathbb{H}^i(Y, \mathbf{B}^\bullet)_{>\nu(a^{-1})} \longrightarrow 0 \tag{4.2.8}$$

given by decomposing $\mathbb{H}^i(Y, \mathbf{B}^\bullet)$ with respect to weights of the local Hecke operator $\bar{\Phi}$ (Cor. 16.5 of [GHM94]). Taking the direct image by $p : \overline{M} \to M^*$ gives (ii). □

Remark 4.2.3. The decomposition of $\mathbb{H}^i(Z, Rj_{d*}L_{d-1})$ (resp. of $\mathbb{H}^i(Y, \mathbf{B}^\bullet)$) into eigenspaces for Φ (resp. for $\bar{\Phi}$) depends on the choice of a lift $A \subset Q$, but once a lift is chosen it is independent of the choice of divisible element $a \in A(\mathbb{Q})$. The sum of eigenspaces with eigenvalues $\leqslant \nu(a^{-1})$ and the short exact sequence (4.2.6) are independent of choices.

4.3. **Pushforward of weighted complexes.** For $d = 1, \dots, r$, let $Q_1, \dots, Q_d$ be the maximal parabolic subgroups containing P_0 (ordered as in 3.2) and let χ_d denote the distinguished generator of the character group of A_d (cf. 3.2). Recall the inclusion $A_d \subset A_0$ induced by the inclusion of split radicals $R_dQ_d \subset R_dP_0$. If $\nu \in X^*(A_0)_{\mathbb{Q}}$ satisfies

$$\nu|_{A_d} = \chi_d^{n_d} \qquad (\text{for } d = 1, \dots, r)$$

(where χ_d^2 is the character of A_d on $\operatorname{Lie} U_d$), we associate with it the sequence $\underline{a} = (a_1, \dots, a_r) \in \mathbb{Q}^r$ defined by

$$a_d = \dim M_d - n_d \qquad (\text{for } d = 1, \dots, r). \tag{4.3.1}$$

Theorem 4.3.1. *Let $\mathbb{E}_M$ be the local system of rational vector spaces on M coming from an irreducible representation of G. Let w be the weight of the associated shifted Hodge module $\mathbb{E}_M^H$. For $(a_1, \dots, a_r)$ associated with $\nu \in X^*(A_0)_{\mathbb{Q}}$ (i.e. satisfying (4.3.1)) and $a_0 = \dim M$ there are natural isomorphisms in $D_c^b(\mathbb{Q}_{M^*})$*

$$Rp_*W^{\geqslant\nu}C(\mathbb{E}_M) = rat(w_{\leqslant(a_0+w, a_1+w, \dots, a_r+w)}Rj_*\mathbb{E}_M^H) \tag{4.3.2}$$

$$Rp_*\big(W^{\geqslant-\infty}C(\mathbb{E}_M)/W^{\geqslant\nu}C(\mathbb{E}_M)\big) = rat(w_{>(a_0+w, a_1+w, \dots, a_r+w)}Rj_*\mathbb{E}_M^H) \tag{4.3.3}$$

[8] The arguments in [GHM94, §16] are for the version of $W^{\geqslant\nu}C(\mathbb{E}_M)$ with $\mathbb{C}$-coefficients, so that the decomposition (4.2.4) is a priori only available with $\mathbb{C}$-coefficients. However, it follows from Lemma 4.3.3 below that it holds with $\mathbb{Q}$-coefficients, as we will see in the proof of Theorem 4.3.1.

compatible with Hecke correspondences. (Here $-\infty$ is any sufficiently negative weight, so that $Rp_* W^{\geqslant -\infty} C(\mathbb{E}_M) = j_! \mathbb{E}_M$).)

The proof of this theorem takes up the rest of 4.3.

Proof. The proof of the first equality is by induction on the stratification $M^* = \coprod_{d=0,\ldots,r} M_d$ in 3.5, for which the objects in question are constructible. To simplify notation in the proof we set:

$$L_d := Rp_* W^{\geqslant \nu} C(\mathbb{E}_M)\big|_{M^*_{\leqslant d}}$$
$$K_d := w_{\leqslant (a_0+w,\ldots,a_r+w)} Rj_* \mathbb{E}^H_M\big|_{M^*_{\leqslant d}} = w_{\leqslant (a_0+w,\ldots,a_d+w)} R(j_d \cdots j_1)_* \mathbb{E}^H_M.$$

(The same notation L_d was used in Lemma 4.2.2.) Further, if we are working at level Γ', i.e. on M'^* for $M' = \Gamma' \backslash D$ we will use L'_d, K'_d etc. to denote the same objects on M'^*. We will use the notation in (3.5.3). Note that $j_d^* L_d = L_{d-1}$ and $j_d^* K_d = K_{d-1}$.

We will prove by induction that for all d there are natural isomorphisms $\alpha_d : L_d \to rat(K_d)$ (at all levels Γ) such that

(i)$_d$ If $\Gamma' = g^{-1}\Gamma g$ then for the isomorphism $(g\cdot) : M'^* \to M^*$ induced by $g\cdot : D^* \to D^*$ the diagram

$$\begin{array}{ccc} (g\cdot)^* L_d & \longrightarrow & L'_d \\ {\scriptstyle (g\cdot)^*(\alpha_d)}\downarrow & & \downarrow{\scriptstyle \alpha'_d} \\ (g\cdot)^* rat(K_d) & \longrightarrow & rat(K'_d) \end{array}$$

commutes. (The horizontal maps are the isomorphisms extending $(g\cdot)^* \mathbb{E}^H_{M'} = \mathbb{E}^H_{M'}$.)

(ii)$_d$ If $b : M'^* \to M^*$ is the finite morphism coming from $\Gamma' \subset \Gamma$ then

$$\begin{array}{ccc} b^* L_d & \longrightarrow & L'_d \\ {\scriptstyle b^*(\alpha_d)}\downarrow & & \downarrow{\scriptstyle \alpha'_d} \\ b^* rat(K_d) & \longrightarrow & rat(K'_d) \end{array}$$

commutes. (Here $b^* L_d \to L'_d$ is as in (4.1.1) and $b^* K_d \to K'_d$ comes from (3.8.2).)

(iii)$_d$ α_d is the unique morphism making the diagram

$$\begin{array}{ccc} L_d & \longrightarrow & Rj_{d*} L_{d-1} \\ {\scriptstyle \alpha_d}\downarrow & & \downarrow{\scriptstyle Rj_{d*}(\alpha_{d-1})} \\ rat(K_d) & \longrightarrow & rat(Rj_{d*} K_{d-1}) \end{array}$$

commute.

(iv)$_d$ If $(c_1, c_2) : M'^*_{\leqslant d} \rightrightarrows M^*_{\leqslant d}$ is a correspondence as in (3.6.2) the diagram

$$\begin{array}{ccc} c_2^* L_d & \longrightarrow & c_1^! L_d \\ {\scriptstyle c_2^*(\alpha_d)}\downarrow & & \downarrow{\scriptstyle c_1^!(\alpha_d)} \\ c_2^* rat(K_d) & \longrightarrow & c_1^! rat(K_d) \end{array} \tag{4.3.4}$$

commutes (cf. (4.1.1) for the top row and 3.8 for the bottom row).

There are some relations between these properties. $(\mathrm{i})_d$ follows from $(\mathrm{i})_{d-1}$ and $(\mathrm{iii})_d$. Furthermore, $(\mathrm{iv})_d$ is implied by the other conditions as follows. (4.3.4) is the outer square of the diagram

$$\begin{array}{ccccc} c_2^*L_d & \longrightarrow & L'_d & \longrightarrow & c_1^!L_d \\ {\scriptstyle c_2^*(\alpha_d)}\downarrow & & \downarrow{\scriptstyle \alpha'_d} & & \downarrow{\scriptstyle c_1^!(\alpha_d)} \\ c_2^*rat(K_d) & \longrightarrow & rat(K'_d) & \longrightarrow & c_1^!rat(K_d) \end{array} \tag{4.3.5}$$

(The top row is (4.1.1). The bottom row is from (3.8.2).) Let $M'^*_g = g\Gamma' g^{-1}\backslash D^*$. Then $c_2 : M'^* \to M^*$ factorizes as $M'^* \xrightarrow{g\cdot} M'^*_g \longrightarrow M^*$ where $g\cdot$ is the isomorphism induced by $g\cdot : D^* \to D^*$ and the second map is induced by $g\Gamma' g^{-1} \subset \Gamma$. Then $(\mathrm{i})_d$ and $(\mathrm{ii})_d$ (for $M'^*_g \to M^*$) imply that the first square commutes. The commutativity of the second square follows by duality from $(\mathrm{ii})_d$.

We start the induction argument. When $d = 0$ both L_0 and $rat(K_0)$ are isomorphic to the homogeneous local system $\mathbb{E}_M$ and an equivariant isomorphism between them α_0 (unique up to a scalar as E is irreducible) satisfies $(\mathrm{i})_0$–$(\mathrm{iv})_0$. The induction hypothesis is that there are natural isomorphisms (at all levels) $\alpha_{d-1} : L_{d-1} \longrightarrow rat(K_{d-1})$ satisfying $(\mathrm{i})_{d-1}$–$(\mathrm{iv})_{d-1}$. Consider the diagram on $M^*_{\leqslant d}$ in which both rows are distinguished triangles:

$$\begin{array}{ccccccc} L_d & \xrightarrow{u} & Rj_{d*}L_{d-1} & \longrightarrow & i_{d*}i_d^!L_d[1] & \xrightarrow{+1} & \\ {\scriptstyle \alpha_d}\downarrow & & \downarrow{\scriptstyle Rj_{d*}(\alpha_{d-1})} & & & & \\ rat(K_d) & \longrightarrow & rat(Rj_{d*}K_{d-1}) & \xrightarrow{v} & i_{d*}rat(w_{>a_d+w}i_d^*Rj_{d*}K_{d-1}) & \xrightarrow{+1} & \end{array} \tag{4.3.6}$$

Here $K_d = w^d_{\leqslant a_d+w}Rj_{d*}K_{d-1}$ by Lemma 2.2.1. We will show that

$$v \circ Rj_{d*}(\alpha_{d-1}) \circ u = 0,$$

so that $Rj_{d*}(\alpha_{d-1}) \circ u$ factors through a morphism α_d as indicated, that α_d is an isomorphism, and that $(\mathrm{i})_d$–$(\mathrm{iv})_d$ hold. This will complete the inductive step.

For a $\mathbb{C}$-vector space V with an endomorphism write $V_{\leqslant\lambda}$ (resp. $V_{>\lambda}$) for the sum of eigenspaces with eigenvalue of modulus $\leqslant \lambda$ (resp. $> \lambda$). We will need the following lemma (from [LR91], cf. especially Prop. 6.4 of loc. cit.) which gives the connection between weights (in mixed Hodge theory) and weights of a local Hecke operator. (The action of correspondences defined in 3.8 gives an action of a local Hecke operator along S on $i_S^*w^d_{\leqslant a}Rj_{d*}K_{d-1}$, see the discussion in [LR91, 4.3].)

