TOPOLOGY II

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These are my notes for the second semester of the first-year topology course taught at TIFR from January to May 2024. Sections marked $(*)$ were not covered (or not covered in detail).

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1. Homology and cohomology of manifolds; Poincaré duality

We assume the following: the basic properties of cohomology (as in the Eilenberg-Steenrod axioms), Mayer-Vietoris sequences, the universal coefficient theorems for homology and cohomology, cross products and the Künneth formula, properties of the cup and cap products. All coefficient rings are assumed to be PIDs. ([1])

1.1. (Topological) manifolds.

Definition 1.1.1. *A* **topological manifold***, or just* **manifold** *for short, is a second countable Hausdorff topological space M which is locally Euclidean, i.e. every point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$.*

By invariance of domain (i.e. $\mathbb{R}^m$ homeomorphic to $\mathbb{R}^n$ implies $n = m$) the dimension of a manifold is constant on each connected component. We will only consider manifolds all of whose connected components are of the same dimension.

Example 1.1.2. An open subset of a manifold is a manifold. A finite product of manifolds is a manifold. Examples of manifolds include Euclidean spaces, spheres, projective spaces (real, complex, quaternionic), Grassmannians (real and complex), and many other spaces which occur naturally.

Example 1.1.3. Here are two examples of the kind of pathologies we want to avoid:

(1) Let M be the real line with two origins, i.e. $\mathbb{R} \times \{0, 1\}$ modulo the identification of $(\lambda, 0)$ and $(\lambda, 1)$ for $\lambda \neq 0$. This is clearly locally Euclidean but it is not a manifold as it is not Hausdorff. It cannot be embedded in $\mathbb{R}^N$. ([2])

(2) Let $M = (0, 1)^2$ with the topology that a set $U \subset M$ is open iff $U \cap (\{a\} \times (0, 1))$ is open in $\{a\} \times (0, 1)$ for all a. This is locally Euclidean (every point has a neighbourhood homeomorphic to an interval in $\mathbb{R}$) but is not second countable (e.g. because it has uncountably many connected components).

The following also holds for noncompact manifolds; we only prove the compact case:

Lemma 1.1.4. *A compact manifold can be embedded as a closed subset of $\mathbb{R}^N$ for some N.*

Proof. Let $\{(U_i, \varphi_i)\}_{i=1,\dots,r}$ be a covering of M by open subsets with $\varphi_i : U_i \to \mathbb{R}^n$ a homeomorphism onto an open subset of $\mathbb{R}^n$. Let $\{f_i\}$ be a partition of unity subordinate to it. ([3]) Then define a mapping $F : M \to \mathbb{R}^{rn+n}$ by $(f_1\varphi_1, \dots, f_r\varphi_r, f_1, \dots, f_r)$, where $f_i\varphi_i$ has been extended by zero outside U_i (this is continuous). It is easy to check that F is continuous and bijective, hence a homeomorphism onto its image. □

This can be used to show that any compact manifold is a Euclidean neighbourhood retract and also that a compact manifold has the homotopy type of a CW complex (see the Appendix of Hatcher).

1.2. Mayer-Vietoris sequences for pairs.

Let X be a topological space and $U, V \subset X$ open subsets. There is a long exact M-V sequence

$$\to H_i(X, U \cap V, R) \to H_i(X, U, R) \oplus H_i(X, V, R) \to H_i(X, U \cup V, R) \to H_{i-1}(X, U \cap V, R) \to \quad (1.2.1)$$

This is proved by the usual argument using "small" chains. Let $C_*^{\{U,V\}}(X)$ be the complex of singular chains spanned by singular simplices which lie in either U or V. By the "small chains argument" we know that $C_*^{\{U,V\}}(U \cup V) \hookrightarrow C_*(U \cup V)$ induces isomorphisms in homology. By the 5-lemma we get that $C_*(X)/C_*^{\{(U,V)\}}(U \cup V) \longrightarrow C_*(X)/C_*(U \cup V)$ induces isomorphisms in homology. Now the short exact sequence of complexes of abelian groups

$$0 \to C_*(X, U \cap V) \to C_*(X, U) \oplus C_*(X, V) \to C_*(X)/C_*^{\{U,V\}}(X) \to 0$$

is termwise split and so $\otimes_{\mathbb{Z}} R$ gives a short exact sequence of complexes of R-modules, which gives the long exact sequence (1.2.1).

---

[1] The material here is treated in Hatcher (Ch. 3) and in Appendix A of Milnor's *Characteristic Classes*, which is the reference we follow for the proof of Poincaré duality, adding a few details. Hatcher's presentation is also modeled on Milnor's. The treatment in Spanier (Ch. 6) is different and perhaps less clear.

[2] Another description of this example: Take the quotient of the action of $\mathbb{R}^*$ on $\mathbb{R}^2 - \{(0, 0)\}$ by $\lambda \cdot (x, y) = (\lambda x, \lambda^{-1}y)$.

[3] Recall that in a compact Hausdorff space any open cover has partitions of unity subordinate to it. In fact, this holds for paracompact Hausdorff spaces, and our manifolds are necessarily paracompact as a second countable locally compact and Hausdorff space is alway paracompact.

Similarly, in the same setup there is a cohomology M-V sequence

$$\to H^i(X, U \cup V, R) \to H^i(X, U, R) \oplus H^i(X, V, R) \to H^i(X, U \cap V, R) \to H^{i+1}(X, U \cup V, R) \to \quad (1.2.2)$$

This comes as follows: $C^*(X, U) \cap C^*(X, V)$ is the complex of singular cochains on X which vanish on chains contained in U or V. There is a short exact sequence

$$0 \to C^*(X, U \cup V) \to C^*(X, U) \cap C^*(X, V) \to C^*(U \cup V, U) \cap C^*(U \cup V, V) \to 0.$$

But the right term is acyclic by the usual small chains type argument. So $C^*(X, U) \cap C^*(X, V)$ has cohomology $H^*(X, U \cup V)$. The short exact sequence of cochain complexes

$$0 \to C^*(X, U) \cap C^*(X, V) \to C^*(X, U) \oplus C^*(X, V) \to C^*(X, U \cap V) \to 0.$$

then gives (1.2.2) for $R = \mathbb{Z}$. For general R applying $\otimes_{\mathbb{Z}} R$ gives a short exact sequence of complexes giving (1.2.2) (because this short exact sequence is termwise split).

1.3. **Homology in degrees $\geq n$.** For a compact set K and $x \in K$ the inclusion of pairs $(M, M - K) \subset (M, M - x)$ induces a homomorphism

$$\rho_x : H_n(M, M - K, R) \to H_n(M, M - x, R).$$

Note that if B is a ball centred at x (i.e. choose a neighbourhood V of x homeomorphic to $\mathbb{R}^n$ by a homeomorphism taking x to 0 and then take B to be the preimage of a ball centred at 0) then excision and deformation retraction give $H_n(M, M - x, R) = H_n(B, B - x, R) = H_n(B, \partial B, R) \cong R$ is a free cyclic R-module of rank one.

Lemma 1.3.1. *If K is a compact set in an n-manifold M then*

(i) $H_i(M, M - K, R) = 0$ *for* $i > n$
(ii) $\alpha \in H_n(M, M - K, R)$ *is zero if and only if* $\rho_x(\alpha) = 0$ *for all* $x \in K$.

Proof. We will drop the coefficients R from the notation. A basic fact used repeatedly in the proof is the following: If (i) and (ii) hold for compact sets K_1, K_2 and for their intersection $K_1 \cap K_2$ then (i) and (ii) hold for $K_1 \cup K_2$. This follows easily from the Mayer-Vietoris sequence (1.2.1) by taking $X = M, U = M - K_1, V = M - K_2$:

$$\to H_{i+1}(M, M - K_1 \cap K_2) \to H_i(M, M - K_1 \cup K_2)$$
$$\to H_i(M, M - K_1) \oplus H_i(M, M - K_2) \to H_i(M, M - K_1 \cap K_2) \to$$

The vanishing for $i > n$ is obvious and property (ii) is also straightforward since ρ_x for $x \in K_i$ factors through $H_n(M, M - K_i)$. Given this we proceed in several steps:

(1) $M = \mathbb{R}^n$, K a compact convex set. In this case for an interior point $x \in K$ there is a deformation retraction of both $(\mathbb{R}^n, \mathbb{R}^n - x)$ and $(\mathbb{R}^n, \mathbb{R}^n - K)$ to a pair $(\mathbb{R}^n, \mathbb{R}^n - B)$ for some large ball $B \supset K$. So we have isomorphisms

$$H_i(\mathbb{R}^n, \mathbb{R}^n - K) = H_i(\mathbb{R}^n, \mathbb{R}^n - x)$$

so that both (i) and (ii) are obvious. Notice that in this case we have a stronger version of (ii), namely $\alpha \in H_n(\mathbb{R}^n, \mathbb{R}^n - K)$ is zero if $\rho_x(\alpha) = 0$ for some $x \in K$, i.e. it is enough to check the image under ρ_x for any one point $x \in K$.

(2) $M = \mathbb{R}^n$, $K = K_1 \cup \cdots \cup K_r$ is a finite union of compact convex sets. In this case use the Mayer-Vietoris sequence and induction on r, using the fact that $(\cup_{i<r} K_i) \cap K_r = \cup_{i<r} K_i \cap K_r$ is a union of $r - 1$ compact convex sets. Note that we get a slightly stronger version of (ii), namely if we choose one point $x_i \in K_i$ for each i, then α is nonzero if and only if $\rho_{x_i}(\alpha) =$ for all i.

(3) $M = \mathbb{R}^n$ and K is an arbitrary compact set. This case requires some argument. Let $\alpha \in H_i(M, M - K)$. Then there exist a compact neighbourhood Ω of K and a class $\tilde{\alpha} \in H_i(M, M - \Omega)$ such that $\tilde{\alpha} \mapsto \alpha$ under the natural map. (To prove this, represent α by a relative cycle, i.e. a chain $\xi \in C_i(M)$ such that $\partial\xi$ lies in $M - K$. Thus $\partial\xi$ is a linear combination of simplices contained in $M - K$, i.e. $supp(\partial\xi) \subset M - K$. Since $supp(\partial\xi)$ is compact we can choose a neighbourhood Ω of the compact set K which is disjoint from $supp(\partial\xi)$. Then ξ also gives a relative cycle in $C_i(M, M - \Omega)$ and the class $\tilde{\alpha}$ of ξ has the necessary property.) Let $B_1, \ldots, B_r$ be a family of closed balls covering K with $B_i \subset \Omega$ and $B_i \cap K \neq \emptyset$ for all i. The map $H_i(M, M - \Omega) \to H_i(M, M - K)$ factors as

$$H_i(M, M - \Omega) \to H_i(M, M - \cup B_i) \to H_i(M, M - K).$$

For $i > n$ the middle group is zero by case (2), so $\alpha = 0$. For $i = n$ suppose that $\rho_x(\alpha) = 0$ for all $x \in K$. Let α_B be the image of $\tilde{\alpha}$ under the first homomorphism. Then we know that $\rho_x(\alpha_B) = \rho_x(\alpha) = 0$ for $x \in K$, and by the stronger version of (ii) in case (2), we conclude that $\alpha_B = 0$. Then $\alpha = 0$.

(4) M arbitrary, K is contained in a neighbourhood U homeomorphic to $\mathbb{R}^n$. In this case the excision $H_i(M, M-K) = H_i(U, U-K)$ reduces it to case (3).

(5) M arbitrary, K arbitrary. In this case write K as a union of compact sets as in (4) and use the same Mayer-Vietoris induction procedure as in (2), using (4) as the input. □

If M is compact then taking $K = M$ we see that $H_i(M, R) = 0$ for $i > n$. For noncompact manifolds we have a stronger statement:

Proposition 1.3.2. *Let M be a noncompact manifold. Then $H_i(M, R) = 0$ for $i \geq n$.*

Proof. We drop the coefficient ring from the notation. We also assume that M is connected without loss of generality.

$\underline{i < n}$: An element $\alpha \in H_i(M)$ can be represented by a cycle with support in some open set U with compact closure $\overline{U}$. Let $K = \overline{U} - U$. The triple $(M, M-K, M-\overline{U})$ has a long exact sequence which is the upper row of the diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_{i+1}(M, M-K) & \longrightarrow & H_i(M-K, M-\overline{U}) & \longrightarrow & H_i(M, M-\overline{U}) & \longrightarrow \cdots \\ & & & \| & & \uparrow & \\ & & & H_i(U) & \longrightarrow & H_i(M) & \end{array} \tag{1.3.1}$$

The first vertical arrow is an isomorphism by excision of $M - \overline{U}$. Now the outer two groups in the top row vanish for $i > n$ by the previous lemma, so $H_i(U) = 0$ for $i > n$. By assumption α is in the image of $H_i(U) \to H_i(M)$, so $\alpha = 0$. This proves $H_i(M) = 0$ for $i > n$.

$\underline{i = n}$: For $\alpha \in H_n(M)$, we represent it as coming from $\alpha \in H_n(U)$ as in the previous case and use the same diagram for $i = n$, which is now

$$\begin{array}{ccccc} 0 \longrightarrow & H_n(M-K, M-\overline{U}) & \longrightarrow & H_n(M, M-\overline{U}) \\ & \| & & \uparrow \\ & H_n(U) & \longrightarrow & H_n(M) \end{array} \tag{1.3.2}$$

by the previous lemma. Consider the image $\bar{\alpha}$ of α in $H_n(M, M-\overline{U})$. Evidently $\rho_x(\bar{\alpha})$ is either nonzero for all x or zero for all x. (Given two points take a path between them and cover it by small open balls. Now use the stronger version of (ii) in case (1) of the previous proof.) Since the cycle representing α has compact support contained in U, it must be the case that $\rho_x(\bar{\alpha}) = 0$ for all $x \in \overline{U}$, as it is certainly so for x outside the support. By the previous lemma $\bar{\alpha}$ is zero in $H_n(M, M-\overline{U})$. By the exact sequence $\alpha = 0$ in $H_n(U)$, and hence in $H_n(M)$. □

1.4. **Orientations.** Let M be an n-manifold.

A **local R-orientation at** $x \in M$ is a choice of generator of the rank one free R-module $H_n(M, M-x, R)$.

An R**-orientation** for M is a choice of local R-orientations at all points, written $x \mapsto \mu_x$, satisfying a continuity condition: For each x, there is a neighbourhood B of x in M such that for $y \in B$, the local orientations μ_x and μ_y are related by the homomorphisms

$$H_n(M, M-x, R) \stackrel{\rho_x}{\leftarrow} H_n(M, M-B, R) \stackrel{\rho_y}{\rightarrow} H_n(M, M-y, R),$$

i.e. there is a $\mu_B \in H_n(M, M-B, R)$ with $\rho_x(\mu_B) = \mu_x$ and $\rho_y(\mu_B) = \mu_y$.

An R-orientation of M may or may not exist, if one exists we say that M is R**-orientable**, and if a specific R-orientation has been chosen we say that M is R**-oriented**.

The R**-orientation manifold** of M is $\tilde{M}_R = \{(x, \mu_x) | \mu_x \text{ is an } R\text{-orientation at } x\}$. It is naturally a manifold and the mapping $(x, \mu_x) \mapsto x$ defines a map $p : \tilde{M}_R \to M$ which is a covering map. (The fibre over any point is (noncanonically) identified with the units $R^\times$ in R.) An R-orientation is simply a section of this covering map. Note that the R-orientation manifold $\tilde{M}_R$ is itself R-oriented, i.e. it comes with an orientation.

If $R = \mathbb{Z}$ (in which case we drop the ring from the notation) then the orientation manifold is a double covering of M. The manifold M is orientable if and only if this is a trivial cover, i.e. $\tilde{M} = M \sqcup M$, and an orientation is a choice of component. (So if M is orientable then it has precisely $2^{|\pi_0(M)|}$ orientations). Since a connected M can have a nontrivial double cover only if $\pi_1(M)$ has an index 2 subgroup, we see that if $\pi_1(M)$ has no such subgroup (e.g. if M is simply connected) then M is orientable. Thus spheres, tori, odd-dimensional real projective spaces, all complex and quaternionic projective spaces, complex Grassmannians etc. are orientable.

If M is oriented (i.e. $\mathbb{Z}$-oriented) then it is R-oriented for any ring R since $H_n(M, M - x, R) = H_n(M, M - x, \mathbb{Z}) \otimes_{\mathbb{Z}} R$ by the universal coefficient theorem.

If $R = \mathbb{Z}/2$ then $\tilde{M}_{\mathbb{Z}/2} = M$ and any manifold is $\mathbb{Z}/2$-orientable.

1.5. **Fundamental class.**

Proposition 1.5.1. *Suppose that M is R-oriented, with local orientations $\mu_x \in H_n(M, M - x, R)$ for each $x \in M$. Then for any compact set $K \subset M$ there is a unique class $\mu_K \in H_n(M, M - K, R)$ such that $\rho_x(\mu_K) = \mu_x$ for all $x \in K$.*

Proof. We drop the ring R from the notation for simplicity. Notice that the previous lemma shows that the class μ_K is unique if it exists, so we have only to prove existence. For K sufficiently small this is contained in the definition of orientation, so we need to patch these together.

Suppose that $K = K_1 \cup K_2$ where μ_{K_1} and μ_{K_2} are known to exist. The M-V sequence is

$$0 \to H_n(M, M - K) \to H_n(M, M - K_1) \oplus H_n(M, M - K_2) \to H_n(M, M - K_1 \cap K_2) \to$$

By the uniqueness (prevous lemma), μ_{K_1} and μ_{K_2} have the same image in $H_n(M, M - K_1 \cap K_2)$, so $(\mu_{K_1}, \mu_{K_2}) \mapsto 0$ in the M-V sequence. Therefore there is a unique $\mu_K \in H_n(M, M - K)$ mapping to it. It is immediate that $\rho_x(\mu_K) = \mu_x$ for all $x \in K$.

For a general K write $K = K_1 \cup \cdots \cup K_r$ with each K_i small enough as in (1) . Induction on r using the Mayer-Vietoris argument gives the existence of μ_K. □

In the case where M is compact and R-oriented, taking $K = M$ we get a class $\mu_M \in H_n(M, R)$ such that $\rho_x(\mu_M)$ is the local R-orientation at each point $x \in M$.

Definition 1.5.2. *The* **fundamental class** *of an R-oriented n-manifold is the class $[M] := \mu_M \in H_n(M, R)$.*

Note that if M is R-orientable and connected then necessarily $H_n(M, R)$ is a free rank one R-module with any fundamental class as a generator. (Different choices of R-orientations give different R-fundamental classes which are multiples of each other by units of R.) For example, this shows that even-dimensional projective spaces can only be R-oriented if R is a $\mathbb{Z}/2$-algebra.

Another consequence of the existence of a fundamental class (at least with $\mathbb{Z}/2$ coefficients) for compact manifolds is that compact manifolds of different dimensions cannot be homotopic to each other. (Intuitively this seems quite plausible, but it is not easy to prove without some machinery.)

1.6. **Compactly supported cohomology.** Define the **compactly supported cohomology** of a topological space X as the direct limit

$$H^i_c(X, R) := \varinjlim_K H^i(X, X - K, R)$$

where the direct limit is taken over compact subsets $K \subset X$ with respect to the pullback maps induced by $(X, X - K) \subset (X, X - K')$ if $K' \subset K$. This can also be defined as the cohomology of the direct limit complex $\varinjlim_K C^*(X, X - K, R)$ since taking direct limits commutes with cohomology. The obvious map $H^*_c(X, R) \to H^*(X, R)$ to cohomology is an isomorphism if X is compact. Compactly supported cohomology is not a homotopy invariant: For example, $H^i_c(\mathbb{R}^n, R)$ is zero for $i \neq n$ and R for $i = n$ (take the limit as K runs over closed balls, which are cofinal among all compacts).

Exercise 1.6.1. Check the following properties of compactly supported cohomology:

(1) If $U \subset X$ is an open set then there is a map $H^*_c(U, R) \to H^*_c(X, R)$.
(2) If $f : X \to Y$ is a proper map then there is a pullback $f^* : H^i_c(Y, R) \to H^i_c(X, R)$.
(3) If $X = U \sqcup Z$ with Z closed then there is a long exact sequence

$$\cdots \to H^i_c(U, R) \to H^i_c(X, R) \to H^i_c(Z, R) \to \cdots$$

in compactly supported cohomology.

Let $X = U \cup V$ where U, V are open. There is a Mayer-Vietoris exact sequence

$$\to H^i_c(U \cap V, R) \to H^i_c(U, R) \oplus H^i_c(V, R) \to H^i_c(X, R) \to . \tag{1.6.1}$$

Indeed, this follows from the cohomology Mayer-Vietoris sequence for pairs (1.2.2) as follows (we drop the coefficients R for notational simplicity): For $K \subset U$ and $L \subset V$ compact sets we have a long exact sequence

$$\to H^i(M, M - K \cap L) \to H^i(M, M - K) \oplus H^i(M, M - L) \to H^i(M, M - K \cup L) \to$$

Using excisions this is the same as

$$\to H^i(U \cap V, U \cap V - K \cap L) \to H^i(U, U - K) \oplus H^i(V, V - L) \to H^i(M, M - K \cup L) \to$$

Now the set of compact subsets of $U \cap V$ which are of the form $K \cap L$ is cofinal among all compact subsets, so taking limits over K and L gives the desired sequence (1.6.1) since direct limits preserve exactness.

1.7. **Poincaré duality.** Recall the cap product

$$H^i(X, A, R) \times H_j(X, A \cup B, R) \to H_{j-i}(X, B, R)$$

which exists if A, B are open in $A \cup B$. (We will generally use it for open sets A, B in X or for sets A, B which can obviously be replaced by open sets without changing the (co)homology groups involved.) It has a simple relationship with cup product, namely

$$\langle a \smile b, c \rangle = \langle b, a \frown c \rangle$$

where $\langle\, , \rangle$ is the pairing of cohomology and homology. (On the level of chains and cochains this is the definition of $\frown$.)

Now suppose that $\mu_K \in H_n(M, M - K, R)$ where M is an n-manifold. Then $a \mapsto a \frown \mu_K$ defines a map

$$\frown \mu_K : H^i(M, M - K, R) \to H_{n-i}(M, R).$$

If M is an R-oriented manifold and μ_K is the class given by the previous proposition, then the maps $\frown \mu_K$ are compatible for varying K (by uniqueness of μ_K and naturality of the cap product with respect to inclusions) and so we get a map from the direct limit

$$PD_M : H^i_c(M, R) \to H_{n-i}(M, R).$$

In the compact case $PD_M(\,\cdot\,) = (\,\cdot\,) \frown [M]$ is capping with the fundamental class.

Theorem 1.7.1. *Let M be an R-oriented manifold of dimension n. Then $PD_M : H^i_c(M, R) \to H_{n-i}(M, R)$ is an isomorphism for all i.*

Proof. The proof is again by a Mayer-Vietoris patching argument from small cases.

(1) $M = \mathbb{R}^n$. Let B be a ball and $\mu_B \in H_n(\mathbb{R}^n, \mathbb{R}^n - B, R)$ a generator. By the universal coefficient theorem $H^n(\mathbb{R}^n, \mathbb{R}^n - B, R)$ is the dual of $H_n(\mathbb{R}^n, \mathbb{R}^n - B, R)$ (because $H_{n-1}(\mathbb{R}^n, \mathbb{R}^n - B, R) = 0$), so it is free of rank one over the coefficient ring with a generator e and $\langle \mu_B, e \rangle = 1$. Now

$$1 = \langle 1 \smile e, \mu_B \rangle = \langle 1, e \frown \mu_B \rangle$$

so that $e \frown \mu_B$ is a generator of $H_0(\mathbb{R}^n, R) = R$. Thus $\frown \mu_B$ maps $H^n(\mathbb{R}^n, \mathbb{R}^n - B, R)$ isomorphically to $H_0(\mathbb{R}^n, R)$. Taking the limit over larger balls B gives the statement.

(2) (M-V principle) Suppose that $M = U \cup V$ and $PD_U : H^*_c(U) \to H_{n-*}(U), PD_V$ and $PD_{U \cap V}$ are known to be isomorphisms. (We drop the coefficients from now on for notational simplicity.) We have the diagram

$$\begin{array}{ccccccccc}
\longrightarrow & H^i_c(U \cap V) & \longrightarrow & H^i_c(U) \oplus H^i_c(V) & \longrightarrow & H^i_c(M) & \longrightarrow & H^{i+1}_c(U \cap V) & \longrightarrow \\
& \downarrow{\scriptstyle PD_{U \cap V}} & & \downarrow{\scriptstyle PD_U \oplus (-PD_V)} & & \downarrow{\scriptstyle PD_M} & & \downarrow{\scriptstyle PD_{U \cap V}} & \\
\longrightarrow & H_{n-i}(U \cap V) & \longrightarrow & H_{n-i}(U) \oplus H_{n-i}(V) & \longrightarrow & H_{n-i}(M) & \longrightarrow & H_{n-i-1}(U \cap V) & \longrightarrow
\end{array} \tag{1.7.1}$$

The naturality of cap product implies that the first two squares commute. For the third square let $K \subset U$ and $L \subset V$ be compact sets and consider

$$\begin{array}{ccc}
H^i(M, M - K \cup L) & \longrightarrow & H^{i+1}(U \cap V, U \cap V - K \cap L) \\
{\scriptstyle \frown \mu_{K \cup L}}\downarrow & & \downarrow{\scriptstyle \frown \mu_{K \cap L}} \\
H_{n-i}(M) & \longrightarrow & H_{n-i-1}(U \cap V).
\end{array}$$

The excision $H^{i+1}(M, M - K \cap L) = H^{i+1}(U \cap V, U \cap V - K \cap L)$ shows that the third square of the previous diagram is the limit of these squares over K and L. That these squares commute up to a sign requires an argument which we skip, referring to Hatcher, p. 246f. The 5-lemma then implies that PD_M is an isomorphism.

(3) If M is homeomorphic to an open ball then (1) applies. If M is the union of finitely many open balls then (2) and an induction on the number of balls implies that PD_M is an isomorphism.

(4) (direct limits) Suppose M is the nested union of subspaces, i.e. $M = \cup_{\alpha \in A} U_\alpha$ where A is directed and $U_\alpha \subset U_\beta$ if $\alpha \leq \beta$ and assume further that every compact set in M is contained in some U_α. Then PD_M is an isomorphism if PD_{U_α} is an isomorphism for all $\alpha \in A$. Indeed, this is because $H^i_c(M) = \varinjlim H^i_c(U_\alpha)$ and $H_{n-i}(M) = \varinjlim H_{n-i}(U_\alpha)$ and the direct limit of isomorphisms is an isomorphism.

(5) If M is an arbitrary open subset of $\mathbb{R}^n$ then (since $\mathbb{R}^n$ is second countable) it can be written as the union of countably many open balls $B_1, B_2, \ldots$. Let $U_i = \cup_{j \leq i} B_j$; then U_i is a nested family of open subsets of M and PD_{U_i} is an isomorphism for all i by (3). By (4) PD_M is an isomorphism.

(6) If M is second countable then cover M by countably many open subsets $\{U_i\}_{i \geq 1}$ homeomorphic to open subsets of $\mathbb{R}^n$. Let $U'_i = \cup_{j \leq i} U_i$. By (2) and induction we know $PD_{U'_i}$ is an isomorphism for all i. By (4) we get that PD_M is an isomorphism.

(7) If M is arbitrary then we use a Zorn's lemma argument. Let

$$\mathscr{U} = \{U \subset M \text{ open} \mid PD_U \text{ is an isomorphism}\}.$$

Since every chain in $\mathscr{U}$ has an upper bound (the union is one, by (4)), Zorn's lemma says that there is a maximal element U. If $U \neq M$ choose $x \in M - U$. Let V be a neighbourhood of x homeomorphic to $\mathbb{R}^n$. Then by (1), (2), and (5) we know that $PD_{U \cup V}$ is an isomorphism, contradicting the maximality of U. □

(Note that the theorem actually holds even without assuming M is second countable.)

The special case when M is compact is of particular importance:

Corollary 1.7.2. *If M is a compact R-oriented n-manifold then $\frown [M] : H^i(M, R) \to H_{n-i}(M, R)$ is an isomorphism for all i.*

It follows that for a compact connected R-oriented n-manifold $H^n(M, R)$ is a free rank one R-module with canonical generator ω_M where $\omega_M \frown [M] = 1 \in H_0(M, R)$.

Note that if $R = \mathbb{Z}/2$ then any manifold is $\mathbb{Z}/2$-orientable, so that the corollary applies. In particular, this "explains" why the homology of $\mathbb{R}P^n$ (n even) looks nice with $\mathbb{Z}/2$ coefficients.

1.8. **Duality and the cup product pairing in cohomology.** Let M be an R-oriented n-manifold. There is a canonical trace homomorphism $Tr : H^n_c(M, R) \to R$ defined by

$$H^n_c(M, R) \stackrel{PD}{\to} H_0(M, R) \to R$$

where $H_0(M, R) = \oplus_{\pi_0(M)} R \to R$ is given by summing components. (The trace should be thought of as integration over the manifold. This will be made precise in the context of de Rham cohomology later.)

Corollary 1.8.1. *Let M be an R-oriented manifold. The pairing*

$$H^i_c(M, R) \times H^{n-i}(M, R) \to R \qquad \text{by } (a, b) \mapsto Tr(a \smile b)$$

is perfect modulo torsion.

Proof. Since we are working modulo torsion the pairing between homology and cohomology is perfect.

$$\begin{array}{ccc} H^i_c(M) \times H^{n-i}(M) & \xrightarrow{PD \times id} & H_{n-i}(M) \times H^{n-i}(M) \\ \downarrow & & \downarrow \\ H^n_c(M) & \longrightarrow & R \end{array}$$

The diagram commutes since $\langle PD(\alpha), \beta \rangle = Tr(\beta \smile \alpha)$, as one checks by working with $H^i(M, M - K)$ instead of $H^i_c(M)$ for a compact set K. □

Now consider the case where M is compact and oriented and $R = \mathbb{Z}$. If the dimension of M is $n = 2m$ then this defines a perfect bilinear form on $H^2(M)/tors$ with values in $\mathbb{Z}$. By the graded-commutivity of cup product this is symmetric if m is even and alternating (=skew-symmetric) if m is odd. We get as a corollary that if m is odd then $H^m(M)/tors$ must be even dimensional. (There cannot be a nondegenerate alternating form on an odd dimensional free abelian group or vector space over a field of characteristic zero.) The case m odd is not so interesting because any two perfect alternating forms on $\mathbb{Z}^r$ are equivalent, so assume that $n = 2m = 4l$. Then the pairing above is a perfect symmetric pairing on $H^{2l}(M, \mathbb{Z})/tors$, i.e. a perfect symmetric bilinear form. Perfect symmetric bilinear forms have many useful invariants, all of which give invariants of the manifold. For example, the signature of the real bilinear form on $H^{2l}(M, \mathbb{R}) = H^{2l}(M, \mathbb{Z}) \otimes \mathbb{R}$ is then an invariant of the manifold.

Now suppose that M is simply-connected and $l = 1$, i.e. M is a compact simply-connected (and hence orientable) 4-manifold. Then $H_1(M) = 0$ and hence $H^1(M) = 0$ (universal coefficients) and $H^3(M) = 0$ (duality). So the only interesting cohomology is $H^2(M)$, which is free (by the universal coefficient theorem), and it has the symmetric bilinear form given by the intersection form. An amazing theorem of M. Freedman (1982) says that there are (up to homeomorphism) at most two simply-connected compact four manifolds (without boundary) with a given $H^2(M, \mathbb{Z})$ and intersection form, and only one of them has a smooth structure.

1.9. **Duality between cohomology and Borel-Moore homology (∗).** For noncompact manifolds, there is another Poincaré duality statement relating cohomology to something called Borel-Moore homology. For now, we will just give the statement:

Theorem 1.9.1. *Let M be an R-oriented n-manifold. Then there is a fundamental class $[M] \in H_n^{BM}(M, R)$ and cap product with $[M]$ gives an isomorphism $PD : H^i(M, R) \to H_{n-i}^{BM}(M, R)$ for all i.*

The definition and properties of Borel-Moore homology and the proof of this theorem will be given in a problem set.

1.10. **Euler characteristics.** The Euler characteristic of a space X is defined as

$$\chi(X) = \sum_i (-1)^i \, rank(H_i(X, \mathbb{Z})).$$

We assume here that the homology groups have finite rank. The mod 2 Euler characteristic is defined as

$$\chi_2(X) = \sum_i (-1)^i \dim H_i(X, \mathbb{Z}/2).$$

Lemma 1.10.1. *If $H_*(X)$ is finitely generated then $\chi(X) = \chi_2(X)$.*

Proof. First note that $\sum_i (-1)^i \dim H^i(X, \mathbb{Z}/2) = \sum_i (-1)^i \dim H_i(X, \mathbb{Z}/2)$ since $\mathbb{Z}/2$ is a field and $H_i(X, \mathbb{Z}/2)^* = H^i(X, \mathbb{Z}/2)$. The universal coefficient theorem for cohomology gives short exact sequences

$$0 \to Ext(H_{i-1}(X, \mathbb{Z}), \mathbb{Z}/2) \to H^i(X, \mathbb{Z}/2) \to Hom(H_i(X, \mathbb{Z}), \mathbb{Z}/2) \to 0.$$

Write $H_i(X, \mathbb{Z})$ as a sum of a free summand $\mathbb{Z}^r$ and its torsion subgroup. Each $\mathbb{Z}$ summand of $H_i(X, \mathbb{Z})$ contributes $\mathbb{Z}/2$ to $H^i(X, \mathbb{Z}/2)$ and $Ext(\mathbb{Z}, \mathbb{Z}/2) = 0$ to $H^{i+1}(X, \mathbb{Z})$. So each $\mathbb{Z}$ summand contributes $(-1)^i$ to both $\chi(X)$ and $\chi_2(X)$. Each torsion summand $\mathbb{Z}/m$ contributes $Hom(\mathbb{Z}/m, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$ to $H^i(X, \mathbb{Z}/2)$ and also $Ext(\mathbb{Z}/m, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$ to $H^{i+1}(X, \mathbb{Z})$. In either case the contribution of a torsion summand of $H_i(X, \mathbb{Z})$ to $H^*(X, \mathbb{Z}/2)$ is zero, proving the lemma. □

Corollary 1.10.2. *The Euler characteristic of a compact odd-dimensional manifold is zero.*

Proof. We will assume the fact (to be proved later) that $H_*(M, \mathbb{Z})$ is finitely generated for a compact manifold. By the lemma, $\chi(M) = \chi_2(M)$. Poincaré duality with $\mathbb{Z}/2$ coefficients implies that

$$\dim H_i(M, \mathbb{Z}/2) = \dim H^i(M, \mathbb{Z}/2) = \dim H_{n-i}(M, \mathbb{Z}/2).$$

Since n is odd, these dimensions appear with opposite signs in χ_2. □

1.11. **Finite generation of homology.** Consider the following trivial remark: If Poincaré duality holds for a zero-dimensional manifold X then X must be finite (because $H^0 \cong H_0$ is required to be finitely generated). This suggests that Poincaré duality should force the homology of a compact manifold to be finitely generated. In fact:

Theorem 1.11.1. *If M is a compact R-oriented manifold then $H^*(M,R)$ and $H_*(M,R)$ are finitely generated R-modules.*

Proof. (from Spanier, Algebraic Topology) With a space X, coefficient ring R, and a coefficient R-module M, one can associate two cochain complexes. The first is the usual cochain complex

$$C^*(X,M) = Hom_R(C_*(X,R),M) = Hom(C_*(X),M)$$

with cohomology $H^*(X,M)$. There is also

$$C^*(X,R)\otimes_R M = Hom(C_*(X),R)\otimes_R M = Hom_R(C_*(X,R),R)\otimes_R M$$

with differential $\delta\otimes 1$. There is a map from the second complex to the first:

$$\mu : Hom_R(C_*(X,R),R)\otimes_R M \to Hom_R(C_*(X,R),M) \qquad \text{by } \mu\left(\sum_i \varphi_i\otimes m_i\right)(\xi) = \sum_i \langle\varphi_i,\xi\rangle\, m_i.$$

This will not usually induce an isomorphism in cohomology, but for a compact manifold X Poincaré duality implies that it does. To see this, recall that the universal coefficient theorem for homology takes the form

$$0\to H_k(X,R)\otimes_R M \to H_k(X,M)\to Tor_1(H_{k-1}(X,R),M)\to 0.$$

Using Poincaré duality isomorphisms $H_k(X)\cong H^{n-k}(X)$ this gives

$$0\to H^k(X,R)\otimes_R M \to H^k(X,M)\to Tor_1(H^{k+1}(X,R),M)\to 0.$$

Now consider the second complex above. Let $K^* = Hom_R(C_*(X),R)$. The definition of Tor_1 gives a short exact sequence

$$0\to H^k(K^*)\otimes_R M \to H^k(K^*\otimes_R M)\to Tor_1(H^{k+1}(K^*),M)\to 0.$$

Since $H^k(K^*) = H^k(X,R)$ we see that the two sequences are the same, and the natural mapping μ induces isomorphisms in cohomology.

Now we see that this property implies finite generation of the homology. Let C_* be a chain complex of R-modules such that

(*) for every R-module M the natural map of cochain complexes

$$\mu : Hom_R(C_*,R)\otimes_R M \to Hom_R(C_*,M)$$

induces isomorphisms in cohomology.

We will show that this implies that $H_*(C_*)$ is finitely generated. The cohomology universal coefficient theorem gives the sequence

$$0\to Ext^1(H_{k-1}(C_*),M)\to H^k(Hom_R(C_*,M))\xrightarrow{h} Hom_R(H^k(C^*),M)\to 0.$$

Taking $M = H_k(C_*)$ the surjectivity of h and the property $(*)$ imply that we can write

$$h\mu\left(\sum_i a_i\otimes b_i\right) = id \qquad \in\ Hom_R(H_k(C_*),H_k(C_*))$$

for some $a_i\in H^k(Hom_R(C_*,R)), b_i\in H_k(C_*)$. Now apply both sides to an element $b\in H_k(C_*)$:

$$b = h\mu\left(\sum a_i\otimes b_i\right)(b) = \left\langle \mu\left(\sum a_i\otimes b_i\right), b\right\rangle = \sum\langle a_i,b\rangle\ b_i.$$

Thus the b_i generate $H_k(C_*)$.

This proves that $H_*(X,R)$ is finitely generated; it follows from the universal coefficient theorem (or Poincaré duality) that $H^*(X,R)$ is finitely generated. □

In fact the homology of a compact manifold is always finitely generated, with no assumption about orientability. To generalize the proof above requires introducing Cech cohomology (cf. Spanier) so we will leave it aside for now. A different, more topological way to prove finite generation is by showing that a compact manifold is homotopy equivalent to a finite CW complex. (See the Appendix of Hatcher for a proof of this.) Note that the theorem would follow immediately if one could show that (a) a compact manifold has a CW structure or (b) a compact manifold has a simplicial structure. (Of course, (b) implies (a).) But (a) is unknown in general and (b) is false in general. A theorem of Whitehead says that (b) (and hence (a)) is true for smooth manifolds, as we shall see later.

1.12. **Manifolds with boundary.** A **manifold with boundary** is a second-countable Hausdorff topological space M such that every point has a neighbourhood homeomorphic to an open subset in the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_n \geq 0\}$. If in such a neighbourhood x is carried to a point with $x_n = 0$ then we say that x is in the boundary of M, denoted ∂M. It is easy to check that ∂M is a manifold of dimension $n-1$. It is also easy to check that the product of manifolds with boundary is naturally a manifold with boundary.

A **collar** for ∂M is a neighbourhood of ∂M in M which is homeomorphic to $\partial M \times [0,1)$ under a homeomorphism taking ∂M to $\partial M \times \{0\}$ by the identity.

Lemma 1.12.1. *If M is a manifold with compact boundary ∂M then ∂M admits a collar in M.*

Proof. (cf. Hatcher p. 253) Let M' be the manifold one gets by attaching a collar $\partial M \times [0,1]$ to ∂M, with $\partial M \times \{0\}$ being glued to ∂M via the identity. It is enough to show that M' is homeomorphic to M.

Choose a partition of unity $\varphi_i : \partial M \to [0,1]$ such that each $supp(\varphi_i)$ has closure contained in an open set U_i in M which is homeomorphic to $\mathbb{R}^n_+$.

$$M_k := M \cup \{(x,t) \in \partial M \times [0,1] : t \leq (\varphi_1 + \cdots + \varphi_k)(x)\}.$$

For k large enough, $M_k = M'$. So it suffices to construct homeomorphisms $h_k : M_{k-1} \cong M_k$ for each k. Now since $U_k \cong \mathbb{R}^n_+$ we can find a homeomorphism of $\partial U_k \times [-1,1]$ onto a neighbourhood of ∂U_k in M'. Define h_k to be the identity outside $\partial U_k \times [-1,1]$, and inside $\partial U_k \times [-1,1]$ we choose a homeomorphism which maps $\{x\} \times \left[-1, \sum_{i \leq k-1} \varphi_k(x)\right]$ onto $\{x\} \times \left[-1, \sum_{i \leq k} \varphi_i(x)\right]$ linearly. This gives the desired h_k. □

(The proof works if ∂M is paracompact.)

We will say that a manifold-with-boundary is R-oriented (or R-orientable) if $M - \partial M$ is R-oriented (or R-orientable).

Lemma 1.12.2. *If M is R-orientable then ∂M is R-orientable.*

Proof. Exercise. □

Now if M is a compact R-oriented manifold with boundary then using the collar above we can write $H_n(M, \partial M, R) = H_n(M - \partial M, \partial M \times (0, \epsilon))$, which has a generator coming from the R-orientation. Thus there is a fundamental class $[M] \in H_n(M, \partial M, R)$.

1.13. **Lefschetz duality.** Let M be an R-oriented n-manifold with boundary. The following theorem can be generalized to the noncompact case but we will be content to state the compact version:

Theorem 1.13.1. *Let M be an R-oriented compact n-manifold with boundary ∂M. Then $\frown [M] : H^i(M, \partial M, R) \to H_{n-i}(M, R)$ and $\frown [M] : H^i(M, R) \to H_{n-i}(M, \partial M, R)$ are isomorphisms.*

Proof. Both homomorphisms exist because $[M] \in H_n(M, \partial M)$. (In the cap product $H^i(X, A) \times H_j(X, A \cup B) \to H_{j-i}(X, B)$ they correspond to the choices $A = \partial M, B = \emptyset$ and $A = \emptyset, B = \partial M$.)

The existence of a collar for ∂M easily implies that $H^k_c(M - \partial M) = H^k(M, \partial M)$ and $H_{n-i}(M) = H_{n-i}(M - \partial M)$. So the first isomorphism is just Poincaré duality for the noncompact manifold $M - \partial M$. For the second, look at the long exact sequences of the pair $(M, \partial M)$

$$\begin{array}{ccccccccc}
\longrightarrow & H^i(M,\partial M) & \longrightarrow & H^i(M) & \longrightarrow & H^i(\partial M) & \longrightarrow & H^{i+1}(M,\partial M) & \longrightarrow \\
& \downarrow{\scriptstyle \frown[M]} & & \downarrow{\scriptstyle \frown[M]} & & \downarrow{\scriptstyle \frown[\partial M]} & & \downarrow{\scriptstyle \frown[M]} & \\
\longrightarrow & H_{n-i}(M) & \longrightarrow & H_{n-i}(M,\partial M) & \longrightarrow & H_{n-i-1}(\partial M) & \longrightarrow & H_{n-i-1}(M) & \longrightarrow
\end{array}$$

The diagram commutes because the image of $[M]$ under the connecting homomorphism $H_{n-i}(M, \partial M) \to H_{n-i-1}(\partial M)$ is a fundamental class, which we have denoted $[\partial M]$. Thus Poincaré duality for the boundary and the first part of the theorem give the second part of the theorem. □

Corollary 1.13.2. *Let M be a compact R-oriented n-manifold with boundary ∂M. Then*

$$H_*(M,R), H_*(M, \partial M, R) \quad \text{and} \quad H^*(M,R), H^*(M, \partial M, R)$$

are finitely generated R-modules.

Proof. Substitute Lefschetz duality in the proof of finite generation above. □

Corollary 1.13.3. *If M is a compact $n+1$-dimensional manifold with boundary then $\chi(\partial M) = (1 + (-1)^n)\chi(M)$. (So $\chi(\partial M)$ is even.)*

Proof. If (X, A) has finitely generated homology then $\chi(X, A) = \chi_2(X, A)$ (this was proved earlier for $A = \emptyset$ and the same proof works). Now for $A \subset X$ we have $\chi_2(X) = \chi_2(A) + \chi_2(X, A)$ by the relative sequence of the pair, assuming finite generation. In our case take $X = M, A = \partial M$ and use the equality $\dim H_i(M, \partial M, \mathbb{Z}/2) = \dim H^i(M, \partial M, \mathbb{Z}/2)$ and duality $H^*(M, \partial M, \mathbb{Z}/2) \cong H_{n+1-*}(M, \partial M, \mathbb{Z}/2)$ to get

$$\chi_2(M) = \chi_2(\partial M) + \chi_2(M, \partial M) = \chi_2(\partial M) + (-1)^{n+1}\chi_2(M) \tag{1.13.1}$$

which gives the corollary. □

(Strictly speaking we have only proved this for M oriented as we have not proved finite generation of $H_*(M, \partial M)$ in the nonorientable case, but assuming that the corollary is proved.)

In particular a manifold of odd Euler characteristic (for example, $\mathbb{C}P^{2k}$) cannot be the boundary of a compact manifold. The oriented topological cobordism ring Ω_*^{top} is defined as follows: Ω_n^{top} consists of classes of oriented manifolds (without boundary) modulo the equivalence relation of cobordism: N_1 and N_2 are oriented cobordant if $N_1 \sqcup (-N_2) = \partial M$ where M is an oriented $n + 1$-manifold with boundary. (Here $-N_2$ is N_2 with orientation reversed. Evidently this relation is weaker than homeomorphism.) This is a group under taking disjoint unions (the identity is the empty set) and $\Omega_*^{top} = \oplus_n \Omega_n^{top}$ is a ring under taking direct products. The corollary tells us that $[N] \neq 0$ if $\chi(N)$ is odd; in particular $[\mathbb{C}P^{2k}] \in \Omega_{4k}^{top}$ is nonzero. A theorem of Sullivan says that $\Omega_{4*}^{top} \otimes \mathbb{Z}[1/2] = \oplus_{k\geq 0}\Omega_{4k}^{top} \otimes \mathbb{Z}[1/2]$ is a polynomial ring generated by the classes of even-dimensional complex projective spaces. (The analogous theorem in the context of smooth manifolds and the corresponding ring Ω_* is a famous theorem of Thom.) The complete structure of the ring Ω_*^{top} is quite subtle.

1.14. **Gysin homomorphisms and Poincaré dual classes.** Let $f : M \to N$ be a continuous map of R-oriented manifolds of dimension m and n respectively. Then using Poincaré duality we get a map

$$f_! : H_c^k(M, R) \to H_c^{k+n-m}(N, R),$$

defined by $f_! := PD_N^{-1} \circ f_* \circ PD_M$. which is called the **Gysin homomorphism**. In the case where M and N are compact the relation to the usual pullback f^* is given by

$$f_!(f^*(\alpha) \smile \beta) = \alpha \smile f_!(\beta) \qquad (\alpha \in H^k(N, R), \beta \in H^*(M, R))$$

(Check this formula.) Thus $f_!$ is a $H^*(N, R)$-module homomorphism if $H^*(M, R)$ is given the $H^*(N, R)$-module structure via f^*.

Now consider the case where $i : M \hookrightarrow N$ is injective (and M and N are compact). Thus i is a continuous bijection onto its image, i.e. a homeomorphism of M with a closed subspace of N. We will refer to M as an embedded submanifold. Then there is a defined a class

$$\xi_M := i_!(1) \in H^{n-m}(M, R)$$

which we call the Poincaré dual class to M. It has the property that if $\alpha \in H^m(N, R)$ and $i^*(\alpha) = \omega_M$ then $\alpha \smile \xi_M = \omega_N$. (Note that ξ_M may be zero. For example if $M = S^1$ is the equator of $N = S^2$ then $\xi_M \in H^1(S^2, R) = 0$.)

There is also a naturally defined homology class $i_*([M]) \in H^m(N, R)$. (Intuitively, M defines a cycle in N (for example, if one had a simplicial complex structure on N in which M is a subcomplex) and this is the class of that cycle.) These two are dual in the sense that

$$\xi_M \frown [N] = i_*([M]).$$

(Note that $[M], [N]$, and ξ_M depend on the choices of orientations. But for example if $R = \mathbb{Z}$ then each of these is independent of the choice up to ± 1.)

1.15. **Cup product and intersection product.** Let M be compact and R-oriented. Then Poincaré duality can be used to define the **intersection product**

$$H_i(M,R) \times H_j(M,R) \to H_{i+j-n}(M,R)$$

by inverting the vertical maps in the following diagram:

$$\begin{array}{ccc} H_i(M,R) \times H_j(M,R) & \longrightarrow & H_{i+j-n}(M,R) \\ {\scriptstyle PD\times PD}\uparrow & & \uparrow{\scriptstyle PD} \\ H^{n-i}(M,R) \times H^{n-j}(M,R) & \xrightarrow{\smile} & H^{2n-i+j}(M,R) \end{array}$$

Intuitively it is given by intersecting homology cycles (think of them, e.g. as simplicial cycles) but it is very difficult to make this rigorous. (Note that this works only for oriented compact manifolds whereas the cup product on cohomology is defined for arbitrary spaces.)

In some cases it is possible to give a more geometric description of the intersection product. Let M be a compact oriented n-manifold. Suppose that $i_A : A \hookrightarrow M, i_B : B \hookrightarrow M$ are closed submanifolds of dimension a, b respectively, each of which is oriented. Suppose also that $A \cap B$ is a connected manifold of dimension $a+b-n$ and moreover, for each $x \in A \cap B$ there is a neighbourhood U of x in M and a homeomorphism $U \cong \mathbb{R}^n$ such that

$$(U \cap A, U \cap B, U \cap A \cap B) \cong (\mathbb{R}^a, \mathbb{R}^b, \mathbb{R}^a \cap \mathbb{R}^b \cong \mathbb{R}^{a+b-n}).$$

We say that A and B **intersect transversely**. The orientations of A and B and the orientation of M together determine an orientation on $A \cap B$. (Exercise: Check this.) We will not prove the following theorem which gives a very geometric description of some cup products. (A proof this version can be found in Dold, *Lectures on Algebraic Topology*, VII.11. We will prove a version later in the context of smooth manifolds.)

Theorem 1.15.1. *Let M be a compact oriented manifold and let A, B be oriented compact submanifolds that intersect transversely (as defined above). Then $\xi_A \smile \xi_B = \pm\xi_{A\cap B}$ and the intersection product of the homology classes is $i_{A_*}([A]) \cdot i_{B*}([B]) = \pm i_{A\cap B*}([A \cap B])$.*

(Here $\pm$ is a sign determined by the orientations of A and B, and can be made $+1$ by choosing orientations correctly, we ignore this for now.)

For example if $\dim A + \dim B = n$ then a transverse intersection is a finite set of points, so that $[M] \frown \xi_{A\cap B} \in H_0(M)$ is a scalar. This is a sum $\sum_{x\in A\cap B} sgn(x)$ where $sgn(x)$ is plus or minus one according to whether the orientations of A and B at x give the orientation of M or its negative.

This theorem is very useful in practice because many cohomology classes (but not all!) can be written as linear combinations of Poincaré dual classes of submanifolds. (Exercise: Use this theorem to determine the cohomology ring of $\mathbb{C}P^n$.)

1.16. **Alexander duality (∗).** This is a duality for a closed subset $X \subset M$ of a "background" manifold, which is taken to be a sphere or Euclidean space in the classical Alexander duality. (The case of manifolds-with-boundary can be reduced to this using a doubling construction.)

Since we are now allowing the subset X to be complicated (e.g. not a manifold), we will need Čech cohomology. We will just state the results for now.

Let $X \subset M$ be a closed subset of a compact n-manifold. Then there are isomorphisms

$$\check{H}^i(M,X) \cong H_{n-i}(M-X)$$

and

$$\check{H}^i(X) \cong H_{n-i}(M, M-X)$$

where $\check{H}^*$ denotes Čech cohomology (to be defined later).

So for example, if X is a manifold of dimension m, embedded as a closed subset of M, then

$$H_i(M, M-X) \cong H^{n-i}(X) \cong H_{n-m+i}(X)$$

The dual version $H^i(M, M-X) \cong H^{i-(n-m)}(X)$ is usually referred to as the Gysin isomorphism.

If X is assumed to be a locally contractible subset of a manifold then $\check{H}^i = H^i$, so that we get simpler statements which do not involve Čech cohomology. If the background manifold M is S^n, then this reduces to a statement called Alexander duality:

$$\tilde{H}_i(S^n - X) \cong \tilde{H}^{n-1-i}(X).$$

Here we have used reduced homology. Note that the homology of $S^n - X$ depends only on X and not on how it sits in S^n.

Remark 1.16.1 (Verdier duality)*.* Wouldn't it be nice if there were a single statement encapsulating all the statements above? There is, namely Verdier duality, which was formulated by Grothendieck and proved by Verdier in the 1960s. The formulation requires more sophisticated homological algebra tools, so maybe later.

2. Smooth manifolds

From now on we will only deal with C^∞ things, and the words "C^∞", "smooth" and "differentiable" will be used interchangeably. A reference for the material here is Warner's *Foundations of Differentiable Manifolds and Lie Groups* or any other book on manifolds.

2.1. Smooth manifolds.

Let M be a paracompact second-countable Hausdorff space. ([4]) ([5])

A **chart** (or **coordinate neighbourhood**) on M is a triple (U, V, φ) consisting of an open subset $U \subset M$, an open subset V of $\mathbb{R}^n$ (for some n), and a homeomorphism $\varphi : U \to V$. Two charts (U, V, φ) and (U', V', φ') are **compatible** if the **transition functions** $\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$ and $\varphi \circ (\varphi')^{-1} : \varphi'(U \cap U') \to \varphi(U \cap U')$ are smooth functions between open subsets of $\mathbb{R}^n$.

An **atlas** for M is a collection of charts such that the open sets cover M. Notice that a space M which has an atlas of charts is necessarily locally compact.

A **smooth atlas** for M is a collection of compatible charts which cover M, i.e. a collection of charts $\mathscr{A} = \{(U_\alpha, V_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ which are pairwise compatible and such that $\cup_\alpha U_\alpha = M$. Any smooth atlas $\mathscr{A}$ is contained in a maximal (with respect to inclusion) smooth atlas: simply include in the collection any chart which is compatible with all the charts in $\mathscr{A}$.

A **smooth (or differentiable or C^∞) manifold** is a pair $(M, \mathscr{A})$ consisting of a paracompact second-countable Hausdorff space M and a maximal smooth atlas $\mathscr{A}$ on M. We will usually drop the atlas from the notation. (Note that a smooth atlas $\mathscr{A}$ determines a (unique) maximal smooth atlas (as above take the collection of all charts compatible with the atlas). The word maximal is just put in so that things are unique, in practice in all arguments with any atlas contained in the maximal atlas will do.)

Note that by our second countability assumption a manifold has at most countably many connected components, and each connected component is second countable. ([6]) ([7])

A continuous function $f : M \to \mathbb{R}$ is called **smooth (or differentiable or C^∞)** if, for each open set $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ of an atlas, the composition $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is a smooth function. (Clearly this holds for one atlas contained in the given maximal atlas if and only if it holds for all atlases contained in the given maximal atlas.) There is an obvious definition for a function defined on an open subset $U \subset M$ to be smooth. The assignment $U \mapsto \{$ smooth functions on $U\}$ is a sheaf (of $\mathbb{R}$-algebras), the sheaf of smooth functions on M.

Let $f : M \to N$ be a continuous map between two smooth manifolds of dimension m and n respectively. It is called a **smooth map** if the following holds at all points p: For each point $p \in M$, using charts around p and $f(p) \in N$, we can get a map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$. This map should be smooth. Again, it is clear that it suffices to check this for any one pair of atlases contained in the maximal atlases defining M and N respectively. Clearly the composition of smooth maps is smooth.

A smooth map $f : M \to N$ is called a **diffeomorphism** if it has a smooth inverse, i.e. there exists a smooth map $g : N \to M$ such that $f \circ g = g \circ f = id$. (A smooth 1-1 map need not have smooth inverse, for example look at $\mathbb{R} \to \mathbb{R}$ by $x \mapsto x^3$.)

The product of two smooth manifolds is a smooth manifold. (• Exercise.)

Smooth manifolds with smooth maps between them form a category with the obvious forgetful functor to the category of topological manifolds and continuous maps and further to *Top*.

2.2. Real analytic manifolds.

By replacing "smooth" everywhere by "real analytic" we get the notion of a real analytic manifold. So a real analytic manifold is a nice topological space M with an open covering by subsets homeomorphic to $\mathbb{R}^n$ such that the transition maps are real analytic functions. It makes sense to talk about real analytic maps between such manifolds, and real analytic functions on them.

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[4] A topological space is paracompact if every open covering admits a locally finite open refinement. (A covering is locally finite if every point has an open neighbourhood which meets only finitely many members of the covering. A refinement of $\{U_\alpha\}$ is a covering $\{V_\beta\}$ such that for each V_β there exists α with $V_\beta \subset U_\alpha$.)

[5] A topological space is second-countable if the topology admits a countable base (i.e. there is a countable collection of open sets such that every open set is a union of members of the collection).

[6] Here is an example to show that the Hausdorff condition is not redundant: Take two copies of $\mathbb{R}$ and identify them outside the origin. (Equivalently, take $\mathbb{R}^2 - \{0\}$ and quotient by the relation $(x, y) \sim (\lambda x, \lambda^{-1} y)$ for $\lambda \in \mathbb{R}^*$, with the quotient topology.) This gives a space which is locally homeomorphic to $\mathbb{R}$ but is not Hausdorff.

[7] Here is another funny example: Take the square $(0, 1)^2$ and put the topology in which a set U is open if $U \cap \{x = a\}$ is open in $\{x = a\} = (0, 1)$ for every $a \in (0, 1)$. This space has uncountably many connected components, each of which is homeomorphic to $\mathbb{R}$. However it is not a smooth manifold because it is not second countable.

2.3. **Complex manifolds.** By replacing $\mathbb{R}^n$ by $\mathbb{C}^n$ and replacing "smooth" everywhere in the above by "holomorphic" (i.e. complex analytic) we get the notion of a complex manifold. So a complex manifold is (determined by) a nice topological space M with a covering by open sets which are homeomorphic to open subsets of $\mathbb{C}^n$ such that the transition functions are holomorphic functions. Since holomorphic functions are smooth functions of the underlying real variables M is also a smooth manifold of (real) dimension $2n$.

(Note that a complex function is differentiable once if and only if it is analytic so there is only one notion of complex manifold, in contrast to the real case.)

Much, but not all, of what is said below about smooth manifolds also holds for real-analytic and complex manifolds. (Exercise: Decide which statements and their proofs are valid in these categories.)

2.4. **Partitions of unity.** Recall that in a paracompact Hausdorff space any open covering has a continuous partition of unity subordinate to it. (Recall that if $\{U_\alpha\}_{\alpha\in A}$ is an open covering of a topological space X then a partition of unity subordinate to it is a collection of continuous functions $\{f_\alpha : X \to [0,1]\}$ such that $supp(f_\alpha) \subset U_\alpha$, $\sum_\alpha f_\alpha \equiv 1$ and $\{supp(f_\alpha)\}_{\alpha\in B}$ is locally finite. It follows that $\{int(supp(f_\alpha)\}_{\alpha\in A}$ is a locally finite open cover refining $\{U_\alpha\}_{\alpha\in A}$ with the same indexing set.) We will not actually use this fact, which is proved using Urysohn's lemma. Instead, for a smooth manifold, the existence of smooth bump functions in $\mathbb{R}^n$ allows us to construct a smooth partition of unity subordinate to any covering.

Lemma 2.4.1. *If $\{U_\alpha\}_\alpha$ is an open covering of a manifold then one can find a countable locally finite refinement by coordinate charts $\{(V_\beta, \varphi_\beta : V_\beta \to \mathbb{R}^n)\}_{\beta\in B}$ and an open covering $\{W_\beta\}_{\beta\in B}$ refining $\{V_\beta\}$ such that $\overline{W}_\beta \subset V_\beta$ and $\overline{W}_\beta$ is compact for all β.*

Proof. Since M is second countable it has a countable base consisting of charts. Since M is also locally compact it has a countable base of open sets with compact closure. (If $\{U_n\}$ is a countable base then for any point x there is a neighbourhood V_x of x with compact closure, and hence a $U_{n=n(x)} \subset V_x$ with $x \in U_n$. Then U_n has compact closure (since $\overline{U}_n \subset \overline{V}_x$ is closed) and contains x. Thus the U_i with compact closure already form a basis at each point, hence a base.) Thus we can assume that the original covering $\{U_\alpha\}$ in the lemma is a countable covering $\{U_n\}$ by charts with compact closure.

So let $\{U_n\}_{n\geq 1}$ be a countable covering of M consisting of charts with compact closure. Replacing U_n by $\cup_{j\leq n} U_j$ we can assume that $\{U_n\}$ is a nested collection of open sets with compact closure. Since $\overline{U}_n$ is compact and is covered by the collection we must have that $\overline{U}_n \subset U_m$ for some m large enough. Replacing U_{n+1} by U_m we can arrange that we have a nested collection of open sets with compact closures and $\overline{U}_n \subset U_{n+1}$. The sets $\Omega_n = \overline{U}_n - U_{n-1}$ (with $\Omega_1 = \overline{U}_1$) are compact and $U_n \cap \Omega_m = \emptyset$ for $m > n$. ([8])

We finish the proof of the lemma as follows: For each $x \in \Omega_n$ choose a coordinate neighbourhood $(V_{x,n}, \varphi_{x,n})$ containing x and contained in $\overline{U}_{n+1} - U_{n-2}$. Let $B_{x,n}$ be a compact subset of $\varphi_{x,n}(V_{x,n})$ with nonempty interior (e.g. a ball in $\mathbb{R}^n$). Then the sets $\varphi_{x,n}^{-1}(int(B_{x,n}))$ form a covering of Ω_n and hence we can choose a finite subcovering, which we call $\{W_\beta\}_{\beta\in B(n)}$ (here $B(n)$ is the finite index set). Then the corresponding $V_{x,n}$ also form a finite covering indexed by $B(n)$. This defines the locally finite refinement $\{V_\beta\}_{\beta\in B}$ indexed by $B = \cup_n B(n)$ and its refinement $\{W_\beta\}_{\beta\in B}$ as in the lemma. ([9]) □

Lemma 2.4.2. *If B is a closed ball in $\mathbb{R}^n$ then there is a smooth bump function supported exactly on B, i.e. a smooth function $f : \mathbb{R}^n \to [0,1]$ with $\{f \neq 0\} = int(B)$.*

Proof. Exercise. □

Lemma 2.4.3. *A smooth manifold has a smooth partition of unity subordinate to any open cover. In other words, if $\{U_\alpha\}_{\alpha\in A}$ is an open covering of M then there exist smooth functions $f_\alpha : M \to [0,1]$ such that*

(i) *$supp(f_\alpha) =$ closure of $\{f_\alpha \neq 0\}$ is contained in U_α*
(ii) *$\{supp(f_\alpha)\}_{\alpha\in A}$ is locally finite*
(iii) $\sum_\alpha f_\alpha \equiv 1$

[8]Since M is the union of the compact sets Ω_i we see that M is σ-compact.

[9]The proof shows the following: A second-countable and locally compact Hausdorff space is paracompact. (Indeed, in the proof just drop the requirement that V_β is a chart and drop the W_β entirely.) In particular, any open subset of a smooth manifold is again a smooth manifold (since this remark ensures paracompactness).

(Note that (iii) ensures that $\{\{f_\alpha \neq 0\}\}_\alpha$ is an open cover and (i) implies that it is a locally finite refinement of $\{U_\alpha\}_\alpha$ (with the same indexing set).)

Proof. Note that if a refinement of an open covering has a partition of unity subordinate to it then so does the original covering. Indeed, if $\{U_\alpha\}_{\alpha\in A}$ is refined by $\{V_\beta\}_{\beta\in B}$ and f_β is the partition of unity subordinate to $\{V_\beta\}_{\beta\in B}$ then for any function $I : B \to A$ such that $V_\beta \subset U_{I(\beta)}$, $g_\alpha = \sum_{\beta\in I^{-1}(\alpha)} f_\beta$ defines a partition of unity subordinate to $\{V_\beta\}_{\beta\in B}$. So it is enough to construct a partition of unity for a refinement.

Starting with an arbitrary open cover $\{U_\alpha\}_\alpha$ we can take a locally finite refinement by charts $\{(V_\beta, \varphi_\beta)\}_\beta$ and a refinement $\{W_\beta\}$ with $\overline{W}_\beta \subset V_\beta$ compact as in the previous lemma. For each $\beta \in B$ cover the compact subset $\varphi_\beta(\overline{W}_\beta)$ of $\mathbb{R}^n$ by finitely many balls $B_{\beta,1}, \ldots, B_{\beta,r}$ which are contained in $\varphi_\beta(V_\beta)$. Choose C^∞ bump functions $f_{\beta,i}$ for $i = 1, \ldots, r$ as in the previous lemma, i.e. with $f_{\beta,i}(x) \neq 0 \Leftrightarrow x \in int(B_{\beta,i})$. Let $f_\beta = \sum_i f_{\beta,i}$. This is a function from $\mathbb{R}^n$ to $[0, \infty)$ which is > 0 on $\varphi_\beta(\overline{W}_\beta)$ and which vanishes outside $\cup_i B_{\beta,i}$. Thus we can consider it as a smooth function on M which is zero outside V_β and is given by $f_\beta(\varphi_\beta(x))$ for $x \in V_\beta$ (and so is > 0 on W_β). Then $supp(f_\beta) \subset V_\beta$ and $0 < \sum_\beta f_\beta < \infty$ on the whole of M (> 0 because W_β cover M and $< \infty$ because $\{V_\beta\}_\beta$ is locally finite) and then dividing each one by $\sum_\beta f_\beta$ gives the partition of unity subordinate to $\{V_\beta\}$. By the remark of the previous paragraph we are finished. □

We will also need to use the following later:

Lemma 2.4.4. *Let M be a smooth manifold, $C \subset M$ a closed subset, and $g : C \to \mathbb{R}$ a function. Suppose that g satisfies:*

(∗) *for each $p \in C$, there is a neighbourhood U of p in M and a smooth function $\tilde{g}$ on U such that $\tilde{g}|_C = g$.*

Then there exists a smooth function $\tilde{g} : M \to \mathbb{R}$ such that $\tilde{g}|_C = g$.

Proof. Let U_α be a covering by charts such that for each point p, the neighbourhood U in property (∗) contains a U_α containing p. Then we have a smooth function g_α on each U_α which restricts to g on $C \cap U_\alpha$. Let f_α be a partition of unity subordinate to the U_α. The function $\tilde{g} = \sum_\alpha g_\alpha f_\alpha$ then agrees with g on C and is smooth on M. □

Another application of partitions of unity:

Proposition 2.4.5. *If $C \subset M$ is a closed subset of a smooth manifold then there exists a smooth function $f : M \to \mathbb{R}$ such that $C = f^{-1}(0)$.*

Proof. First consider the case where $M = U$ is an open subset of $\mathbb{R}^n$. The complement $U - C$ is a countable union of open balls B_i. Choose smooth functions $f_i : U \to [0, \infty)$ such that $f_i(x) > 0 \Leftrightarrow x \in B_i$ and such that the first i derivatives of f_i are bounded on B_i (uniformly) by $1/2^i$. (The second condition can always be achieved by multiplying by a scalar.) Now $f = \sum_i f_i$ converges uniformly on U, so that the function f is smooth. Obviously $f(x) = 0$ iff $x \in C$.

Now consider the general case. Take a locally finite covering by charts $\{U_\alpha\}$ and a partition of unity $\{g_\alpha\}$ subordinate to it. For each α the intersection $C \cap U_\alpha$ is a closed subset so using the previous case let f_α be a smooth function on U_α with $f_\alpha^{-1}(0) = C \cap U_\alpha$. Consider the sum $f = \sum_\alpha g_\alpha f_\alpha$, which makes sense as a smooth function on M. Then $f^{-1}(0) = C$. □

This shows that the preimage of a point under a smooth map can be arbitrarily bad. We will soon see a condition ensuring that the preimage is good (e.g. a manifold).

(Nothing in this section holds for real-analytic or complex manifolds as there cannot be real-analytic or holomorphic partitions of unity on them, for obvious reasons.)

2.5. **Tangent and cotangent spaces.** Let M be a smooth manifold and $p \in M$. A **germ** at p is an equivalence class of pairs (U, f) consisting of an open neighbourhood U of p and a smooth function f on U, modulo the equivalence relation that $(U, f) \sim (V, g)$ if there is an open neighbourhood $W \subset U \cap V$ on which f and g agree. Let G_p be space of germs of smooth functions at p. This is an $\mathbb{R}$-algebra. The ideal $\mathfrak{m}_p$ of germs vanishing at p has codimension one, i.e. $G_p/\mathfrak{m}_p = \mathbb{R}$ (by evaluation at p). The subspace of germs with vanishing first partial derivatives (in some local coordinate system) at p is denote S_p and called the space of **stationary germs**. (It does not depend on the coordinate system because under a local change of coordinates the first partial derivatives change by the Jacobian matrix of the coordinate change, so they vanish in any other coordinate system).