Lemma 4.3.2. *Let S be a component of M_d, and Q the maximal parabolic subgroup stabilizing a boundary component covering S. Let $A \subset Q$ be a lift of the split centre and let Φ be given by a divisible element $a \in A(\mathbb{Q})$. Let $K_{d-1} = w_{\leqslant(a_0+w,\dots,a_r+w)}Rj_*\mathbb{E}^H_M\big|_{M^*_{\leqslant d-1}}$ (as above). Then applying ${}^pH^{i+\dim M_d} \circ i_S^*$ to the triangle*

$$rat(w^d_{\leqslant a_d+w}Rj_{d*}K_{d-1}) \longrightarrow rat(Rj_{d*}K_{d-1}) \longrightarrow i_{d*}rat(w_{>a_d+w}i_d^*Rj_{d*}K_{d-1}) \xrightarrow{+1}$$

gives the short exact sequence of local systems on S

$$\begin{aligned} 0 \longrightarrow H^i(rat(i_S^*Rj_{d*}K_{d-1}))_{\leqslant\chi(a)^{a_d-\dim M_d}} \longrightarrow H^i(rat(i_S^*Rj_{d*}K_{d-1})) \longrightarrow \\ \longrightarrow H^i(rat(i_S^*Rj_{d*}K_{d-1}))_{>\chi(a)^{a_d-\dim M_d}} \longrightarrow 0 \end{aligned} \tag{4.3.7}$$

given by decomposing $H^i(i_S^*rat(Rj_{d*}K_{d-1}))$ with respect to weights of Φ.

Proof. Applying i_S^* to $w^d_{\leqslant a_d+w}Rj_{d*}K_{d-1} \to Rj_{d*}K_{d-1} \to i_{d*}w_{>a_d+w}i_d^*Rj_{d*}K_{d-1} \xrightarrow{+1}$ gives the triangle for the t-structure $({}^wD^{\leqslant a_d+w}(S), {}^wD^{>a_d+w}(S))$, i.e.

$$w_{\leqslant a_d+w}i_S^*Rj_{d*}K_{d-1} \to i_S^*Rj_{d*}K_{d-1} \to w_{>a_d+w}i_S^*Rj_{d*}K_{d-1} \xrightarrow{+1} .$$

Now if $\chi(a)^2 = q$ then the results in [LR91, §2] (specifically, (2.14) and (2.13) of loc. cit., noting also that the truncation functors $w_{\leqslant a_k}$ make sense in the setting of q-Hodge modules of loc. cit.) show that $\chi(a)^w\Phi^* = q^{w/2}\Phi^*$ acts as a q-endomorphism on each term in the triangle, making each into a (mixed) q-Hodge module in the terminology of [LR91]. Thus for $x \in S$, $\chi(a)^w\Phi^*$ splits the weight filtration on $H^i(i_x^*i_S^*Rj_{d*}K_{d-1})$, i.e. it acts by eigenvalues of modulus $q^{k/2} = \chi(a)^k$ on $\mathrm{Gr}_k^W(\,\cdot\,)$ of this space. By Lemma 2.2.4, for each i, $H^i(i_x^*w^d_{\leqslant a_d+w}Rj_{d*}K_{d-1})$ is the subspace of $H^i(i_x^*Rj_{d*}K_{d-1})$ with Hodge weights $\leqslant a_d + w - \dim M_d$. Since $\chi(a)^w\Phi^*$ is a q-endomorphism, this is precisely the subspace on which Φ^* has eigenvalues of modulus $\leqslant \chi(a)^{a_d-\dim M_d}$. Applying rat gives the statement of the lemma. □

We will also need the following lemma of Laumon and Ngô (the analogue of Jordan decomposition in the derived category):

Lemma 4.3.3. *(*[LN08, Lemme 3.2.5]*) Let E be a field, $\mathscr{A}$ an E-linear abelian category, K an object of $D^b(\mathscr{A})$, and Γ an abelian group acting E-linearly on K. Assume that for each integer n the cohomology object $H^n(K)$ admits in $\mathscr{A}$ a Γ-equivariant decomposition*

$$H^n(K) = \bigoplus_\chi H^n(K)_\chi$$

where χ runs over characters of Γ with values in E^*, and for each χ and each $\gamma \in \Gamma$, $\gamma - \chi(\gamma)$ operates nilpotently on $H^n(K)_\chi$, and $H^n(K)_\chi = 0$ for all but finitely many χ.

Then there exists a unique Γ-equivariant decomposition

$$K = \bigoplus_\chi K_\chi$$

in $D^b(\mathscr{A})$ where χ runs over E^*-valued characters of Γ, such that for each χ and each $\gamma \in \Gamma$, $\gamma - \chi(\gamma)$ operates nilpotently on K_χ, and $K_\chi = 0$ for all but finitely many χ. Moreover, $H^n(K_\chi) = H^n(K)_\chi$ for any integer n and character χ.

Let $i_S : S \hookrightarrow M^*_{\leqslant d}$ be a component of $M_d = M^*_{\leqslant d} - M^*_{\leqslant d-1}$. Consider the triangle

$$i_S^*w^d_{\leqslant a_d+w}Rj_{d*}K_{d-1} \longrightarrow i_S^*Rj_{d*}K_{d-1} \longrightarrow w_{>a_d+w}i_S^*Rj_{d*}K_{d-1} \xrightarrow{+1} . \tag{4.3.8}$$

Fix a local Hecke operator Φ as in Lemmas 4.2.2 and 4.3.2. The action of $\Gamma = \Phi^{\mathbb{Z}}$ on $i_S^*Rj_{d*}K_{d-1}$ satisfies the condition of Lemma 4.3.3 (by the faithfulness of rat on mixed Hodge modules (applied to cohomology objects), the induction hypothesis, and Lemma 4.2.2(ii)). This gives a decomposition

$$i_S^*Rj_{d*}K_{d-1} = \bigoplus_\lambda (i_S^*Rj_{d*}K_{d-1})_\lambda \tag{4.3.9}$$

where the λ are rational and $\Phi - \lambda$ is nilpotent on $(i_S^*Rj_{d*}K_{d-1})_\lambda$.

Lemma 4.3.4. *There are natural isomorphisms*

$$w_{\leqslant a_d+w} i_S^* Rj_{d*} K_{d-1} = \bigoplus_{\lambda \leqslant \chi(a)^{a_d - \dim M_d}} (i_S^* Rj_{d*} K_{d-1})_\lambda$$

and

$$w_{> a_d+w} i_S^* Rj_{d*} K_{d-1} = \bigoplus_{\lambda > \chi(a)^{a_d - \dim M_d}} (i_S^* Rj_{d*} K_{d-1})_\lambda .$$

The triangle (4.3.8) is isomorphic to the split triangle

$$\bigoplus_{\lambda \leqslant \chi(a)^{a_d - \dim M_d}} (i_S^* Rj_{d*} K_{d-1})_\lambda \longrightarrow i_S^* Rj_{d*} K_{d-1} \longrightarrow \bigoplus_{\lambda > \chi(a)^{a_d - \dim M_d}} (i_S^* Rj_{d*} K_{d-1})_\lambda \xrightarrow{0} \tag{4.3.10}$$

coming from the decomposition of $i_S^* Rj_{d*} K_{d-1}$.

Proof. Let $J = \bigoplus_{\lambda \leqslant \chi(a)^{a_d - \dim M_d}} (i_S^* Rj_{d*} K_{d-1})_\lambda$. By Lemma 4.3.2 the obvious morphism $J \to i_S^* Rj_{d*} K_{d-1}$ induces isomorphisms $H^k(J) \to w_{\leqslant a_d+w} H^k(i_S^* Rj_{d*} K_{d-1})$ for all k. It follows (since the cohomology functors $\{H^k\}_{k \in \mathbb{Z}}$ are conservative) that $J \to i_S^* Rj_{d*} K_{d-1}$ induces an isomorphism $J \cong w_{\leqslant a_d+w} i_S^* Rj_{d*} K_{d-1}$. A similar argument (or the octahedral axiom) shows that if $J' := \bigoplus_{\lambda > \chi(a)^{a_d - \dim M_d}} (i_S^* Rj_{d*} K_{d-1})_\lambda$ then the obvious morphism $i_S^* Rj_{d*} K_{d-1} \to J'$ induces an isomorphism $w_{> a_d+w} i_S^* Rj_{d*} K_{d-1} \cong J'$.

That (4.3.8) and (4.3.10) are isomorphic then follows from the fact that (4.3.8) is the unique (up to unique isomorphism) triangle $A \to i_S^* Rj_{d*} K_{d-1} \to B \xrightarrow{+1}$ with A in ${}^w D^{\leqslant a_d - \dim M_d}$ and B in ${}^w D^{> a_d - \dim M_d}$. □

Let us return to the diagram (4.3.6) and complete the proof of (4.3.2) by completing the inductive step. By adjunction

$$\begin{aligned} v \circ Rj_{d*}(\alpha_{d-1}) \circ u = 0 \;&\Leftrightarrow\; i_d^*(v \circ Rj_{d*}(\alpha_{d-1}) \circ u) = 0 \\ &\Leftrightarrow\; i_S^*(v \circ Rj_{d*}(\alpha_{d-1}) \circ u) = 0 \text{ for each component } S \subset M_d. \end{aligned}$$

Fix a component S and apply i_S^* to get:

$$\begin{array}{ccccccc} i_S^* L_d & \xrightarrow{i_S^*(u)} & i_S^* Rj_{d*} L_{d-1} & \longrightarrow & i_S^! L_d[1] & \xrightarrow{+1} & \\ \downarrow & & \downarrow{\scriptstyle i_S^* Rj_{d*}(\alpha_{d-1})} & & & & \\ rat(i_S^* w^d_{\leqslant a_d+w} Rj_{d*} K_{d-1}) & \longrightarrow & rat(i_S^* Rj_{d*} K_{d-1}) & \xrightarrow{i_S^*(v)} & rat(w_{> a_d+w} i_S^* Rj_{d*} K_{d-1}) & \xrightarrow{+1} & \end{array} \tag{4.3.11}$$

Lemma 4.3.3 applies to the action of Φ on $i_S^* Rj_{d*} L_{d-1}$ (by Lemma 4.2.2(ii)) and gives a decomposition in $D_c^b(\mathbb{Q}_S)$:

$$i_S^* Rj_{d*} L_{d-1} = \bigoplus_\lambda (i_S^* Rj_{d*} L_{d-1})_\lambda .$$

By uniqueness of the decomposition this is the same as that given in Lemma 4.2.2(i) (showing, in particular, that (4.2.4) is rational). By Lemmas 4.2.2 and 4.3.4, (4.3.11) is isomorphic to

$$\begin{array}{cccccc}
\bigoplus_{\lambda\leqslant\nu(a)^{-1}}(i_S^*Rj_{d*}L_{d-1})_\lambda & \longrightarrow & i_S^*Rj_{d*}L_{d-1} & \longrightarrow & \bigoplus_{\lambda>\nu(a)^{-1}}(i_S^*Rj_{d*}L_{d-1})_\lambda & \xrightarrow{0} \\
\downarrow & \searrow & \downarrow & & & \\
\bigoplus_{\lambda\leqslant\chi(a)^{a_d-\dim M_d}} rat(i_S^*Rj_{d*}K_{d-1})_\lambda & \longrightarrow & rat(i_S^*Rj_{d*}K_{d-1}) & \longrightarrow & \bigoplus_{\lambda>\chi(a)^{a_d-\dim M_d}} rat(i_S^*Rj_{d*}K_{d-1})_\lambda & \xrightarrow{0}
\end{array} \tag{4.3.12}$$

The isomorphism $i_S^*Rj_{d*}(\alpha_{d-1})$ from the induction hypothesis is Φ-equivariant and so (by uniqueness) the two decompositions are related by this isomorphism, i.e. $(i_S^*Rj_{d*}L_{d-1})_\lambda = rat((i_S^*Rj_{d*}K_{d-1})_\lambda)$ for all λ. It follows that if

$$\chi(a)^{a_d-\dim M_d} \geqslant \nu(a^{-1}) = \chi(a)^{-n_d}$$

then the diagonal arrow factors through the first morphism in the bottom triangle. Then $i_S^*(v\circ Rj_{d*}(\alpha_{d-1})\circ u)$ factors through the composition of successive morphisms in a triangle, hence is zero. So the arrow α_d in (4.3.6) exists (over $M^*_{\leqslant d-1}\sqcup S$) and $i_S^*(\alpha_d)$ is an isomorphism if

$$\chi(a)^{a_d-\dim M_d} = \nu(a^{-1}) = \chi(a)^{-n_d}.$$

Varying $S\subset M_d$ completes the construction of an isomorphism α_d over $M^*_{\leqslant d}$.