The **cotangent space** to M at p is the vector space

$$T_p^*M := G_p/S_p = \mathfrak{m}_p/\mathfrak{m}_p \cap S_p$$

of germs modulo stationary germs (or germs vanishing at p modulo stationary germs vanishing at p). The **tangent space** to M at p is the dual vector space

$$T_pM := (T_p^*M)^*.$$

Lemma 2.5.1. *Let $x_1, \ldots, x_n$ be local coordinates centred at $p \in M$. Any germ $f \in \mathfrak{m}_p$ can be written as $f = \sum_{i=1}^n x_i g_i$ for some smooth germs g_i with $\frac{\partial f}{\partial x_i}(0) = g_i(0)$. If f is also stationary, i.e. $f \in \mathfrak{m}_p \cap S_p$ then the g_i can be chosen to vanish at 0, i.e. $g_i \in \mathfrak{m}_p$, so that $\mathfrak{m}_p \cap S_p = \mathfrak{m}_p^2$.*

Proof. Since we are dealing with germs we can always restrict to neighbourhoods of p which, in the local coordinates, are starlike in $\mathbb{R}^n$. ($U \subset \mathbb{R}^n$ is starlike if it is invariant under dilation by elements of $[0, 1]$.) Then one can write, for $(x_1, \ldots, x_n) \in U$

$$\begin{aligned} f(x_1, \ldots, x_n) &= \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n)\, dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) \cdot x_i\, dt \\ &= \sum_{i=1}^n x_i \cdot \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\, dt \end{aligned}$$

Both assertions of the lemma follow. □

(It follows that we have an equivalent description of the tangent space as: $T_pM = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*$. (If you have seen the definition of Zariski tangent space in algebraic geometry you will recognize this.))

This lemma implies that the images of the x_i give a basis of T_p^*M. (Indeed, for $f \in G_p$ write $f - f(p) = \sum_i x_i g_i = \sum_i x_i(g_i - g_i(p)) + \sum_i x_i g_i(p)$. The germ $f(p) + \sum_i x_i(g_i - g_i(p))$ is stationary. They are obviously linearly independent.) The dual basis of T_pM consists of the linear functionals on germs $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ defined by:

$$\frac{\partial}{\partial x_i}(f) = \frac{\partial f}{\partial x_i}(0).$$

Summarizing:

Lemma 2.5.2. *The tangent space T_pM is a vector space of dimension n. If $x_1, \ldots, x_n$ are local coordinates centred at p then it is spanned by the tangent vectors $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ which are defined by*

$$\frac{\partial}{\partial x_i}(f) = \frac{\partial f}{\partial x_i}(0).$$

The dimension of the cotangent space T_p^*M is n and the dual basis to $\left\{\frac{\partial}{\partial x_i}\right\}$ is given by the germs of the coordinate functions x_i.

A **derivation** on M at p is a linear map $D : G_p \to \mathbb{R}$ which satisfies the Leibniz rule: for two germs f and g, $D(fg) = f(p)D(g) + D(f)g(p)$. It is clear that D must vanish on the constant germs, and also on $\mathfrak{m}_p^2$. So it is determined by its restriction to $\mathfrak{m}_p$ and gives an element of the tangent space. Let Der_p be the space of derivations at p. We have defined an injective map

$$Der_p \to T_pM.$$

It follows also that $D = \sum_i a_i \frac{\partial}{\partial x_i}$ in local coordinates, where $a_i = D(x_i)$. This is an isomorphism because $\left\{\frac{\partial}{\partial x_i}\right\}$ is exactly the dual basis to the basis $\{x_i\}$ of $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Another description of the tangent space is the most intuitive. A smooth map $\gamma : \mathbb{R} \to M$ is called a **smooth curve in** M. Suppose that $\gamma(0) = p$. Then we can define a derivation D_γ as follows: For any smooth germ f at p, let

$$D_\gamma(f) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma.$$

(i.e. pullback using the map γ and then take the derivative at 0.) We will call this derivation the **tangent vector to** γ **at** p. If we look at smooth curves in $\mathbb{R}^n$ this corresponds to the usual notion of "directional derivative along the curve γ at p". By choosing local coordinates at p we know that all derivations are linear combinations of the $\frac{\partial}{\partial x_i}$, and $\frac{\partial}{\partial x_i} = D_{\gamma_i}$ where γ_i is the curve given by $\gamma_i(t) = (0, \ldots, 0, t, 0, \ldots, 0)$ in the local coordinates. So we have:

Lemma 2.5.3. *Any derivation at* p *is of the form* D_γ *for some curve* γ*, so that every element of the tangent space at* p *is the tangent to a smooth curve.*

Remark 2.5.4. It is useful to keep in mind the following fact: If $V \subset \mathbb{R}^n$ is an open set, then choosing coordinates $x_1, \ldots, x_n$, gives tangent vectors $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ form a basis for the tangent space T_pV at any point $p \in V$, and give an isomorphism $T_pV \cong \mathbb{R}^n$. So all tangent vectors at different points in U can be thought of as belonging to the same $\mathbb{R}^n$. (You are already used to this, you often compare vectors at different points in $\mathbb{R}^n$ by "translation to the origin".)

A more invariant way of saying this is the following: If V is a finite-dimensional real vector space, considered as a manifold via any linear isomorphism $V \cong \mathbb{R}^n$, then there are canonical identifications $T_vV = V$ for any $v \in V$.

2.6. **Functoriality.** Given a smooth map $f : M \to N$ between smooth manifolds, there is a linear map of vector spaces $df_p : T_pM \to T_{f(p)}N$ for each $p \in M$. This can be seen in any of the equivalent descriptions above, but let us see it using derivations. Given $D \in Der_p$, define a derivation $df(D) \in Der_{f(p)}$ by $a \mapsto D(a \circ f)$ for a germ a at $f(p)$. Check that this is a derivation.

What is the map df_p in more concrete terms? If you write it out in local coordinates $x_1, \ldots, x_m$ near p and $y_1, \ldots, y_n$ near $f(p)$, and use the bases $\frac{\partial}{\partial x_i}$ of T_pM and $\frac{\partial}{\partial y_i}$ of $T_{f(p)}N$, then df_p is just the Jacobian matrix of partial derivatives, i.e.

$$(df_p)_{ij} = \frac{\partial f_i}{\partial x_j}$$

where $f = (f_1, \ldots, f_n)$ are the component functions of f, i.e. $f_i = y_i \circ f$.

From this local calculation and the chain rule it is clear that if $g : N \to X$ is another smooth map, then $d(g \circ f)_p = dg_{f(p)} \circ df_p$.

Note that a diffeomorphism $f : M \to N$ induces an isomorphism $df_p : T_pM \to T_{f(p)}N$, so they must be of the same dimension. In particular, $\mathbb{R}^n$ and $\mathbb{R}^m$ are not diffeomorphic if $m \neq n$, so that the dimension of a manifold is a diffeomorphism invariant. (Note that to prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ are nonhomeomorphic for $n \neq m$ ("invariance of domain") requires nontrivial algebraic topology arguments.)

Note that if $\gamma : \mathbb{R} \to M$ is a smooth curve in M with $\gamma(0) = p$ then in local coordinates x_i around p,

$$d\gamma_0\left(\frac{\partial}{\partial t}\right) = D_\gamma = \sum_i \frac{d\gamma_i}{dt}\Big|_{t=0} \frac{\partial}{\partial x_i}$$

where $\gamma_i(t) = x_i(\gamma(t))$.

A geometric way to think about the map $df_p : T_pM \to T_{f(p)}N$ is using tangents to smooth curves. The map df_p simply sends the tangent vector to $\gamma : (-\epsilon, \epsilon) \to M$ at $\gamma(0) = p$ to the tangent vector to $f \circ \gamma : (-\epsilon, \epsilon) \to N$ at $f(p)$.

2.7. **Tangent and cotangent bundles.** Let TM be the set of all pairs (p, v) where $p \in M$ and $v \in T_pM$ is a tangent vector at p. There is a map $\pi : TM \to M$ by $(p, v) \mapsto p$. The space TM is called the **tangent bundle** of M; let us see that it is itself a smooth manifold of dimension $2n$ ($n = dim(M)$). For this we need to put a topology (second countable, Hausdorff and paracompact) and provide an smooth atlas of charts. Choose an atlas for M and let $\varphi : U \to V \subset \mathbb{R}^n$ be a chart of the atlas giving local coordinates $x_1, \ldots, x_n$. A tangent vector at any point of U is then of the form $v = \sum_i q_i(v)\frac{\partial}{\partial x_i}$. This gives a homeomorphism $\pi^{-1}(U) \cong U \times \mathbb{R}^n \cong V \times \mathbb{R}^n$ by sending (p, v) to $(x_1(p), \ldots, x_n(p), q_1(v), \ldots q_n(v))$. Put on TM the topology generated by the open sets of the form $W \times W'$ where $W \subset \pi^{-1}(U)$ and $W' \subset \mathbb{R}^n$ are open. (You must check that finite intersections of such sets are again of this form.) It is easy to see that TM is second countable, Hausdorff and locally compact (and hence also paracompact). Moreover there is an obvious atlas and the transition maps are smooth, so that the sets $\pi^{-1}(U)$ give a smooth atlas for TM, making it into a smooth manifold of dimension $2n$.

Given a smooth map $f : M \to N$ there is an induced map on the tangent bundle $Tf : TM \to TN$ defined by $Tf(p, v) = (f(p), df_p(v))$. It is smooth. If $g : N \to W$ is another smooth map then it satisfies $T(gf) = Tg \circ Tf$. So T is a functor.

• Check that $T_{(p,q)}(M \times N) = T_pM \times T_qN$ (natural isomorphism of vector spaces) and $T(M \times N) = TM \times TN$ (natural diffeomorphism of manifolds).

The tangent bundle of $\mathbb{R}^n$ is identified with $\mathbb{R}^n \times T_0\mathbb{R}^n \cong \mathbb{R}^{2n}$ by the remark earlier.

Similar remarks apply to the cotangent bundle, which is denoted T^*M.

2.8. **Examples.**

Hypersurfaces in $\mathbb{R}^{m+1}$. Let $f : \mathbb{R}^{1+m} \to \mathbb{R}$ be a smooth function. When is the zero-set $M := \{f(x_0, \ldots, x_m) = 0\}$ a smooth manifold? (This is a special case of a more general situation discussed in the next section.) It might be useful to keep in mind a situation like the function $f(x, y, z) = x^2 + y^2 - z^2$ in $\mathbb{R}^3$. For each point $p \in M$ we must produce a chart containing p, so that the collection of charts form a smooth atlas. If the partial derivative

$$\frac{\partial f}{\partial x_0}(p) \neq 0$$

then by the implicit function theorem there is a mapping g from an open neighbourhood U of $(p_1, \ldots, p_m)$ to an open neighbourhood of $p_0 \in \mathbb{R}$, i.e. a smooth function $g(x_1, \ldots, x_m)$ with $f(g(x_1, \ldots, x_m), x_1, \ldots, x_m) = 0$. Now the map $U \to M$ by

$$(x_1, \ldots, x_m) \mapsto (g(x_1, \ldots, x_m), x_1, \ldots, x_m)$$

is a chart at p. On the other hand, if $\frac{\partial f}{\partial x_0}(p) = 0$ but $\frac{\partial f}{\partial x_i}(p) \neq 0$ for $i > 0$ then we can do the same argument with x_i playing the role of x_0 to produce a chart at p. This means that for every $p \in M$ such that

$$\nabla f(p) := \left(\frac{\partial f}{\partial x_0}(p), \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}(p)\right) \neq (0, \ldots, 0)$$

we have produced a chart at p. I leave it to you to check that this is a smooth atlas. So M is a manifold if $\nabla f(p) \neq 0$ at every point of M.

• Show that the tangent space map $df_p : \mathbb{R}^{m+1} \to \mathbb{R}$ is given by $v \mapsto \nabla f(p) \cdot v$.

Now we can use this to show that several standard examples are smooth manifolds: The sphere S^n is the zero-set of $f(x_0, \ldots, x_n) = x_0^2 + x_1^2 + \cdots + x_n^2 - 1$. Since $\nabla f = 2(x_0, \ldots, x_n)$ is nonzero when $f = 0$, this is a smooth manifold. (Note that another way to produce a smooth atlas on S^n is by using the homeomorphisms $S^n - \{(0, 0, \ldots, 1)\} \cong \mathbb{R}^n$ and $S^n - \{(0, 0, \ldots, -1)\} \cong \mathbb{R}^n$ via stereographic projection. Check that this atlas is compatible with any one defined by the implicit function argument above, so that we get the same smooth structure because the maximal atlas containing both is the same.)

Example: Real projective spaces. Consider the real projective space $\mathbb{R}P^n$. One description of it is as $\mathbb{R}P^n = S^n/\sim$ where $x \sim -x$.

• Use this description to get an atlas.

Another description is $\mathbb{R}P^n = \mathbb{R}^{n+1} - \{0\}/\mathbb{R}^*$ or rather $n + 1$-tuples of numbers $(x_0, \ldots, x_n) \neq (0, \ldots, 0)$ upto scaling by nonzero reals. Let

$$U_i = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} - \{0\} : x_i \neq 0\}/\mathbb{R}^*.$$

Then $\varphi_i : U_i \to \mathbb{R}^n$ by $\varphi_i(x_0, \ldots, x_n) = \left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right)$ is a homeomorphism, so it gives a chart. The U_i form an open covering of $\mathbb{R}P^n$, so this is an atlas. Let us check that this is a smooth atlas. The transition function $\varphi_i \circ \varphi_j^{-1}$ is defined on the open subset $\{t_i \neq 0\} \subset \mathbb{R}^n$ and maps to $\{t_j \neq 0\} \subset \mathbb{R}^n$. It is given by multiplication by $t_i^{-1}t_j$, which is smooth.

Example: Complex projective spaces. The complex projective space $\mathbb{C}P^n = \mathbb{P}^n$ is a smooth manifold of dimension $2n$. Since $\mathbb{P}^n = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$ we can describe an atlas of charts by

$$U_i = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} - \{0\} : z_i \neq 0\}/\mathbb{C}^*.$$

The isomorphism $U_i \cong \mathbb{C}^n = \mathbb{R}^{2n}$ by $(z_0, \ldots, z_n) \mapsto \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right)$ gives the chart. As in the previous example, the transition functions are z_i/z_j, so that this is a smooth atlas. (Note that in this case the charts of the atlas are modeled on (open subsets of) $\mathbb{C}^n$ and the transition functions are holomorphic. This is an example of a complex manifold of dimension n. A complex manifold of dimension n is also a real manifold of dimension $2n$.)

Example: Hypersurfaces in $\mathbb{P}^n$. Let f be a homogeneous polynomial in $n+1$ variables. Its zero-set in $\mathbb{C}^{n+1}$ is invariant under dilations, so it defines a subset $V \subset \mathbb{P}^n$ which is in fact closed. So we can ask: under what condition on f is V smooth manifold? (If f has real coefficients then we can also ask the same question for the subset of $\mathbb{R}P^n$ defined by f.) We can also ask the question for several homogeneous polynomials $f_1, \ldots, f_k$. These questions will be answered in the next section.

Example: Grassmannians. For $F = \mathbb{R}$ or $\mathbb{C}$ (or $\mathbb{H}$) let $Gr(k, F^n)$ be the set of k-dimensional F-subspaces (left F-submodules in the case $F = \mathbb{H}$) of F^n. This is a smooth manifold of dimension $d\,k(n-k)$. Recall how to produce an atlas. Fix $E \in Gr(k, F^n)$ and choose a complement $E^\perp$ to E, i.e. $F^n = E \oplus E^\perp$, and let $\pi_E, \pi_{E^\perp}$ be the projections. Then consider the subset

$$U_E := \{E' \in Gr(k,n) : E' \cap E^\perp = \{0\}\}.$$

For any $E' \in U_E$ the restriction $\pi_E|_{E'}$ is an isomorphism. So

$$\varphi_{E'} := \pi_{E^\perp} \circ (\pi_E|_{E'})^{-1} \in Hom(E, E^\perp).$$

It is easy to check that $E' \mapsto \varphi_{E'}$ gives an identification of U_E with $Hom(E, E^\perp) \cong Mat(k \times n-k, F) = F^{k(n-k)} \cong \mathbb{R}^{dk(n-k)}$. We leave it to be checked that these are the charts of a smooth atlas making $Gr(k, F^n)$ into a smooth manifold. (In the case $F = \mathbb{C}$ this also shows that $Gr(k, F^n)$ is a smooth complex manifold of dimension $k(n-k)$).)

2.9. **C^k-manifolds.** Instead of requiring the transition functions of an atlas to be smooth, one can require them to be only C^k for some k. The formal definition is the following: A **C^k-atlas** on a space M is a covering of M by charts $\{(U_\alpha, \varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n)\}_\alpha$ such that if $U_\alpha \cap U_\beta \neq \phi$ then the transition function $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a C^k map between open subsets of $\mathbb{R}^n$. (Recall that C^k means that all derivatives of order $\leq k$ exist and are continuous.) A **C^k-manifold** is a pair of a space M and a maximal C^k-atlas on M (as before, the maximal atlas is unique). The atlas allows us to talk about the C^k functions on a C^k manifold, and also C^k maps between C^k manifolds.

In general, if $k \geq 1$ things work quite similarly to the C^∞ case. (In particular, it is a theorem that every $C^k(k \geq 1)$ atlas on a C^k manifold contains at least one C^∞ atlas. It is also a theorem that if two C^∞ manifolds are diffeomorphic as C^k manifolds for $k \geq 1$ then they are actually smoothly diffeomorphic.)

If $k = 0$ a C^0 manifold is usually called a **topological manifold** and a C^0 map is simply a continuous map. The world of topological manifolds is quite different from the smooth case. (For example, it is known that there are topological manifolds which admit no smooth atlas at all! Also, two smooth manifolds may be homeomorphic without being smoothly diffeomorphic. Unfortunately the simplest examples are already nonelementary.)

Note that for $k \geq 1$ it makes sense to talk about the tangent space to a C^k manifold: It is defined as before as germs of C^k functions modulo stationary germs of C^k functions. The same proof as above shows that the tangent space T_pM to a n-dimensional C^k manifold has dimension n. A C^k map $f : M \to N$ induces a linear map $df : T_pM \to T_{f(p)}N$ which, in local coordinates, is given by the Jacobian.

From now on we will only work in the smooth category, so that the words "manifold, function, mapping" etc. will always refer to smooth things, unless indicated otherwise.

Remark 2.9.1. A useful fact which we will not prove is the following: Let $f : M \to N$ be a continuous map between C^k manifolds for $k \geq 1$ or $k = \infty$. If C is a closed subset such that f is C^k on some open neighbourhood of C then there exists a continuous map $F : [0,1] \times M \to M$ such that $F|_{\{0\} \times M} = f$, $F|_{\{1\} \times M}$ is C^k, and $F_{\{t\} \times C} = f|_C$ for all $0 \leq t \leq 1$. In particular, taking C to be empty, we see that any continuous map between smooth manifolds is homotopic to a smooth map. In particular, every homotopy class of loops in a smooth manifold contains a smooth loop.

2.10. **Inverse and implicit function theorems.** In addition to the chain rule we will need to use the inverse and implicit function theorems of multivariable calculus. These show that the linear approximation to a map (i.e. its derivative) controls the behaviour of the map up to change of coordinates, under an assumption on the derivative (basically that it should have the maximal possible rank). (For proofs of these see e.g. the book *Ordinary Differential Equations* by Arnol'd.)

Inverse function theorem: *Let $f : U \to V$ be a map between open subsets U, V of $\mathbb{R}^n$ which is smooth, i.e. all derivatives exist and are continuous. f is given by n differentiable functions $f_1, \ldots, f_n$,* *each of n variables. Let $p \in U$ be a point at which the Jacobian matrix*

$$df_p = \left(\frac{\partial f_i}{\partial x_j}\bigg|_p \right)$$

is nonsingular. Then there is a neighbourhood $U' \subset U$ of p such that $f : U' \to V = f(U')$ is an isomorphism, i.e. the inverse function $g : V \to U'$ defined by $g(y) = x$ if $f(x) = y$ is smooth and has derivative $dg_y = (df_{g(y)})^{-1}$ for $y \in V$.

If you are given n functions $f_1, \dots, f_n$ of $n+m$ variables $x_1, \dots, x_n, y_1, \dots, y_m$ you can ask whether in the system of n equations

$$\begin{gathered} f_1(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \end{gathered}$$

it is possible to solve for the variables $x_1, \dots, x_n$ in terms of the $y_1, \dots, y_m$. The implicit function theorem gives a sufficient condition to do this:

Implicit function theorem: *Suppose we are given n differentiable functions $f_1, \dots, f_n$ of the $n+m$ variables $x_1, \dots, x_n, y_1, \dots, y_m$. If $(p,q) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $f(p,q) = 0$ and if the matrix*

$$d(f|_{\mathbb{R}^n \times \{q\}})_{(p,q)} = \left(\frac{\partial f_i}{\partial x_j}\bigg|_{(p,q)} \right)_{i=1,\dots,n; j=1,\dots,n}$$

is nonsingular at (p,q) then there exist open neighbourhoods U of q in $\mathbb{R}^m$ and V of p in $\mathbb{R}^n$ and a (unique, differentiable) map $g : U \to V$ such that $g(q) = p$ and $f(g(y), y) = 0$ for all $y \in U$. The derivative of g is given by

$$dg_y = -\left(\frac{\partial f_i}{\partial x_j}\bigg|_{(g(y),y)} \right)^{-1} \left(\frac{\partial f_i}{\partial y_j}\bigg|_{(g(y),y)} \right) = -\left(d(f|_{\mathbb{R}^n \times \{y\}})_{(g(y),y)} \right)^{-1} \left(d(f_{\{g(y)\} \times \mathbb{R}^n})() \right)_{(g(y),y)}$$

for $y \in U$. Moreover, there is a neighbourhood W of (p,q) such that

$$W \cap \{(x,y) : f(x,y) = 0\} = \{(g(y), y) : y \in U\},$$

i.e. every solution of $f(x,y) = 0$ near (p,q) is of the form $(g(y), y)$.

These two theorems are equivalent: To show that the implicit function theorem implies the inverse function theorem, suppose $f : U \to V$ is a smooth map between open subsets of $\mathbb{R}^n$ with df_p (=the Jacobian matrix at p) nonsingular. Consider the mapping $F : U \times \mathbb{R}^n \to \mathbb{R}^n$ by $F(x,y) = y - f(x)$. The n functions $F_i(x_1, \dots, x_n, y_1, \dots, y_n) = y_i - f_i(x_1, \dots, x_n, y_1, \dots, y_n)$ of $2n$ variables satisfy the hypotheses of the implicit function theorem. Therefore there is a mapping $g = (g_1, \dots, g_n)$ from an open neighbourhood of $f(p) \in \mathbb{R}^n$ to the open subset U such that $F(g(y), y) = 0$, i.e. $f(g(y)) = y$, i.e. $f \circ g$ is the identity.

Conversely, given a smooth mapping $f : U \to V$ with $U \subset \mathbb{R}^{n+m}$ and $V \subset \mathbb{R}^n$, we will apply the inverse function theorem to $F : U \to \mathbb{R}^{n+m}$ given by $F(x,y) = f(x,y) + y$ where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. The differential $dF_{(x,y)}$ is block upper-triangular with the matrix $\left(\frac{\partial f_i}{\partial x_j} \right)_{i=1,\dots,n; j=1,\dots,n}$ and the $m \times m$ identity on the diagonal. Thus under the hypothesis of the implicit function theorem it is invertible at (p,q), and so by the inverse function theorem F has a local inverse, i.e. there is a neighbourhood W of (p,q) in which it has a smooth inverse G. Now let $g(y) = G(0,y)$, defined for y in the open neighbourhood $U = \{y \in \mathbb{R}^m : (0,y) \in W\}$ of q. The identities $FG = id$ and $GF = id$ lead to the conclusions of the theorem.

The following immediate consequence of the inverse function theorem is useful: A bijective smooth map $f : M \to N$ such that df_p is an isomorphism everywhere is a diffeomorphism.

Remark 2.10.1. Note that the real-analytic version of the inverse function theorem holds, i.e. are real-analytic map with nonsingular differential at a point has real-analytic inverse in a neighbourhood of the point. Similarly for complex-analytic. Also the version with "smooth" replaced by C^r for any $r \geq 1$.)

2.11. **Immersions and submersions.** Consider a smooth map $f : M \to N$ of smooth manifolds of dimension m and n respectively. The map f is an **immersion** at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is injective. It is a **submersion** at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is surjective. These are both open conditions, i.e. if they hold at a point p then they must hold in a neighbourhood of the point. To see this, if f is an immersion at p, then df_p, considered as a linear transformation from $T_pM \cong \mathbb{R}^m$ to $T_{f(p)}N \cong \mathbb{R}^n$ is an $n \times m$ matrix with rank m. An $n \times m$ matrix has rank m if and only if it has a nonsingular $m \times m$ submatrix. Since determinant is continuous in the entries, that same submatrix is nonsingular at points close to p, which means that the rank of df_q is still m if q is close to p. The same reasoning applies to submersions, because f is a submersion at p means df_p has rank n.

Consider the example of $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, \ldots, x_n) = -x_1^2 + x_2^2 + \cdots + x_n^2$. The map df_p at $p = (x_1, \ldots, x_n)$ is the $1 \times n$ matrix $(-2x_1, 2x_2, \ldots, 2x_n)$, so f is a submersion at all points except $0 \in \mathbb{R}^n$. Note also that $f^{-1}(t)$ is a manifold except at $t = 0$. We will see below that this happens exactly because f is submersive at all points of $f^{-1}(t)$ if $t \neq 0$.

We say that $f : M \to N$ is a **submersion** (resp. **immersion**) if it is a submersion (resp. immersion) at all $p \in M$.

2.12. **Submanifolds.** Let M be a smooth manifold. There are various possible notions of submanifold which need to be distinguished.

A **submanifold** is a pair (W, i) consisting of a smooth manifold W and a smooth map $i : W \to M$ which is a 1-1 immersion. (This is sometimes also called an immersed submanifold.)

A submanifold $i : W \to M$ which induces a homeomorphism of M with the subspace $i(W) \subset N$ (with the induced topology) an **embedded submanifold** and we say that i is an **embedding**.

If $i : W \to M$ is a 1-1 immersion which is proper then then $i : W \to M$ will be called a **closed submanifold**. (The terminology makes sense because a proper map between manifolds is necessarily closed, so that $i(W)$ is closed in N.) If a 1-1 immersion i is proper then $i : W \to i(W)$ is a homeomorphism, so that a closed submanifold is embedded. In particular, if W is compact then this holds, so a 1-1 immersion $i : W \to M$ with W compact is an embedding.

Remark 2.12.1. It is important to remember that a submanifold of M is not simply a subset of M. Here are some examples: For an irrational $\alpha \in \mathbb{R} - \mathbb{Q}$ consider the map $i : \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ by $t \mapsto (t, \alpha t) \mod \mathbb{Z}^2$. This is an immersion with dense image (by Kronecker's theorem). It is evidently not an embedded submanifold since the induced topology on $i(W)$ is not the usual one. An example of an embedded submanifold which is not closed is $(0, 1) \subset \mathbb{R}$.

2.13. **Normal forms for immersions and submersions.** The inverse and implicit function theorems imply some properties of immersions and submersions. The first one says that an immersed submanifold can be "straightened out" by a suitable choice of local coordinates.

Proposition 2.13.1 (Local normal form for immersions)**.** *Let $f : M^m \to N^n$ be an immersion at $p \in M$. Then there exist charts U at p and U' at $f(p)$ and systems of coordinates $x_1, \ldots, x_m$ on U and $y_1, \ldots, y_n$ on U' such that $f : U \to U'$ is given by $f(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$, i.e. f looks locally like the inclusion $\mathbb{R}^m \subset \mathbb{R}^n$.*

Proof. Choose local charts $\phi : U \to V \subset \mathbb{R}^m$ and $\psi : U' \to V' \subset \mathbb{R}^n$ at p and $f(p)$ respectively, so that $\phi(p) = 0$ and $\psi(f(p)) = 0$. We want to show that by changing the local coordinates on U' we can make $\psi \circ f \circ \phi^{-1}$ the map induced by the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$ as the first m coordinates. So we are in the following situation: $f : V \to V'$ is a smooth map between open neighbourhoods of 0 in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. It is enough to find an open $W' \subset V'$ and a diffeomorphism $T : W' \to T(W')$ such that $T^{-1} \circ f' = i$ on W'.

First of all we can clearly assume that $df_0 : T_0V \to T_0V'$ is the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$. (Indeed, by a linear change of coordinates $S : \mathbb{R}^n \to \mathbb{R}^n$ we can arrange this by replacing f by $S^{-1} \circ f$.) Now define a smooth mapping $T : V \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ by

$$T(x, y) := f(x) + (0, y)$$

where $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ are coordinates on $\mathbb{R}^n$. At $y = 0$ we get the map f, so that $f = T \circ i$. Then

$$dT_{(0,0)}(x, y) = df_0(x) + (0, y) = (x, y)$$

so $dT_{(0,0)}$ is the identity $T_{(0,0)}(V \times \mathbb{R}^{n-m}) \to T_{(0,0)}\mathbb{R}^n = T_0V'$. By the inverse function theorem, there is a small neighbourhood $W' \subset V'$ of $(0, 0)$ on which there is a smooth inverse $T^{-1} : W' \to V \times \mathbb{R}^{n-m}$ and $T^{-1} \circ f = i$ on W'. ☐

This implies (and is essentially the same as) the statement that any smooth function on M extends, locally on M, to a smooth function on N. Note that this is only true locally on M (consider the immersion $\mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ given by a line of irrational slope, which is not an embedding). If M is an embedded submanifold of N, then it says that the smooth functions on M are exactly the restrictions of smooth functions from N.

Proposition 2.13.2 (Local normal form for submersions). *Let $f : M^m \to N^n$ be a submersion at $p \in M$. Then there is a chart U at p and a chart U' of N at $f(p)$ such that $f(U) \subset U'$ and $f|_U$ looks like the projection $\mathbb{R}^m = \mathbb{R}^{m-n} \times \mathbb{R}^n \to \mathbb{R}^n$.*

Proof. Since the question is local we may assume that f maps an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$. Write $\mathbb{R}^m = \mathbb{R}^{m-n} \times \mathbb{R}^n$ and write (x, y) for the coordinates in each factor. Let p be a point at which f is submersive. We may arrange by a change of coordinates in $\mathbb{R}^m$ that $df_p|_{\{0\}\times\mathbb{R}^n}$ is nonsingular. Define $F : \mathbb{R}^m \to \mathbb{R}^m$ by $F(x, y) = (x, f(x, y))$. The differential dF_p is nonsingular. By the inverse function theorem we know that F is invertible in a neighbourhood W of p. Let $W' \times V'$ be a product neighbourhood (w.r.t. $\mathbb{R}^m = \mathbb{R}^{n-m} \times \mathbb{R}^m$) in $F(W)$. Then $F^{-1} : W' \times V' \to W$ is given by $F^{-1}(x, y) = (x, g(x, y))$ for some smooth function g from m variables to n variables, and we have

$$(x, y) = F \circ F^{-1}(x, y) = F(x, g(x, y)) = (x, f(x, g(x, y))$$

so that $f(x, g(x, y)) = y$, i.e. $fF^{-1}(x, y) = y$ and for $y \in V'$, $f^{-1}(y) \cap W$ is the graph of $W' \ni x \mapsto g(x, y)$. In particular, in the coordinates given by F^{-1}, f is the projection. □

(Remark: Show that this proposition implies the implicit function theorem. This again shows that the inverse function theorem implies the implicit function theorem.)

Remark 2.13.3. A C^k map $f : M \to N$ induces a linear map $df : T_pM \to T_{f(p)}N$ which, in local coordinates, is given by the Jacobian. The notions of C^k immersion, C^k submersion are defined in the obvious way and they have normal forms as above (the same proofs given above work because for $k \geq 1$ one can replace the word "smooth" by "C^k" everywhere in the inverse and implicit function theorems and they remain true).

2.14. **Critical point, critical value, regular value.** Let $f : M \to N$ be a smooth map. A point $p \in M$ is called a **critical point** for f if df_p has rank $< n = \dim N$, i.e. f is not a submersion at p. A point $q \in N$ is called a **critical value** if $f^{-1}(q)$ contains a critical point. The complement of the set of critical values is the set of **regular values**. Thus if $q \in N$ is a regular value and $p \in f^{-1}(q)$ then f is a submersion at p. (The fibre over a regular value is allowed to be empty.) For example, if $m < n$ then any point is critical and hence every point of $f(M)$ is a critical value.

The normal form for a submersion implies the following (fill in the necessary details):

Corollary 2.14.1. *If $f^{-1}(q) \neq \emptyset$ and $f : M^m \to N^n$ is a submersion at all points in $f^{-1}(q)$ (i.e., q is a regular value of f), then $f^{-1}(q)$ is an embedded submanifold of M of dimension $m - n$. The tangent space to $f^{-1}(q)$ at p is identified with the subspace $ker(df_p : T_pM \to T_qN)$.*

Examples 2.14.2. Let us use the corollary to get some interesting examples of manifolds as "level sets" of smooth maps.

(i) Let $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2$. Then f is submersive at all points of $f^{-1}(\lambda)$ for $\lambda > 0$, so that the fibres $f^{-1}(\lambda)$ are smooth manifolds of dimension $n - 1$, which are all diffeomorphic to the sphere S^{n-1}.

(ii) For $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x_1, \dots, x_n) = -x_1^2 + x_2^2 + \dots + x_n^2$ we showed that f is submersive at all points of $f^{-1}(t)$ for any $t \neq 0$. So $f^{-1}(t)$ is a manifold for $t \neq 0$.

(iii) (Hypersurfaces) Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. The induced map $\mathbb{R}^{n+1} \to \mathbb{R}$ on tangent spaces is given by $v \mapsto \nabla f(p) \cdot v$. (Check this.) So a level set $\{f = \lambda\}$ is a manifold if $\nabla f(p)$ is nonzero at all points $p \in \{f = \lambda\}$. This is exactly what we saw earlier. Note also that since M is a closed submanifold the tangent space T_pM is a subspace of $T_p\mathbb{R}^{m+1} = \mathbb{R}^{m+1}$. What is this subspace? By the corollary it is $T_pM = \{v \in \mathbb{R}^{m+1} : v \cdot \nabla f(p) = 0\}$.

(iv) Let $GL(n, \mathbb{R})$ be the group of nonsingular $n \times n$ matrices with real entries. It is the complement in $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ of the closed set $\det = 0$. So it is a manifold. To show that $SL(n, \mathbb{R}) = \det^{-1}(1)$ is a manifold we can directly check the gradient condition $\nabla\det(g) \neq 0$ for $g \in SL(n, \mathbb{R})$. We will instead compute the tangent map $d\det$ and show that $\det : GL(n, \mathbb{R}) \to \mathbb{R}^*$ has 1 as a regular value, which will also give us the tangent space as a subspace of $M(n, \mathbb{R})$.

Since $GL(n,\mathbb{R})$ is identified with an open subset of the real vector space $M(n,\mathbb{R})$ of all $n \times n$ matrices with real entries, its tangent space at the identity I is naturally $M(n,\mathbb{R})$. Then $T_I SL(n,\mathbb{R}) \subset T_I GL(n,\mathbb{R}) = M(n,\mathbb{R})$ is $ker(d\det)$, so we must compute the differential $d\det : M(n,\mathbb{R}) \to \mathbb{R}$. To do so, for $A \in M(n,\mathbb{R})$ choose a path through I which will have as tangent vector a given matrix $A \in M(n,\mathbb{R})$. There are many ways to choose one; the easiest is $\gamma(t) = I + tA$. For small enough t this is invertible and the tangent vector to γ at I is A. To compute the image of A under $d\det$ we compute

$$\left.\frac{d}{dt}\right|_{t=0} \det(I + tA) = \operatorname{tr}(A).$$

So $d\det_I = \operatorname{tr} : M(n,\mathbb{R}) \to \mathbb{R}$ is surjective and $T_I SL(n,\mathbb{R}) = ker(\operatorname{tr})$ is the space of trace zero matrices.

For $g \in SL(n,\mathbb{R})$ one can make a similar computation of the map $d\det_g : T_g GL(n,\mathbb{R}) \to T_1\mathbb{R}^* = \mathbb{R}$ by using the curve $\gamma(t) = g + tA$. Another way to do this is to use the diffeomorphism left translation $L_{g^{-1}} : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ by $h \mapsto g^{-1}h$. Then $\det = \det \circ L_{g^{-1}}$ and hence $d\det_g = d\det_I \circ (dL_{g^{-1}})_g$ must also be surjective. (This argument also computes $T_g SL(n,\mathbb{R})$ as a subspace of $M(n,\mathbb{R})$.)

(iv) Consider the map $f : GL(n,\mathbb{R}) \to M(n,\mathbb{R})$ given by $f(x) = {}^t xx$. It is surjective onto the space of symmetric positive definite matrices, and it is submersive at all points. (Exercise.) In particular, the identity matrix $I \in M(n,\mathbb{R})$ is a regular value, so the preimage $f^{-1}(I) = O(n)$ is a smooth manifold.

(v) • Show that $SU(n)$ has a manifold structure by doing something similar to (iv).

2.15. **Transversality.** Let $f : M \to N$ be a smooth map, and suppose that $W \subset N$ is an embedded submanifold. We say that f **is transverse along** W if

$$T_{f(p)}N = df_p(T_pM) + T_{f(p)}W \quad \text{for all } p \in f^{-1}(W).$$

(The sum need not be direct.)

Two special cases of this definition are: (1) if f is an embedding of M in N then we say M **intersects** W **transversely** if f is transverse along W (2) if f is submersive at all points of $f^{-1}(q)$ for $q \in N$, then f is transverse along $\{q\}$. The appropriate generalization of the previous corollary is

Proposition 2.15.1. *If $f : M^m \to N^n$ is a smooth map transverse along an embedded submanifold $W^w \subset N^n$ then $f^{-1}(W)$ is an embedded submanifold of M of dimension $m - n + w$.*

• Prove this by generalizing the proof of the previous proposition.

Here are two simple examples illustrating the proposition: First consider the intersection in $\mathbb{R}^3$ of the surface $M = \{z = xy\}$ and the plane $W = \{z = \lambda\}$. It is easy to check that for $\lambda \neq 0$ the intersection is transverse and $M \cap W$ is a manifold. For $\lambda = 0$ the intersection fails to be transverse at $(0,0,0)$ (the two tangent spaces are the same plane) and the intersection is $\{xy = 0 = z\}$, which is a pair of crossing lines. This is not a manifold. Second consider the intersection in $\mathbb{R}^3$ of the manifolds given by equations $x^2 + y^2 + z^2 = 1$ (2-sphere) and $x^2 + y^2 - z^2 = \lambda$ ($\lambda \neq 0$). For $\lambda \neq 1$ the intersection is transverse and consists of two circles. For $\lambda = 1$ the intersection is not transverse, but nonetheless it is a manifold, though not of the dimension given in the theorem.

2.16. **Whitney's embedding theorems.** With our abstract definition of manifold it is not obvious whether every manifold can be embedded as a submanifold of $\mathbb{R}^N$ for some N.

Theorem 2.16.1. *A compact smooth manifold can be embedded in $\mathbb{R}^N$ for some N.*

Proof. We will use a partition of unity. From Lemma 2.4.1 we can assume the following: We have a finite (because M is compact) covering by charts $(V_i, \phi_i : V_i \to \mathbb{R}^n)$ and an open refinement W_i with $\overline{W}_i \subset V_i$ (here $i = 1, \ldots, r$). Let $\{f_i\}_i$ be a partition of unity subordinate to the V_i with $f_i|_{\overline{W}_i} \equiv 1$ (as was constructed in the proof of existence of partitions of unity). Define a map $\Phi : M \to \mathbb{R}^{N=nr+r}$ by

$$\Phi = (f_1\phi_1, \ldots, f_r\phi_r, f_1, \ldots, f_r).$$

Then $d\Phi = (d(f_1\phi_1), \ldots, d(f_r\phi_r), df_1, \ldots, df_r)$. Since, for any $p \in M$, some f_i is identically one near p, and ϕ_i is a chart near p, we see that Φ is an immersion at p. So it is an immersion. Let us see that it is 1-1. If $\Phi(p) = \Phi(q)$ then $f_i(p) = f_i(q)$ for all i. In particular, there is some i for which these are both equal to one (since the V_i also cover M), and then p and q belong to the closure of V_i. But $\overline{V}_i \subset U_i$, so $p, q \in U_i$. But then $\phi_i(p) = \phi_i(q)$ and hence $p = q$.

This proves that Φ is a 1-1 immersion of M in $\mathbb{R}^N$. Since M is compact it is a homeomorphism onto its image, i.e. Φ is an embedding. □

(We will improve this to $N = 2n + 1$ below using Sard's theorem.)

Whitney's embedding theorem tells us that two ways of defining manifolds – abstractly as we have or as submanifolds of Euclidean space – are equivalent. For example, this tells us that our manifolds are metric spaces. (Of course this also follows from the Urysohn metrization theorem, which is much more general.)

Recall that $q \in N$ is a critical value of a smooth map $f : M \to N$ if there is a critical point in $f^{-1}(q)$, i.e. a point $p \in f^{-1}(q)$ such that f is not submersive at p.

Theorem 2.16.2 (Sard's theorem)**.** *The set of critical values of a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ is of measure zero.*

In particular, the set of critical values has no interior. The set of regular values is of full measure, so in particular it is nonempty. (There is another version of Sard's theorem which says that the set of critical values is nowhere dense, i.e. its closure has empty interior. This also implies that the set of regular values is nonempty.)

We will only use the case $m < n$ below so let's only prove that case, which is very easy. If $m < n$ then every point is critical, so we must show that $im(f)$ is of measure zero.

Lemma 2.16.3. *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^n$ be smooth (C^1 is enough). If $C \subset U$ has measure zero then $f(C)$ has measure zero.*

Proof. It is enough to consider the case where the closure of C is contained in a compact set, otherwise we could write it as a countable union of subsets with closure contained in compact sets and prove that each has measure zero. For C with closure contained in compact set $\Omega \subset U$, since df is continuous, there is a constant μ such that

$$|f(x) - f(y)| \leq \mu\, |x - y| \qquad (x, y \in \Omega).$$

(Use the mean value theorem and let μ be the supremum of $\|df_x\|$ for $x \in \Omega$.) For $\epsilon > 0$ cover C by balls B_i (contained in Ω) such that $\sum_i vol(B_i) < \epsilon/\mu^n$. By the previous estimate $f(B_i)$ is contained in a ball B_i' of radius $\mu \cdot radius(B_i)$, hence $f(C) \subset \cup_i B_i'$ and $\sum_i vol(B_i') \leq \mu^n \sum_i vol(B_i) < \epsilon$. Thus $f(C)$ has measure zero. □

To prove Sard's theorem for $f : \mathbb{R}^m \to \mathbb{R}^n$ with $m < n$ apply the lemma to the composition of f with the projection $\mathbb{R}^n \to \mathbb{R}^m$, i.e. to the map $\mathbb{R}^m \times \mathbb{R}^{n-m} \ni (x, y) \mapsto f(x)$. Since $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$ has measure zero we see that the image of f has measure zero. (The general version of Sard's theorem requires a little more work, see any standard text for a proof.)

Now we say that a subset $Z \subset M$ of a smooth manifold has measure zero if there is a countable covering by charts $\{(U_i, \varphi_i : U_i \to \mathbb{R}^n)\}$ such that $\varphi_i(U_i \cap Z)$ has Lebesgue measure zero in $\mathbb{R}^n$. (This notion is independent of the choice of $\{U_i\}$ by the lemma above.) With this definition, we have:

Corollary 2.16.4. *The set of critical values of a smooth map $f : M \to N$ is of measure zero. In particular, it has no interior.*

We will use the consequence that the set of regular values is nonempty (if $m < n$).

Theorem 2.16.5 (Whitney)**.** *A smooth compact n-manifold can be embedded in $\mathbb{R}^{2n+1}$ and immersed in $\mathbb{R}^{2n}$.*

Proof. We start with an embedding $M \subset \mathbb{R}^N$ for some N. Suppose that v is a vector in $\mathbb{R}^N$ which is never a tangent vector to M. (We are identifying the tangent spaces of all points of $\mathbb{R}^N$ with $\mathbb{R}^N$ simultaneously.) Let $\pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the orthogonal projection along v. It is easy to check that $d\pi_p : T_p\mathbb{R}^N \to T_{\pi(p)}\mathbb{R}^N$ is, under the identifications $T_pR^N = \mathbb{R}^N$ and $T_{\pi(p)}\mathbb{R}^{N-1}$, simply the projection $\pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$. Then $d\pi_p$ is injective for $p \in M$, so that $\pi|_M : M \to \mathbb{R}^{N-1}$ is an immersion. If, in addition, v is never a multiple of the vector $p - q$ for $p, q \in M$, then it is also 1-1. Since M is compact it is an embedding, i.e. a homeomorphism onto its image. So it remains to find such a vector v.

For a pair of points $p, q \in M$, $p \neq q$, let $\ell(p, q)$ denote the line in $\mathbb{R}^N$ spanned by $p - q$. Define a map $f : M \times M - \Delta \to \mathbb{R}P^{N-1} = \mathbb{R}^N - \{0\}/\mathbb{R}^*$ by $(p, q) \to \ell(p, q)$. (Here $\Delta \subset M \times M$ denotes the diagonal $\{(p, p) : p \in M\}$.) This is evidently a smooth map. If $2n < N - 1$ then every point of $image(f)$ is critical, and so by Sard's theorem $image(f)$ is of measure zero.

Define another map $g : TM - M \to \mathbb{R}P^{N-1}$ by $(p, v) \mapsto \mathbb{R}v$. Again, since $2n+1 < N$, $image(g)$ is of measure zero. So the union $image(f) \cup image(g) \subset \mathbb{R}P^{N-1}$ is of measure zero, so there is a point $\tilde{v}$ in the complement. Any vector in $\mathbb{R}^N$ projecting to $\bar{v}$ is the one we want.

For the immersion we argue a little more carefully with the map g. It is easy to see that g factors through a map $X \to \mathbb{R}P^{N-1}$ where $X = TM - M/\mathbb{R}^*$ is the quotient space. Now this quotient space is naturally a manifold of dimension $2n-1$ making the map g smooth. Thus if $2n-1 < N-1$ we know (by Sard's theorem) that the image is of measure zero. Projecting along a direction not in the image will still give an immersion. (Instead of this quotient you can put a Riemannian metric on M and take the unit sphere bundle in the tangent bundle for X and make the same argument.) □

Notice that the argument proves slightly more: If an n-manifold has a finite cover by open neighbourhoods homeomorphic to $\mathbb{R}^n$ then it has a 1-1 immersion in $\mathbb{R}^{2n+1}$. (The projection argument nowhere uses compactness.) This will be used below in the proof of the embedding theorem for noncompact manifolds.

It is clear that this argument cannot work to get an embedding in $\mathbb{R}^{2n}$ since we have no control on the initial embedding. For example, if we start with $M = S^1$ we may in the first step end up with an embedding like the embedding of the trefoil knot in $\mathbb{R}^3$. There is no vector in $\mathbb{R}^3$ along which we can project and get a 1-1 map to the plane. On the other hand, it is obvious that the circle can be embedded in $\mathbb{R}^2$. So Whitney's theorem that M^n can be embedded in $\mathbb{R}^{2n}$ requires a new idea, the famous "Whitney trick".

Whitney proved some much stronger theorems: If $k \geq 2n+1$ then the embeddings are dense (in a suitable topology) among all smooth maps $M^n \to \mathbb{R}^k$. If $k = 2n$ then the immersions are dense. (See R. Narasimhan's book *Analysis on real and complex manifolds* or his TIFR lecture notes *Topics in Analysis* for some of this.)

We can now prove the noncompact version of the embedding theorem.

Theorem 2.16.6 (Whitney). *A smooth manifold can be embedded in $\mathbb{R}^{2n+1}$ and immersed in $\mathbb{R}^{2n}$.*

Proof. Recall that in the proof of Lemma 2.4.1 we showed that there are open sets U_n with $\overline{U}_n \subset U_{n+1}$ and $\overline{U}_n$ compact such that $M = \cup_n U_n$. Using this we constructed countable open covers $\{V_\beta\}_{\beta \in B}$ and refinement $\{W_\beta\}_{\beta \in B}$ with $B = \cup_n B(n)$ such that $\{W_\beta\}_{\beta \in B(n)}$ is a finite cover of $\overline{U}_n - U_{n-1}$ and $\overline{W}_\beta \subset V_\beta \subset \overline{U}_{n+1} - U_{n-2}$ if $\beta \in B(n)$. For $i = 0, 1, 2$ let

$$B_i = \cup_{k \geq 0} B(i + 3k).$$

Then M has an open covering by three sets $M = M_0 \cup M_1 \cup M_2$ where

$$M_i = \cup_{\beta \in B_i} V_\beta = \cup_{k \geq 0} M_{i,k} \qquad \text{where } M_{i,k} = \cup_{\beta \in B(i+3k)} V_\beta.$$

We will first immerse each M_i in a Euclidean space and then use the immersions to immerse M. To immerse M_i note that on each finite union $M_{i,k} = \cup_{\beta \in B(i+3k)} V_\beta$ one can define an immersion $\Phi_{i,k}$ using the construction in the compact Whitney theorem (all that was required was a finite cover as in Lemma 2.4.1 and then a partition of unity as in Lemma 2.4.3). Thus $M_{i,k}$ has an injective immersion $\Phi_{i,k}$ in $\mathbb{R}^N$ and the image, being contained in the union of the finitely many compact sets $\Phi_{i,k}(\overline{W}_\beta)$ for $\beta \in B(i+3k)$, is contained in some ball. Moreover, the projection argument as in the $2n+1$ theorem says that we can take $N = 2n+1$. Translating the immersion of $M_{i,k}$ by a multiple of a suitable fixed vector so that the images of the various $\Phi_{i,k}$ for $k \geq 0$ are disjoint, we get injective immersions $\Phi_i : M_i \to \mathbb{R}^{N=2n+1}$ for $i = 0, 1, 2$.

Now take a partition of unity $\{f_0, f_1, f_2\}$ subordinate to the covering $\{M_0, M_1, M_2\}$ and consider the map

$$\Phi := (f_0\Phi_0, f_1\Phi_1, f_2\Phi_2, f_0, f_1, f_2).$$

It is easy to see that this is an injective immersion in $\mathbb{R}^{3N+3}$. It only remains to see that it is an embedding, which would follow if we show that the map Φ is closed. Now any closed subset $C \subset M$ can be written as a union $C = C_0 \cup C_1 \cup C_2$ where $C_i = \cup_{\beta \in B_i} C \cap (\overline{U}_i - U_{i-1})$. This is a disjoint union of compact sets, which maps under Φ to a disjoint union of compact sets which can be seen to be closed. □

Once we have a closed embedding in $\mathbb{R}^{3(2n+1)+3}$, the same projection idea as in the compact case reduces the dimension to $2n+1$ and $2n$ for an immersion. This can be improved to embedding in $\mathbb{R}^{2n+1}$ using the same argument as in the compact case:

Theorem 2.16.7 (Whitney $2n+1$ embedding)**.** *A smooth n-manifold admits a closed embedding in $\mathbb{R}^{2n+1}$.*

• Modify the previous proof to get that any smooth manifold embeds in $\mathbb{R}^{2n+1}$. (Check the details of this argument, in particular you have to ensure that the map is proper.)

Remark 2.16.8 (Real and complex analytic cases). It is natural to ask about a real-analytic version of Whitney's theorem. It is true that a real-analytic manifold has a real-analytic embedding in Euclidean space (Grauert, Morrey), but it's difficult. For complex analytic manifolds there can be no such theorem, because e.g. a compact complex manifold has no everywhere-holomorphic functions, so that any map to $\mathbb{C}^n$ has to be constant. A **Stein manifold** is a complex manifold which admits a closed embedding in $\mathbb{C}^N$ for some N. (This is one of many equivalent definitions.) Bishop and Narasimhan proved that a Stein manifold of (complex) dimension n admits an embedding in $\mathbb{C}^{2n+1}$.

Another theorem of Whitney is the following: A smooth n-manifold admits an embedding in $\mathbb{R}^{2n+1}$ such that the image is a real-analytic submanifold. Thus any smooth manifold has real-analytic structures (but not a canonical one).

2.17. **Tubular neighbourhoods.** We must use the notion of **vector bundle** on a smooth manifold; we will consider it again in more detail later. A **(smooth) vector bundle** on a smooth manifold M is a smooth manifold E with a smooth mapping $\pi : E \to M$ such that there exists an open covering $\{U_\alpha\}_\alpha$ of M and diffeomorphisms $\{\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r)\}_\alpha$ such that π is given on $\pi^{-1}(U_\alpha)$ by the second projection composed with φ_α. The integer r is called the **rank** of the vector bundle. An example is the tangent bundle, which is a rank n vector bundle over M. A **section** of $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = id$. The **zero section** is the mapping defined by $s(x) = 0$ for all $x \in M$. Any linear algebra construction that can be applied to vector spaces can be applied to smooth vector bundles, e.g. duals, tensor products, exterior powers, quotients etc. If $f : M \to N$ is a smooth map and E is a vector bundle on N then there is a pullback bundle f^*E with fibres $(f^*E)_p = E_{f(p)}$ for $p \in M$.

Let $i : M \to N$ be an embedded submanifold of N. A **tubular neighbourhood** of M in N is an open neighbourhood U which is diffeomorphic to a neighbourhood of the zero section in some vector bundle on M. It is a general theorem that an embedded submanifold of a manifold has a tubular neighbourhood, with the vector bundle being the normal bundle $\mathcal{N} = i^*TN/TM$ which has fibre $T_pN/di(T_pM)$ at $p \in M$. We will prove this when the submanifold is compact. To simplify the notation we will think of M as a subset of N, which causes no problems since it is an embedded submanifold. First we deal treat the case of a manifold embedded in $\mathbb{R}^n$.

Proposition 2.17.1. *Let M be a smooth manifold embedded in $\mathbb{R}^n$. Then M admits a tubular neighbourhood in $\mathbb{R}^n$.*

Proof. Let $\mathbb{R}^n$ be given the standard inner product. Then at each point $p \in M$ there is a decomposition $T_p\mathbb{R}^n = T_pM \oplus (T_pM)^\perp$. It is straightforward to check that

$$\mathcal{N} := \{(p, v) \in M \times \mathbb{R}^n : v \in (T_pM)^\perp\}$$

with the obvious map $\mathcal{N} \to M$ is a smooth vector bundle of rank $n - \dim M$ and is isomorphic to the normal bundle.

Now define a map $f : \mathcal{N} \to \mathbb{R}^n$ by $(p, v) \mapsto p + v$. This map is the identity on the zero section ($= M$) and in the identification $T_{(p,v)}\mathcal{N} = T_pM \oplus \mathcal{N}_p$ the map $df_{(p,v)}$ is given by $(v, w) \mapsto v + w$. Thus df is a diffeomorphism isomorphism along the zero section, in particular it is a local homeomorphism along the zero section $M \subset \mathcal{N}$. We must show that there is a neighbourhood U of M in $\mathcal{N}$ on which f is a homeomorphism. It is enough to show that there is neighbourhood on which f is injective. Now for each x and $\delta > 0$ let $V_{p,\delta} = \{(p, v) : |x - p| + |v| < \delta\}$. Since f is a local homeomorphism there exists $V_{p,\delta}$ on which f is injective. Let $\delta(p) = sup\ \{\delta : f$ is injective on $V_{p,\delta}\}$. Then $x \mapsto \delta(x)$ is a continuous function. (Indeed, since $V_{y,\delta-|x-y|} \subset V_{x,\delta}$ for all $\delta > 0$, we have $\delta(y) \geq \delta(x) - |x - y|$. The same holds with x, y reversed. This shows that $|\delta(x) - \delta(y)| \leq |x - y|$, so that δ is continuous. In fact it is Lipschitz with Lipschitz constant one.)

Now let $U = \{(p, v) \in \mathcal{N} : |v| < \delta(p)/4\}$. It will suffice to show that f is injective on U. Let $(p, v), (q, w) \in U$ and $f(p, v) = p + v = q + w = f(q, w)$ and assume $\delta(p) \leq \delta(q)$. Then $|p - q| = |w - v| \leq |w| + |v| \leq \delta(p)/2$, so $|p - q| + |w| = |w - v| + |w| \leq \delta(p)$. Thus both (p, v) and (q, w) lie in $V_{p,\delta}$ for some $\delta < \delta(p)$. But on such a neighbourhood the map f is injective, so $(p, v) = (q, w)$. □

Theorem 2.17.2 (Tubular neighbourhood theorem)**.** *Let N be a smooth manifold and $M \subset N$ a smooth embedded compact submanifold. Then there exists a neighbourhood U of the zero section in the normal bundle and a diffeomorphism of U with an open neighbourhood of M in N.*

Proof. We first embed N in $\mathbb{R}^n$ and use the previous proposition to get a neighbourhood W of N which is tubular, or more precisely, as in that proof the mapping $(p, v) \mapsto p + v$ is a diffeomorphism onto W. Inverting this gives a map from W to the neighbourhood of N in the normal bundle of N in $\mathbb{R}^n$ and then projecting to N defines a retraction $r : W \to N$ which is the identity on N. Let

$$\mathcal{N} = \{(p, v) \in M \times \mathbb{R}^n : v \in (T_pM)^\perp \cap T_pN\}.$$

(This is clearly isomorphic to the normal bundle as defined earlier.) Define a map $F : \mathcal{N} \to N$ by $(p, v) \mapsto r(p+v)$ where $r : W \to N$ is the retraction. It is easy to check that F is a local diffeomorphism along M and hence a local homeomorphism, so it remains to check that it is injective on some neighbourhood of M when M is compact. Suppose not. Then we can find sequences of points (p_n, v_n) and (q_n, w_n) such that $(p_n, v_n) \neq (q_n, w_n)$ and $v_n \to 0, w_n \to 0$ but $F(p_n, v_n) = F(q_n, w_n)$ for all n. By compactness of N we have subsequences converging to p and q with $F(p, 0) = F(q, 0)$, i.e. $p = q$. But then for large enough n, (p_n, v_n) and (q_n, w_n) must both lie in a neighbourhood $V_{p,\delta}$ for some $\delta > 0$, on which F is injective, contradicting that $F(p_n, v_n) = F(q_n, w_n)$. □

Notice that a tubular neighbourhood W of M in N admits a retraction $r : W \to M$.

The theorem also holds if the embedded submanifold is noncompact, but requires more work.

2.18. **Ehresmann's fibration theorem.** We first define the notion of fibre bundle. Let $\pi : E \to B$ be a continuous mapping of topological spaces. Let F be a topological space. We say that π is a **locally trivial fibre bundle with fibre** F or just **fibre bundle with fibre** F for short, if there is a covering $\{U_\alpha\}_\alpha$ of B and homeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ (called local trivializations) such that

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times F \\ \pi \big\downarrow & & \big\downarrow pr \\ U_\alpha & = & U_\alpha \end{array} \tag{2.18.1}$$

commutes for all α. When E and B are smooth we can speak of a **smooth fibre bundle** by requiring the local trivializations to be smooth maps.

The domain and range of a fibre bundle $\pi : E \to B$ are referred to as the **total space** and **base** respectively. The space F is referred to as the **fibre**.

The following important theorem gives a general criterion for a smooth map to be a fibre bundle:

Theorem 2.18.1 (Ehresmann)**.** *A proper submersion of smooth manifolds is a smooth fibre bundle.*

Proof. This will follow easily from the tubular neighbourhood theorem. Let $f : M \to N$ be a proper submersion. A fibre $f^{-1}(y)$ is a smooth embedded submanifold of M (by the normal form for submersions). Therefore there is a neighbourhood W of $f^{-1}(y)$ in M and a smooth retraction $r : W \to f^{-1}(y)$ coming from the tubular neighbourhood theorem. Consider the map

$$(r, f) : W \to f^{-1}(y) \times N$$

by $w \mapsto (r(w), f(w))$. This is a diffeomorphism along $f^{-1}(y)$ and since this is compact, there is an open neighbourhood $W' \subset W$ of $f^{-1}(y)$ on which this map is a diffeomorphism onto its image. Since f is proper, W' contains an open set of the form $f^{-1}(U)$ for some open set $U \subset N$ containing y ("tube lemma"). Then (r, f) restricts to a diffeomorphism $f^{-1}(U) \cong f^{-1}(y) \times U$. We leave it to be checked that this gives local trivializations showing that $M \to N$ is a smooth fibre bundle. □

In particular, all the fibres of such a map are diffeomorphic to each other if the base N is connected.

Example 2.18.2. Recall that a complex hypersurface of degree d is the subset of $\mathbb{C}P^n$ defined by a homogeneous polynomial in $n + 1$ variables of degree d. Ehresmann's theorem can be used to show that any two smooth hypersurfaces of the same degree are diffeomorphic. (The details will be worked out in the problem set.)

Example 2.18.3. Here is a simple example to keep in mind showing that properness is essential: Consider the map $\mathbb{R}^2 - \{0\} \to \mathbb{R}$ by $(x, y) \mapsto x$. (How does the proof of the theorem fail in this case?)

Later we will give a different proof of Ehresmann's theorem using flows.

Remark 2.18.4. The tubular neighbourhood theorem has no analogue in the complex setting, i.e. a complex submanifold $M \subset N$ need not have an open neighbourhood W (which is again a complex manifold) with a *holomorphic* retraction $r : W \to N$. The analogue of Ehresmann's theorem also fails: The fibres of a smooth submersion $f : M \to N$ of complex manifolds, which are complex submanifolds if f is holomorphic, are not biholomorphic to each other. Put another way, there are interesting families of complex manifold structures on the same underlying smooth manifold. The simplest example is that of complex tori of dimension one, or equivalently, elliptic curves, or equivalently, plane cubics (i.e. hypersurfaces of degree 3 in $\mathbb{C}P^2$).

2.19. **Triangulation of smooth manifolds** (∗)**.** We will not prove the following theorem:

Theorem 2.19.1 (Whitehead)**.** *A smooth compact manifold admits a triangulation (i.e. there is a (finite) simplicial complex with topological realization homeomorphic to the manifold).*

The analogue for topological manifolds is *false*: In every dimension ≥ 4 there is a topological manifold which does not admit a triangulation. In dimension 4 this was known in the 1980s (Freedman, Casson) but in (all) dimensions ≥ 5 it was proved only recently (Manolescu, 2013).

As a corollary, we see that the (simplicial or singular) homology and cohomology groups of a compact smooth manifold are finitely generated. We will prove this in a different way in the next subsection.

2.20. **Good covers and finite generation of homology.** A **good cover** of an n-manifold is a locally finite cover $\{U_i\}_{i\in I}$ such that all intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ for $i_0, \ldots, i_n \in I$ are homeomorphic to $\mathbb{R}^n$ (or empty).

Proposition 2.20.1. *Let M be a smooth manifold. Then M has a good cover.*

Proof. (Weil ([10])) The idea is that in an embedding in $\mathbb{R}^N$ we can find a covering by neighbourhoods which are

Using Whitney's embedding theorem we may embed M in $\mathbb{R}^N$. For each $x \in M$, orthogonal projection ϕ_x in $\mathbb{R}^N$ to the n-plane T_xM, when restricted to a sufficiently small neighbourhood U_x of $x \in M$ homeomorphic to $\mathbb{R}^n$, is actually a chart around x. The collection $\{(U_x, \phi_x)\}_x$ is an atlas for M. The transition function $\phi_y \circ \phi_x^{-1} : \phi_x(U_x) \to \phi_y(U_y)$ between open subsets of T_xM and T_yM is close to the identity (here we fix a reference linear isomorphism between T_xM and T_yM which is the orthogonal projection to T_yM). So we may assume that $\phi_y \circ \phi_x^{-1} - id$ has sufficiently small first (Jacobian) and second (Hessian) derivative matrices (in the sense that all entries have absolute value as small as we want). But a diffeomorphism of $\mathbb{R}^n$ which has such small Jacobian and Hessian takes a convex set to a convex set. In particular, the intersection $U_x \cap U_y$ is again convex. It follows that the cover $\{U_x\}_x$ is good. □

In the compact case the Mayer-Vietoris sequence immediately gives:

Corollary 2.20.2. *The homology and cohomology of a compact smooth manifold (with arbitrary coefficients) are finitely generated.*

Recall that in the lectures on Poincaré duality we had assumed this fact for some applications, e.g. to prove $\chi(M) = \chi_2(M)$ etc., so now we have justified all that, at least for smooth manifolds. In fact the corollary is true with the word "smooth" omitted, i.e. for topological manifolds, but this is harder to prove. (I am not sure if any compact topological manifold has a good cover, so one has to prove finite generation of homology in a different way.)

[10]Weil, Sur les theorèmes de de Rham, *Commentarii Math. Helvetici* 26 (1952)).

3. Vector fields and Frobenius' theorem

3.1. **Vector fields.** A **vector field** on a smooth manifold M is a smooth section of the tangent bundle, i.e. a smooth map $V : M \to TM$ such that $\pi \circ V = id$. Informally, $V(p) = (p, v(p))$, i.e. we are given, for each point p, a tangent vector $v(p) \in T_pM$ and it "varies smoothly as we vary p". We will usually ignore the first factor and speak of "the vector field v."

Let $\varphi : M \to M'$ be a diffeomorphism and $T\varphi : TM \to TM'$ the induced diffeomorphism of the tangent bundle. Given a vector field v on M and v' on M' we can define new vector fields $\varphi_*(v)$ and $\varphi^*(v')$, called the **pushforward** of v and **pullback** of v' respectively, by

$$\begin{aligned}\varphi_*(v) &:= T\varphi \circ v \circ \varphi^{-1}\\ \varphi^*(v') &:= T\varphi^{-1} \circ v' \circ \varphi.\end{aligned}$$

These are both sections of the respective tangent bundles, i.e. vector fields. (There is a mild abuse of notation here as $T\varphi$ must be applied to $V(p) = (p, v(p))$.) It is immediate that

$$\begin{aligned}\varphi^*\varphi_*(v) &= v\\ \varphi_*\varphi^*(v') &= v'\end{aligned}$$

In concrete terms we have:

$$\begin{aligned}\varphi_*(v)(q) &= d\varphi_p(v(p)) \qquad \text{for } \varphi(p) = q\\ \varphi^*(v')(p) &= d\varphi_q^{-1}(v'(q)) \qquad \text{for } \varphi(p) = q.\end{aligned}$$

(Here again we abuse notation by thinking of $v(p)$ as a tangent vector rather than a pair (point, tangent vector at the point).) Thinking of tangent vectors as derivations on (germs of) smooth functions, we have:

$$\begin{aligned}\varphi_*(v)(f')(x') &= \varphi^{-1*}(v(\varphi^*(f'))(x')\\ &= v(f' \circ \varphi)(\varphi^{-1}(x')) && (x \in M', f' \text{ smooth function on } M')\\ \varphi^*(v')(f)(x) &= \varphi^*(v'(\varphi^{-1*}(f))(x)\\ &= v'(f \circ \varphi^{-1})(\varphi(x)) && (x \in M, f \text{ smooth function on } M)\end{aligned}$$

where $\varphi^*(f') = f' \circ \varphi$ is the pullback of smooth functions from M' to M.

- If f' is a smooth function on M' then $\varphi^*(f'v') = \varphi^*(f')\varphi^*(v')$ (i.e. pullback of vector fields is a module over pullback of functions)

In general, it is only possible to pushforward and pullback vector fields by diffeomorphisms, not by general smooth maps. (There are some special classes of maps for which pullback of vector fields exists.)

Consider what a vector field looks like "locally". Let $p \in M$ and let $(U, \varphi, U' \subset \mathbb{R}^n)$ be a chart containing p. The smooth map $V : M \to TM$ restricts to a smooth map $V : U \to TU$ which is a section of $\pi : TU \to U$. Now using the diffeomorphism $\varphi : U \to U'$ we can get a vector field $\varphi_*(v)$ on $U' \subset \mathbb{R}^n$, i.e. a section of $TU' \to U'$. The basis $\left\{\frac{\partial}{\partial x_i}\right\}$ gives a trivialization $TU' = U' \times T_{\varphi(p)}U'$, in terms of which we can write

$$\varphi_*(v) = \sum_i a_i \frac{\partial}{\partial x_i} \qquad \text{or} \qquad v = \varphi^*\left(\sum_i a_i \frac{\partial}{\partial x_i}\right) \tag{3.1.1}$$

for smooth functions $a_i = a_i(x_1, \ldots, x_n) \in C^\infty(U')$. This is what a vector field looks like locally, i.e. in a coordinate chart. In the sequel we will frequently write $v = \sum_i a_i \frac{\partial}{\partial x_i}$ as a local expression for the vector field v; it is to be understood as meaning (3.1.1). (In other words we will usually drop the diffeomorphism φ from the notation.)