Next we note that (iii)$_d$ holds, i.e. α_d is the unique morphism in (4.3.6) making the square commute. The difference of two lifts of $Rj_{d*}(\alpha_{d-1})\circ u$ in (4.3.6) is a morphism

$$\beta : L_d \longrightarrow i_{d*}rat(w_{>a_d+w}i_d^*Rj_{d*}K_{d-1})[-1].$$

For a component $S\subset M_d$, the restriction $i_S^*(\beta)$ is the difference of two lifts of $i_S^*(Rj_{d*}(\alpha_{d-1})\circ u)$ in (4.3.11). But a lift of $i_S^*(Rj_{d*}(\alpha_{d-1})\circ u)$ (when it exists) is unique because the lower triangle in (4.3.11) is split (by Lemma 4.3.4). Thus $i_S^*(\beta)=0$ for all $S\subset M_d$ and hence $i_d^*(\beta)=0$. By adjunction, $\beta=0$.

As remarked earlier, (i)$_{d-1}$ and (iii)$_d$ imply (i)$_d$, and (iv)$_d$ is a consequence of (i)$_d$–(iii)$_d$. So to complete the inductive step it remains to prove (ii)$_d$. The diagram of (ii)$_d$ is the back face of the following cube:

$$\begin{array}{ccccc}
b^*L_d & & \longrightarrow & & L'_d \\
\downarrow & \searrow & & & \downarrow \quad \searrow \\
 & Rj'_{d*}b^*L_{d-1} & \longrightarrow & & Rj'_{d*}L'_{d-1} \\
b^*rat(K_d) & \downarrow & \longrightarrow & rat(K'_d) & \downarrow \\
 & \searrow & & \searrow & \\
 & Rj'_{d*}b^*rat(K_{d-1}) & \longrightarrow & & Rj'_{d*}rat(K'_{d-1})
\end{array} \tag{4.3.13}$$

The maps from the back face to the front face are all adjunctions induced by $id\to Rj'_{d*}{j'_d}^*$. The commutativity of the top, bottom, left, and right faces follow by functoriality of adjunction. The front face is the result of applying Rj'_{d*} to the square of (ii)$_{d-1}$, which commutes by the induction hypothesis. Thus all faces, except possibly the back one, commute. Consider the two compositions $b^*L_d \longrightarrow rat(K'_d)$ given by the back face. Their difference δ, when composed with $rat(K'_d)\longrightarrow Rj'_{d*}rat(K'_{d-1})$, vanishes (since all other faces of the cube commute). Thus the difference lifts to a morphism

$$\tilde{\delta} : b^*L_d \longrightarrow i'_{d*}rat(w_{>a_d+w}{i'_d}^*Rj'_{d*}K'_{d-1}).$$

Now for $S' \subset M'_d$, $i^*_{S'}(\tilde{\delta}) = 0$ since the morphism $i^*_{S'} rat(K'_d) \longrightarrow i^*_{S'} Rj'_{d*} rat(K'_{d-1})$ is part of a split triangle (the triangle (4.3.8) on $M'^*_{\leqslant d}$, split by Lemma 4.3.4). By adjunction $\tilde{\delta} = 0$ and hence $\delta = 0$. This proves (ii)$_d$ and completes the proof of (4.3.2) in Theorem 4.3.1.

The second assertion (4.3.3) is deduced as follows: It is evident from the construction of $\alpha = \alpha_r$ that the square

$$\begin{array}{ccc} Rp_* W^{\geqslant \nu} C(\mathbb{E}_M) & \longrightarrow & Rp_* W^{\geqslant -\infty} C(\mathbb{E}_M) \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ rat(w_{\leqslant (a_0+w,\dots,a_r+w)} Rj_* \mathbb{E}^H_M) & \longrightarrow & Rj_* \mathbb{E}_M \end{array}$$

commutes, where the quasiisomorphism $Rp_* W^{\geqslant -\infty} C(\mathbb{E}_M) = Rj_* \mathbb{E}_M$ extends $\alpha_0 = \alpha|_M$. It can be completed to a morphism of triangles

$$\begin{array}{ccccccc} Rp_* W^{\geqslant \nu} C(\mathbb{E}_M) & \longrightarrow & Rp_* W^{\geqslant -\infty} C(\mathbb{E}_M) & \longrightarrow & Rp_* \left(W^{\geqslant -\infty} C(\mathbb{E}_M) / W^{\geqslant \nu} C(\mathbb{E}_M) \right) & \xrightarrow{+1} & \\ \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle \alpha'} & & \\ rat(w_{\leqslant (a_0+w,\dots,a_r+w)} Rj_* \mathbb{E}^H_M) & \longrightarrow & Rj_* \mathbb{E}_M & \longrightarrow & rat(w_{>(a_0+w,\dots,a_r+w)} Rj_* \mathbb{E}^H_M) & \xrightarrow{+1} & \end{array} \tag{4.3.14}$$

in which α' is necessarily an isomorphism, but may not, a priori, be unique. The source and target of α' are supported on the minimal boundary $M^* - M$, so that if $i : M^* - M \hookrightarrow M^*$ denotes the inclusion, then applying $i_! i^!$ gives (since $i^! Rj_* = 0$) a square

$$\begin{array}{ccc} Rp_* \left(W^{\geqslant -\infty} C(\mathbb{E}_M) / W^{\geqslant \nu} C(\mathbb{E}_M) \right) & \xrightarrow{\ y\ } & i_! i^! Rp_* W^{\geqslant \nu} C(\mathbb{E}_M)[1] \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle i_! i^! \alpha[1]} \\ rat(w_{>(a_0+w,\dots,a_r+w)} Rj_* \mathbb{E}^H_M) & \xrightarrow{\ z\ } & i_! i^! rat(w_{\leqslant (a_0+w,\dots,a_r+w)} Rj_* \mathbb{E}^H_M)[1] \end{array}$$

in which the horizontal maps y, z are isomorphisms (and evidently Hecke-equivariant). Thus

$$\alpha' = z^{-1} \circ i_! i^! \alpha[1] \circ y$$

is unique and Hecke-equivariant. □

Remarks 4.3.5. (i) The proof of the theorem shows that if $a_1, \dots, a_r$ are defined by (4.3.1) then

$$Rp_* W^{\geqslant \nu} C(\mathbb{E}_M) = rat(w_{\leqslant (a_0+w, a_1+w, \dots, a_r+w)} Rj_* \mathbb{E}^H_M) \tag{4.3.15}$$

for any $a_0 \geqslant \dim M$. By [M08, Prop. 3.4.2] (i.e. Lemma 2.1.4 above) we also have

$$Rp_* W^{\geqslant \nu} C(\mathbb{E}_M) = rat(w_{\geqslant (a_0+w, a_1+1+w, \dots, a_r+1+w)} j_! \mathbb{E}^H_M) \tag{4.3.16}$$

for any $a_0 \leqslant \dim M$. The proof also shows that (4.3.2) is the unique morphism extending $\alpha_0 : \mathbb{E}_M \to rat(\mathbb{E}^H_M)$.

(ii) If ν is very positive (or very negative) the theorem is trivial as we have $j_! \mathbb{E}_M$ (or $Rj_* \mathbb{E}_M$) on both sides.

If $\nu = -\rho$ is minus the half-sum of positive roots then $-\rho|_{A_d} = \chi_d^{-\operatorname{codim} M_d}$ for $d = 1, \dots, r$ and hence $a_d = \dim M_d + \operatorname{codim} M_d = \dim M$ for all d. In this case there are natural isomorphisms

$$Rp_* W^{\geqslant -\rho} C(\mathbb{E}_M) = (j_{!*} \mathbb{E}_M[\dim M])[-\dim M] = rat(w_{\leqslant \dim M + w} Rj_* \mathbb{E}^H_M) \tag{4.3.17}$$

by [GHM94, Theorem 23.2] and [M08, Theorem 3.1.4], which gives the theorem. (In fact, either one of these equalities, together with (4.3.1), implies the other. The reader will note that the proof of the theorem (especially the use of the splitting property Lemma 4.2.2) is similar to the proof of Theorem 23.2 of [GHM94], except that in loc. cit. things are made easier by the characterization of the intersection complex in $D_c^b(\mathbb{Q}_X)$ by local cohomology vanishing conditions. For the general complex $rat(w_{\leqslant \underline{a}+w} Rj_* \mathbb{E}_M^H)$ there is no such characterization, so that one has to argue differently.)

For $\nu = 0$ and E trivial we get

$$Rp_* \mathbb{Q}_{\overline{M}} = rat(w_{\leqslant(\dim M, \dim M_1, \dots, \dim M_r)} Rj_* \mathbb{Q}_M^H)$$

where we use the fact that $W^{\geqslant 0} C(\mathbb{Q}_M) = \mathbb{Q}_{\overline{M}}$ (cf. [GHM94, §19]). For a discussion of related results see 4.6 below.

For general ν, the resulting equality in the Grothendieck group of $D_c^b(\mathbb{Q}_{M^*})$ is in [M08, Rem. 4.2.4].

Remark 4.3.6. We have assumed in §3 that G is simply connected and almost $\mathbb{Q}$-simple but this is unnecessary. The theorem remains true for general semisimple G provided that in (4.3.1) we let $\chi_d \in X^*(A_d)_{\mathbb{Q}}$ be the square root of the character of A_d appearing in $\mathrm{Lie}\, U_d$. (This need not belong to $X^*(A_d)$ unless G is simply connected, cf. the appendix.) The statement of the theorem is then compatible with taking products and coverings, hence holds for any connected semisimple group. The generalization to connected reductive groups is straightforward.

Remark 4.3.7. The theorem and its proof work in some other categories of mixed sheaves.

If (1) $\mathbb{E}_M$ underlies an object in the theory and (2) the forgetful functor For to perverse complexes factors through rat, i.e. through mixed Hodge modules, then the proof is exactly the same since Lemma 4.3.2 is available. This is the case with the theory $\mathrm{MHM}(\cdot/k)$ of 2.3 of mixed Hodge modules with de Rham structure.

In the theory of Morel [M11], if $\mathbb{E}_M \otimes \mathbb{Q}_l$ underlies an object in $\mathscr{M}_l(M)$ (e.g. if $\mathbb{E}_M$ is of geometric origin), one can argue as follows: By definition we have a mixed l-adic complex on a model of M^* over an open subset of $Spec(\mathscr{O}_k)$. The weights of such an object are given by reducing modulo a prime and then looking at Frobenius weights. Now the arguments in 6.9–6.11 of [LR91] show how to deduce the analogue of Lemma 4.3.2 in this situation. The rest of the argument is the same and the result is the following version of Theorem 4.3.1: There is a natural isomorphism

$$Rp_* W^{\geqslant \nu} C(\mathbb{E}_M) \otimes \mathbb{Q}_l = For(w_{\leqslant(a_0+w, a_1+w, \dots, a_r+w)} Rj_* \mathbb{E}_M^{\mathscr{M}_l})$$

(where $For(\mathbb{E}_M^{\mathscr{M}_l}) = \mathbb{E}_M \otimes \mathbb{Q}_l$) in the derived category of $\mathbb{Q}_l$-sheaves in the classical topology.