Note that a vector field is determined by its action as a derivation on smooth functions (i.e., if v, w are vector fields and $v(f) = w(f)$ for all smooth functions f then $v = w$, e.g. because of the local expression). In other words, a vector field is a linear differential operator of order one on smooth functions.

3.2. **Local flows and vector fields.** Suppose that we have an action $\alpha : \mathbb{R} \times M \to M$ of the group $\mathbb{R}$ on M such that the map $\alpha : \mathbb{R} \times M \to M$ is smooth. Such a thing is called a **flow** on M. A flow gives a vector field: For each point p let $v(p)$ be the tangent vector to the smooth curve $t \mapsto \alpha(t, p)$.

In the other direction, given a vector field v, a smooth curve $\gamma : (a, b) \to M$ is called an **integral curve** of v if, at points p along the curve, the tangent vector at p is $v(p)$. This means that for any smooth function f,

$$\frac{d}{dt}\Big|_{t=s} f \circ \gamma = v(\gamma(s))(f)$$

for $s \in (a, b)$, or, in different notation, that:

$$d\gamma_s\Big(\frac{\partial}{\partial t}\Big) = v(\gamma(s)).$$

We shall now see that an integral curve of a vector field through a given point p always exists, at least locally, and it is unique if we require $\gamma(0) = p$. This will be a consequence of the existence-uniqueness theorem for ordinary differential equations.

Since the question is local, we may assume using a coordinate chart that we are in an open subset O of $\mathbb{R}^n$. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a smooth curve in our open subset with $\gamma(0) = q \in O$. The condition that γ is an integral curve of v is that

$$d\gamma_s\Big(\frac{\partial}{\partial t}\Big) = v(\gamma(s))$$

holds for s in a neighbourhood of 0. In the local coordinates $x_1, \ldots, x_n$ this is

$$\sum_i \frac{d\gamma_i}{dt}\Big|_{t=s} \frac{\partial}{\partial x_i} = \sum_i a_i(\gamma_1(s), \ldots, \gamma_n(s)) \frac{\partial}{\partial x_i}$$

which (dropping the dummy variable s), is the same as

$$\sum_i \frac{d\gamma_i}{dt} \frac{\partial}{\partial x_i} = \sum_i a_i(\gamma_1(t), \ldots, \gamma_n(t)) \frac{\partial}{\partial x_i}$$

with the initial conditions $\gamma_i(0) = q_i$. (Here q_i are the local coordinates of q.) Solving this equation is equivalent to solving the following system of n ordinary differential equations with initial conditions:

$$\frac{d\gamma_i}{dt} = a_i(\gamma_1(t), \ldots, \gamma_n(t))$$
$$\gamma_i(0) = q_i.$$

Since the a_i are smooth functions, the existence-uniqueness theorem for o.d.e. (Cauchy-Kovalevskaya therem) says that:

(i) there exists a relatively compact neighbourhood U of p and an ϵ (depending on U) such that for any $q \in U$, the system of differential equations with initial condition has a unique solution $\gamma_q(t)$ which is defined for $t \in (-\epsilon, \epsilon)$
(ii) the solution is smooth and depends smoothly on $q \in U$, i.e. the mapping

$$\alpha : (-\epsilon, \epsilon) \times U \to \mathbb{R}^n \qquad \alpha(t, q) = \gamma_q(t)$$

("follow the integral curve starting at q for a time t") is smooth.

(For a discussion and proof of the existence-uniqueness theorem see Arnol'd's book.) Let us see that α satisfies

$$\alpha(t, \alpha(s, p)) = \alpha(s + t, p)$$

whenever both sides make sense. This equation is the same as $\gamma_{\gamma_p(s)}(t) = \gamma_p(s + t)$. Fixing s, we see that both these satisfy the same system of differential equations with the same initial condition, so by uniqueness they are the same.

The map α defined here is not a flow, because $\alpha(t, p)$ is only defined for small t using the integral curve. Such a thing is called a "local flow".

A **local flow** (Φ, α) on a manifold M is a pair consisting of an open neighbourhood Φ of $\{0\} \times M$ in $\mathbb{R} \times M$ and a map $\alpha : \Phi \to M$ such that:

(i) $\Phi \cap \mathbb{R} \times \{p\}$ is an interval containing 0
(ii) $\alpha(0, x) = x$ for all $x \in M$
(iii) $\alpha(s, \alpha(t, x)) = \alpha(s + t, x)$ whenever both sides make sense.

(The open set Φ will usually be dropped from the notation.)

The earlier discussion shows that a vector field gives rise to a local flow and vice-versa. This evidently sets up a bijection

$$\{\,\text{vector fields}\,\} \quad\longleftrightarrow\quad \{\,\text{local flows}\,\}\,/\sim \qquad \text{where } (\alpha,\Phi)\sim(\alpha',\Phi') \text{ if } \alpha|_{\Phi\cap\Phi'} = \alpha'|_{\Phi\cap\Phi'}.$$

The following property of a local flow (α, Φ) associated with the vector field v is key:

$$d\alpha_{(t,x)}\left(\frac{\partial}{\partial t}\right) = v(\alpha(t,x)) \qquad (\text{if } (t,x)\in\Phi). \tag{3.2.1}$$

In terms of the action on smooth functions one has:

$$v(f)(x) = \frac{d}{dt}\Big|_{t=0}\alpha_t^*(f)(x) = \frac{d}{dt}\Big|_{t=0} f(\alpha_t(x)) \tag{3.2.2}$$

The right-hand side has the following meaning: $(t, x) \mapsto f(\alpha_t(x))$ is a smooth function of t and x, so we can differentiate in t and set $t = 0$. We will also write this as:

$$v(f) = \frac{d}{dt}\Big|_{t=0}\alpha_t^*(f). \tag{3.2.3}$$

(This expression could have a different meaning, namely consider $t \mapsto \alpha_t^*(f)$ as a function from $\mathbb{R}$ to $C^\infty(M)$ and take the derivative (in the usual limit definition) at 0. This requires putting a topology on $C^\infty(M)$, which can certainly be done (uniform convergence of all derivatives on compact sets), but we will avoid this and simply take (3.2.3) to mean (3.2.2) for all $x \in M$.)

Note that v is a vector field on M and if $V \subset M$ is relatively compact, then there is an ϵ such that $(-\epsilon, \epsilon)\times V$ is contained in the local flow of v, i.e. $\alpha : (-\epsilon, \epsilon)\times V \to M$ is defined (since any neighbourhood of the closure of V contains a product neighbourhood).

One cannot always get a (global) flow from a vector field: Take $M = \mathbb{R}, v(x) = x^2 \frac{\partial}{\partial x}$. The integral curve starting at $a \in \mathbb{R}$ is $\gamma(t) = \frac{a}{1-at}$. If $a < 1$ this is defined for $t \in (-\infty, a^{-1})$. In particular, it is not possible to choose ϵ such that $(-\epsilon, \epsilon) \times \mathbb{R}$ is contained in the local flow. In fact, there is no ϵ which will work for any unbounded interval.

Lemma 3.2.1. *If v is a vector field which vanishes outside a compact set, then its integral curves are defined for all t. In particular, if M is compact then every vector field v arises from a flow $\alpha : \mathbb{R}\times M \to M$.*

Proof. • Exercise. (Do this in two steps: First show that if M is compact there is an ϵ such that the local flow contains $(-\epsilon, \epsilon) \times M$. Next show that on any manifold M if a local flow contains $(-\epsilon, \epsilon) \times M$ then it extends (uniquely) to a flow $\mathbb{R} \times M \to M$.) □

3.3. **A useful lemma.** Suppose we have a vector field v on a one-dimensional manifold M. Let $p \in M$. The integral curve of v through p is a map $\gamma : (-\epsilon, \epsilon) \to M$ with the property that $d\gamma_t\left(\frac{\partial}{\partial t}\right) = v(\gamma(t))$. If we assume that $v(p) \neq 0$ then there is a neighbourhood of $0 \in (-\epsilon, \epsilon)$ on which γ is a diffeomorphism onto a neighbourhood of p, say $\gamma : (-\epsilon', \epsilon') \cong W$. Then in the coordinate t on W given by $\gamma^{-1} : W \to \mathbb{R}$, the vector field v becomes $d\gamma^{-1}_{\gamma(t)}\left(v(\gamma(t)\right) = \frac{\partial}{\partial t}$, i.e. $\gamma^*(v) = \frac{\partial}{\partial t}$.

Given a vector field on a general manifold, locally one can choose coordinates in which it is simple:

Lemma 3.3.1. *Let v be a vector field on M and $p \in M$ a point such that $v(p) \neq 0$. Then we can find a neighbourhood U of p and local coordinates $x_1, \dots, x_n$ on U such that $v = \frac{\partial}{\partial x_1}$ on U. (More precisely, there is a chart $(U, \varphi, U' \subset \mathbb{R}^n)$ with $p \in U$ such that $v = \varphi^*\left(\frac{\partial}{\partial x_1}\right)$ on U.)*

Proof. Let U be a neighbourhood of p in a chart on which the local flow is defined for $t \in (-\epsilon, \epsilon)$. Let N be any submanifold of dimension $n - 1$ in U such that $p \in N$ and $v(p) \notin T_pN$. (For example, choose any local coordinates in which $v(p) = \frac{\partial}{\partial x_1}$ and take $N = \{x_1 = 0\}$.) The flow map $\alpha : (\epsilon, \epsilon) \times N \to M$ is a diffeomorphism in some neighbourhood of $(0, p) \in (-\epsilon, \epsilon) \times N$ (by the inverse function theorem). Choose coordinates $y_2, \dots, y_n$ in N near p and set

$$x_i = y_i \circ (\text{projection to } N) \circ \alpha^{-1} \qquad (i \geq 2)$$
$$x_1 = (\text{projection to } (-\epsilon, \epsilon)) \circ \alpha^{-1}$$

on a neighbourhood on which α is invertible. In these coordinates the vector field v is simply $\frac{\partial}{\partial x_1}$, because of the fact (3.2.1) that

$$d\alpha_{(t,x)}\left(\frac{\partial}{\partial t} = \frac{\partial}{\partial x_1}\right) = v(\alpha(t,x))$$

for all t, x in some neighbourhood of $(0, p)$. □

This leads naturally to the next question: if we have k vector fields, all nonvanishing and linearly independent at p, when can we find local coordinates so that they are the $\frac{\partial}{\partial x_i}$ for $i = 1, \ldots, k$? (More precisely, we should ask, does there exist a chart (U, φ, U') containing p such that $v_i = \varphi^* \left(\frac{\partial}{\partial x_i}\right)$ on U.) The answer is given by the Lie bracket of vector fields and Frobenius' theorem.

3.4. **Lie bracket.** Let v and w be vector fields on M. The **Lie bracket** of v and w is $[v, w]$ defined by its action on smooth functions:

$$[v, w](f) := v(w(f)) - w(v(f)).$$

Defined in this way it is not obvious that this is a vector field. We claim that this is actually a derivation (linearity is clear, one must check the Leibniz rule) on germs of smooth functions at p, and that it varies smoothly with p, so that it defines a vector field. To check these statements it will suffice to check in local coordinates that it can be written as $\sum_i c_i \frac{\partial}{\partial x_i}$ for smooth functions $c_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$. Fix a point of M and fix local coordinates $x_1, \ldots, x_n$ in a neighbourhood of the point. Write the vector fields as $v = \sum_i a_i \frac{\partial}{\partial x_i}$ and $w = \sum_i b_i \frac{\partial}{\partial x_i}$. Then

$$(vw - wv) = \sum_i c_i \frac{\partial}{\partial x_i} \quad \text{for} \quad c_i = \sum_j \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right).$$

Remark 3.4.1. On a C^k (or better) manifold one has the notion of C^k vector field. The bracket of two C^k vector fields is only C^{k-1}, as the previous local formula suggests.

If the manifold is real analytic then there is a notion of real analytic vector field, and the bracket of two real analytic vector fields is again real analytic.

On a complex manifold there is a notion of holomorphic vector field, and the bracket of holomorphic vector fields is holomorphic.

The bracket has the following properties, which follow from the definition:

(i) $[v, v] = 0$ (antisymmetry)
(ii) $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$ (Jacobi identity)

These two conditions mean that the space of vector fields on M with the Lie bracket is a Lie algebra over $\mathbb{R}$ (of infinite dimension if $\dim M > 0$). ([11]) We also have:

(iii) for functions f, g and vector fields v, w, we have

$$[fv, gw] = fg\,[v, w] + fv(g)\,w - gw(f)\,v$$

There is also the following very important naturality property:

(iv) If $\varphi : M \to M$ is a diffeomorphism then

$$\varphi_*[v, w] = [\varphi_* v, \varphi_* w]$$
$$\varphi^*[v, w] = [\varphi^* v, \varphi^* w]$$

This is easy from the fact that $\varphi_*(u)(f) = {\varphi^{-1}}^*(u(\varphi^* f))$. Indeed if $u = [v, w]$ we have

$$\varphi_*([v, w])(f) = {\varphi^{-1}}^*([v, w](\varphi^* f)) = {\varphi^{-1}}^*(vw(\varphi^* f) - wv(\varphi^* f)).$$

Now

$$\varphi_* v((\varphi_* w)(f)) = {\varphi^{-1}}^*(v(\varphi^*((\varphi_* w)(f))) = {\varphi^{-1}}^*(v(\varphi^*({\varphi^{-1}}^* w(\varphi^* f)))) = {\varphi^{-1}}^*(vw(\varphi^* f))$$

(the last equality comes from ${\varphi^{-1}}^* \varphi^* f = f$), so that we have $\varphi_*[v, w](f) = [\varphi_* v, \varphi_* w](f)$ for all smooth functions. The case of φ^* follows since $\varphi^* \varphi_* = id$.

More generally, if $f : M \to N$ is a smooth map then we say vector fields v on M and v' on N are ***f*-related** if we have $df_p(v(p)) = v'(f(p))$ for all $p \in M$. The same proofs show that

(v) If $f : M \to N$ is a smooth mapping and v, w are vector fields on M which are f-related to vector fields v', w' on N respectively, then $[v, w]$ is f-related to $[v', w']$.

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[11] By definition, a **Lie algebra** over a field k is a k-vector space V with a k-bilinear operation $[\ ,\ ] : V \times V \to V$ which is antisymmetric and satisfies the Jacobi identity.

3.5. **Local flows and brackets.** Let (α, Φ) be a local flow on M. For $p \in M$ let U be a relatively compact neighbourhood of p and let $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times U \subset \Phi$, i.e. $\alpha(t, x)$ makes sense for $t \in (-\epsilon, \epsilon)$ and $x \in U$. For such t, $\alpha_t(-) := \alpha(t, -)$ is a diffeomorphism of U onto an open set in M (indeed, it has a smooth inverse, namely α_{-t}). So for $|t| < \epsilon$, the pullback $\alpha_t^*(v)$ and the pushforward $(\alpha_t)_*(v)$ make sense as vector fields on neighbourhoods of p. Writing things out in local coordinates shows that these depend smoothly on t. We also have

$$\alpha_t^*(v) = (\alpha_{-t})_*(v)$$

(from the definitions).

A more geometric picture of the bracket can be given using the local flow of one of the vector fields. Let α_t be the local flow associated with v. Then we have the alternate description:

$$[v, w] = \left.\frac{d}{dt}\right|_{t=0} \alpha_t^*(w) = -\left.\frac{d}{dt}\right|_{t=0} (\alpha_t)_*(w).$$

(If v_t is a vector field depending smoothly on a parameter t, then $\frac{dv_t}{dt}$ is the vector field which acts on smooth functions by $\frac{dv_t}{dt}(f) = \frac{d}{dt}(v_t(f))$. That this is actually a vector field can be checked by a local calculation.) Let us prove this identity. It is enough to do this in local coordinates centred at p. By the last lemma, we may assume that coordinates have been chosen so that $v = \frac{\partial}{\partial x_1}$, so $\frac{\partial a_j}{\partial x_i} = 0$ for all i, j. Then the definition of the bracket gives

$$(vw - wv)(p) = \sum_i \frac{\partial b_i}{\partial x_1} \frac{\partial}{\partial x_i}.$$

On the other hand, the integral curve of v through $(p_1, \ldots, p_n)$ is simply $t \mapsto (p_1 + t, p_2, \ldots, p_n)$, i.e. $x_1(t) = p_1 + t, x_i(t) = p_i$ for $i \neq 1$. The local flow is given by $\alpha_t(x_1, x_2, \ldots, x_n) = (x_1 + t, x_2, \ldots, x_n)$. Write $w = \sum_i b_i \frac{\partial}{\partial x_i}$. Using the fact that $\alpha_t^*\left(b_i \frac{\partial}{\partial x_i}\right) = \alpha_t^*(b_i) \frac{\partial}{\partial x_i}$ (• Exercise) we get:

$$\alpha_t^*(w) = \sum_i \alpha_t^*\left(b_i \frac{\partial}{\partial x_i}\right) = \sum_i b_i(x_1 + t, x_2, \ldots, x_n) \frac{\partial}{\partial x_i}.$$

So then $[v, w]$ is

$$[v, w] := \left.\frac{d}{dt}\right|_{t=0} \alpha_t^*(w) = \sum_i \left.\frac{\partial b_i}{\partial x_1}(x_1 + t, x_2, \ldots, x_n)\right|_{t=0} \frac{\partial}{\partial x_i} = \sum_i \frac{\partial b_i}{\partial x_1} \frac{\partial}{\partial x_i}$$

and we have justified the alternate description.

There is a nicer formula for the bracket in terms of both the associated local flows:

Lemma 3.5.1. *Let v, w be vector fields with local flows α_t, β_s respectively. Then*

$$[v, w] = \left.\frac{\partial^2}{\partial s\, \partial t}\right|_{s=t=0} \alpha_t^*\, \beta_s^* - \beta_s^*\, \alpha_t^*$$

(as operators on smooth functions).

Proof. By definition,

$$(\beta_s^*\, v(f))(x) = v(f)(\beta_s(x)) = \left.\frac{\partial}{\partial t}\right|_{t=0} (\alpha_t^* f)(\beta_s(x)) = \left.\frac{\partial}{\partial t}\right|_{t=0} f(\alpha_t \beta_s(x)).$$

Taking $\left.\frac{\partial}{\partial s}\right|_{s=0}$ and using $w(g)(x) = \left.\frac{\partial}{\partial s}\right|_{s=0} (\beta_s^* g)(x)$ for $g = v(f)$ gives

$$w(v(f))(x) = \left.\frac{\partial}{\partial s}\right|_{s=0} \left.\frac{\partial}{\partial t}\right|_{t=0} f(\alpha_t \beta_s(x)) = \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} (\beta_s^* \alpha_t^* f)(x).$$

This implies the lemma. □

Let α and β be two local flows on M. Let $U \subset M$. We say that α and β **commute** on U, if, for $p \in U$,

$$\alpha(t, \beta(s, p)) = \beta(s, \alpha(t, p)) \quad \text{(equivalently, } \alpha_t \circ \beta_s = \beta_s \circ \alpha_t)$$

whenever s, t are small enough so that both sides make sense. Note that if this equation holds for s, t in some neighbourhood of $(0, 0)$, then it holds for all s, t where it makes sense.

The previous lemma shows that if local flows of v and w commute then $[v, w] = 0$. The converse is also true, i.e. we will show that if $[v, w] = 0$ then the local flows of v and w commute.

Let $\varphi : V \to M$ be a diffeomorphism onto a relatively compact $V' \subset M$. If w is a vector field on M, we say that w **is invariant under** φ if $\varphi_*(w) = w$ (or, equivalently, if $\varphi^*(w) = w$). Concretely, this means $d\varphi_a(v(a)) = v(\varphi(a))$ for all $a \in W$.

Let v, w be vector fields, with (α, Φ) a local flow of v. We say that w **is invariant under** α if, for any $(-\epsilon, \epsilon) \times V \subset \Phi$ we have $\alpha_{t*}(w) = w$ for $|t| < \epsilon$ (equivalently, $\alpha_t^*(w) = w$ for $|t| < \epsilon$).

Lemma 3.5.2. *Let v and w be vector fields with local flows α and β.*

(i) $\alpha_t^*(v) = v$, *i.e. a vector field is invariant under its own local flow.*
(ii) *w is invariant under α if and only if α and β commute.*
(iii) *For t in some interval around 0 we have:*

$$\frac{d}{dt}\,\alpha_t^*(w) = [v, \alpha_t^*(w)].$$

(At $t = 0$ this recovers the definition of $[v, w]$.)

Proof. (i) If we choose local coordinates in which $v = \frac{\partial}{\partial x_1}$ then $\alpha_t(x_1, \ldots, x_n) = (x_1 + t, x_2, \ldots, x_n)$ and this is immediate.

(ii) We show that if $\varphi : V \to M$ is a diffeomorphism onto a relatively compact open subset of M then $\varphi_*(w) = w$ if and only if $\varphi \circ \beta_t = \beta_t \circ \varphi$ for t in an interval around zero. (Applying this to $\varphi = \alpha_s$ for small enough s gives (ii).) To see this note that $\varphi \circ \beta_t \circ \varphi^{-1}$ is again a local flow on $V' = \varphi(V)$. For a smooth function f on V' we compute:

$$\begin{aligned}
\varphi_*(w)(f)(x) &= {\varphi^{-1}}^*(w(\varphi^* f))(x) \\
&= w(f \circ \varphi)(\varphi^{-1}(x)) \\
&= \frac{d}{dt}\Big|_{t=0} \beta_t^*(f \circ \varphi)(\varphi^{-1}(x)) \\
&= \frac{d}{dt}\Big|_{t=0} (f \circ \varphi \circ \beta_t \circ \varphi^{-1})(x)
\end{aligned}$$

where in the third equality we have used (3.2.3). By (3.2.3) again, $\varphi \circ \beta_t \circ \varphi^{-1}$ induces the vector field $\varphi_*(v)$ on V'. So if $\varphi_*(v) = v$ then the flows $\varphi \circ \beta_t \circ \varphi^{-1}$ and β_t must agree for t small enough. This proves (ii).

(iii) This is a local statement, so we will check it in a coordinate chart with local coordinates $x_1, \ldots, x_n$ in which $v = \frac{\partial}{\partial x_1}$. Write $w = \sum_i b_i \frac{\partial}{\partial x_i}$. Then

$$\alpha_t^*(w) = \sum_i \alpha_t^* \left(b_i \frac{\partial}{\partial x_i} \right) = \sum_i b_i(x_1 + t, x_2, \ldots, x_n) \frac{\partial}{\partial x_i}.$$

Then

$$\frac{d}{dt}\,\alpha_t^*(w) = \sum_i \frac{\partial b_i}{\partial x_1}\Big|_{(x_1+t, x_2, \ldots, x_n)} \frac{\partial}{\partial x_i} = [v, \alpha_t^*(w)]$$

by the local calculation of the bracket. □

Proposition 3.5.3. *Let v, w be vector fields on M with associated local flows α and β. Then $[v, w] = 0$ in a neighbourhood U of p if and only if the local flows commute on U.*

Proof. ($\Leftarrow$) This implication follows from the formula $[v, w] = \frac{\partial^2}{\partial s\, \partial t}\big|_{s=0,t=0} \alpha_t^* \beta_s^* - \beta_s^* \alpha_t^*$ given above, but let us prove it using the previous lemma instead. If α and β commute then by (i) w is invariant under α, i.e. $\alpha_t^*(w) = w$ for $|t|$ small enough. By (iii), for small enough $|t|$ we have:

$$0 = \frac{d}{dt}\,\alpha_t^*(w) = [v, \alpha_t^*(w)] = [v, w].$$

($\Rightarrow$) By (i) and (iii) of the previous lemma and properties of the bracket, for $|t|$ small enough we have:

$$\frac{d}{dt}\,\alpha_t^*(w) = [v, \alpha_t^*(w)] = [\alpha_t^*(v), \alpha_t^*(w)] = \alpha_t^*[v, w].$$

So if $[v, w] = 0$ then $\alpha_t^*(w)$ is constant in t (for t near zero), hence is equal to w (since $\alpha_0 = id$), i.e. w is invariant under α. By (ii) above α and β must commute. □

3.6. **Local Frobenius theorem.** Let M be a manifold. A (smooth) **subbundle** of rank k of the tangent bundle TM is a union $E = \cup_{p\in M} E_p$, where $E_p \subset T_pM$ is a k-dimensional subspace which "varies smoothly in p": More precisely, we require that for any $p \in M$ there is an open set $U \subset M$ containing p and k vector fields $v_1, \ldots, v_k$ such that $v_1(q), \ldots, v_k(q)$ form a basis of E_q for all $q \in U$. ([12]) It is easy to check that E is itself a smooth manifold of dimension $n+k$, in fact an embedded submanifold of the tangent bundle.

An example is the rank one subbundle $\cup_{p\in M} \mathbb{R}v(p)$ where v is a smooth nonvanishing vector field on M. More generally we could take the subbundle spanned by k vector fields, provided they are linearly independent everywhere. (Not every subbundle is of this form.)

A k-dimensional smooth subbundle $E \subset TM$ of the tangent bundle is called **integrable** (or **involutive**) on an open set $U \subset M$ if, for any vector fields v, w such that $v(p), w(p) \in E_p$ for all $p \in U$, the bracket has the same property, i.e. $[v, w](p) \in E_p$ for all $p \in U$. We say that $E \subset TM$ is integrable (or involutive) if each point has a neighbourhood on which E is integrable.

• Prove (using a partition of unity) that E is integrable if it is integrable on $U = M$. (This is false in the real analytic or complex analytic setting.)

Let $E \subset TM$ be a subbundle. A submanifold $i : I \to M$ is called an **integral manifold** of E if

$$di_p(T_pI) = E_{i(p)} \quad \text{for all } p \in I.$$

Sometimes we will suppress the notation di_p and simply identify T_pI with its image in $T_{i(p)}M$; we will also suppress the inclusion i and think of points of I as points of M. So we will write the above condition simply as: $T_pI = E_p \quad$ for all $p \in I$.

Remark 3.6.1. Note that an integral manifold is not necessarily a closed or embedded submanifold. Here is an example: Fix $\alpha \in \mathbb{R} - \mathbb{Q}$. The integral manifolds (=integral curves=orbits) of the vector field associated with the flow on $S^1 \times S^1$ defined by $\alpha(t, (z_1, z_2)) = (e^{it}z_1, e^{i\alpha t}z_2)$ are dense in $S^1 \times S^1$, hence are not embedded submanifolds.

Theorem 3.6.2 (Local Frobenius theorem)**.** *A subbundle $E \subset TM$ of the tangent bundle of a manifold M is integrable if and only if it admits an integral manifold through any point of M.*

This will be a consequence of two lemmas.

Lemma 3.6.3. *Any integrable subbundle $E \subset TM$ can be locally spanned by k commuting vector fields, i.e. given p, there is a neighbourhood U of p and there exist vector fields $w_1, \ldots, w_k$ on U such that*

(1) *E_x is the span of $w_1(x), \ldots, w_k(x)$ for all $x \in U$ and*
(2) *$[w_i, w_j](x) = 0$ for all i, j and $x \in U$.*

Proof. Let U be a neighbourhood of p with local coordinates $x_1, \ldots, x_n$ and let $v_1, \ldots, v_n$ be k vector fields which span E_x at each point of U. Write $v_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j}$ for smooth functions $a_{ij} = a_{ij}(x_1, \ldots, x_n)$. The matrix $A = (a_{ij})$ has rank k near p, so some $k \times k$ submatrix of it is nonsingular. By renumbering we may assume that $A_0 = (a_{ij})_{i=1,\ldots,k, j=1,\ldots,k}$ is the nonsingular submatrix, and by shrinking U we may assume that A_0 is nonsingular on all of U. Define

$$\begin{pmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_k \end{pmatrix} = A_0^{-1} \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_k \end{pmatrix} = A_0^{-1} A \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial}{\partial x_n} \end{pmatrix}.$$

Then $w_1, \ldots, w_k$ are still a basis for E on U (since they are related to the basis $v_1, \ldots, v_k$ by the nonsingular matrix A_0). Since A is of the form $(A_0|*)$, it follows that $A_0^{-1}A$ is of the form $(I|*)$ and so we have

$$w_i = \frac{\partial}{\partial x_i} + \sum_{\ell > k} c_{i\ell} \frac{\partial}{\partial x_\ell}.$$

Now $\left[\frac{\partial}{\partial x_i}, f\frac{\partial}{\partial x_\ell}\right] = \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_\ell}$, so $[w_i, w_j]$ involves only the $\frac{\partial}{\partial x_\ell}$ for $\ell > k$. As E is integrable this must be a linear combination (with smooth function coefficients) of the w_i, so it must vanish.. So we have found vector fields satisfying (1) and (2) as in the lemma. □

[12]In older books, the assignment $p \mapsto E_p$ is called a smooth k-**distribution** on M. Since distribution has a well-established meaning in analysis it's best to avoid this terminology.

The next idea is that the local flows of these commuting vector fields, which commute among themselves, provide coordinates.

Lemma 3.6.4. *Let $v_1, \dots, v_k$ be linearly independent vector fields on $V \subset \mathbb{R}^n$ satisfying $[v_i, v_j] = 0$ for all i, j. For any $p \in V$, there exists a neighbourhood U of p and coordinates $x_1, \dots, x_n$ on U such that $v_i = \frac{\partial}{\partial x_i}$ for $1 \le i \le k$ (on U).*

Proof. Choose a chart W containing p and local coordinates $z_1, \dots, z_n$ centred at p. Using a linear change of coordinates, we can arrange that

$$v_1(p), v_2(p), \dots, v_k(p), \frac{\partial}{\partial z_{k+1}}, \dots, \frac{\partial}{\partial z_n} \text{ are linearly independent at } p. \tag{3.6.1}$$

Let $\alpha_i : (-\epsilon, \epsilon) \times W \to \mathbb{R}^n$ be the local flows associated with the v_i (we shrink W so that one ϵ works for all α_i).

Let B_δ denote the open ball of radius $\delta < \epsilon$ in $\mathbb{R}^n$. Define a map

$$h : B_\delta \to \mathbb{R}^n$$

by

$$h(t_1, \dots, t_k; y_{k+1}, \dots, y_n) = \alpha_1(t_1, \alpha_2(t_2, \dots, \alpha_k(t_k, (0, \dots, 0, y_{k+1}, \dots, y_n)) \dots)).$$

We will show that this is a coordinate system at p for sufficiently small δ. For this it suffices (by the inverse function theorem) to show that the differential $dh_0 : T_0 B_\delta \to T_p V$ has rank n. By definition, since α_1 is the flow associated with v_1, we have (by (3.2.1)):

$$\begin{aligned} dh_{(t_1,\dots,t_k;y_{k+1},\dots,y_n)}\left(\frac{\partial}{\partial t_1}\right) &= (d\alpha_1)_{(t_1, h(0,t_2,\dots,t_k;y_{k+1},\dots,y_n))}\left(\frac{\partial}{\partial t_1}\right) \\ &= v_1(\alpha_1(t_1, h(0, t_2, \dots, t_k; y_{k+1}, \dots, y_n)) \\ &= v_1(\alpha_1(t_1, \alpha_2(t_2, \dots, \alpha_k(t_k, (0, \dots, 0, y_{k+1}, \dots, y_n)) \dots)) \end{aligned}$$

or

$$dh_{(t_1,\dots,t_k;y_{k+1},\dots,y_n)}\left(\frac{\partial}{\partial t_1}\right) = v_1(h(t_1, \dots, t_k; y_{k+1}, \dots, y_n)).$$

Since the vector fields v_i commute among themselves, the flows α_i also do so, so we could change the order of the α_is in the definition of h and get that for any $j \le k$:

$$dh_{(t_1,\dots,t_k;y_{k+1},\dots,y_n)}\left(\frac{\partial}{\partial t_j}\right) = v_j(\alpha_1(t_1, \alpha_2(t_2, \dots, \alpha_k(t_k, (0, \dots, 0, y_{k+1}, \dots, y_n)) \dots))$$

i.e.

$$dh_{(t_1,\dots,t_k;y_{k+1},\dots,y_n)}\left(\frac{\partial}{\partial t_j}\right) = v_j(h(t_1, \dots, t_k; y_{k+1}, \dots, y_n)) \qquad \text{for } 1 \le j \le k. \tag{3.6.2}$$

Putting $(t_1, \dots, t_k; y_{k+1}, \dots, y_n) = (0, \dots, 0) = 0$ we have

$$dh_0\left(\frac{\partial}{\partial t_j}\right) = v_j(h(0, \dots, 0)) = v_j(p) \qquad \text{for } j \le k.$$

We also have

$$dh_0\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial z_j} \quad \text{for } \ j > k$$

(since $h(0, \dots, 0; y_{k+1}, \dots, y_n) = (0, \dots, 0, y_{k+1}, \dots, y_n)$). By (3.6.1), dh_0 has rank n and hence h is diffeomorphism onto $h(B_\delta)$ for small enough δ by the inverse function theorem.

Now let $U = h(B_\delta)$ and let us check that in the coordinates $t_1, \dots, t_k, y_{k+1}, \dots, y_n$ given by $h^{-1} : U \to B_\delta$, the vector fields v_j are as we want. But this is precisely the equation (3.6.2). This proves the proposition. ☐

Putting these together we prove the local Frobenius theorem.

Proof. Suppose first that $E \subset TM$ admits integral manifolds through any point. We must show that it is integrable. Let U be an open set and v, w vector fields with $v(p), w(p) \in E_p$ for all $p \in U$. Let $p \in U$ and let $i : I \to M$ be an integral manifold through p. Now there are vector fields $\tilde{v}$ and $\tilde{w}$ on I that are i-related to v and w. Indeed, they are defined by

$$di_x(\tilde{v}(x)) = v(i(x)) \quad (x \in I)$$

and similarly for $\tilde{w}$. It is easy to see that they are smooth vector fields on I e.g. using the normal form for immersions. Now the fact that the bracket of i-related vector fields is i-related shows that

$$di_x([\tilde{v}, \tilde{w}](x) = [v, w](x) \quad (x \in I).$$

In particular for x such that $i(x) = p$ we get that $[v, w](p) \in di_x(T_x I) = E_p$. Thus E is integrable.

Conversely, suppose $E \subset TM$ is integrable and $p \in M$. There is a neighbourhood U of p on which E is spanned by k commuting vector fields (by the first lemma above). By the second lemma, we may assume that there are local coordinates on U centred at p in which E is the span of the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$. Clearly for small enough $c = (c_{k+1}, \ldots, c_n)$ the "slices"

$$U_c := \{(x_1, \ldots, x_n) \in U : x_{k+1} = c_{k+1}, \ldots, x_n = c_n\}$$

are integral manifolds of E, and the choice $c = (0, \ldots, 0)$ gives an integral manifold through p. □

(This proof of the local Frobenius theorem is from R. Narasimhan's book *Analysis on Real and Complex Manifolds* (or see his TIFR lectures *Topics in Analysis*). For a slightly different proof see Warner's book.)