4.4. **Compatibility.** The isomorphisms in Theorem 4.3.1 for various μ are compatible:

Proposition 4.4.1. *Let $\mu, \nu \in X^*(A_0)_{\mathbb{Q}}$ with associated sequences $(a_1, \dots, a_r), (b_1, \dots, b_r) \in \mathbb{Q}^r$ and assume that $\mu|_{A_d} \geqslant \nu|_{A_d}$ for all d (with respect to $X^*(A_d)_{\mathbb{Q}} = \mathbb{Q}\chi_d \cong \mathbb{Q}$), so that $a_d \leqslant b_d$ for all d. Let $a_0 = b_0 = \dim M$.*

If α and β in the diagram

$$\begin{array}{ccc} Rp_* W^{\geqslant \mu} C(\mathbb{E}_M) & \longrightarrow & Rp_* W^{\geqslant \nu} C(\mathbb{E}_M) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ rat(w_{\leqslant(a_0+w, \dots, a_r+w)} Rj_* \mathbb{E}_M^H) & \longrightarrow & rat(w_{\leqslant(b_0+w, \dots, b_r+w)} Rj_* \mathbb{E}_M^H) \end{array}$$

are isomorphisms such that $\alpha|_M = \beta|_M$*, then the diagram commutes.*

For the isomorphisms α' *and* β' *determined by* α *and* β *(as in the proof of (4.3.3)), the diagram*

$$\begin{array}{ccc} Rp_*\left(W^{\geqslant -\infty}C(\mathbb{E}_M)/W^{\geqslant \mu}C(\mathbb{E}_M)\right) & \longrightarrow & Rp_*\left(W^{\geqslant -\infty}C(\mathbb{E}_M)/W^{\geqslant \nu}C(\mathbb{E}_M)\right) \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta'} \\ rat(w_{>(a_0+w,\ldots,a_r+w)}Rj_*\mathbb{E}_M^H) & \longrightarrow & rat(w_{>(b_0+w,\ldots,b_r+w)}Rj_*\mathbb{E}_M^H) \end{array}$$

commutes.

Proof. For the proof of the first part we use the notation

$$L_d^\mu := Rp_* W^{\geqslant \mu}C(\mathbb{E}_M)\big|_{M^*_{\leqslant d}}$$
$$K_d^{\underline{a}} := w_{\leqslant(a_0+w,\ldots,a_d+w)}R(j_d\cdots j_1)_*\mathbb{E}_M^H$$

and, similarly, L_d^ν and $K_d^{\underline{b}}$. We will show by induction on d that

$$\begin{array}{ccc} L_d^\mu & \longrightarrow & L_d^\nu \\ \downarrow{\scriptstyle \alpha_d} & & \downarrow{\scriptstyle \beta_d} \\ rat(K_d^{\underline{a}}) & \longrightarrow & rat(K_d^{\underline{b}}) \end{array}$$

commutes for all d. If $d = 0$, $L_0^\mu = L_0^\nu$ and $K_d^{\underline{a}} = K_d^{\underline{b}}$ and $\alpha_0 = \beta_0$ by assumption. Assume that the diagram for $d-1$ commutes. We argue as in the proof of (ii)$_d$ in the inductive step in the proof of the first part of Theorem 4.3.1: The diagram for d is the back face of the cube

$$\begin{array}{ccccc} L_d^\mu & & \longrightarrow & & L_d^\nu \\ & \searrow & & & & \searrow \\ \downarrow & & Rj_{d*}L_{d-1}^\mu & \longrightarrow & \downarrow & & Rj_{d*}L_{d-1}^\nu \\ rat(K_d^{\underline{a}}) & & \longrightarrow & & rat(K_d^{\underline{b}}) \\ & \searrow & \downarrow & & & \searrow & \downarrow \\ & & Rj_{d*}rat(K_{d-1}^{\underline{a}}) & \longrightarrow & & & Rj_{d*}rat(K_{d-1}^{\underline{b}}). \end{array}$$

The front face commutes by the induction hypothesis, while the top, bottom, left, and right faces commute by functoriality of adjunction. Thus the two compositions on the back face agree after composing with $rat(K_d^{\underline{b}}) \to Rj_{d*}rat(K_{d-1}^{\underline{b}})$, and so their difference lifts to a morphism $\delta : L_d^\mu \longrightarrow i_{d*}rat(w_{>b+d+w}i_d^*Rj_{d*}K_{d-1}^{\underline{b}})$. For each $S \subset M_d$ the restriction $i_S^*(\delta)$ vanishes because $i_S^*rat(K_d^{\underline{b}}) \longrightarrow i_S^*Rj_{d*}rat(K_{d-1}^{\underline{b}})$ is part of the split triangle (4.3.8). Then $\delta = 0$.

The second assertion follows from the first by considering the diagrams (4.3.14) for μ and ν and the morphism between them. □

The horizontal morphisms in the first diagram have natural cones, namely $Rp_*\left(W^{\geqslant\nu}C(\mathbb{E}_M)/W^{\geqslant\mu}C(\mathbb{E}_M)\right)$ for the upper one and $rat(w_{>\underline{a}+w}w_{\leqslant\underline{b}+w}Rj_*\mathbb{E}_M^H)$ for the lower. The proposition gives an isomorphism between them, but does not fix a canonical (or unique) one. (In the special case $\nu = -\infty$ this was done in (4.3.3) of Thm 4.3.1.)

4.5. **Mixed structures in cohomology.** To translate the results in 2.4 into results about weighted cohomology groups we note some facts about roots and weights which come from explicit calculations with the $\mathbb{Q}$-root system to be found in the appendix. At some points in the discussion below we mix additive and multiplicative notation for characters, but this should cause no confusion.

To fix some notation, recall that we have fixed a minimal $\mathbb{Q}$-parabolic subgroup P_0, and fix a maximal $\mathbb{Q}$-split torus $A_0 \subset P_0$. This fixes in $\check{\mathfrak{a}}_0 = X^*(A_0) \otimes \mathbb{R}$ a root system $\Phi_0 = \Phi(A_0, G)$, a system of positive roots Φ_0^+, a set of simple roots $\Delta_0 \subset \Phi_0$, a positive cone ${}^+\check{\mathfrak{a}}_0$, a positive Weyl chamber $\check{\mathfrak{a}}_0^+$ $(\subset {}^+\check{\mathfrak{a}}_0)$, and their closures $\overline{{}^+\check{\mathfrak{a}}_0}$ and $\overline{\check{\mathfrak{a}}_0^+}$. We use the partial order $\leqslant$ on $X^*(A_0)_\mathbb{Q}$ given by $\lambda \leqslant \mu$ if and only if $\mu - \lambda \in \overline{{}^+\check{\mathfrak{a}}_0}$. (This is the same as the condition that $\lambda|_{A_d} \leqslant \mu|_{A_d}$ for all d, where an isomorphism $X^*(A_d)_\mathbb{Q} \cong \mathbb{Q}$ is fixed by the character in $\operatorname{Lie} U_d$.) We will call $\nu \in X^*(A_0)_\mathbb{Q}$ *dominant* if belongs to the closed positive Weyl chamber $\overline{\check{\mathfrak{a}}_0^+}$. The half-sum of positive roots in Φ_0 is denoted ρ; it is dominant.

As a consequence of the fact that $G(\mathbb{R})$ gives rise to a Hermitian symmetric domain and the fact that G is almost $\mathbb{Q}$-simple, the set of simple roots comes with a canonical ordering (see A.1 in the appendix; this is the same as that given by the ordering of standard maximal parabolics in 3.2). It determines a set of simple coroots (with ordering) and the dual basis to the simple coroots is the set of *relative fundamental weights* $\{\varpi_1, \dots, \varpi_r\} \subset X^*(A_0)_\mathbb{Q}$ (cf. the appendix for more details). They are dominant and span the Weyl chamber, i.e. $\check{\mathfrak{a}}_0^+ = \sum_i \mathbb{R}_{>0}\, \varpi_i$. ([9])

First we note that there is a natural condition on $\nu \in X^*(A_0)_\mathbb{Q}$ ensuring that the associated sequence $(a_1, \dots, a_r)$ is as in Lemma 2.4.1(i):

Lemma 4.5.1. *If $\nu + \rho$ is dominant (resp. antidominant) then the sequence $(a_1, \dots, a_r)$ associated with ν is nonincreasing (resp. nondecreasing).*

Proof. It is enough to show that for each relative fundamental weight ϖ_i the sequence $m_1, \dots, m_r$ defined by $\varpi_i|_{A_d} = \chi_d^{m_d}$ is nondecreasing, because the sequence associated with $-\rho + \varpi_i$ is $(\dim M, \dots, \dim M) - (m_1, \dots, m_r)$ and every dominant weight is a nonnegative linear combination of the ϖ_i. An explicit calculation with the system $\Phi(A_0, G)$ of rational roots shows this (cf. Lemma A.3.2 in the Appendix). □

Let $\chi_d \in X^*(A_d)_\mathbb{Q}$ be such that χ_d^2 is the character of A_d on $\operatorname{Lie} U_d$. (In the simply connected case χ_d is a generator of $X^*(A_d)$, cf. the appendix.) The relative fundamental dominant weight ϖ_1 has a special property:

Lemma 4.5.2. $\varpi_1|_{A_d} = \chi_d$ *for* $d = 1, \dots, r$.

Proof. This is proved in the appendix, cf. Lemma A.3.1. □

In the simply connected case $\varpi_1 \in X^*(A_0)$ (cf. the appendix), but in general it is in $X^*(A_0)_\mathbb{Q}$. For a general group G the (quasi)character defined by this property (namely, that for each rank one split torus given by a standard maximal parabolic the restriction is the positive generator of the character group) is not always dominant (e.g. if G is split of type G_2), so this property is particular to groups giving rise to Shimura varieties.

The following theorem summarizes the properties of the mixed structures on weighted cohomology groups. We state it for mixed Hodge structures, but see Remark 4.5.4 (i) below.

[9] The term "relative fundamental weight" seems to have multiple meanings in the literature. We will always mean the ϖ_i defined here.

Theorem 4.5.3. *(i) The weighted cohomology groups*

$$W^{\geqslant\nu}H^i(\Gamma,E) := \mathbb{H}^i(\overline{M}, W^{\geqslant\nu}C(\mathbb{E}_M))$$

carry canonical mixed Hodge structures with polarized graded quotients such that the morphisms

$$W^{\geqslant\mu}H^i(\Gamma,E) \to W^{\geqslant\nu}H^i(\Gamma,E) \qquad (\text{for } \mu \geqslant \nu),$$

the cup products

$$W^{\geqslant\mu}H^i(\Gamma,E_1)\otimes W^{\geqslant\nu}H^j(\Gamma,E_2) \to W^{\geqslant\mu+\nu}H^{i+j}(\Gamma,E_1\otimes E_2),$$

and the duality pairings

$$W^{\geqslant\nu}H^i(\Gamma,E)\otimes W^{>-2\rho-\nu}H^{2\dim M-i}(\Gamma,\check{E}) \to \mathbb{Q}(-\dim M)$$

are all morphisms of mixed Hodge structures.

(ii) If $\nu+\rho$ is dominant then the mixed Hodge structure on $W^{\geqslant\nu}H^i(\Gamma,E)$ is like that of a complete variety, i.e. $W^{\geqslant\nu}H^i(\Gamma,E)$ has weights $\leqslant i+w$.

If $\nu+\rho$ is antidominant then the mixed Hodge structure on $W^{\geqslant\nu}H^i(\Gamma,E)$ is like that of a smooth variety, i.e. $W^{\geqslant\nu}H^i(\Gamma,E)$ has weights $\geqslant i+w$.

(iii) If $\nu=-\rho+k\varpi_1$ for $k\geqslant 0$ or if $\nu=0$ then the top weight quotient is isomorphic to the image of $W^{\geqslant\nu}H^i(\Gamma,E)$ in $W^{\geqslant-\rho}H^i(\Gamma,E)=IH^i(M^*,\mathbb{E}_M)$, i.e.

$$Gr^W_{i+w}W^{\geqslant\nu}H^i(\Gamma,E) = \operatorname{im}\left(W^{\geqslant\nu}H^i(\Gamma,E)\to IH^i(M^*,\mathbb{E}_M)\right).$$

If $\nu=-\rho+k\varpi_1$ for $k\leqslant 0$ then the lowest weight subspace is isomorphic to the image of $IH^i(M^*,\mathbb{E}_M)=W^{\geqslant-\rho}H^i(\Gamma,E)$ in $W^{\geqslant\nu}H^i(\Gamma,E)$, i.e.

$$W_{i+w}\left(W^{\geqslant\nu}H^i(\Gamma,E)\right) = \operatorname{im}\left(IH^i(M^*,\mathbb{E}_M)\to W^{\geqslant\nu}H^i(\Gamma,E)\right).$$

Proof. The first part of (i) follows from Theorem 4.3.1 and Prop. 4.4.1 and the duality statement follows because the isomorphisms constructed in Thm 4.3.1 are compatible with duality. (We leave the case of cup products to the reader.)

(ii) follows from Lemma 4.5.1 and Lemma 2.4.1.

(iii) follows from Lemma 4.5.2 and Proposition 2.4.2 after noticing that the sequence associated with $-\rho-k\varpi_1$ is $\underline{a}=(\dim M+k,\dots,\dim M+k)$. □

Remarks 4.5.4. (i) The theorem holds in the theories of mixed sheaves for which Theorem 4.3.1 holds. In particular by Remark 4.3.7 it holds for mixed Hodge-de Rham structures and for mixed l-adic complexes. If one were to work in the context of Shimura varieties all this would obviously work over the reflex field.

(ii) The mixed HdR structures in the theorem are effective (Remark 2.3.1).

(iii) The condition that $\nu+\rho$ is antidominant is natural from the automorphic point of view, for under this condition the groups $W^{\geqslant\nu}H^*(\Gamma,E)$ are computable using automorphic forms (by results of [N99, F98], which will be quoted in 5.3 below). So the theorem provides a supply of mixed structures related to automorphic forms. One might expect that the relation between extensions and automorphic forms is easier in some of these than in $H^*(M,\mathbb{E}_M)$.

(iv) It is an interesting question whether the Hodge structures in weighted cohomology groups can be seen in more classical terms, e.g. in terms of explicit bifiltered complexes of differential forms (as in classical mixed Hodge theory for smooth varieties [D74]).

4.6. **Remarks on the RBS compactification.** As remarked earlier, if $\nu = 0$ and E is trivial then we have an isomorphism

$$rat(w_{\leqslant(\dim M_0,\dots,\dim M_r)} Rj_*\mathbb{Q}^H_M) = Rp_*\mathbb{Q}_{\overline{M}}. \tag{4.6.1}$$

Thus we get a mixed Hodge structure on $H^*(\overline{M},\mathbb{Q})$ with various good properties. (A natural mixed Hodge structure on $H^*(\overline{M},\mathbb{Q})$ was first constructed by Zucker [Z04] by different means.) Theorem 4.5.3(iii) says that the mixed Hodge structure is like that of a complete variety, and

$$Gr^W_i H^i(\overline{M}) = \operatorname{im}\left(H^i(\overline{M}) \to IH^i(M^*)\right). \tag{4.6.2}$$

Let $\pi : M_\Sigma \to M^*$ be a smooth toroidal compactification of M. In [NV12] we prove the formula

$$rat(w_{\leqslant(\dim M_0,\dots,\dim M_r)} R\pi_*\mathbb{Q}^H_{M_\Sigma}) = Rp_*\mathbb{Q}_{\overline{M}} \tag{4.6.3}$$

which shows that there is a natural map $H^*(\overline{M}) \to H^*(M_\Sigma)$. [NV12] also shows that

$$Gr^W_i H^i(\overline{M}) = \operatorname{im}\left(H^i(\overline{M}) \to H^i(M_\Sigma)\right). \tag{4.6.4}$$

(4.6.4) and (4.6.2) are equivalent by the decomposition theorem. (4.6.4) is reminiscent of the fact from classical Hodge theory that for a complete variety X and resolution of singularities $Y \to X$, $Gr^W_i H^i(X) = \operatorname{im}\left(H^i(X) \to H^i(Y)\right)$ (cf. [D74, III.8.2.5]), which combined with the decomposition theorem gives $Gr^W_i H^i(X) = \operatorname{im}\left(H^i(X) \to IH^i(X)\right)$. (The identification $IH^i(M^*) = IH^i(\overline{M})$ given by [Sap05] makes the resemblance of $\overline{M}$ to a "partial" resolution of singularities of M^* stronger.)

In fact, the particular truncation $w_{\leqslant(\dim M_0,\dots,\dim M_r)}$ (suitably modified so that it becomes independent of the stratification) has many nice properties (in either l-adic or mixed Hodge contexts) for general algebraic varieties, as we show in [NV12].

We discuss some other results related to (4.6.1). Ayoub and Zucker [AZ12] proved a formula analogous to (4.6.1) in the category of motivic sheaves on M^*, using a functor ω^0 (the relative Artin motive functor of loc. cit.) in place of $w_{\leqslant(\dim M_0,\dots,\dim M_r)}$. Pushforward to a point gives a Voevodsky motive for $\overline{M}$ and applying realizations one gets a mixed Hodge structure on $H^*(\overline{M},\mathbb{Q})$ (or l-adic Galois representations etc.)

Following the outline of [NV12], Vaish [V13] gives another construction of the RBS motive of Ayoub and Zucker, which in fact shows slightly more, namely that it is truncation with respect to a t-structure (the motivic analogue of Morel's t-structure $({}^wD^{\leqslant \underline{dim}}(M^*), {}^wD^{>\underline{dim}}(M^*))$ defined on a suitable subcategory of the category of motivic sheaves on M^*. Thus [V13], together with the l-adic analogue of (4.6.1) (available by Remark 4.3.7), recovers the main results of [AZ12] by a different route. (Realization functors from the categories of motivic sheaves used in [AZ12] to the derived category of mixed Hodge modules have not been constructed, so that (4.6.1) itself does not follow from [AZ12] and one has to use the l-adic realizations. Vaish also shows that, assuming the existence of suitable realization functors to mixed Hodge modules, the mixed Hodge structure coming from (4.6.1) agrees with that coming from [AZ12].)

In another direction, in [N10] we applied (4.6.1)–(4.6.4), the C^∞ de Rham description of $\overline{M}$ from [N99], and analytic results about automorphic forms from [F98] to study Chern classes of automorphic vector bundles and to the study of restriction maps between locally symmetric varieties, providing the first generalization to noncompact Shimura varieties of some well-known Lefschetz-type properties.

5. Spectral sequences and automorphic forms

5.1. Postnikov systems and spectral sequences. Let $\mathscr{D}$ be a triangulated category. A (right) *Postnikov system* for an object K of $\mathscr{D}$ is a diagram

$$\begin{array}{ccccccccccc} & G^0 & & G^1 & & & & & G^n & & \\ \nearrow & & \searrow^{+1} \nearrow & & \searrow^{+1} & & & \nearrow & & \searrow^{+1} & \\ K = F^0 & \longleftarrow & F^1 & \longleftarrow & F^2 & \longleftarrow & \cdots & \longleftarrow & F^n & \longleftarrow & F^{n+1} = \{0\} \end{array}$$

in which all the triangles are distinguished in $\mathscr{D}$. The following lemma is standard (cf. e.g. [GeM96, p.262f]):

Lemma 5.1.1. *For a cohomological functor H from $\mathscr{D}$ to an abelian category there is a spectral sequence with*

$$E_1^{p,q} = H^{p+q}(G^p) \quad \Rightarrow \quad H^{p+q}(K)$$

(with E_1 differential induced by $G^p \to F^{p+1}[1] \to G^{p+1}[1]$). The E_∞ term computes the graded of the limit filtration of the spectral sequence, i.e. the filtration of $H^*(K)$ given by

$$F^pH^i(K) = \operatorname{im}\left(H^i(F^p) \to H^i(K)\right) = \ker\left(H^i(K) \to H^i(K/F^p)\right).$$

Exact functors take Postnikov systems to Postnikov systems. A morphism of Postnikov systems is a morphism of diagrams; it gives a morphism of spectral sequences. An isomorphism of Postnikov systems induces an isomorphism of spectral sequences from the E_1 term on.

Example 5.1.2. (i) If K is a complex of objects in an abelian category $\mathscr{A}$ and $F^\bullet$ is a decreasing filtration of K by subcomplexes, then $F^p = F^pK$ and $G^p = F^pK/F^{p+1}K$, together with the triangles given by the exact sequences $0 \to F^{p+1}K \to F^pK \to F^pK/F^{p+1}K \to 0$ give a Postnikov system in $\mathscr{D} = D^b(\mathscr{A})$. If H is a cohomological functor then we get the usual spectral sequence for a filtered complex.

(ii) If $\mathscr{D}$ has a t-structure with truncation functors $\tau_{\leqslant 0}$ and $\tau_{>0}$ then for any object K, $F^pK = \tau_{\leqslant -p}K$ and $G^p = \tau_{\geqslant -p}\tau_{\leqslant p}K$ with the t-structure triangles defines a Postnikov system for K. A cohomological functor $H : \mathscr{D} \to \mathscr{A}$ to an abelian category gives a spectral sequence in $\mathscr{A}$ with $E_1^{p,q} = H^{p+q}(\tau_{\leqslant -p}\tau_{\geqslant -p}K) \Rightarrow H^{p+q}(K)$. In this example the Postnikov system is functorial in the obvious sense.

In the locally symmetric setting, an increasing sequence $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_n$ of elements of $X^*(A_0)_\mathbb{Q}$ gives a filtration of $W^{\geqslant \nu_1}C(\mathbb{E}_M$ by subcomplexes

$$W^{\geqslant \nu_1}C(\mathbb{E}_M) \supset W^{\geqslant \nu_2}C(\mathbb{E}_M) \supset \cdots \supset W^{\geqslant \nu_n}C(\mathbb{E}_M) \supset \{0\}$$

and hence a Postnikov system for $W^{\geqslant \nu_1}C(\mathbb{E}_M)$ in $D_c^b(\mathbb{Q}_{\overline{M}})$ (as in Ex. 5.1.2(i)). Applying Rp_* gives a Postnikov system in $D_c^b(\mathbb{Q}_{M^*})$ with the triangles

$$\begin{array}{ccc} & Rp_*\left(W^{\geqslant \nu_p}C(\mathbb{E}_M)/W^{\geqslant \nu_{p+1}}C(\mathbb{E}_M)\right) & \\ \nearrow & & \searrow^{+1} \\ Rp_*W^{\geqslant \nu_p}C(\mathbb{E}_M) & \longleftarrow & Rp_*W^{\geqslant \nu_{p+1}}C(\mathbb{E}_M) \end{array} \qquad (5.1.1)$$

for $p = 1, \dots n$. (We make the convention that $W^{\geqslant \nu_p}C(\mathbb{E}_M) = \{0\}$ for $p > n$ and $W^{\geqslant \nu_p}C(\mathbb{E}_M) = W^{\geqslant \nu_1}C(\mathbb{E}_M)$ for $p \leq 0$.) The r-tuples $\underline{a}^i = (a_1^i, \dots, a_n^i) \in \mathbb{Q}^r$ associated with ν_i by (4.3.1) satisfy $\underline{a}^1 \geqslant \dots \geqslant \underline{a}^n$. With the convention $a_0^i = \dim M$ for all i, we get a Postnikov system for $K = w_{\leqslant(a_0^1+w,\ldots,a_r^1+w)}Rj_*\mathbb{E}_M^H$ in $D^b\mathrm{MHM}(M^*)$ with the triangles

$$\begin{array}{ccc} & w_{>(a_0^{p+1}+w,\ldots,a_r^{p+1}+w)}w_{\leqslant(a_0^p+w,\ldots,a_r^p+w)}Rj_*\mathbb{E}_M^H & \\ \nearrow & & \searrow^{+1} \\ w_{\leqslant(a_0^p+w,\ldots,a_r^p+w)}Rj_*\mathbb{E}_M^H & \longleftarrow & w_{\leqslant(a_0^{p+1}+w,\ldots,a_r^{p+1}+w)}Rj_*\mathbb{E}_M^H \end{array} \tag{5.1.2}$$

for $p = 1, \ldots, n$. Applying rat gives a Postnikov system in $D_c^b(\mathbb{Q}_{M^*})$.