3.7. **Examples.** (i) Consider the subbundle in $\mathbb{R}^3 - \{0\}$ given by

$$E_{x=(x_1,x_2,x_3)} = \{v \in T_x\mathbb{R}^3 : v \perp x\}.$$

(Here we are using identifications $T_x(\mathbb{R}^3 - \{0\}) = T_x\mathbb{R}^3 = \mathbb{R}^3$.) This subbundle is integrable and the integral manifolds are spheres centred at the origin.

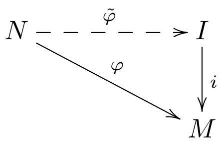
(ii) Consider the subbundle in $\mathbb{R}^3 - \{0\}$ given by

$$E_{x=(x_1,x_2,x_3)} = \{v \in T_x\mathbb{R}^3 : v \perp (x_2, x_3, x_1)\}.$$

Check that this is not integrable. This is an example of a **completely nonintegrable hyperplane field** or **contact structure**: A subbundle of rank $n-1$ such that for every pair of vector fields $v, w \in F$ which are nonzero and linearly independent near a point x, the bracket $[v, w]$ does not lie in F. This has a nice consequence: starting from any point p in $\mathbb{R}^3 - \{0\}$, you can get to any other point by the following process: choose $v_1 \in F$ and follow its integral curve for a time t_1 to get to $q \in \mathbb{R}^3 - \{0\}$. Then choose $v_2 \in F$ in a neighbourhood of q and follow its integral curve for a time t_2. Choose v_3 ...

3.8. **Integral manifolds and the global Frobenius theorem.** Integral manifolds of integrable subbundles $E \subset TM$ need not be embedded but they have the following "factoring" property which holds if $i : I \hookrightarrow M$ is embedded:

Theorem 3.8.1. *Suppose that $i : I \hookrightarrow M$ is an integral manifold of an integrable subbundle $E \subset TM$ and suppose that $\varphi : N \to M$ is a smooth mapping such that $\varphi(N) \subset i(I)$. Then there is a smooth mapping $\tilde{\varphi} : N \to I$ such that $i \circ \tilde{\varphi} = \varphi$:*



Proof. The mapping $\tilde{\varphi}$ is already given as a set map, since i is injective, and it is given by $\tilde{\varphi}(n) = i^{-1}(\varphi(n))$ for $n \in N$. So we have only to prove that it is smooth. The main point is actually to prove continuity.

Let $p \in I$ and let U be a neighbourhood of p in I. For $q \in N$ with $\tilde{\varphi}(q) = p$, i.e. with $\varphi(q) = i(p)$ we must find a neighbourhood W of q such that $\tilde{\varphi}(W) \subset U$, i.e. $\varphi(W) \subset i(U)$. Choose a neighbourhood V of $i(p)$ in M such that $i(U) \subset V$ and V has local coordinates $x_1, \ldots, x_{m=\dim M}$ such that the slices

$$V_c = \{x_{k+1} = c_{k+1}, \ldots, x_m = c_m\} \qquad (\text{for } c = (c_{k+1}, \ldots, c_m) \in (-\epsilon, \epsilon)^{m-k})$$

are integral manifolds of E. We further assume the coordinates are centred at $i(p)$, so that $i(U) \cap V$ is contained in the slice $V_0 = \{x_{k+1} = \cdots = x_m = 0\}$ of V. By possibly shrinking V further we can arrange that $i(U) \cap V = V_0$.

Let us remark that each connected component of $i^{-1}(V)$ is mapped by i into in a single slice of V. Indeed, this is because the functions $\bar{x}_i = x_i \circ i$ for $i \geq k+1$ are locally constant on $i^{-1}(V)$. (If $i > k$ then $d\bar{x}_i|_{T_p I} = 0$ for $j \leq k$ and $p \in i^{-1}(V)$ because $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ span $di_p(T_p I)$ and $dx_j\left(\frac{\partial}{\partial x_i}\right) = 0$ if $j > k$ and $i \leq k$).

Now we remark that the connected components of $i^{-1}(V)$ map (under i) onto connected components of $i(I) \cap V$ in the topology of V (i.e. of M). To see this note that since I is second countable the set $i^{-1}(V)$ has countably many connected components, each mapping into a single slice, hence $i(I) \cap V$ maps under the projection $V \to \mathbb{R}^{m-k}$ to a countable collection of points. The connected components of a countable set in $\mathbb{R}^n$ are points, hence the V-connected components of $i(I) \cap V$ are contained in slices. Now on a single slice the two topologies agree because the slices are embedded submanifolds, so the assertion follows.

Let $q \in \varphi^{-1}(i(p))$. Let

$$W = \text{ connected component of } \varphi^{-1}(V) \text{ containing } q$$

(this is an open neighbourhood of q in N). To prove continuity of $\tilde{\varphi}$ it will suffice to show that $\tilde{\varphi}(W) \subset U$, i.e. that $\varphi(W) \subset i(U)$. By continuity of φ, $\varphi(W)$ is a connected set containing $i(p)$, so to prove that $\varphi(W) \subset i(U)$ it will suffice to show that $\varphi(W)$ is contained in the single slice $V_0 = \{x_{k+1} = \cdots = x_m = 0\}$. But $\varphi(W) \subset i(I) \cap V$, and the components of $i(I) \cap V$ are contained in slices (as remarked earlier), so $\varphi(W)$ is contained in a single slice, which must be the slice V_0 since $i(p) \in \varphi(W)$. So $\varphi(W) \subset V_0 = i(U) \cap V$ and hence $\tilde{\varphi}(W) \subset U$.

The smoothness of $\tilde{\varphi}$ is left as an exercise. □

Note that the integrability of the subbundle is used essentially in the proof to have information on all the integral manifolds in the neighbourhood V, not just on I. Also, the second countability of I played a key role in the proof.

The following example is useful to think about: You can construct a 1-1 immersion $i : \mathbb{R} \to \mathbb{R}^2$ with image the figure eight in the plane and an immersion $\varphi : (0, 1) \to \mathbb{R}^2$ with image inside the figure eight but which does not factor through i even continuously.

Integral manifolds of an integral subbundle through a given point are obviously not unique: an open subset of an integral manifold is again an integral manifold. Taking the maximal integral manifold through a point fixes this:

Theorem 3.8.2 (Frobenius theorem). *Let E be an integrable subbundle of TM and let $p \in M$. There exists a unique maximal connected integral manifold of E containing p.*

Maximality means that every other connected integral manifold of E through p is contained in this one. Uniqueness means that if $i : I \hookrightarrow M$ and $j : J \hookrightarrow M$ are two maximal connected integral manifolds of E through p then there exists a diffeomorphism $\varphi : I \to J$ such that $i = j \circ \varphi$. It is easy to check that maximality implies uniqueness by the previous theorem (factorization property) so it is left as an exercise.

Proof. Let $\mathscr{I}$ be the collection of all subsets of M which are unions of integral manifolds of M. This defines a topology on M, which it is easy to check is finer (i.e. has more open sets) than the topology on M. Write $M_{\mathscr{I}}$ for M with this topology. Then $M_{\mathscr{I}}$ is Hausdorff and locally connected. Write

$$M_{\mathscr{I}} = \bigsqcup_{\alpha \in A} M_\alpha$$

as a union of connected components. Then any connected integral manifold $i : I \to M$ has $i(I) \subset M_\alpha$ for some α. We will show that the M_α are integral manifolds; this will prove the existence part of the theorem.

To prove that the M_α are second countable cover M by countably many neighbourhoods V with coordinates $x_1, \ldots, x_m$ such that slices V_c for $c = (c_{k+1}, \ldots, c_n)$ are integral manifolds of E. Since M_α and V_c are both open in the topology $\mathscr{I}$, we known that $M_\alpha \cap V_c$ is open in V_c for all $c = (c_{k+1}, \ldots, c_m)$. Now $M_\alpha \cap V$ is open in M_α and is the disjoint union of $M_\alpha \cap V_c$, all of which are open and closed in it, so each connected component of $M_\alpha \cap V$ is open in some V_c and hence is second countable. By the lemma below we conclude that M_α is second countable.

Now there is clearly a unique smooth manifold structure on M_α such that each integral manifold of E contained in M_α is an open submanifold. □

Let us prove the lemma used above:

Lemma 3.8.3. *Let X be a connected, locally connected Hausdorff space. Suppose that $X = \cup_{n \geq 1} X_n$ where $X_n \subset X$ are open and each connected component of X_n is second countable. Then X is second countable.*

Proof. Let $\mathscr{C}_n$ be the collection of connected components of X_n and let $\mathscr{C} = \cup_{n\geq 1}\mathscr{C}_n$. If $C \in \mathscr{C}$, then there are countably many $C' \in \mathscr{C}$ with $C \cap C' \neq \emptyset$. (If there were uncountably many such then there would exist an n for which $\{C' \in \mathscr{C}_n : C' \cap C \neq \emptyset\}$ is uncountable. But then $\{C \cap C' : C' \in \mathscr{C}_n, C \cap C' \neq \emptyset\}$ gives an uncountable collection of pairwise disjoint open sets in C. Since C is assumed 2nd countable this is impossible.)

Let $C_0 \in \mathscr{C}$ and set $J_0 = \{C_0\}$. Inductively define countable sets J_k by

$$J_k = \{C \in \mathscr{C} : C \text{ meets an element of } J_{k-1}\}.$$

Let $Y = \cup_{k\geq 0} \cup_{C \in J_k} C$. This a countable union of second countable open sets, hence second countable and open in X. If $x \in \overline{Y}$ there is a $C \in \mathscr{C}$ such that $x \in C$. Since $C \cap Y \neq \emptyset$ there is a k and $C' \in J_k$ such that $C \cap C' \neq \emptyset$, whence $C \in J_{k+1}$, and hence $x \in Y$. Thus Y is open and closed in X, so $Y = X$ is second countable. □

(The proof of the global Frobenius theorem here is originally from Chevalley's *Theory of Lie Groups*, as modified slightly in Varadarajan's *Lie Groups and Lie Algebras*. Warner has a (very) slightly different version.)

Here is a different description of the maximal connected integral manifold I through $p \in M$: A point $x \in M$ belongs to I if there exists a piecewise-smooth path joining p and x, each smooth piece of which is an integral curve of E (in the sense that the tangent vector belongs to E pointwise). (Exercise: Show that this is the same as the definition given above.)

Example 3.8.4. It is useful to keep in mind the example of the irrational slope flow on the torus $M = \mathbb{R}^2/\mathbb{Z}^2$. The corresponding vector field defines a one-dimensional integrable subbundle of the tangent bundle, and the integral manifolds are unions of segments. If we take a small square neighbourhood U of a point then the topology induced on U by $M_{\mathscr{I}}$ is homeomorphic to the following: $(0,1)^2$ with the topology that a set is open if its intersection with $(0,1) \times \{y_0\}$ is open for every y_0. In this topology $(0,1)^2$ has uncountably many connected components, each homeomorphic to $(0,1)$. So it satisfies all the conditions to be a manifold (in fact, an integral manifold) except 2nd countability. The second countability of maximal connected integral manifolds in the theorem tells us that the intersection $I \cap U$ must have countably many connected components.

Belonging to the same maximal connected integral manifold is an equivalence relation on M. The decomposition of M into equivalence classes is called a **foliation** and the maximal integral manifolds are the **leaves** of the foliation.

Finally, note that the local and global Frobenius theorems hold also in the real analytic and the complex analytic settings, with the same proofs. (All that we really used was the inverse function theorem, which holds in those settings.)

4. Basics of Lie groups

We will use the Frobenius theorems to establish the basic correspondence between Lie groups and Lie algebras, which is the starting point of Lie theory.

4.1. **Lie groups, Lie algebras.** A **Lie group** is a smooth manifold G with a group structure in which the multiplication map $\mu : G \times G \to G$ by $\mu(g,h) = gh$ and the inversion map $\iota : G \to G$ by $\iota(g) = g^{-1}$ are smooth maps. (Obviously it is enough to require that $\alpha(g,h) = gh^{-1}$ is smooth.)

Some examples: $\mathbb{R}, \mathbb{R}^*, \mathbb{C}^*, \mathbb{C}, S^1, GL(n,\mathbb{R})$, the group of affine transformations $x \mapsto Ax + b$ of $\mathbb{R}^n$.

• If the multiplication $\mu : G \times G \to G$ is smooth then the inversion $\iota : G \to G$ is smooth.

If G, H are Lie groups a **Lie group homomorphism** is a smooth map $\varphi : H \to G$ which is also a group homomorphism. Lie groups with homomorphisms form a category.

A **Lie algebra** over a field k is a k-vector space $\mathfrak{g}$ with a bilinear operation $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is antisymmetric:

$$[x,x] = 0 \qquad \text{for all } x \in \mathfrak{g}$$

and satisfies the Jacobi identity:

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0 \qquad \text{for all } x,y,z \in \mathfrak{g}.$$

We will mainly take $k = \mathbb{R}$ or $k = \mathbb{C}$ here.

Examples 4.1.1. Some examples:

$\mathbb{R}^n$ with the trivial bracket ($[x,y] = 0$ for all x,y). (More generally, any vector space with the trivial bracket.)

$M(n,\mathbb{R})$ with the bracket $[A,B] = AB - BA$. (More generally, if A is an associative algebra over k we can define $[a,b] := ab - ba$ for $a,b \in A$ and this is a Lie algebra.)

$\mathbb{R}^3$ with the bracket $[v,w] = v \times w$ (cross product of vectors) where we set $v \times v = 0$.

Remark 4.1.2. Obviously $\mathbb{R}$ is the unique real Lie algebra of dimension one. There are two real Lie algebras of dimension two (up to isomorphism), namely $\mathbb{R}^2$ with the trivial bracket and the Lie algebra $\mathfrak{g} = \mathbb{R}x + \mathbb{R}y$ with bracket $[x,y] = y$.

A **Lie algebra homomorphism** is a linear map $\psi : \mathfrak{h} \to \mathfrak{g}$ which respects the bracket, i.e. $\psi([x,y]) = [\psi(x),\psi(y)]$ for all $x,y \in \mathfrak{h}$. Lie algebras with homomorphisms between them form a category.

4.2. **Lie algebra of a Lie group.** Let G be a Lie group. It has diffeomorphisms $L_g : G \to G$ and $R_g : G \to G$ defined by $L_g(x) = gx$ and $R_g(x) = xg$.

For the moment we drop the assumption that vector fields are smooth, i.e "vector field" means "possibly discontinuous section of $\pi : TM \to M$". A vector field v on G is **left-invariant** if

$$L_g^* v = v \qquad \text{for all } g \in G$$

(equivalently, $L_{g*}(v) = v$ for all $g \in G$). In other notation,

$$v(gx) = (dL_g)_x(v(x)) \qquad \text{for all } g,x \in G.$$

Let $\mathfrak{g}$ or $Lie(G)$ denote the space of left-invariant vector fields on a Lie group G. Since G acts transitively on itself a left-invariant vector field is determined by $v(e) \in T_eG$; since the action is simply transitive the map $v \mapsto v(e)$ gives an isomorphism of vector spaces $\mathfrak{g} \cong T_eG$.

Lemma 4.2.1. *(1) Left-invariant vector fields are smooth.*

(2) The bracket of two left-invariant vector fields is left-invariant.

Therefore the space $\mathfrak{g}$ of left-invariant vector fields is a Lie algebra under bracket.

Proof. (1) It is enough to show that if $f \in C^\infty(M)$ and v is left-invariant then $v(f) \in C^\infty(M)$. (Here $v(f)$ means: apply $v(g)$ to the germ of f at g.) We have

$$v(f)(g) = v(g) \cdot f = (dL_g)_e(v(e)) \cdot f = v(e) \cdot L_g^* f = v(e) \cdot f \circ L_g.$$

Let $i_g : G \to G \times G$ by $i_g(x) = (x, g)$ and let $j_g : G \to G \times G$ by $j_g(x) = (g, x)$. Choose a smooth vector field w on G such that $w(e) = v(e)$. Then $(0, w)$ is a smooth vector field on $G \times G$ and we have

$$\begin{aligned}((0,w)(f\circ\mu))(i_e(g)) &= ((0,w)(g,e))(f\circ\mu)\\ &= (0(g))(f\circ\mu\circ i_e) + (w(e))(f\circ\mu\circ j_g)\\ &= w(e)(f\circ\mu\circ j_g)\\ &= w(e)\cdot f\circ L_g\\ &= v(e)\cdot f\circ L_g.\end{aligned} \tag{4.2.1}$$

(Here in the second equality we have used that for tangent vectors $w_1, w_2 \in T_{(g_1,g_2)}(G \times G)$ and $h \in C^\infty(G\times G)$, we have $(w_1, w_2)(h) = (0, w_2)(h)+(w_1, 0)(h) = w_1(h\circ i_{g_2})+w_2(h\circ j_{g_1})$.) Since the left-hand side is evidently smooth as a function of g we conclude that the right-hand side is smooth in g. By the previous identity $v(f)(g)$ is smooth in g.

(2) Follows from $[v, w] = [L_g^* v, L_g^* w] = L_g^*[v, w]$. □

Thus with any Lie group G is canonically associated a finite-dimensional Lie algebra, namely the Lie algebra $\mathfrak{g} = Lie(G)$ of left-invariant vector fields. Next we will show that this is functorial.

The following remark is often useful: The tangent bundle of a Lie group is trivial. ([13]) More precisely, there is an isomorphism $TG \cong G \times T_eG$ given by $(g, v) \mapsto (g, (dL_{g^{-1}})_g(v))$. (Exercise: Check that this is an isomorphism.)

4.3. **Homomorphisms.** A homomorphism $\varphi : G \to H$ gives a map $d\varphi_e : T_eG \to T_eH$ on tangent spaces at the identity, and hence defines a linear map $d\varphi : \mathfrak{g} \to \mathfrak{h}$. By definition, for $X \in \mathfrak{g}$, $d\varphi(X)$ is the unique left-invariant vector field on H with value $d\varphi_e(X(e))$ at $e \in H$. Thus we have

$$d\varphi(X)(h) = (dL_h)_e d\varphi_e(X(e)) \qquad (h \in H).$$

Recall the following definition: If $f : M \to N$ is a smooth map and X, X' are vector fields on M and N respectively, we say X and X' are **f-related** if $df_p(X(p)) = X'(f(p))$ for all $p \in M$. The following is left as an exercise:

- If X is f-related to X' and Y is f-related to Y', then $[X, X']$ is f-related to $[Y, Y']$.

Lemma 4.3.1. *The map $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras.*

Proof. Let us first see that any $X \in \mathfrak{g}$ is φ-related to $d\varphi(X)$. Note that because φ is a group homomorphism, $\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$. This gives that

$$\begin{aligned}d\varphi_g(X(g)) &= d\varphi_g((dL_g)_e X(e))\\ &= d(\varphi\circ L_g)_e X(e)\\ &= d(L_{\varphi(g)}\circ\varphi)_e X(e)\\ &= (dL_{\varphi(g)})_e (d\varphi)_e X(e)\\ &= (d\varphi(X))(\varphi(g)).\end{aligned}$$

Thus X and $d\varphi(X)$ are φ-related.

Let $X, Y \in \mathfrak{g}$. We must show that $d\varphi([X, Y]) = [d\varphi(X), d\varphi(Y)]$. By the previous observation and the exercise above we know that $[X, Y]$ and $d\varphi([X, Y])$ are φ-related. Then we have:

$$d\varphi_g([X, Y](g)) = (d\varphi[X, Y])(\varphi(g)). \tag{4.3.1}$$

Now $d\varphi[X, Y]$ is the unique left-invariant vector field on G with value $d\varphi_e([X, Y](e))$ at $e \in G$. On the other hand $[d\varphi(X), d\varphi(Y)]$ is a left-invariant vector field on G with this same value at e by (4.3.1). This proves $d\varphi[X, Y] = d\varphi(X), d\varphi(Y)]$. □

We thus have a functor *Lie* from the category of Lie groups (and Lie group homomorphisms) to the category of Lie algebras (and Lie algebra homomorphisms). This functor is not an equivalence, as the following simple example shows: The Lie groups $\mathbb{R}$ and S^1 have the same Lie algebra, namely $\mathfrak{g} = \mathbb{R}$.

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[13] A manifold M for which the tangent bundle is isomorphic to the trivial bundle $M \times \mathbb{R}^{\dim M}$ is called **parallelizable**. It is easy to see that an n-manifold M is parallelizable if and only if it admits n vector fields which are linearly independent at every point (in particular, nowhere-vanishing). Since even-dimensional spheres do not admit nowhere-vanishing vector fields they can never be Lie groups. (It is a famous theorem of Adams from the 1960s that the only parallelizable spheres are S^1, S^3, S^7.)

4.4. **Examples.** Let us look at some simple examples. There are two one-dimensional connected Lie groups, namely $\mathbb{R}$ (with addition) and S^1 (with multiplication). (This can be deduced e.g. from the fact that there are two connected one-manifolds up to diffeomorphism.) Their Lie algebras are both the unique one-dimensional Lie algebra $\mathbb{R}$. In the case S^1 the left-invariant vector field $\frac{\partial}{\partial\theta}$ spans the Lie algebra. In the case $\mathbb{R}$ the Lie algebra is spanned by $\frac{\partial}{\partial t}$. The covering map $\mathbb{R} \to S^1$ by $t \mapsto exp(2\pi i t)$ induces an isomorphism of Lie algebras.

Another example is $\mathbb{R}^*$ with multiplication. In this case the Lie algebra is again $\mathbb{R}$, spanned by the invariant vector field $t\frac{\partial}{\partial t}$. Note that the identity component $\mathbb{R}_+$ is diffeomorphic to $\mathbb{R}$ under the map $\mathbb{R} \to \mathbb{R}_+$ by $t \mapsto e^t$ which also induces an isomorphism of Lie algebras.

Now consider the following examples of Lie subgroups in the torus $T = \mathbb{R}^2/\mathbb{Z}^2$. For $\alpha \in \mathbb{R}$ define a homomorphism of Lie groups $\mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ by

$$\varphi_\alpha(t) = t\,(1,\alpha) \mod \mathbb{Z}^2.$$

If $\alpha = p/q$ is rational then φ_α factors through the covering $\mathbb{R} \to S^1$ and the image is a compact circle in the torus. If α is irrational then $\varphi_\alpha : \mathbb{R} \to T$ is a non-embedded submanifold and injective group homomorphism. In either case the Lie algebra map is $\mathbb{R} \to \mathbb{R}^2$ by $1 \mapsto (1,\alpha)$.

4.5. **Lie subgroups and Lie subalgebras.** A **Lie subgroup** of G is a Lie group H which is a submanifold of G by an injective immersion $i : H \hookrightarrow G$ which is also a group homomorphism.

A **Lie subalgebra** of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ which is closed under the bracket, i.e. $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$.

If $i : H \to G$ is a Lie subgroup then $di(Lie(H))$ is an subspace of $Lie(G)$ closed under the bracket, i.e. a Lie subalgebra. The following theorem says that this gives a bijection between Lie subgroups and Lie subalgebras:

Theorem 4.5.1 (Lie's first theorem)**.** *Let G be a Lie group. There is a one-to-one correspondence between connected Lie subgroups of G and Lie subalgebras of the Lie algebra of G, given by $H \mapsto Lie(H)$.*

Proof. The main point is to show that a Lie subalgebra of $Lie(G)$ comes from a (unique) Lie subgroup. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g} = Lie(G)$. Define a subbundle of the tangent bundle on G by

$$F_g = (dL_g)_e(\mathfrak{h}) \qquad (g \in G).$$

This is a smooth subbundle because it is (globally) spanned by $\dim\mathfrak{h}$ smooth vector fields: Choose a basis $X_1,\dots,X_{\dim\mathfrak{h}}$ of $\mathfrak{h}$; the left-invariant vector fields $v_i(g) = dL_g(X_i)$ are smooth and span F at every point. Note that $[v_i,v_j](g) = (dL_g)_e([X_i,X_j]) \in F_g$ for all g.

Let us check that F is integrable. Every vector field v such that $v(p) \in F_p$ for all p can be written as $v = \sum_i a_i v_i$ for $a_i \in C^\infty(G)$. Now

$$[a_i v_i, b_j v_j] = a_i b_j [v_i, v_j] + a_i v_i(b_j) v_j - b_j v_j(a_i) v_i$$

which lies (pointwise) in F since $[v_i,v_j](g) \in F_g$ for all g. Thus F is integrable.

Let $i : H \to G$ be the maximal connected integral manifold of F through e. Then

$$di_h(T_hH) = F_{i(h)} = (dL_{i(h)})_e(\mathfrak{h}) \qquad (\text{for all } h \in H).$$

We must show that H is a Lie subgroup. Let us first see that it is closed under the group operation of G. For this note that for $h \in H$, $L_{i(h)^{-1}} \circ i : H \to G$ is again an integral submanifold of F through e. Indeed, for any $x \in H$,

$$d(L_{i(h)^{-1}} \circ i)_x(T_xH) = (dL_{i(h)^{-1}})_{i(x)} di_x(T_xH) = (dL_{i(h)^{-1}})_{i(x)}(dL_{i(x)})_e(\mathfrak{h}) = (dL_{i(h)^{-1}i(x)})_e(\mathfrak{h}).$$

By maximality of $i : H \to G$ we have that $L_{i(h)^{-1}} i(H) \subset i(H)$. This proves that $i(h)^{-1} \in i(H)$ and hence that $i(H)$ is a subgroup of G.

To prove that $i : H \to G$ is a Lie subgroup it remains only to show that the map defined by $\alpha(h_1,h_2) := i^{-1}(i(h_1)i(h_2)^{-1})$ is smooth. Consider the diagram

$$\begin{array}{ccc} H \times H & \xrightarrow{\alpha} & H \\ \Vert & & \downarrow i \\ H \times H & \xrightarrow{i\circ\alpha} & G \end{array} \tag{4.5.1}$$

By the factorization property of integral manifolds of integrable subbundles the map α is smooth.

Finally we must prove uniqueness. Suppose $i' : H' \to G$ is another Lie subgroup with $di'(Lie(H')) = \mathfrak{h}$. Then $i' : H' \to G$ is an integral manifold of the subbundle F. By maximality of H we have $i'(H') \subset i(H)$. Consider the diagram

$$\begin{array}{ccc} H' & \xrightarrow{\exists f} & H \\ \Vert & & \downarrow i \\ H' & \xrightarrow{i'} & G \end{array} \tag{4.5.2}$$

A smooth $f : H' \to H$ exists by the factorization property. It is a local diffeomorphism at e, hence it contains an open neighbourhood of $e \in H$ in its image. Therefore it is surjective by the following elementary fact:

- A neighbourhood of the identity in a topological group generates the connected component of identity. (Exercise)

Hence f is a bijective homomorphism and is a local diffeomorphism everywhere (because it is a group homomorphism). Therefore it is an isomorphism of Lie groups. □

As the example of the Lie subgroup $\varphi_\alpha : \mathbb{R} \to T$ with image the dense winding line on the torus for $\alpha \notin \mathbb{Q}$ shows, the Lie subgroup will not necessarily be embedded.

4.6. **Example.** Let us compute the Lie algebra of $G = GL(n, \mathbb{R})$. Let $\beta : \mathfrak{g} \to M(n, \mathbb{R})$ be the isomorphism defined by evaluating a left-invariant vector field at e:

$$\beta(X) = X(e) \in T_e M(n, \mathbb{R}) = M(n, \mathbb{R}).$$

We will show that β is a Lie algebra isomorphism if $M(n, \mathbb{R})$ is given the Lie algebra structures $[A, B] = AB - BA$. Let x_{ij} be the coordinate functions on G, so that $x_{ij}(g) = g_{ij}$. The essential property of β is that

$$\beta(X)_{ij} = X(e)(x_{ij}).$$

We will show that $\beta([X, Y]) = [\beta(X), \beta(Y)]$ (this will be enough since β is an isomorphism of vector spaces). For $Y \in \mathfrak{g}$ we have:

$$\begin{aligned} Y(x_{ij})(h) &= Y(h)(x_{ij}) \\ &= (dL_h)_e(Y(e))(x_{ij}) = Y(e)(x_{ij} \circ L_h) \\ &= Y(e)\left(\sum_k x_{ik}(h) x_{kj}\right) = \sum_k x_{ik}(h) Y(e)(x_{kj}) \\ &= \sum_k x_{ik}(h) \beta(Y)_{kj} \end{aligned} \tag{4.6.1}$$

where we have used the formula for matrix multiplication:

$$(x_{ij} \circ L_g)(h) = x_{ij}(gh) = \sum_{k=1}^{n} x_{ik}(g) x_{kj}(h).$$

Using this we have:

$$\begin{aligned} \beta([X, Y])_{ij} = [X, Y](e)(x_{ij}) &= X(e)Y(x_{ij}) - Y(e)X(x_{ij}) \\ &= \sum_k X(e)(x_{ik})\beta(Y)_{kj} - Y(e)(x_{ik})\beta(X)_{kj} \\ &= [\beta(X), \beta(Y)]_{ij}. \end{aligned} \tag{4.6.2}$$

It follows that for the Lie algebra of any Lie subgroup of $GL(n, \mathbb{R})$ the Lie bracket on the corresponding subalgebra of $M(n, \mathbb{R})$ is given by $[X, Y] = XY - YX$.

In more basis-invariant terms this shows that the Lie algebra of $GL(V)$ is identified with $End(V)$ with the bracket $[A, B] = AB - BA$.

4.7. **Linear algebraic groups.** ([14]) A large class of groups (in fact, by far the most important ones), for example groups like $SO(n), Sp(2n)$ etc. which are defined by polynomial conditions on matrices can be shown to be Lie groups by a direct method. We will prove this theorem first and then use it to discuss many examples.

Let $U \subset \mathbb{R}^n$ be open. Suppose that I is an ideal in $\mathbb{R}[x_1, \ldots, x_n]$, which we know to be finitely generated by the Hilbert basis theorem. The set

$$V(I) = \{x \in U : f(x) = 0 \text{ for all } f \in I\}$$

is called an **algebraic subset** of U. Conversely, given a set $S \subset U$ let $I(S)$ be the ideal of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ which vanish on S. Then a set is algebraic if and only if $V(I(S)) = S$. ([15])

(The same definition makes sense in the complex case: If $U \subset \mathbb{C}^n$ is open then a subset $S \subset U$ is called a **complex algebraic subset** if the ideal I functions in $\mathbb{C}[x_1, \ldots, x_n]$ vanishing on S has $V(I) = S$, where $V(I) \subset U$ is defined as before.)

While an algebraic subset need not be a manifold we will see that a large open subset of it is a manifold. Let M be an algebraic subset of $U \subset \mathbb{R}^n$. For $x \in M$ consider the differentials $df_x : T_xU = \mathbb{R}^n \to T_{f(x)}\mathbb{R} = \mathbb{R}$. Thinking of these as linear functionals on T_xU, let

$$r(x) = \text{ dimension of span of } \{df_x \in (T_xU)^* : f \in I\}.$$

This is finite because if $I = (f_1, \ldots, f_N)$ then the $\{df_{i,x}\}_{i=1,\ldots,N}$ span the same space as $\{df_x\}_{f \in I}$ because of the identity

$$d(fg)_x = f(x)dg_x + g(x)df_x.$$

Now let

$$r_M = \max_{x \in M} r(x).$$

(The maximum exists because I is finitely generated.) Let

$$M^{\text{reg}} = \{x \in M : r(x) = r_M\}$$

be the **regular locus** of M. Note that the regular locus is open and nonempty.

(The same definition can be given for $U \subset \mathbb{C}^n$, except that we we replace T_xU by the complexified tangent space (which is spanned over $\mathbb{C}$ by the $\frac{\partial}{\partial x_i}$ and consider the elements $df_x \in T_xU \to T_{f(x)}\mathbb{C} = \mathbb{C}$ as elements of $Hom_{\mathbb{C}}(T_xU, \mathbb{C})$.)

Proposition 4.7.1 (Whitney)**.** *The open subset M^{reg} is a smooth manifold of dimension $n - r$ and an embedded submanifold of U. (In fact, M^{reg} is an embedded real-analytic submanifold of U.)*

(The proof below will also apply to the complex case, showing that if M is a complex algebraic subset of $U \subset \mathbb{C}^n$ then M^{reg} is a complex manifold of (complex) dimension $n - r$ and hence also a real manifold of dimension $2(n - r)$.)

Proof. Let $p \in M^{\text{reg}}$. Let $x_1, \ldots, x_n$ be coordinates centred at p and assume that $df_{1,p}, \ldots, df_{r,p}$ are linearly independent. Thus the $r \times n$ matrix of partials

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$$

has rank r at 0; by renumbering coordinates we may assume that the first $r \times r$ columns are independent.

Define a map from $\varphi : U \to \mathbb{R}^n$ by

$$\varphi(x_1, \ldots, x_n) = (f_1, \ldots, f_r, x_{r+1}, \ldots, x_n)$$

This is a real-analytic diffeomorphism at 0, hence there is a neighbourhood V of $0 \in U$ such that $\varphi : V \to \mathbb{R}^n$ is a real-analytic diffeomorphism onto an open subset of $\mathbb{R}^n$. (Here we must use the real-analytic version of the inverse function theorem.) Let $y_1, \ldots, y_n$ be the coordinates on V given by φ and let

$$V_0 = \{x \in V : y_1(\varphi(x)) = \cdots = y_r(\varphi(x)) = 0\}.$$

Clearly this is a connected manifold of dimension $n - r$. If we show that $V_0 \subset M$ (in other words, the vanishing of $f_1, \ldots, f_r$ on V_0 guarantees the vanishing of all $f \in I$ on V_0) then we will have shown that

[14] This section is a bit of a diversion in that we use a real-analytic argument, following Chapter I of Varadarajan's *Lie Groups, Lie Algebras, and their Representations*. It is the quickest way I know of to prove that the various standard examples of matrix groups are in fact Lie groups all at once, i.e. without working out cases separately.

[15] Note that for an ideal J, $I(V(J)) \supset J$, but the inclusion may be strict. For example, if $J = (x^2) \subset \mathbb{R}[x]$ then $I(V(J)) = (x)$. In general, $I(V(J)) = rad(J)$ (this is known as Hilbert's Nullstellensatz).

$V_0 = V \cap M$. Since V_0 is a manifold of dimension $n-r$ and $V_0 = V \cap M$ is an open subset of M, this will prove that M is a manifold at 0.