Proposition 5.1.3. *The Postnikov system for $Rp_*W^{\geqslant\nu_1}C(\mathbb{E}_M^H)$ with triangles (5.1.1) and the Postnikov system for $rat(w_{\leqslant(a_0^1+w,\ldots,a_r^1+w)}Rj_*\mathbb{E}_M^H)$ with triangles given by applying rat to (5.1.2) are isomorphic.*

Proof. This follows from Prop. 4.4.1. □

We have not shown that the Postnikov systems are uniquely isomorphic, as one might expect, nor that there is an isomorphism compatible with Hecke correspondences. We leave this for another occasion.

5.2. **Statement of the main theorem.** We now introduce the Postnikov systems giving the spectral sequence of the introduction. Consider the increasing sequence of weights $\cdots \leqslant \nu_{-2} \leqslant \nu_{-1} \leqslant \nu_0$ where

$$\nu_p = -\rho + p\varpi_1 \qquad \text{for } p \leq 0.$$

It gives a Postnikov system for $W^{\geqslant-\infty}C(\mathbb{E}_M) = Rj_*\mathbb{E}_M$ with the triangles

$$\begin{array}{ccc} & Rp_*\left(W^{\geqslant\nu_p}C(\mathbb{E}_M)/W^{\geqslant\nu_{p+1}}C(\mathbb{E}_M)\right) & \\ \nearrow & & \searrow^{+1} \\ Rp_*W^{\geqslant\nu_p}C(\mathbb{E}_M) & \longleftarrow & Rp_*W^{\geqslant\nu_{p+1}}C(\mathbb{E}_M) \end{array} \tag{5.2.1}$$

for $p \leq 0$ (with the convention $W^{\geqslant\nu_1}C(\mathbb{E}_M) = \{0\}$). The corresponding decreasing sequence $\cdots \geqslant \underline{a}^{-1} \geqslant \underline{a}^0$ with

$$\underline{a}^p = (d_0 - p, \ldots, d_0 - p) \qquad (d_0 = \dim M)$$

gives a Postnikov system for $rat(Rj_*\mathbb{E}_M^H) = Rj_*\mathbb{E}_M$ with triangles

$$\begin{array}{ccc} & rat(w_{\geqslant d_0-p+w}w_{\leqslant d_0-p+w}Rj_*\mathbb{E}_M^H) & \\ \nearrow & & \searrow^{+1} \\ rat(w_{\leqslant d_0-p+w}Rj_*\mathbb{E}_M^H) & \longleftarrow & rat(w_{\leqslant d_0-p-1+w}Rj_*\mathbb{E}_M^H) \end{array} \tag{5.2.2}$$

for $p \leq 0$.

The spectral sequence given by the Postnikov system with triangles (5.2.2) is the spectral sequence (1) of the introduction. It has

$$E_1^{p,q} = \mathbb{H}^{p+q}(M^*, w_{\leqslant\dim M+w-p}w_{\geqslant\dim M+w-p}\,Rj_*\mathbb{E}_M^H) \quad\Rightarrow\quad H^{p+q}(M, \mathbb{E}_M^H). \tag{5.2.3}$$

With our indexing conventions this is a second-quadrant spectral sequence with edge terms given by $E_1^{0,q} = IH^q(M^*, \mathbb{E})$. The following summarizes our results about it:

Theorem 5.2.1. *The spectral sequence (5.2.3) has the following properties:*

(i) It is a spectral sequence of split mixed Hodge(-de Rham) structures, i.e. the E_1 term is a direct sum of pure structures. The E_1 term is (canonically) a sum of intersection cohomology groups of the minimal compactifications of strata of M^* with homogeneous coefficients.

(ii) It is automorphic (after $\otimes\mathbb{C}$), i.e. in the isomorphism

$$H^*(M, \mathbb{E}_M)_{\mathbb{C}} = H^*(\operatorname{Lie} G(\mathbb{R}), K; \mathscr{A}(\Gamma\backslash G(\mathbb{R})) \otimes E)$$

of [F98] *it is isomorphic to the spectral sequence associated with an explicit decreasing filtration on the space of automorphic forms.*

(iii) If the highest weight of E is regular then it degenerates at the E_1 term.

(i) was proved in 3.9–3.10, while (ii) and (iii) are proved below in 5.3.

Remark 5.2.2. The spectral sequence exists and (i) holds in any theory of A-mixed sheaves (in the sense of Saito) if $\mathbb{E}_M \otimes A$ underlies an object in the theory, in particular if $\mathbb{E}_M$ is of geometric origin. In Morel's theory $\mathscr{M}(\cdot, \mathbb{Q}_l)$ of 2.3 the spectral sequence exists under the geometric origin condition and the E_1 term continues to be a direct sum of pure Galois modules. (The object $w_{\leqslant \dim M+w-p} w_{\geqslant \dim M+w-p} Rj_* \mathbb{E}_M^{\mathscr{M}}$ breaks up as a sum of twisted intersection complexes with pure local systems as coefficients, but one does not know if these local systems are semisimple or homogeneous.)

5.3. **Automorphic forms.** In this section we prove the relation of the spectral sequence (5.2.3) with automorphic forms and complete the proof of Theorem 5.2.1. This is a simple matter of applying results of [F98] using the translation provided by [N99], but we must recall some notation and facts about automorphic forms from [F98] (cf. also [W97]) required for this. *Since* [F98] *is written in an adelic context we must restrict to congruence subgroups* Γ *in the sequel.*

We fix some notation. Recall that we have fixed a minimal $\mathbb{Q}$-parabolic P_0 and a maximal $\mathbb{Q}$-split torus A_0. This fixes in $\check{\mathfrak{a}}_0 = X^*(A_0) \otimes \mathbb{R}$ a root system $\Phi_0 = \Phi(A_0, G)$, a system of positive roots Φ_0^+, a positive cone ${}^+\check{\mathfrak{a}}_0$, a positive Weyl chamber $\check{\mathfrak{a}}_0^+$ ($\subset {}^+\check{\mathfrak{a}}_0$), and their closures $\overline{{}^+\check{\mathfrak{a}}_0}$ and $\overline{\check{\mathfrak{a}}_0^+}$. Elements of $\overline{\check{\mathfrak{a}}_0^+}$ and $\check{\mathfrak{a}}_0^+$ are called *dominant* and *strictly dominant*, respectively. The centralizer M_0 of A_0 is a minimal Levi subgroup. For any standard Levi subgroup $M \supset M_0$ the split centre is a torus $A_M \subset A_0$ and this gives the dual vector spaces $\mathfrak{a}_M = \operatorname{Lie} A_M(\mathbb{R}) = X_*(A_M) \otimes \mathbb{R}$ and $\check{\mathfrak{a}}_M = X^*(M) \otimes \mathbb{R} = X^*(A_M) \otimes \mathbb{R}$. Restriction of characters by $M_0 \subset M$ gives an embedding $\check{\mathfrak{a}}_M \subset \check{\mathfrak{a}}_0$. Restriction by $A_M \subset A_0$ gives a projection $\check{\mathfrak{a}}_0 \twoheadrightarrow \check{\mathfrak{a}}_M$ inverse to $\check{\mathfrak{a}}_M \subset \check{\mathfrak{a}}_0$. ([10])

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbb{C}}$ containing $\mathfrak{a}_0 = \operatorname{Lie} A_0(\mathbb{R})$ (hence $\mathfrak{h} \subset (\mathfrak{m}_0)_{\mathbb{C}}$). This gives a root system $\Phi = \Phi(\mathfrak{h}, \mathfrak{g}_{\mathbb{C}})$ and Weyl group $W = W(\mathfrak{h}, \mathfrak{g}_{\mathbb{C}})$. Fix a system of positive roots $\Phi^+ \subset \Phi = \Phi(\mathfrak{h}, \mathfrak{g}_{\mathbb{C}})$ compatible with Φ_0^+ (i.e. if $\beta \in \Phi^+$ then $\beta|_{\mathfrak{a}_0} \in \Phi_0^+ \cup \{0\}$). The half-sum of roots in Φ^+ will be denoted $\rho_{\mathfrak{h}}$; so $\rho_{\mathfrak{h}}|_{\mathfrak{a}_0} = \rho$, where ρ is the half-sum of roots in Φ_0^+. The Harish-Chandra isomorphism $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$ identifies infinitesimal characters with W-orbits in $\check{\mathfrak{h}}$ and ideals of finite codimension in $Z(\mathfrak{g})$ with W-invariant subschemes in $\check{\mathfrak{h}}$ of dimension zero.

We need an elementary construction. For $\lambda \in \check{\mathfrak{a}}_0$ denote by λ_+ the closest point to λ in the closed positive Weyl chamber $\overline{\check{\mathfrak{a}}_0^+}$ (for an inner product on $\check{\mathfrak{a}}_0$ invariant under the

[10] The notation here is slightly different from that in §3: M is a Levi subgroup of P (not the Levi quotient) and so A_M is a lift of the split centre A of the Levi quotient from 3.2.

stabilizer of $\check{\mathfrak{a}}_0$ in W). For a finite set $\Theta \subset \check{\mathfrak{h}}$ define another finite set $\Theta_+ \subset \overline{\check{\mathfrak{a}}_0^+}$ as follows:

$$\Theta_+ := \bigcup\nolimits_M \left\{ \operatorname{Re}(\theta|_{\mathfrak{a}_M})_+ : \theta \in \Theta \right\}.$$

(The union is over standard Levis and we have used the inclusions $\mathfrak{a}_M \subset \mathfrak{a}_0 \subset \mathfrak{h}$ to restrict and the inclusion $\check{\mathfrak{a}}_M \subset \check{\mathfrak{a}}_0$ to consider all elements as lying in $\check{\mathfrak{a}}_0$.) Note that associated with each $\lambda \in \Theta_+$ is a canonical standard parabolic subgroup P_λ and standard Levi M_λ, maximal with the property that $\lambda \in \check{\mathfrak{a}}_{M_\lambda} \subset \check{\mathfrak{a}}_0$. (The root group of $\alpha \in \Phi_0$ is contained in P_λ (resp. in M_λ) if and only if $(\alpha, \lambda) \geq 0$ (resp. $(\alpha, \lambda) = 0$.)

Let $\mathscr{A}(\Gamma\backslash G(\mathbb{R}))$ be the space of automorphic forms for Γ; these are the $Z(\mathfrak{g})$-finite and K-finite smooth functions on $\Gamma\backslash G(\mathbb{R})$ which satisfy a moderate growth condition. For a finite W-invariant set $\Theta \subset \check{\mathfrak{h}}$, let

$$\mathscr{A}_\Theta(\Gamma\backslash G(\mathbb{R}))$$

denote the subspace of automorphic forms killed by a power of the ideal of $Z(\mathfrak{g})$ given by Θ. Then $\mathscr{A}(\Gamma\backslash G(\mathbb{R})) = \bigcup_\Theta \mathscr{A}_\Theta(\Gamma\backslash G(\mathbb{R}))$ as Θ runs over W-orbits in $\check{\mathfrak{h}}$.

Let $f \in \mathscr{A}(\Gamma\backslash G(\mathbb{R}))$. For each standard rational parabolic P, the constant term of f along P admits a Fourier expansion in terms of characters of $\mathfrak{a}_M$ (cf. [F98, §6]); the characters appearing form a finite set, the *P-exponents of f*:

$$\operatorname{Exp}_P(f) \ \subset \ (\check{\mathfrak{a}}_M)_{\mathbb{C}} \subset (\check{\mathfrak{a}}_0)_{\mathbb{C}}.$$

The exponents of automorphic forms in $\mathscr{A}_\Theta(\Gamma\backslash G(\mathbb{R}))$ satisfy:

$$f \in \mathscr{A}_\Theta(\Gamma\backslash G(\mathbb{R})) \text{ and } \lambda \in \operatorname{Exp}_P(f) \implies \operatorname{Re}(\lambda)_+ \in \Theta_+. \tag{5.3.1}$$

Let $\mathscr{A}_{\log}(\Gamma\backslash G(\mathbb{R})) \subset \mathscr{A}(\Gamma\backslash G(\mathbb{R}))$ be the subspace of automorphic forms which have all their exponents in the closed negative cone $-\overline{{}^+\check{\mathfrak{a}}_0}$, i.e. $f \in \mathscr{A}_{\log}(\Gamma\backslash G(\mathbb{R}))$ if and only if $\operatorname{Re}(\lambda)_+ = 0$ for all $\lambda \in \operatorname{Exp}_P(f)$. (This is not the definition of $\mathscr{A}_{\log}(\Gamma\backslash G(\mathbb{R}))$ in [F98] but is equivalent. The functions in $\mathscr{A}_{\log}(\Gamma\backslash G(\mathbb{R}))$ are square-integrable up to some logarithmic terms; in particular, square-integrable automorphic forms belong to $\mathscr{A}_{\log}(\Gamma\backslash G(\mathbb{R}))$.)

From now on, let

$$\Theta := W \cdot (\rho_{\mathfrak{h}} + \lambda_{\check{E}}) = -\, W \cdot (\rho_{\mathfrak{h}} + \lambda_E)$$

where $\lambda_{\check{E}} \in \check{\mathfrak{h}}$ is the highest weight of the contragredient $\check{E}$ of E. Thus Θ is the infinitesimal character of $\check{E}$. We will filter $\mathscr{A}_\Theta(\Gamma\backslash G(\mathbb{R}))$ by conditions on the exponents. (This is a degenerate version of the filtrations used by Franke [F98], which we will also recall below.) Let ϖ_1 be the relative fundamental dominant weight defined in 4.5 (or the appendix). Define an indexing function $T : \Theta_+ \to \mathbb{Z}$ by

$$T(\lambda) = p \quad \text{if} \quad -(p+1)\varpi_1 < \lambda \leqslant -p\,\varpi_1. \tag{5.3.2}$$

Define a decreasing filtration $F^\bullet$ of $\mathscr{A}_\Theta(\Gamma\backslash G(\mathbb{R}))$ by the condition

$$f \in F^p\mathscr{A}_\Theta \iff \text{for all } P \text{ and } \lambda \in \operatorname{Exp}_P(f) \text{ we have } T(\operatorname{Re}(\lambda)_+) \geqslant p. \tag{5.3.3}$$

The nonzero graded quotients are for $p \leq 0$, and

$$\begin{aligned} F^p &= \mathscr{A}_\Theta(\Gamma\backslash G(\mathbb{R})) \quad \text{for } p \ll 0 \\ F^0 &= \mathscr{A}_{\Theta,\log}(\Gamma\backslash G(\mathbb{R})) \\ F^1 &= \{0\} \end{aligned}$$

so that $F^\bullet$ is finite and exhaustive. Here $\mathscr{A}_{\Theta,\log} := \mathscr{A}_\Theta \cap \mathscr{A}_{\log}$. (If N is the minimal integer such that $\lambda \in \Theta_+ \Rightarrow \lambda \leqslant N\varpi_1$, then $F^{-N} = \mathscr{A}_\Theta(\Gamma\backslash G(\mathbb{R}))$. The integer N depends on the representation E; if E is trivial then $N = \operatorname{codim} M_r$. The number of nontrivial graded pieces of the filtration is independent of E.)

By Franke's fundamental theorem [F98, Theorem 18] (formerly Borel's conjecture), the cohomology of M can be computed using automorphic forms:

$$H^*(M, \mathbb{E}_M)_{\mathbb{C}} = H^*(\mathfrak{g}, K; \mathscr{A}_\Theta(\Gamma\backslash G(\mathbb{R})) \otimes E).$$

$F^\bullet$ induces a filtration of the relative Lie algebra cohomology complex $C^\bullet(\mathfrak{g}, K; \mathscr{A}_\Theta \otimes E)$ and hence a spectral sequence with

$$E_1^{p,q} = H^{p+q}(\mathfrak{g}, K; F^p\mathscr{A}_\Theta/F^{p+1}\mathscr{A}_\Theta \otimes E) \quad \Rightarrow \quad H^*(M, \mathbb{E}_M)_{\mathbb{C}}. \tag{5.3.4}$$

We can now complete the proof of Theorem 5.2.1.

Proof of (ii) of Theorem 5.2.1. We will show that (5.2.1) is identified with (5.3.4) after tensoring with $\mathbb{C}$, using the results of [N99, F98]. By Prop. 5.1.3 we need to consider the Postnikov system with triangles

$$Rp_*W^{\geqslant -\rho+(p+1)\varpi_1}C(\mathbb{E}_M) \longrightarrow Rp_*W^{\geqslant -\rho+p\varpi_1}C(\mathbb{E}_M) \longrightarrow Rp_*\left(\frac{W^{\geqslant -\rho+p\varpi_1}C(\mathbb{E}_M)}{W^{\geqslant -\rho+(p+1)\varpi_1}C(\mathbb{E}_M)}\right) \xrightarrow{+1}$$

for $p < 0$. Now we will use a particular version of the weighted complexes, namely the one constructed using special differential forms in [GHM94, II]. The hypercohomology long exact sequence of the triangle above is that of the following short exact sequence of complexes of sheaves on $\overline{M}$:

$$0 \longrightarrow W^{\geqslant -\rho+(p+1)\varpi_1}C(\mathbb{E}_M) \longrightarrow W^{\geqslant -\rho+p\varpi_1}C(\mathbb{E}_M) \longrightarrow \frac{W^{\geqslant -\rho+p\varpi_1}C(\mathbb{E}_M)}{W^{\geqslant -\rho+(p+1)\varpi_1}C(\mathbb{E}_M)} \longrightarrow 0.$$

The explicit weighted complexes of [GHM94, II] are complexes of fine sheaves (they are sheaves of modules over the fine sheaf of special functions) so that applying $\Gamma(\overline{M}, \cdot)$ gives the short-exact sequence of complexes

$$0 \longrightarrow \Gamma(\overline{M}, W^{\geqslant -\rho+(p+1)\varpi_1}C(\mathbb{E}_M)) \longrightarrow \Gamma(\overline{M}, W^{\geqslant -\rho+p\varpi_1}C(\mathbb{E}_M)) \longrightarrow \frac{\Gamma(\overline{M}, W^{\geqslant -\rho+p\varpi_1}C(\mathbb{E}_M))}{\Gamma(\overline{M}, W^{\geqslant -\rho+(p+1)\varpi_1}C(\mathbb{E}_M))} \longrightarrow 0.$$

The associated long-exact sequence in cohomology is the hypercohomology long-exact sequence of the triangle (because all three complexes of sheaves are fine).

Consider the following diagram of complexes in which each row is short-exact:

$$\begin{array}{ccccc}
\Gamma(\overline{M}, W^{\geqslant -\rho+(p+1)\varpi_1}C(\mathbb{E}_M)) & \longrightarrow & \Gamma(\overline{M}, W^{\geqslant -\rho+p\varpi_1}C(\mathbb{E}_M)) & \longrightarrow & \frac{\Gamma(\overline{M}, W^{\geqslant -\rho+p\varpi_1}C(\mathbb{E}_M))}{\Gamma(\overline{M}, W^{\geqslant -\rho+(p+1)\varpi_1}C(\mathbb{E}_M))} \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (3)} \\
C^\bullet(\mathfrak{g}, K; S_{\rho-\tau_{p+1}+\log} \otimes E) & \longrightarrow & C^\bullet(\mathfrak{g}, K; S_{\rho-\tau_p+\log} \otimes E) & \longrightarrow & C^\bullet\left(\mathfrak{g}, K; \frac{S_{\rho-\tau_p+\log}}{S_{\rho-\tau_{p+1}+\log}} \otimes E\right) \\
{\scriptstyle (1')}\uparrow & & {\scriptstyle (2')}\uparrow & & {\scriptstyle (3')}\uparrow \\
C^\bullet(\mathfrak{g}, K; F^{p+1}\mathscr{A}_\Theta \otimes E) & \longrightarrow & C^\bullet(\mathfrak{g}, K; F^p\mathscr{A}_\Theta \otimes E) & \longrightarrow & C^\bullet\left(\mathfrak{g}, K; \frac{F^p\mathscr{A}_\Theta}{F^{p+1}\mathscr{A}_\Theta} \otimes E\right)
\end{array}$$

Here in the second row $S_{\rho-\tau_p+\log}(\Gamma\backslash G(\mathbb{R}))$ is the log-modified weighted L^2 space appearing in Franke [F98] for the element $\tau_p := -p\varpi_1 \in \check{\mathfrak{a}}_0$.

The maps (1), (2) were constructed in [N99, §2], essentially by checking that suitably weighted special differential forms satisfy suitable weighted L^2 conditions; (3) is the induced map on quotients. By [N99, Theorem A], (1) and (2) are quasiisomorphisms, and hence so is (3).

The maps ($1'$) and ($2'$) are the inclusions of subspaces of automorphic forms with generalized infinitesimal character Θ. (This follows from [F98, Theorem 15], using the dominance of $-p\varpi_1$ for $p \leq 0$.) By [F98, Theorem 16], ($1'$) and ($2'$) are quasiisomorphisms, again using the dominance of $-p\varpi_1$ for $p \leq 0$, and hence so is ($3'$).

(3) and ($3'$) give an isomorphism of the E_1 terms of (5.3.4) and (5.2.1)$\otimes\mathbb{C}$ and these are compatible with differentials. □

Proof of (iii) of Theorem 5.2.1. We will use the elementary fact that for a complex K with (finite, exhaustive) filtration $F^\bullet$ the spectral sequence converging to $H^*(K)$ satisfies $\sum_{p,q} E_1^{p,q} \leq \sum_n H^n(K)$, with equality holding if and only if it degenerates at E_1.