Let $\mathscr{A} = \mathscr{A}(V)$ be the algebra of all real-analytic functions on V. Let $\mathscr{I} = I \cdot \mathscr{A}$ be the ideal in A generated by $I = I(M)$. For $k > r$ the operator $\frac{\partial}{\partial y_k}$ leaves $\mathscr{I}$ invariant. Indeed, it is enough to prove that

$$\frac{\partial}{\partial y_k} I \subset \mathscr{I} \tag{4.7.1}$$

for $k > r$. Let $F \in I$. Define an n-tuple $(f_1, \ldots, f_n)$ by $f_i = y_i = x_i$ if $i > r$ and $i \neq k$ and let $f_k = F$. (Recall that $f_1, \ldots, f_r$ were already fixed as $y_i = f_i$ for $i \leq r$.) Then the Jacobians of partials are related by

$$\det\left(\frac{\partial f_i}{\partial y_j}\right)_{1\leq i,j\leq n} = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\leq i,j\leq n} \cdot \det\left(\frac{\partial x_i}{\partial y_j}\right)_{1\leq i,j\leq n}. \tag{4.7.2}$$

Now the left-hand side is simply $\frac{\partial F}{\partial y_k}$ (recall that $y_i = f_i$ for $i \leq r$) while on the right-hand side the second factor $\det\left(\frac{\partial x_i}{\partial y_j}\right)$ belongs to $\mathscr{A}$. The first factor can be simplified as an $r+1 \times r+1$ determinant:

$$\det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\leq i,j\leq n} = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\leq i\leq r, i=k, 1\leq j\leq r, j=k}$$

Now since $f_i \in I$ and $F \in I$, this vanishes on M. (If not there would be a point $q \in M$ at which $df_{1,q}, \ldots, df_{r,q}, dF_q$ are linearly independent, violating the maximality of r.) Thus (4.7.2) shows that the left-hand side, namely $\frac{\partial}{\partial y_k}(F)$, belongs to $I \cdot \mathscr{A} = \mathscr{I}$. This proves (4.7.1) and hence that $\frac{\partial}{\partial y_k}\mathscr{I} \subset \mathscr{I}$ for $k > r$.

Now it follows that for $F \in \mathscr{I}$ and any $(a_{r+1}, \ldots, a_n)$, we have

$$\left(\frac{\partial}{\partial y_{r+1}}\right)^{a_{r+1}} \cdots \left(\frac{\partial}{\partial y_n}\right)^{a_n} (F) \in \mathscr{I}.$$

Since every element of I vanishes at 0, any $F \in \mathscr{I}$ has all such derivatives vanishing at 0. Since F is an analytic function and V_0 is connected, we see that F vanishes on all of V_0. In particular, elements of I vanish on V_0, proving that $V_0 \subset M$. □

Remark 4.7.2. In fact, a theorem of Nash says that any connected compact smooth manifold is diffeomorphic to a connected component of the nonsingular part of an algebraic subset of $\mathbb{R}^n$. More precisely, Nash (*Annals of Math.* 56 (1952)) proved that a connected compact smooth n-manifold can be embedded in $\mathbb{R}^{2n+1}$ as a connected component of an algebraic subset. As a corollary, any such manifold admits a real analytic structure, which was an earlier theorem of Whitney.

Now define a (real) **linear algebraic group** to be a subgroup $G \subset GL(n, \mathbb{R})$ which is an algebraic subset. (A complex linear algebraic group is a subgroup of $GL(n, \mathbb{C})$ which is an algebraic subset). Then we have:

Corollary 4.7.3. *A linear algebraic subgroup of $GL(n, \mathbb{R})$ (resp. $GL(n, \mathbb{C})$) is a closed Lie subgroup of $GL(n, \mathbb{R})$ (resp. of $GL(n, \mathbb{C})$).*

Proof. We will apply Whitney's theorem with $U = GL(n, \mathbb{R})$, an open subset of $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ with polynomial functions in x_{ij}. If $I = I(G)$ then for $f \in I$ we also have $L_g^* f \in I$ for $g \in G$. It is easy to see from this that the number $r(x)$ defined above is the same at all points of the group because of the left translation. So $G^{\mathrm{reg}} = G$ and we are done. □

The corollary gives us a number of examples of Lie groups which are linear algebraic. For example, the group of 3×3 upper triangular matrices $\left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\}$ is defined by the ideal $(x_{21}, x_{31}, x_{32}, x_{11} - 1, x_{22} - 1, x_{33} - 1)$ and so is a linear algebraic group. (That this is a Lie group can be proved directly, of course.) The following more interesting examples are usually referred to as the **classical groups**:

(i) (special linear groups) $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det(A) = 1\}$ and $SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : \det(A) = 1\}$. (The topology is the one induced from $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.)

(ii) (compact orthogonal groups) $O(n) = \{A \in GL(n, \mathbb{R}) : AA^t = A^tA = I\}$ and $SO(n) = O(n) \cap SL(n, \mathbb{R})$.

(iii) (compact unitary groups) $U(n) = \{A \in GL(n,\mathbb{C}) : AA^* = A^*A = I\}$ and $SU(n) = U(n) \cap SL(n,\mathbb{C})$.

(iv) (compact symplectic groups) Let $\mathbb{H}$ denote the quaternions, i.e. the unique four-dimensional skew-field over $\mathbb{R}$. Then $Sp(g)$ is the group of matrices in $GL(g,\mathbb{H})$ which preserve the standard Hermitian form on $p,q \in \mathbb{H}^g$ given by $H(p,q) = \sum_{i=1}^{g} p_i \bar{q}_i$, i.e.

$$Sp(g) = \{g \in GL(g,\mathbb{H}) : g\bar{g}^t = I\}.$$

(Here $\bar{q}$ denotes the quaternion conjugation: If $q = a + bi + cj + dk$ then $\bar{q} = a - bi - cj - dk$.) Another way to define the compact symplectic group is as

$$Sp(2g) = Sp(2g,\mathbb{C}) \cap U(2g,\mathbb{C}),$$

i.e. the group of $2g \times 2g$ unitary matrices which preserve the standard symplectic form on $\mathbb{C}^{2g}$. (See (vii) below for the definition of $Sp(2g,\mathbb{C})$.)

(Exercise: Check equivalence of the two definitions as follows: As $\mathbb{C}$-vector spaces we can write $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ (write $a+bi+cj+dk = a+bi+(c+di)j$). Check that the standard Hermitian form can be written as $H = h + Bj$ where h is the standard Hermitian form on $\mathbb{C}^{2g}$ and B is an alternating form on $\mathbb{C}^{2g}$. Then a matrix in $GL(g,\mathbb{H})$ preserving H becomes a $GL(2g,\mathbb{C})$ matrix preserving both h and B.)

(v) (orthogonal groups) Let $p + q = n$ for $p,q > 0$. Let B be the symmetric bilinear form on $\mathbb{R}^n$ given by $B(x,y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{q} x_i y_i$. Let $O(p,q) \subset GL(n,\mathbb{R})$ be the subgroup $\{A \in GL(n,\mathbb{R}) : B(Ax,Ay) = B(x,y)\}$. Let $SO(p,q) = O(p,q) \cap SL(n,\mathbb{R})$.

(On a real vector space of dimension n, there is up to equivalence a unique nondegenerate quadratic form of signature (p,q) for $p+q=n$ (Sylvester's theorem). So if Q is any quadratic form, the group $O(Q)$ of matrices preserving it is conjugate in $GL(n,\mathbb{R})$ $O(p,q)$ for some p,q.)

(vi) (unitary groups) Let $p + q = n$ for $p,q > 0$. Let B be the Hermitian form on $\mathbb{C}^n$ given by $B(z,w) = \sum_{i=1}^{p} z_i \bar{w}_i - \sum_{i=p+1}^{q} z_i \bar{w}_i$. Let $U(p,q) \subset GL(n,\mathbb{C})$ be the subgroup $\{A \in GL(n,\mathbb{C}) : B(Az,Aw) = B(z,w)\}$. Let $SU(p,q) = U(p,q) \cap SL(n,\mathbb{C})$.

(vii) (symplectic groups) Let $Sp(2g,\mathbb{R})$ be the subgroup of matrices in $GL(g,\mathbb{R})$ preserving the anti-symmetric form $B : \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}$ given by $B(x,y) = \sum_{i=1}^{g} x_i y_{i+g} - x_{i+g} y_i$ for $x = (x_1,\ldots,x_{2g})$ and $y = (y_1,\ldots,y_{2g})$.

Similarly, looking at elements of $GL(2g,\mathbb{C})$ preserving B (extended linearly to $\mathbb{C}^{2g} \times \mathbb{C}^{2g}$) gives the complex symplectic group $Sp(2g,\mathbb{C})$.

(On a vector space of even dimension over any field there is a unique nondegenerate alternating form. So for any such form on $\mathbb{R}^{2g}$ the group of matrices preserving the form is conjugate in $GL(2g,\mathbb{R})$ to $Sp(2g,\mathbb{R})$. Also, any matrix preserving an alternating form must have determinant one, so $Sp(2g,\mathbb{R}) \subset SL(2g,\mathbb{R})$.)

For each of these groups, the Lie algebra is a subalgebra of $M(n,\mathbb{R})$ (or $M(n,\mathbb{C})$ in the complex cases) and so the bracket is given by $[A,B] = AB - BA$.

There are linear algebraic groups which do not fall into the above classes, but aside from finitely many examples, any linear algebraic group which is simple (as a linear algebraic group, i.e. does not contain any proper nontrivial linear algebraic subgroup) falls into one of the above classical examples.

4.8. **Coverings of Lie groups.** If M is a smooth manifold then it has a universal cover $\tilde{M}$, which again has a natural structure of smooth manifold for which $\tilde{M} \to M$ is smooth. (It is obvious that $\tilde{M}$ is Hausdorff and locally Euclidean; the only non-obvious part is that it is second-countable. This follows from the fact that $\pi_1(M)$ is countable. ([16]))

Proposition 4.8.1. *The universal cover $\tilde{G}$ of a Lie group G has a natural structure of Lie group such that $\pi : \tilde{G} \to G$ is smooth.*

[16] Here is a proof for a second countable topological manifold: Take a countable cover $\{U_n\}_n$ of M by coordinate charts homeomorphic to $\mathbb{R}^n$. For each m,n the intersection $U_m \cap U_n$ has at most countably many components; choose a point in each component and let C be the set of all such points (over all m,n). For each n and $x,y \in C$ such that $x,y \in U_n$ choose a path $\gamma_{n,x,y}$ from x to y in U_n. For a fixed basepoint $p \in C$ there are countably many loops which are finite products of paths of the form $\gamma_{n,x,y}$. It suffices to show that every element of $\pi_1(M,p)$ is of this form.

For a loop $\gamma : [0,1] \to M$ based at p there is an N such that when $[0,1]$ is subdivided into N equal intervals, each subinterval $[(k-1)/N, k/N]$ has image in some U_n. Reparametrizing $\gamma|_{[(k-1)/N,k/N]}$ we get paths γ_k such that $\gamma = \gamma_1 \cdots \gamma_N$. For each k, the point $\gamma(k/N)$ lies in some component $U_n \cap U_m$, in which we have a chosen point, call it $x_k \in C$. Now choose a path δ_k in $U_m \cap U_n$ from x_k to $\gamma(k/N)$ and let $\tilde{\gamma}_k = \delta_{k-1}\,\gamma\,\delta_k$. Then $[\gamma] = [\tilde{\gamma}_1] \cdots [\tilde{\gamma}_N]$. But for each k, $\tilde{\gamma}_k$ is a path in U_m from x_{k-1} to x_k, hence homotopic to the reference path γ_{n,x_{k-1},x_k}. Thus γ is homotopic to a product of paths of the form $\gamma_{n,x,y}$.

Proof. We have remarked above that $\tilde{G}$ has a smooth manifold structure for which π is smooth, so we must define the group structure. Fix $\tilde{e} \in \pi^{-1}(e)$. Let $\alpha : \tilde{G} \times \tilde{G} \to G$ be the map $\alpha(\tilde{x}, \tilde{y}) = xy^{-1}$. Consider the diagram of maps of pointed spaces

$$\begin{array}{ccc} (\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e})) & \xrightarrow{\exists \tilde{\alpha}} & (\tilde{G}, \tilde{e}) \\ \Vert & & \downarrow \pi \\ (\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e})) & \xrightarrow{\alpha} & (G, e). \end{array}$$

By the simple connectedness of $\tilde{G} \times \tilde{G}$ there is a unique lift $\tilde{\alpha}$ of α such that $\tilde{\alpha}(\tilde{e}, \tilde{e}) = \tilde{e}$. It is easy to see that this lift is locally smooth, hence smooth. Define the inversion map $\iota : \tilde{G} \to \tilde{G}$ by

$$\iota(\tilde{g}) := \tilde{\alpha}(\tilde{e}, \tilde{g}).$$

Define the group operation $m : \tilde{G} \times \tilde{G} \to \tilde{G}$ by

$$m(\tilde{g}, \tilde{h}) = \tilde{\alpha}(\tilde{g}, \iota(\tilde{h})).$$

Verifying the group axioms is left as an exercise. □

Lemma 4.8.2. *If $p : G' \to G$ is a covering of topological groups and G' is connected then $ker(p)$ is contained in the centre of G'.*

Proof. If $p : G' \to G$ is a covering then $ker(p)$ is a discrete normal subgroup of G'. For $g \in ker(p)$ the map $h \mapsto hgh^{-1}$ defines a continuous map $G' \to ker(p)$, which must then have constant value e if G' is connected. □

Theorem 4.8.3. *A homomorphism $\varphi : H \to G$ of connected Lie groups is a covering map if and only if $d\varphi : Lie(H) \to Lie(G)$ is an isomorphism (equivalently, $d\varphi_e : T_eH \to T_eG$ is an isomorphism).*

Proof. Suppose first that the homomorphism $\varphi : H \to G$ is a covering map. We will show first that $d\varphi_e$ is injective. Indeed, if not then consider the subbundle defined by $F_h := ker(d\varphi_h)$ on H. It is smooth and integrable. Thus the integral manifolds of F_h are submanifolds of H such that $d\varphi$ is uniformly zero on each one. Thus each integral manifold is contracted to a point in G. This contradicts the local one-to-one property of φ. Next let us see that $d\varphi_e$ is surjective. If not then $\dim H < \dim G$ and so (by Sard's theorem) the map $\varphi : H \to G$ cannot be surjective.

Conversely suppose that $\varphi : H \to G$ is a homomorphism of Lie groups with $d\varphi_e$ an isomorphism. Then $ker(\varphi)$ is a discrete normal subgroup of H. Let V be a neighbourhood of e in H such that

$$V^{-1}V \cap ker(\varphi) = \{e\}.$$

We will show that $\varphi(V)$ is a regularly covered neighbourhood of $e \in G$. Firstly $\varphi|_V$ is one-to-one. (Indeed, if $\varphi(h_1) = \varphi(h_2)$ for $h_1, h_2 \in V$ then $h_1^{-1}h_2 \in V \cap ker(\varphi)$ and hence $h_1 = h_2$.) Since $d\varphi_h$ is an isomorphism for all $h \in H$ we see that $\varphi|_V$ is a diffeomorphism of V with the open neighbourhood $\varphi(V)$ of e in G. Next we claim that

$$\varphi^{-1}(\varphi(V)) = \bigsqcup_{\theta \in ker(\varphi)} V\theta.$$

(Indeed, the inclusion $\supset$ is obvious, so consider $\subset$. Let $h \in H$ with $\varphi(h) \in \varphi(V)$. Suppose $\varphi(h) = \varphi(h')$ for $h' \in V$. Then $\theta := (h')^{-1}h \in ker(\varphi)$ and $h = h'(h')^{-1}h \in V\theta$. Moreover the union above is disjoint: if $h \in V\theta_1 \cap V\theta_2 \neq \emptyset$ then $h = h_1\theta_1 = h_2\theta_2$ for $h_1, h_2 \in V$ and then $h_1^{-1}h_2 \in V^{-1}V \cap ker(\varphi)$ so $h_1 = h_2$ and hence $\theta_1 = \theta_2$.) This proves that $\varphi(V)$ is a regularly covered neighbourhood since $\varphi|_{V\theta} : V\theta \to \varphi(V)$ is a diffeomorphism for each θ. For a general $g = \varphi(h) \in G$ the neighbourhood $g\varphi(V) = \varphi(hV)$ is evenly covered by the disjoint union $\bigsqcup_{\theta \in ker(\varphi)} hV\theta$. □

We have also shown the following: For any Lie group G, the universal cover $\pi : \tilde{G} \to G$ is a Lie group, and $G = \tilde{G}/ker(\pi : \tilde{G} \to G)$. Moreover, $\pi_1(G) = ker(\pi : \tilde{G} \to G)$.

4.9. **Example:** $SU(2), SO(3), Sp(1)$**, quaternions.** The group $SU(2)$ of complex matrices $A \in SL(2,\mathbb{C})$ satisfying $AA^* = I$ (for A^*= conjugate transpose of A) is isomorphic to S^3. This follows because any such matrix can be written as $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ where $z, w \in \mathbb{C}$ with $|z|^2 + |w|^2 = 1$, so that $SU(2)$ is the unit ball in $\mathbb{C}^2$. (Alternately, the natural action of $SU(2)$ on $\mathbb{C}^2$ preserves the standard Hermitian form, so it preserves the unit ball S^3 of vectors of norm one in $\mathbb{C}^2$. It can be checked that the action of $SU(2)$ is simply transitive, so that there is an identification of $SU(2)$ with S^3.)

There is a double covering $SU(2) \to SO(3)$. To see this, consider the space $su(2)$ of traceless skew-Hermitian matrices, i.e. A satisfying $A + A^* = 0$ and $\mathrm{tr}(A) = 0$. This is the Lie algebra of $SU(2)$. This is a real vector space of dimension 3 and it has a natural bilinear form $B(A,B) = -\frac{1}{2}\mathrm{tr}(AB)$.

- The form $B(-,-)$ is nondegenerate and positive definite.

There is an action of $SU(2)$ on $su(2)$ by linear transformations by $(g, A) \mapsto gAg^{-1}$. In other words, there is a homomorphism $\rho : SU(2) \to GL(su(2))$. The action preserves the symmetric bilinear form $B(A,B) = -\frac{1}{2}\mathrm{tr}(AB)$. In other words, the homomorphism ρ has image inside $SO(q_B)$ where q_B is the inner product $q_B(v) = B(v,v)$ on the three-dimensional space $su(2)$. So we have produced a Lie group homomorphism $SU(2) \to SO(q_B)$, with kernel exactly $\pm I$. To see the homomorphism ρ explicitly, you must choose an orthonormal (w.r.t. q_B) basis of $su(2)$, which gives an isomorphism $su(2) \cong \mathbb{R}^3$. A popular choice among physicists is the basis $i\sigma_1, i\sigma_2, i\sigma_3$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are called the Pauli matrices. To show that ρ is surjective one has to only check that $d\rho_e$ is an isomorphism, for then the image contains a neighbourhood of e; since $SO(3)$ is connected this neighbourhood generates the whole group, so ρ is surjective. ([17]) As $SU(2) = S^3$, this establishes that $SO(3)$ is $\mathbb{R}P^3$. ([18]) In physics, the fact that the group of rotational symmetries of physical space (= $\mathbb{R}^3$) is $SO(3)$, which has fundamental group $\mathbb{Z}/2\mathbb{Z}$, is responsible for the existence of particles of different "spin" in nature.

A third way to see the natural double cover $SU(2) \to SO(3)$ is using quaternions. Let $\mathbb{H}$ denote the skew-field of quaternions, i.e. $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ where i, j, k are symbols satisfying the relations $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. The norm of a quaternion $a+bi+cj+dk$ is $(a^2+b^2+c^2+d^2)^{1/2}$. The norm one quaternions are identified with $SU(2)$ by $a + bi + cj + dk \mapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$. The group of norm one quaternions is topologically the 3-sphere, and acts by multiplication on quaternions in $\mathbb{H}$ with zero real part, preserving the norm. This gives the map $SU(2) \to SO(3)$ again.

The group $SU(2)$ acts naturally on $\mathbb{C}^2$, and hence on $\mathbb{C}P^1 = S^2$. The isotropy of any vector is a closed subgroup isomorphic to $U(1)$, namely a conjugate of the group of matrices of the form $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-\theta} \end{smallmatrix}\right)$. The orbit map $SU(2) \to \mathbb{C}P^1$ is then a map $S^3 \to S^2$ with $U(1) = S^1$ fibres, which is called the Hopf fibration. (This map is interesting because it is a map $S^3 \to S^2$ which can be shown to be not homotopic to a constant map (cf. Hatcher's book for a proof). This shows that $\pi_3(S^2) \neq 0$, and in fact $\pi_3(S^2) = \mathbb{Z}$, the Hopf map is a generator.)

Finally, note that there is still one more description of S^3: It is homeomorphic to the compact symplectic group $Sp(1)$. Indeed, $Sp(1) = SU(2)$ since both are identified with the unit quaternions.

4.10. **Exponential map.** Let G be a Lie group and $X \in Lie(G)$ a left-invariant vector field. Let $\gamma_X(t)$ be the integral curve of X with $\gamma_X(0) = e$, defined for $|t| < \epsilon$. So for any $|t| < \epsilon$, the tangent vector to γ_X at $\gamma_X(t)$ is $X(\gamma_X(t))$. Then the tangent vector to $t \mapsto L_g\gamma_X(t)$ at $g\gamma_X(t)$ is $(dL_g)_{\gamma_X(t)}X(\gamma_X(t)) = X(g\gamma_X(t))$. Then $L_g\gamma_X(t)$ is the integral curve of X which is at g at $t = 0$.

Applying this to $g = \gamma_X(s)$ (for s small enough) we conclude that $t \mapsto \gamma_X(s)\gamma_X(t)$ is the integral curve of X starting at $\gamma_X(s)$. On the other hand $\gamma_X(s+t)$ is also an integral curve of X, and at $t = 0$ is at $\gamma_X(s)$. We conclude that

$$\gamma_X(s)\gamma_X(t) = \gamma_X(s+t) \tag{4.10.1}$$

[17] One can also show that ρ is surjective onto $SO(q_B) \cong SO(3)$ by using that $SO(3)$ is generated by rotations and showing that every rotation is in the image of $SU(2)$.

[18] Here is a direct proof that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$. Define a map $D^3 \to SO(3)$ by sending a point x to the matrix of the rotation by $|x|\pi$ around the axis through x (and sending 0 to the identity). This map is continuous and injective on the interior of D^3, and is surjective since any element of $SO(3)$ is a rotation around an axis (Exercise). On the boundary ∂D^3 we see that two points go to the same matrix in $SO(3)$ if they are antipodal. This gives a bijective continuous map from the quotient space of D^3 by the relation $x \sim -x$ on ∂D^3 to $SO(3)$. This quotient space is $\mathbb{R}P^3$, and since the domain is compact the map is a homeomorphism.

for s, t small enough that both sides make sense. But then γ_X can be extended to a mapping $\gamma_X : \mathbb{R} \to G$ which is in fact a group homomorphism. It is also clear that γ_X is an integral curve of X for all t. Thus we see that *the integral curves of a left-invariant vector field on a Lie group are defined for all* $t \in \mathbb{R}$.

Define the **exponential map** $exp : Lie(G) \to G$ by

$$exp(X) := \gamma_X(1).$$

This has the following properties:

Lemma 4.10.1. *The exponential map* $exp : Lie(G) \to G$ *has the following properties:*

(i) $\gamma_X(t) = exp(tX)$ *for all* t
(ii) $exp(tX)exp(sX) = exp((s+t)X)$ *and* $exp(-tX) = exp(tX)^{-1}$
(iii) exp *is a smooth mapping*
(iv) $d\,exp_0 : Lie(G) \to Lie(G)$ *is the identity and hence* exp *is a local diffeomorphism at* 0
(v) *The flow of* $X \in Lie(G)$ *is given by* $\alpha(t, g) = g\,exp(tX)$ *(i.e.* $\alpha_t = R_{exp(tX)}$*).*

Proof. (i) The integral curve for tX is $s \mapsto \gamma_X(st)$.

(ii) follows from (i) and (4.10.1).

(iii) Consider the smooth vector field on $G \times \mathfrak{g}$ defined by $v(g, X) = (X(g), 0)$. The integral curve of v through (g, X) is then $t \mapsto (g\,exp(tX), X)$ and the flow is $\beta_t(g, X) = (g\,exp(tX), X)$. Both these are defined for all t, hence β_1 is defined on $G \times \mathfrak{g}$ and then $exp(X) = \pi_1 \circ \beta_1(e, X)$. But then $exp(X)$ is a composition of smooth maps, hence smooth.

(iv) tX is a curve in $\mathfrak{g}$ with tangent vector X at 0; by (i) $\gamma_X(t) = exp(tX)$ is a curve in G with tangent vector X at e. Thus $d\,exp_0(X) = X$.

(v) restates the fact that $L_g \circ \gamma_X$ is the integral curve of X starting at g. □

Lemma 4.10.2. *Every Lie group homomorphism* $\mathbb{R} \to G$ *(i.e. every one-parameter subgroup of* G*) is of the form* $t \mapsto exp(tX)$ *for some* $X \in Lie(G)$.

Proof. Let X_e be the tangent vector to $\delta : \mathbb{R} \to G$ at e, i.e $d\delta_0\left(\frac{\partial}{\partial t}\right) = X_e$. Let X be the associated left-invariant vector field on G. It will be enough to show that δ is an integral curve for X, i.e. that $X(\delta(s)) = d\delta_s\left(\frac{\partial}{\partial s}\right)$. We compute:

$$\begin{aligned} X(\delta(s)) &= d\,L_{\delta(s)}X(e) = dL_{\delta(s)}d\delta_0\left(\frac{\partial}{\partial s}\right) \\ &= d(L_{\delta(s)} \circ \delta)_0\left(\frac{\partial}{\partial t}\right) = d(\delta \circ L_s)_0\left(\frac{\partial}{\partial s}\right) = d\delta_s\left(\frac{\partial}{\partial s}\right) \end{aligned}$$

where we have used that $\delta \circ L_a = L_{\delta(a)} \circ \delta$ (because δ is a group homomorphism) and that $(dL_s)_0\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial s}$ because $\frac{\partial}{\partial s}$ is translation-invariant on $\mathbb{R}$. □

Proposition 4.10.3. *Let* $\varphi : H \to G$ *be a homomorphism of Lie groups. The diagram*

$$\begin{array}{ccc} Lie(H) & \xrightarrow{d\varphi} & Lie(G) \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ H & \xrightarrow{\varphi} & G \end{array} \tag{4.10.2}$$

is commutative.

Proof. Let $X \in Lie(H)$. Then $t \mapsto exp(td\varphi(X))$ is an integral curve of $d\varphi(X)$ with tangent vector $d\varphi_e(X(e))$ at e. On the other hand $t \mapsto \varphi(exp(tX))$ is a homomorphism $\mathbb{R} \to G$ and as a curve has tangent vector $d\varphi_e(X)$ at e. Therefore (by the previous lemma) it is an integral curve, in fact we must have

$$\varphi(exp(tX)) = exp(td\varphi(X)).$$

Putting $t = 1$ gives the proposition. □

Corollary 4.10.4. *Let* H *be a connected Lie group. If* $\varphi_1, \varphi_2 : H \to G$ *are Lie group homomorphisms such that* $d\varphi_1 = d\varphi_2$ *then* $\varphi_1 = \varphi_2$.

Remark 4.10.5. For $X \in M(n,\mathbb{R})$ we have the familiar exponential series $e^X = \sum_{n\geq 0} \frac{X}{n!}^n$ which gives an element of $GL(n,\mathbb{R})$. Once we know that the exponential series converges absolutely, $t \mapsto e^{tX}$ is a one-parameter subgroup of G with derivative X at $t = 0$. By the previous proposition we conclude that $exp(X) = e^X$. The same applies for any subgroup of $GL(n,\mathbb{R})$.

Proposition 4.10.6 (No small subgroups property)**.** *Let G be a Lie group. There is a neighbourhood of the identity which contains no nontrivial subgroup.*

Proof. Put a Euclidean norm $||\cdot||$ on the finite-dimensional vector space $\mathfrak{g}$ and for $\epsilon > 0$ let $B_\epsilon := \{X \in \mathfrak{g} : ||X|| < \epsilon\}$ be the ϵ-ball around 0. Since $d\,exp_0 : \mathfrak{g} \to \mathfrak{g}$ is an isomorphism, we know that there exists δ such that exp is a diffeomorphism on B_δ. Now consider the neighbourhood $U = exp(B_{\delta/2})$. Let $e \neq g \in U$ and let $X \in B_{\delta/2}$ such that $exp(X) = g$. There exists n such that $nX \in B_\delta - B_{\delta/2}$. But then $g^n = exp(nX) \notin U$ since exp is bijective on B_δ. Thus there is no subgroup contained in U except $\{e\}$. □

Remark 4.10.7. An immediate consequence of this is that any continuous homomorphism from a profinite group Γ to a real Lie group factors through a finite quotient of Γ. For example, if Γ is a Galois group with the Krull topology then any continuous representation of Γ on a finite-dimensional real vector space factors through a finite quotient.

Thus, for example, a group like $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, known as the absolute Galois group of $\mathbb{Q}$ does not have a faithful finite-dimensional real or complex representation. To get such representations we have to consider homomorphisms into $GL(n,\mathbb{Q}_p)$, which is an p-adic version of a Lie group. A good reference where the theory of p-adic Lie groups and real Lie groups is developed together is Serre's *Lie Groups and Lie Algebras*.

Remark 4.10.8. One can ask if the exponential map is surjective for a connected group G. Here are some exercises in linear algebra:

- Show that $exp : M(n,\mathbb{C}) \to GL(n,\mathbb{C})$ is surjective. (Hint: Jordan normal form.)
- Show that if $g \in SL(2,\mathbb{R})$ is in the image of exp then $trace(g) \geq -2$. Conclude that exp is not surjective.
- Show that $exp : Lie(SU(n)) \to SU(n)$ and $exp : Lie(SO(n)) \to SO(n)$ are surjective.

In fact exp is surjective for any compact Lie group G, but we will not prove this. It is also surjective for connected Abelian Lie groups (because exp is a group homomorphism (exercise) and hence $exp(\mathfrak{g})$ is a connected subgroup which contains a neighbourhood of the identity).

4.11. **Simply connected groups.**

Theorem 4.11.1. *Let H be a simply connected Lie group and let G be a Lie group. Then for any Lie algebra homomorphism $\psi : Lie(H) \to Lie(G)$ there is a unique Lie group homomorphism $\varphi : H \to G$ such that $d\varphi = \psi$.*

Proof. Note that giving a Lie group homomorphism $\varphi : H \to G$ is the same as giving its graph $\hat{H} := \{(h,\varphi(h)) : h \in H\} \subset H \times G$. Then the homomorphism $\varphi : H \to G$ is the composition of the second projection restricted to $\hat{H}$ with the isomorphism $H \cong \hat{H}$. We will use this idea.

Let $\psi : \mathfrak{h} \to \mathfrak{g}$ be a Lie algebra homomorphism. Let $\hat{\mathfrak{h}} \subset \mathfrak{h} \times \mathfrak{g}$ be its graph, i.e. $\hat{\mathfrak{h}} = \{(X,\psi(X)) : X \in \mathfrak{h}\}$. This is a Lie subalgebra of $Lie(H \times G) = \mathfrak{h} \times \mathfrak{g}$, so there is a unique Lie subgroup $\hat{H} \subset H \times G$ with $Lie(\hat{H}) = \hat{\mathfrak{h}}$. The first projection $H \times G \to H$ restricts to a homomorphism $\hat{H} \to H$ which is an isomorphism on Lie algebras. By a previous theorem it must be a covering of Lie groups. Since H is simply connected we have that $\hat{H} \cong H$. Let φ be the restriction of $H \times G \to G$ to $\hat{H}$. This is the required homomorphism, which is unique by the previous corollary. □

At this stage we have proved:

Theorem 4.11.2 (Lie's third theorem)**.** *The functor Lie from the category of connected and simply connected Lie groups to finite-dimensional real Lie algebras is fully faithful.*

The following theorem of Ado is not easy (for a proof see Jacobson's *Lie Algebras*):

Theorem 4.11.3 (Ado)**.** *Every finite-dimensional Lie algebra (over $\mathbb{R}$) is linear, i.e. has an injective Lie algebra homomorphism to $M(n,\mathbb{R})$ for some n.*

Applying Lie's first theorem we see that every finite-dimensional Lie algebra is the Lie algebra of a Lie group. This gives the strong form of Lie's third theorem:

Theorem 4.11.4 (Cartan)**.** *The functor Lie is an equivalence of categories from connected and simply-connected real Lie groups to finite-dimensional real Lie algebras.*

Remarks 4.11.5. (1) It is usually the weak form of Lie's third theorem that is used, because it is usually Lie groups which arise naturally.

(2) The essential surjectivity of *Lie* can be proved directly, i.e. without using Ado's theorem. Such a proof is given in Appendix B of Knapp's *Lie Groups Beyond an Introduction.*)

Let us summarize our picture of the situation: Any Lie group G is the quotient $\tilde{G}/C$ of its simply connected cover by a subgroup C of the centre of $\tilde{G}$. Different groups with the same Lie algebra come from choosing different subgroups of the centre. So in practice many questions about Lie groups reduce to considering simply connected groups (which can often be reduced to a purely Lie algebra question) and then considering the effect of the centre (which is a purely "discrete" question).

4.12. **Example: Universal cover of $SL(2,\mathbb{R})$.** Let $G = SL(2,\mathbb{R})$ and let $\pi : \tilde{G} \to G$ be its universal cover. Note that $\pi_1(G) = \pi_1(SO(2)) = S^1) \cong \mathbb{Z}$, so the kernel of $\tilde{G} \to G$ is infinite cyclic. We will see that any Lie group homomorphism $\rho : \tilde{G} \to GL(n,\mathbb{R})$ must be trivial on $ker(\pi)$, i.e. must factor through π. In particular it cannot be injective. Thus not every Lie group can be thought of as a subgroup of a general linear group.

Suppose that $\rho : \tilde{G} \to GL(n,\mathbb{C})$ is a Lie group homomorphism. Let $d\rho : Lie(\tilde{G}) \to M(n,\mathbb{C})$ be the induced Lie algebra homomorphism. Composing with the inverse of the isomorphism $d\pi : Lie(\tilde{G}) \to sl(2,\mathbb{R})$ and complexifying gives a Lie algebra homomorphism

$$\psi := \left(d\rho \circ (d\pi)^{-1}\right) \otimes id_{\mathbb{C}} : sl(2,\mathbb{C}) \to M(n,\mathbb{C}).$$

Since $SL(2,\mathbb{C})$ is simply connected this can be integrated to give a Lie group homomorphism

$$\varphi : SL(2,\mathbb{C}) \to GL(n,\mathbb{C})$$

with $d\varphi = \psi$. Then $\varphi|_{SL(2,\mathbb{R})} \circ \pi$ and ρ are two Lie group homomorphisms $\tilde{G} \to GL(n,\mathbb{C})$ and

$$d\left(\varphi|_{SL(2,\mathbb{R})} \circ \pi\right) = d\varphi|_{SL(2,\mathbb{R})} \circ d\pi = \psi|_{sl(2,\mathbb{R})} \circ d\pi = d\rho.$$

Thus $\rho = \varphi|_{SL(2,\mathbb{R})} \circ \pi$ and thus ρ factors through π.