It suffices to work $\otimes\mathbb{C}$. Let $T' : \Theta_+ \to \mathbb{Z}_{\geq 0}$ be a function satisfying the condition

$$T'(\lambda) < T'(\mu) \quad \text{if } \mu \leqslant \lambda, \mu \neq \lambda. \tag{5.3.5}$$

This defines a filtration $'F^\bullet$ of the space of automorphic forms by the condition (5.3.3) which refines $F^\bullet$ in the obvious sense. Let $'E_1^{p,q}$ be the E_1 term of the corresponding spectral sequence. Then

$$\begin{aligned}\sum_n H^n(M, \mathbb{E}_M) = \sum_{p,q} \dim E_\infty^{p,q} &\leq \sum_{p,q} \dim E_1^{p,q} \\ &\leq \sum_{p,q} \dim {}'E_1^{p,q} = \sum_n H^n(M, \mathbb{E}_M).\end{aligned}$$

The inequality in the second line holds because $F^\bullet$ is refined by $'F^\bullet$. (The column $E_1^{p,*-p} = H^*(Gr^p)$ is the abutment of a spectral sequence which involves the columns $'E_1^{r,*-r} = H^*('Gr^r)$ for $r \in I_p := \{s \in \mathbb{Z} : F^{p+1} \subset {}'F^s \subset F^p\}$ (an interval of integers). Hence $\sum_k H^k(Gr^p) \leq \sum_{r\in I_p} \sum_k H^k('Gr^r)$.)

The equality in the second line holds because the spectral sequence of T' degenerates at E_1 when the highest weight of E is regular by [F98, Theorem 19].

It follows that $\sum_n H^n = \sum_{p,q} \dim E_1^{p,q}$, i.e. that the spectral sequence degenerates at E_1. □

Remarks 5.3.1. (i) For the filtration $'F^\bullet$ coming from an indexing function T' on Θ_+ satisfying (5.3.5), Theorem 14 of [F98] gives a detailed decomposition of the graded pieces of this filtration. This decomposition has the following coarsening:

$$'F^p/'F^{p+1} = \bigoplus_{\nu\in\Theta_+, T(\nu)=p} \operatorname{Ind}_{P_\nu}^G \left(\mathbb{C}_\nu^{M_\nu} \otimes \mathscr{A}_{\Theta-\nu,\log}\left(\Gamma_{M_\nu} A_\nu(\mathbb{R})^0 \backslash M_\nu(\mathbb{R})\right)\right) \tag{5.3.6}$$

(Here Ind_P^G is a suitable induction functor on $(\mathfrak{g}, K)$-modules, and $\mathbb{C}_\nu^{M_\nu}$ is the one-dimensional space on which M_ν acts via the character $\nu \in \check{\mathfrak{a}}_{M_\nu}$.) This admits a further decomposition using Eisenstein series, as does the E_1 term of the associated spectral sequence. There are similar decompositions of the E_1 term of the distinguished spectral sequence (5.2.1). This does not follow immediately from the results of [F98, §6] because the function T in (5.3.2) does not satisfy (5.3.5), so some adaptation of arguments in [F98] is necessary. It is an interesting question to what extent the finer decompositions respect mixed Hodge-de Rham structures.

(ii) It seems that our methods can be used to show that all of Franke's spectral sequences (i.e. those associated with a filtration of the space of automorphic forms satisfying (5.3.5)) are spectral sequences of mixed structure. This would require (1) a generalization of Morel's construction in the category of automorphic complexes on a locally symmetric variety, (2) a refinement of the weighted cohomology construction of [GHM94] (along the lines suggested in [GHM94, 35.4]), (3) a version of Theorem 4.3.1 relating the two constructions, and (4) a refinement of the main result of [N99] relating these constructions to [F98]. All of these elements are straightforward generalizations of the existing results. Notice that these spectral sequences need not have the important property of (5.2.1) that the E_1 term is a direct sum of pure structures, which is why we do not pursue this here.

Appendix A. Relative roots

For a connected reductive group G over a field k, by the k-root system or the relative root system of G we mean the system of roots with respect to a maximal k-split torus. This root system may be reducible and nonreduced.

A.1. Suppose that G is a connected, semisimple, almost $\mathbb{R}$-simple real algebraic group. If the symmetric space of $G(\mathbb{R})$ is Hermitian then the $\mathbb{R}$-root system of G is of type BC_s or C_s. We choose a maximal set of strongly orthogonal roots $\gamma_1, \dots, \gamma_s$ in the manner specified in [BB66, 1.2] or [AMRT75, III.2.3]. The roots of G are the nonzero elements of $\left\{\frac{\pm\gamma_i\pm\gamma_j}{2}\right\}_{1\leq i\leq j\leq s}$ in the C_s case and the nonzero elements of $\left\{\frac{\pm\gamma_i\pm\gamma_j}{2}, \frac{\pm\gamma_i}{2}\right\}_{1\leq i\leq j\leq s}$ in the BC_s case (with multiplicities) (cf. [AMRT75, p.186]).

A.2. Suppose now that G is connected, semisimple and almost $\mathbb{Q}$-simple $\mathbb{Q}$-algebraic group such that $G(\mathbb{R})$ has a Hermitian symmetric space. Fix a maximal $\mathbb{R}$-split torus T which contains a maximal $\mathbb{Q}$-split torus A_0. The real root system $\Phi(T,G)$ is a union of irreducible root systems of type BC and C (by A.1). Choosing a maximal set of strongly orthogonal roots in each irreducible factor as above gives a set of $s = \dim T$ roots $\gamma_1, \dots, \gamma_s \in X^*(T)$ such that the roots of $\Phi(T,G)$ are among the nonzero elements of $\left\{\frac{\pm\gamma_i\pm\gamma_j}{2}, \frac{\pm\gamma_i}{2}\right\}_{1\leq i\leq j\leq s}$. Consider the restriction of characters from T to A_0. Arguments of [BB66, 2.9] (cf. [AMRT75, III.2.5, p.195f]) show that there is a partition $\{1,\dots,s\} = \sqcup_{k=0,\dots,r} I_k$ such that $A_0 \subset T$ is the identity component of the subgroup defined by setting the characters in I_0 to be trivial and equating all characters in I_k for each $k \geq 1$, i.e.:

$$A_0 = \left(\bigcap_{i\in I_0} \ker(\gamma_i) \ \cap \ \bigcap_{k\geq 1} \bigcap_{i,j\in I_k} \ker(\gamma_i - \gamma_j) \right)^0 .$$

Setting $\beta_k = \gamma_i|_{A_0}$ for any $i \in I_k$ (for $k \geq 1$) we get a collection of r roots $\beta_1, \dots, \beta_r \in \Phi(A_0, G)$ (r the $\mathbb{Q}$-rank of G) such that $\Phi_0 := \Phi(A_0, G)$ is a subset of the nonzero elements of $\left\{\frac{\pm\beta_i\pm\beta_j}{2}, \frac{\pm\beta_i}{2}\right\}_{1\leq i\leq j\leq r}$. Since G is almost $\mathbb{Q}$-simple, Φ_0 is irreducible, and hence of type BC_r or C_r. Thus Φ_0 consists of the nonzero elements of

$$\left\{\frac{\pm\beta_i \pm \beta_j}{2}\right\}_{1\leq i\leq j\leq r} \qquad \text{in the } C_r \text{ case,}$$

$$\left\{\frac{\pm\beta_i \pm \beta_j}{2}, \frac{\pm\beta_i}{2}\right\}_{1\leq i\leq j\leq r} \qquad \text{in the } BC_r \text{ case}$$

(with multiplicities). A set of simple roots $\{\alpha_1, \ldots, \alpha_r\}$ is given by

$$\alpha_i := \frac{\beta_i - \beta_{i+1}}{2} \quad (\text{for } 1 \le i < r) \quad \text{and} \quad \alpha_r := \begin{cases} \beta_r & \text{for } C_r \\ \frac{\beta_r}{2} & \text{for } BC_r. \end{cases}$$

The basis dual to the simple coroots $\{\alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i)\}_{1 \le i \le r}$ is

$$\varpi_i = \frac{\beta_1 + \cdots + \beta_i}{2} \quad (\text{for } 1 \le i < r) \quad \text{and} \quad \varpi_r = \begin{cases} \frac{\beta_1 + \cdots + \beta_r}{2} & \text{for } C_r \\ \frac{\beta_1 + \cdots + \beta_r}{4} & \text{for } BC_r. \end{cases}$$

The abstract weight lattice of Φ_0 is

$$P(\Phi_0) := \{\lambda \in X^*(A_0)_{\mathbb{Q}} | (\lambda, \alpha^\vee) \in \mathbb{Z} \text{ for } \alpha \in \Phi_0\}.$$

The lattice $P(\Phi_0)$ is generated by $\varpi_1, \ldots, \varpi_r$ in the C_r case and by $\varpi_1, \ldots, \varpi_{r-1}, 2\varpi_r$ in the BC_r case. (In the BC_r case $2\alpha_r = \beta_r$ is a root and $(\varpi_r, (2\alpha_r)^\vee) = (\varpi_r, \beta_r^\vee) = 2\frac{(\varpi_r, \beta_r)}{(\beta_r, \beta_r)} = 1/2$, so that $\varpi_r \notin P(\Phi_0)$.) Since Φ_0 is a root system, $X^*(A_0) \subset P(\Phi_0)$.

Lemma A.2.1. *If G is simply connected or Φ_0 is of type BC_r then $X^*(A_0) = P(\Phi_0)$.*

Proof. This follows from [BT72, Cor. 4.4 to Prop. 4.3]. □

A.3. Continuing in the setting of A.2, let

$$A_d = \left(\bigcap_{i \ne d} \ker(\alpha_i) \right)^0 \qquad (\text{for } d = 1, \ldots, r).$$

On A_d we have $\beta_1 = \cdots = \beta_d$ and $\beta_{d+1} = \cdots = \beta_r = 0$. For C_r we have:

$$\varpi_i|_{A_d} = \begin{cases} i\, \varpi_1|_{A_d} & \text{if } i \le d \\ d\, \varpi_1|_{A_d} & \text{if } i > d. \end{cases} \tag{A.3.1}$$

For BC_r we have the same identities for $i < r$ plus the identity

$$2\varpi_r|_{A_d} = d\, \varpi_1|_{A_d} \tag{A.3.2}$$

for each $d = 1, \ldots, n$.

Lemma A.3.1. *(i) The restriction of $\beta_1 = 2\varpi_1 \in X^*(A_0)$ to A_d is the character appearing in* Lie U_d *for $d = 1, \ldots, r$.*

(ii) If G is simply connected or Φ_0 is of type BC_r then $\varpi_1 = \beta_1/2 \in X^*(A_0)$ is a character and its restriction to A_d generates $X^*(A_d)$ for $d = 1, \ldots, r$.

Proof. (i) Since $\beta_1|_{A_d} = \beta_d|_{A_d} = \alpha_d|_{A_d} \ne 0$ its root space must be contained in Lie W_d. Since β_1 is the highest root it must be contained in Lie U_d.

(ii) Since $X^*(A_0) \twoheadrightarrow X^*(A_d)$ this follows from Lemma A.2.1, (A.3.1), and (A.3.2). □

Let

$$\chi_d := \varpi_1|_{A_d} = \frac{1}{2}\, \beta_1|_{A_d} \qquad (\in X^*(A_d)_{\mathbb{Q}}).$$

(By Lemma A.3.1 this is consistent with the notation χ_d introduced in 3.2 and used in the main body of the paper.) From (A.3.1) and (A.3.2) we have:

Lemma A.3.2. *For each $i = 1, \ldots, r$, the sequence of integers $(m_1, \ldots, m_r)$ defined by $\varpi_i|_{A_d} = \chi_d^{m_d}$ is nondecreasing.*

Remark A.3.3. The example of $G = PGL(2)$ shows that the conclusions of Lemma A.2.1 and A.3.1(ii) can fail if G is not simply connected.

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