Exercise. Show using a similar argument that the universal cover of $SL(n,\mathbb{R})$ has no injective Lie group homomorphism to a general linear group. (In contrast, the universal cover of the subgroup $SO(n) \subset SL(n,\mathbb{R})$ is necessarily linear since it is a compact group, and compact groups are always linear (as a consequence of the Peter-Weyl theorem).)

4.13. **Closed subgroups ($*$).** The following theorem will not be used, but it is important and often useful:

Theorem 4.13.1 (Cartan)**.** *A closed subgroup of a Lie group is a Lie subgroup. (More precisely, if $A \subset G$ is a closed subgroup then there exists a smooth submanifold structure on A such that $A \subset G$ is a Lie subgroup. This structure is unique.)*

The correspondence between Lie subgroups and Lie subalgebras means that a closed subgroup must have a Lie algebra. It is easy to define it: If $A \subset G$ is a closed subgroup then $\mathfrak{a} = \{X \in Lie(G) : exp(tX) \in A \text{ for all } t\}$. The proof requires showing that $exp : \mathfrak{a} \to A$ is a homeomorphism near the identity, so that it can be used to define charts on A. For the proof see e.g. Warner p. 110ff.

Note that this theorem can also be used to see that the examples of linear algebraic groups given above are indeed Lie groups, since they are closed in $GL(n,\mathbb{R})$ (or $GL(n,\mathbb{C})$).

(In the sequel, the phrase "closed Lie subgroup" will be used; in fact by this theorem we can just say "closed subgroup".)

However, we will need one property of closed Lie subgroups. Here by a **closed Lie subgroup** we will mean a Lie subgroup $\iota : H \to G$ such that $\iota(H)$ is closed in G. As usual, we will drop ι from the notation and write $H \subset G$ etc.

Lemma 4.13.2. *A closed Lie subgroup of a Lie group is a embedded submanifold (and in fact a closed submanifold).*

Proof. Let H be a Lie subgroup of G. It is an integral manifold of the subbundle defined by $Lie(H) \subset Lie(G)$. Recall that in the proof of the factorization property for integral manifolds we showed that there is a neighbourhood U of $e \in G$ with coordinates $x_1, \ldots, x_n$ in which the integral manifolds are contained in slices, in which $U \cap H$ is given by a slice $\{x_{k+1} = \cdots = x_n = 0\}$, and under the mapping $U \to \mathbb{R}^{n-k}$ given by $\pi(x_1, \ldots, x_n) = (x_{k+1}, \ldots, x_m)$ the image $\pi(U \cap H)$ is a countable subset. Now since H is closed, the image $\pi(U \cap H)$ is locally closed. A countable locally closed set in $\mathbb{R}^{n-k}$ has at least one isolated point. Thus there is some component of $U \cap H$ on which the induced topology agrees with the topology of H. Translating using H implies this holds everywhere on H. This shows that H is embedded. □

(The converse of this lemma also holds.)

4.14. **Group actions and homogeneous manifolds (∗).** We will consider only left actions: A left action of a Lie group G on a smooth manifold M is a smooth map $\alpha : G \times M \to M$ such that (1) $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$ and $x \in M$ and (2) $\alpha(e, x) = x$ for all $x \in M$.

Theorem 4.14.1. *If H is a closed Lie subgroup of G then there is a manifold structure on the space of cosets G/H such that $\pi : G \to G/H$ is a smooth submersion and a fibre bundle (with fibres isomorphic to H). If H is a normal closed Lie subgroup then with this manifold structure G/H is a Lie group. The translation action $G \times G/H \to G/H$ is a transitive smooth left action.*

Conversely, if $G \times M \to M$ is a transitive smooth left action of G on M and $x \in M$ then the orbit mapping $G/G_x \to M$ is a diffeomorphism.

Let $\varphi : H \to G$ be a homomorphism of Lie groups. Applying this theorem to the action $H \times G \to G$ by $(h, g) \mapsto \varphi(h)g$ gives:

Corollary 4.14.2. *The kernel of a Lie group homomorphism $H \to G$ is a closed normal Lie subgroup of H. The image of a Lie group homomorphism $H \to G$ is a Lie subgroup of G.*

4.15. **Examples: Homogeneous spaces (∗).** The previous theorem gives a large number of examples of manifold structures on homogeneous spaces.

(1) Consider the action of $SO(n)$ on S^{n-1} via its action on $\mathbb{R}^n$. The isotropy subgroup of a point is isomorphic to $SO(n-1)$ and hence we have $SO(n)/SO(n-1) \cong S^{n-1}$.
(2) Consider the action of $SU(n)$ on S^{2n-1} via its action on $\mathbb{C}^n$. The isotropy subgroup of a point is isomorphic to $SU(n-1)$ and hence we have $SU(n)/SU(n-1) \cong S^{2n-1}$.
(3) Consider the action of $Sp(n)$ on S^{4n-1} via its action on $\mathbb{H}^n$. The isotropy subgroup of a point is isomorphic to $Sp(n-1)$ and hence we have $Sp(n)/Sp(n-1) \cong S^{4n-1}$.
(4) (real Grassmannians) Let $G = SL(n, \mathbb{R})$ and let H be the subgroup of matrices stabilizing the subspace $\mathbb{R}^k \subset \mathbb{R}^n$ given by the first k coordinates. The space G/H is identified with the space $Gr^{\mathbb{R}}(k, n)$ of k-dimensional subspaces of $\mathbb{R}^n$ and is called the **real Grassmannian** of k-planes in n-space. Note that there is a map $Gr^{\mathbb{R}}(k, n) \to Gr^{\mathbb{R}}(n-k, n)$ given by $V \mapsto V^\perp$ ($\perp$ taken w.r.t. the standard Euclidean inner product on $\mathbb{R}^n$). This map is smooth and hence diffeomorphism (exercise).

   Another description of the Grassmannian is
$$Gr^{\mathbb{R}}(k, n) = O(n)/O(k) \times O(n-k) = SO(n)/S(O(k) \times O(n-k))$$
   which shows that it is compact. In this description the isomorphism $Gr^{\mathbb{R}}(k, n) \cong Gr^{\mathbb{R}}(n-k, n)$ becomes obvious. When $k = 1$ we have $\mathbb{R}P^{n-1} = SO(n)/O(n-1)$.
(5) (complex Grassmannians) Let $G = SL(n, \mathbb{C})$ and let H be the subgroup of matrices stabilizing the subspace $\mathbb{C}^k \subset \mathbb{C}^n$ given by the first k coordinates. The space G/H is identified with the space of k-dimensional complex subspaces of $\mathbb{C}^n$ and is called the **complex Grassmannian** of k-planes in $\mathbb{C}^n$ and denoted $Gr(k, n)$. Once again, taking the $\perp$ w.r.t. a Hermitian form on $\mathbb{C}^n$ gives a diffeomorphism $Gr(k, n) \to Gr(n-k, n)$.

   Another description of the complex Grassmannian is
$$Gr(k, n) = U(n)/U(k) \times U(n-k) = SU(n)/S(U(k) \times U(n-k))$$
   which shows that it is compact. In this description the isomorphism $Gr(k, n) \cong Gr(n-k, n)$ becomes obvious. When $k = 1$ we have $\mathbb{C}P^{n-1} = SU(n)/U(n-1)$.

(6) (symplectic or Lagrangian Grassmanians) Let V be a $2n$-dimensional vector space over $F = \mathbb{R}$ or $F = \mathbb{C}$ with a nondegenerate F-bilinear alternating form $V \times V \to F$. A subspace $I \subset V$ is called **isotropic** if $L^\perp \supset L$, in which case $\dim I \leq n$ and **Lagrangian** if $L^\perp = L$. A Lagrangian subspace is necessarily of dimension n. Let $\Lambda_F(V)$ be the set of Lagrangian subspaces of v, which is naturally a closed subspace of the Grassmannian $Gr_F(n, V)$. Then $\Lambda_F(V)$ is a homogeneous manifold for $Sp(2n, F) = \{g \in GL(2n, F) : (gv, gw) = (v, w) \text{ for all } v, w \in V\}$. (Equivalently, $Sp(2n, F)$ acts transitively on the Lagrangian subspaces of V.)

Alternate descriptions of $\Lambda_\mathbb{R}(V), \Lambda_\mathbb{C}(V)$ are:

$$\Lambda_\mathbb{R}(n) = U(n)/O(n), \qquad \Lambda_\mathbb{C}(n) = Sp(n)/U(n)$$

which show that they are compact. (Alternately, they are closed subsets of $Gr_\mathbb{R}(n, V)$ and $Gr(k, V)$, hence compact.)

(7) (Stiefel manifolds) The space St_k of k-tuples of orthonormal vectors in $\mathbb{R}^n$ is a homogeneous manifold for $O(n)$. The isotropy subgroup in $O(n)$ of the standard k-tuple is identified with $O(n-k)$. Thus $St_k = O(n)/O(n-k) = SO(n)/SO(n-k)$.

The corresponding complex Stiefel manifold classifying k-tuples of Hermitian-orthonormal vectors in $\mathbb{C}^n$ is $U(n)/U(n-k) = SU(n)/SU(n-k)$.

(8) (flag manifolds) Let $F = \mathbb{R}$ or $F = \mathbb{C}$. Let V be a vector space of dimension n over F. A (full) flag in V is a chain of F-subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

such that $\dim_F V_i = i$. Show that the space X of full flags (naturally a closed subset of the product of Grassmannians $Gr_F(1, n) \times \cdots \times Gr_F(n-1, n)$) is a homogeneous manifold for $SL(n, F)$ (or $GL(n, F)$).

(9) (symplectic flag manifolds) Let V be an F-vector space of dimension $2n$. Let X be the set of chains of isotropic subspaces

$$\{0\} = I_0 \subset I_1 \subset \cdots \subset I_n$$

where $\dim_F I_k = k$. (So I_n is Lagrangian.) This is a closed subset of a product of n Grassmannians. This is a homogeneous manifold for $Sp(2n, F)$. Note that there is an $Sp(2n, F)$-equivariant map $X \to \Lambda(n)$ to the Lagrangian Grassmannian.

4.16. π_0 and π_1 of some classical groups (∗).

The following lemma is easy:

Lemma 4.16.1. *If G is a topological group and H is a closed connected subgroup such that G/H is connected (in the quotient topology) then G is connected.*

Corollary 4.16.2. *$SO(n), SU(n), Sp(n)$ are connected.*

Using $det : O(n) \to \{\pm 1\}$ we see that $O(n)$ has two components. Using $det : U(n) \to U(1)$ we see that $U(n)$ is connected.

Proposition 4.16.3. *$SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$ are connected.*

Proof. The Gram-Schmidt orthogonalization procedure, with parameters, shows that $O(n)$ is a deformation retract of $GL(n, \mathbb{R})$, which retracts the identity component $GL(n, \mathbb{R})^0$ onto $SO(n)$. The analogue in the Hermitian setting shows that $GL(n, \mathbb{C})$ has a deformation retraction onto $U(n)$ and $SL(n, \mathbb{C})$ has a deformation retraction onto $SU(n)$.

Instead of using this one can show that there is a diffeomorphism $GL(n, \mathbb{R})^+ \cong SO(n) \times Pos(n, \mathbb{R})$ where $GL(n, \mathbb{R})^+$ is the invertible matrices with determinant > 0 and $Pos(n)$ is the space of positive definite symmetric $n \times n$ real matrices. See Warner p. 131 for a proof of this. The analogue in the complex case is that $GL(n, \mathbb{C})$ is diffeomorphic to $U(n)$ times the space of positive definite Hermitian matrices. □

Lemma 4.16.4. *If $H \subset G$ is a subgroup and H^0 is the identity component of H then $p : G/H \to G/H^0$ is a covering. (In particular, if G/H is simply-connected then H must be connected.)*

Proof. Since H^0 is open in H there is a neighbourhood V of e in G such that $V^{-1}V \cap H \subset H^0$. Refer to the following diagram to fix notation:

$$\begin{array}{ccc} G & \xrightarrow{\pi^0} & G/H^0 \\ \| & & \downarrow p \\ G & \xrightarrow{\pi} & G/H. \end{array}$$

Now I claim that for $g \in G$ the nhbd $\pi(gV)$ of gH in G/H is evenly covered by p.

Consider the open sets $\pi^0(gV\theta)$ as θ runs over a set of representatives $\Theta \subset H$ for the cosets of H^0. Each of them maps homeomorphically onto $\pi(gV)$ under p, so it will be enough to show that

$$p^{-1}(\pi(gV)) = \bigsqcup_{\theta \in \Theta} \pi^0(gV\theta).$$

The inclusion $\supset$ is clear, so let us prove $\subset$. If $p(xH^0) = \pi(gv) = gvH$ for $v \in V$ then $xH = gvH$ hence $v^{-1}g^{-1}x \in H$. Then $gvH^0v^{-1}g^{-1}x = xH^0$ and hence $xH^0 \in \pi^0(gV\theta)$. We leave the disjointness of the union as an exercise. □

Lemma 4.16.5. *Let G be a connected Lie group and H a connected closed Lie subgroup.*

(i) If H is simply-connected and G/H is simply-connected then G is simply-connected.

(ii) More generally, if G/H is simply-connected then $\pi_1(H) \to \pi_1(G)$ is surjective.

Proof. (i) follows from (ii), so we prove (ii). Let $\pi : \tilde{G} \to G$ be the universal cover of G. Then π induces a map $\tilde{G}/\pi^{-1}(H) \to G/H$ which is easily seen to be a diffeomorphism (it is a bijective map which is a local diffeomorphism). Since $\tilde{G}/\pi^{-1}(H)$ is simply connected, by the previous lemma $\pi^{-1}(H)$ is connected. Thus $\pi : \pi^{-1}(H) \to H$ is a connected covering and hence the universal cover $\pi_H : \tilde{H} \to H$ of H factors as

$$\tilde{H} \xrightarrow{p} \pi^{-1}(H) \xrightarrow{\pi} H.$$

Now p is surjective (it is a covering) and hence induces a surjection of $ker(\pi_H) = \pi_1(H)$ onto $ker(\pi : \pi^{-1}(H) \xrightarrow{\pi} H) = ker(\pi : \tilde{G} \to G) = \pi_1(G)$. (We leave as an exercise that this is the natural map $\pi_1(H) \to \pi_1(G)$, although it is not needed.) □

Corollary 4.16.6. *The groups $SU(n), Sp(n), SL(n, \mathbb{C})$ are simply-connected. There is a surjection $\mathbb{Z}/2\mathbb{Z} \to \pi_1(SO(n)) = \pi_1(SL(n, \mathbb{R}))$ for $n \geq 3$.*

In fact we can do more if we use the long exact sequence of homotopy groups of a fibration. (We are assuming here that the quotient map $G \to G/H$ is a fibration. In fact it is even better; it is a fibre bundle.) We have seen that $SO(3)$ admits a double cover, namely $SU(2) \cong S^3$, so that $\pi_1(SO(3)) = \mathbb{Z}/2$. The homotopy long-exact sequence for the fibration $SO(n+1) \to S^n$, with fibres isomorphic to $SO(n)$ is:

$$\cdots \to \pi_3(SO(n)) \to \pi_3(SO(n+1)) \to \pi_3(S^n) \to \pi_2(SO(n)) \to$$
$$\to \pi_2(SO(n+1)) \to \pi_2(S^n) \to \pi_1(SO(n)) \to \pi_1(SO(n+1)) \to 0$$

One sees that if $n \geq 3$ then $\pi_1(SO(n+1)) = \pi_1(SO(3))$. By induction, we conclude that

$$\pi_1(SO(n)) = \mathbb{Z}/2 \quad (n \geq 3)$$

and more generally that $\pi_k(SO(n)) = \pi_k(SO(n+1))$ if $k+1 < n$. Since $SO(3) = \mathbb{R}P^3$, we get $\pi_2(SO(3)) = 0$ and hence $\pi_2(SO(4)) = 0$. Using the previous assertions we see that

$$\pi_2(SO(n)) = 0 \quad (\text{all } n).$$

One also sees from the sequence with $n = 3$ that the map $\pi_2(S^2) \to \pi_1(SO(2))$ must be $\mathbb{Z} \to \mathbb{Z}$ by multiplication by 2. and hence that $\pi_2(SO(3)) = 0$. Since $\pi_2(S^1) = \pi_3(S^1) = 0$, we get $\pi_3(SO(3)) = \pi_3(S^2) = \mathbb{Z}$. (The last group is generated by the Hopf map.)

- Use the fibrations $SU(n) \to S^{2n-1}$ and $Sp(n) \to S^{4n-1}$ to show that $\pi_2(SU(n)) = \pi_2(Sp(n)) = 0$.

In fact for any compact Lie group with zero-dimensional centre, $\pi_1(G)$ is finite, $\pi_2(G)$ is trivial, and $\pi_3(G) = \mathbb{Z}$.

The fact that $\pi_1(SO(n)) = \mathbb{Z}/2$ tells us that there is another sequence of simply-connected compact groups, namely the universal (=double) cover of $SO(n)$. These are called the **spin groups** and denoted

$$Spin(n).$$

In the case of $n \leq 6$ they are in fact already on our list above:

4.17. **Example:** $Spin(4)$ **($*$).** There is an action of $S^3 \times S^3$ on $\mathbb{H}$ given by thinking of S^3 as quaternions of norm one:

$$(\alpha, \beta) \cdot \theta = \alpha\, \theta\, \beta \qquad (\theta \in \mathbb{H}).$$

This action (evidently) preserves the norm quadratic form on $\mathbb{H}$, which is positive definite. So we have a map

$$S^3 \times S^3 \to SO(4)$$

which is easily seen to be a group homomorphism. Check that the kernel is precisely $\{\pm(1,1)\} = \mathbb{Z}/2$. By computing the Lie algebra map we can check that this is nonsingular at the identity, hence a double covering. It follows that

$$Spin(4) = S^3 \times S^3 = SU(2) \times SU(2).$$

There are two more isomorphisms

$$Spin(5) = Sp(4)$$

and

$$Spin(6) = SU(4).$$

But for $n \geq 7$ the spin groups $Spin(n)$ are new simply connected compact groups, not in the list of examples above. These groups are linear, i.e. they can be embedded in $GL(n, \mathbb{R})$ but this requires the theory of Clifford algebras. They do not have a "canonical" embedding in $GL(n)$ like the series $SU(n)$ and $Sp(n)$, however.

Here is another application of the simple connectedness of $SU(n)$: (There are other ways to prove this result, of course.)

Proposition 4.17.1. *If G is a finite group then there is a compact manifold with $\pi_1(M) = G$.*

Proof. Suppose that there is an injective homomorphism $G \hookrightarrow SU(n)$ for some $n \neq 2$. Then $M = SU(n)/G$ is a compact manifold with $\pi_1(M) \cong G$.

To show that $G \hookrightarrow SU(n)$ it is enough to treat the case of the symmetric group $\mathfrak{S}_k$ on k letters (since there is an embedding $G \hookrightarrow \mathfrak{S}_k$ by somebody's theorem in elementary group theory). We can embed $\mathfrak{S}_k$ in $U(k)$ as the matrices of the permutation of the coordinates, i.e. σ is sent to the matrix of the linear transformation $\mathbb{C}^n \to \mathbb{C}^n$ defined by $z_i \mapsto z_{\sigma(i)}$. The image is not in $SU(k)$, however, so we compose with the homomorphism $U(k) \hookrightarrow SU(2k)$ by $g \mapsto \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix}$ to get an injective homomorphism $\mathfrak{S}_k \hookrightarrow SU(2k)$. $\square$

Remark 4.17.2. The question of when a group can be realized as the fundamental group of a smooth projective complex variety is very interesting. Every finite group can be, by a result of Serre. If we drop the smoothness requirement then any finitely presented group is the fundamental group of a complex projective variety (Simpson), but there are nontrivial restrictions in the smooth case.

4.18. **Hopf's theorem and compact Lie groups ($*$).** The cohomology algebra $H^*(G, k)$ of a Lie group G with coefficients in a field k of characteristic zero, is a graded-commutative Hopf algebra of finite dimension. By the Hopf structure theorem (proved in the exercises) we have that $H^*(G, k)$ is an exterior algebra generated by elements of odd degree (equivalently, the cohomology algebra of a product of odd-dimensional spheres). In fact it is possible, for the compact Lie groups, to compute the degrees directly from the structure of the Lie group. The integral cohomology of the compact Lie groups is still not fully calculated (as far as I know).

5. Differential forms and de Rham cohomology

The material in this section is in many books, e.g. Warner, Bott-Tu etc. We will use the language of sheaves and presheaves although that will only appear formally in the next section.

5.1. **Vector bundles.**

A (smooth) **vector bundle** of rank r on a manifold M is a smooth manifold E with a smooth map $\pi : E \to M$ together with:

(i) an open covering $M = \cup_\alpha U_\alpha$ of M, together with diffeomorphisms
$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$
such that $\pi|_{\pi^{-1}(U_\alpha)}$ is given by the first projection

(ii) for α and β such that $U_\alpha \cap U_\beta \neq \phi$, a smooth map
$$h_{\alpha\beta} : U_\beta \cap U_\alpha \to GL(r, \mathbb{R})$$
such that the composite
$$\mathbb{R}^r \times U_\alpha \cap U_\beta \xrightarrow{\psi_\beta^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\psi_\alpha} \mathbb{R}^r \times U_\alpha \cap U_\beta$$
is given by $(h_{\alpha\beta}, \mathrm{id})$.

The manifolds E and M are called the **total space** and the **base space**, respectively. Each U_α, ψ_α is called a **local trivialization**, the collection $\{(U_\alpha, \psi_\alpha)\}_\alpha$ is an **atlas** for the vector bundle. The $h_{\beta\alpha}$ are called the **transition functions** of the vector bundle. The **fibres**
$$E_x := \pi^{-1}(x)$$
are naturally vector spaces of dimension r. There are obvious notions of vector bundle homomorphisms and vector bundle isomorphisms. Note that the transition functions are the key data. The satisfy the following **cocycle condition**: If $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ then on this set we have the identity
$$h_{\alpha\beta} h_{\beta\gamma} = h_{\alpha\gamma}. \tag{5.1.1}$$
The vector bundle is determined up to isomorphism by the covering of M and the corresponding transition functions, as the following lemma shows:

Lemma 5.1.1. *Let $\{E_x\}_{x\in M}$ be a collection of vector spaces of dimension r indexed by points of M. Suppose that we are given a covering $\{U_\alpha\}$ of M and vector space isomorphisms $\rho_{\alpha,x} : E_x \to \mathbb{R}^r$ when $x \in U_\alpha$. If the functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ defined by*
$$h_{\alpha\beta}(x) := \rho_{\alpha,x} \circ \rho_{\beta,x}^{-1} \in GL(r, \mathbb{R})$$
are smooth then there is a unique structure of rank r vector bundle on $E := \coprod_x E_x \to M$ for which
$$\left(E_{U_\alpha} = \coprod_{p\in U_\alpha} E_p, \quad \psi_\alpha : \coprod_{p\in U_\alpha} E_p \to U_\alpha \times \mathbb{R}^r \qquad \text{by } \psi_\alpha(v) = (p, \rho_{\alpha,p}(v)) \text{ if } v \in E_p \right)$$
make an atlas and the $h_{\alpha\beta}$ are the transition functions.

Proof. First we must put a smooth manifold structure on the total space $E = \coprod_x E_x$. The proof is similar to the proof that the tangent bundle has a structure of a smooth manifold. Namely, for each member of the covering U_α we have a map
$$\psi_\alpha : E_{U_\alpha} := \coprod_{x\in U_\alpha} E_x \to U_\alpha \times \mathbb{R}^r \qquad \text{by } \psi_\alpha|_{E_x} = \rho_{\alpha,x}.$$
We declare this to be a diffeomorphism, i.e. we give E_{U_α} the topology and smooth structure given by ψ_α. We leave to be checked that the topology on $E = \cup_\alpha E_{U_\alpha}$ defined in this way is second countable and paracompact. To check that we have a smooth atlas we must check that transition functions are smooth, i.e. if $U_\alpha \cap U_\beta \neq \emptyset$ then the map
$$U_\beta \cap U_\alpha \times \mathbb{R}^r \xrightarrow{\psi_\beta^{-1}} E_{U_\alpha \cap U_\beta} \xrightarrow{\psi_\alpha} U_\beta \cap U_\alpha \times \mathbb{R}^r$$
is smooth. By construction this is given by $(id, h_{\alpha\beta})$ which is smooth by hypothesis. This puts a structure of smooth manifold on E, and the map $\pi : E \to M$ is defined in the obvious way on each chart.

We leave the remaining verifications in the lemma to be checked. $\square$

The lemma allows standard constructions of linear algebra to be extended to vector bundles. If $E \to M$ and $E' \to M$ are vector bundles of rank r and r' then there are vector bundles

$$E^*, \quad E^{\otimes k}, \quad \Lambda^k E, \quad E \otimes E', \quad \mathrm{Hom}(E, E'), \quad E \oplus E'$$

over M which, fibrewise, are

$$E_x^*, \quad E_x^{\otimes k}, \quad \Lambda^k E_x, \quad E_x \otimes E_x', \quad \mathrm{Hom}(E_x, E_x'), \quad E_x \oplus E_x'.$$

By the lemma, it is enough to give smooth transition functions. If $h_{\alpha\beta}$ (resp. $h'_{\alpha\beta}$) are transition functions for E (resp. E') then the transition functions for these bundles (with respect to an atlas which trivializes both E and E') are given by

$${}^t h_{\alpha\beta}^{-1} \in GL(r, \mathbb{R}), \quad h_{\alpha\beta} \otimes \cdots \otimes h_{\alpha\beta} \in GL(rr', \mathbb{R}), \quad \wedge^k h_{\alpha\beta} \in GL(\Lambda^k \mathbb{R}^r), \quad \ldots$$

In many books (e.g. Bott-Tu) you will find an alternate way to construct a vector bundle out of transition functions satisfying the cocycle condition. Let $\{U_\alpha\}$ be a covering of M and let $\{h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})\}_{\alpha,\beta}$ be a collection of smooth maps satisfying the cocycle condition (5.1.1). Define the space

$$E := \coprod_\alpha U_\alpha \times \mathbb{R}^r \Big/ \sim$$

where the relation $\sim$ is as follows: For $(x, v) \in U_\alpha \times \mathbb{R}^r$ and $(y, w) \in U_\beta \times \mathbb{R}^r$ we have

$$(x, v) \sim (y, w) \iff x = y \text{ and } v = h_{\alpha\beta}(w).$$

This is an equivalence relation and the quotient defines a topological space with a map $\pi : E \to M$.

• Show that E defined in this way is a smooth manifold. (This amounts to showing that E is diffeomorphic to the construction in Lemma 5.1.1.)

Now suppose that E is a vector bundle, trivialized over $\{U_\alpha\}$ by ψ_α and suppose that ψ'_α is another family of trivializations. It is easy to check that the transition functions $\{h_{\alpha\beta}\}$ and $\{h'_{\alpha\beta}\}$ are related as follows: There exist maps $\lambda_\alpha : U_\alpha \to GL(r, \mathbb{R})$ such that

$$h_{\alpha\beta} = \lambda_\alpha \, h'_{\alpha\beta} \, \lambda_\beta^{-1} \qquad \text{on } U_\alpha \cap U_\beta. \tag{5.1.2}$$

• Show that isomorphism classes of vector bundles on M which admit a trivialization over the cover $\{U_\alpha\}$ are in bijection with collections of maps $\{h_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{R})\}_{\alpha,\beta}$ satisfying the cocycle condition (5.1.1) modulo the relation (5.1.2). This implies the following:

Lemma 5.1.2. *Two vector bundles on M are isomorphic if and only if their transition functions with respect to some open cover of M are equivalent (in the sense of (5.1.2)).*

A **homomorphism** between the vector bundles $\pi : E \to M$ and $\pi' : E' \to M$ is a smooth map $f : E \to E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & = & M \end{array}$$

commutes and which is fibrewise linear, i.e. $E_x \to E'_x$ is linear. Note that the rank of $f|_{E_x}$ may not be constant in x, so that the kernel and cokernel are not necessarily vector bundles. An isomorphism of vector bundles is an invertible homomorphism. (Exs: A vector bundle homomorphism is an isomorphism if and only if it is an isomorphism fibrewise.)

A **subbundle** of a vector bundle $E \to M$ is a vector bundle $E' \to M$ with an injective homomorphism $E' \to E$. Put another way, a subbundle is a submanifold $E' \subset E$ such that there exists an atlas (U_α, ψ_α) for E such that for each α, $\pi^{-1}(U_\alpha) \cap E' = \psi_\alpha^{-1}(\mathbb{R}^{r'} \times \{0\})$. (It is clear that such an E' is a vector bundle and there is an injective vector bundle homomorphism $E' \to E$.) The **quotient bundle** E/E' can be defined as follows: Take the union $\coprod_x E_x/E'_x$ with the isomorphisms $\bar{\rho}_{\alpha,x} : E_x/E'_x \to \mathbb{R}^{r-r'}$ given by

$$E_x/E'_x \to \mathbb{R}^r/(\mathbb{R}^{r'} \times \{0\}) \cong \mathbb{R}^{r-r'}.$$

and apply Lemma 5.1.1. There is an obvious exact sequence of vector bundles

$$0 \to E' \to E \to E/E' \to 0.$$

A **section** of a vector bundle $\pi : E \to M$ over $U \subset M$ is a smooth map $s : U \to \pi^{-1}(U) \subset E$ such that $\pi \circ s = \mathrm{id}$. A section over $U = M$ will be called a **global section**.

Examples 5.1.3. (i) The trivial bundle of rank r, where $\pi : \mathbb{R}^r \times M \to M$ by projection. The transition functions of this bundle are $h_{\alpha\beta} = Id$ for any chart. (It follows from (5.1.2) that a bundle is trivial (i.e. isomorphic to the trivial bundle) if there exists a covering $\{U_\alpha\}_\alpha$ and smooth maps $\lambda_\alpha : U_\alpha \to GL(r,\mathbb{R})$ such that the transition functions are given by $h_{\alpha\beta} = \lambda_\alpha \lambda_\beta^{-1}$.)

(ii) The tangent bundle $TM \to M$ and the cotangent bundle $T^*M \to M$ are vector bundles of rank n. This gives a vector bundle $TM^{\otimes r} \otimes T^*M^{\otimes s}$ for any r, s. A section of this bundle is called a tensor field of type (r, s) on M.

The tangent bundle is trivialized by any atlas on the manifold. In an atlas $\{(U_\alpha, \varphi_\alpha)\}$ the transition functions of the tangent bundle are given by $h_{\alpha\beta} = Jac(\varphi_\alpha \circ \varphi_\beta^{-1})$. Since the cotangent bundle is its dual it has transition functions $g_{\alpha\beta} = {}^t\Big(Jac(\varphi_\alpha \circ \varphi_\beta^{-1})\Big)^{-1}$ in this atlas.

(iii) The top exterior power $\Lambda^n T^*M$ of the cotangent bundle on a smooth n-manifold M is a line bundle (= rank one vector bundle). The transition functions are given by $h_{\alpha\beta} = det(Jac(\varphi_\alpha \circ \varphi_\beta^{-1})^{-1}$ in an atlas.

(iv) On any manifold M (not necessarily oriented) the **orientation line bundle** is defined, with respect to any atlas, by the transition functions

$$h_{\alpha\beta} = \frac{det(Jac(\varphi_\alpha \circ \varphi_\beta^{-1})}{|det(Jac(\varphi_\alpha \circ \varphi_\beta^{-1})|} \qquad (= \text{ sign of } det(Jac(\varphi_\alpha \circ \varphi_\beta^{-1}))$$

i.e. it is the sign of the transition functions for $\Lambda^n T^*M$. The resulting line bundle is denoted or_M.

(v) The **tautological bundle** on $M = \mathbb{R}P^n$ is a line bundle whose fibre at a point in $\mathbb{R}P^n$ is the corresponding one-dimensional subspace of $\mathbb{R}^{n+1}$. Explicitly, it is the subbundle of the trivial $n+1$-dimensional bundle defined by:

$$E := \{(v, x) \in \mathbb{R}^{n+1} \times \mathbb{R}P^n : v \in x\}.$$

with $\pi : E \to \mathbb{R}P^n$ by $\pi(v, x) = x$.

• Show that this is a vector bundle, i.e. find local trivializations and transition functions for it.

(iv) The **Moebius bundle** on $\mathbb{R}P^1$ is defined as the quotient of $S^1 \times \mathbb{R}$ by the action of $\mathbb{Z}/2$ by $(x, v) \mapsto (-x, -v)$.

• The tautological bundle over $\mathbb{R}P^1 = S^1$ is isomorphic to the Moebius bundle.

(vi) The tautological complex line bundle over $\mathbb{C}P^n$ is

$$E := \{(v, x) \in \mathbb{C}^{n+1} \times \mathbb{C}P^n : v \in x\}.$$

This is a vector bundle in which the fibres are of complex dimension one, so as a real vector bundle it has rank two. The associated circle bundle (in each fibre one takes the circle $S^1 \subset \mathbb{C} = \mathbb{R}^2$) has total space $S^{2n+1} \subset \mathbb{C}^{n+1}$.

(vii) There is a tautological bundle of rank k over the real Grassmannian $Gr(k, n)$. As in the case $Gr(1, n+1) = \mathbb{R}P^n$ it can be realized as the subbundle $\{(v, E) \in F^n \times Gr(k, n) : v \in E\}$ of the trivial bundle $\mathbb{R}^n \times Gr(k, n)$. Similarly for the complex Grassmannian.

Remark 5.1.4. The following obvious remark will be used repeatedly: A vector bundle of rank r is isomorphic to the trivial bundle if and only if it has r sections which are linearly independent at every point. If $E \to M$ is such a bundle and $s_1, \ldots, s_r$ are r such sections then $s_1(x), \ldots, s_n(x)$ are a basis for E_x, so the map sending $v = \sum_i c_i s_i(x) \in E_x$ to $(x, c_1, \ldots, c_n)$ defines a map $E \to M \times \mathbb{R}^r$ which is an isomorphism.

5.2. **Pullback.** A useful construction with vector bundles is the pullback. Given a smooth map $f : M \to N$ and a vector bundle $\pi : E \to N$ the **pullback** of E by f is a vector bundle $f^*E \to M$ which has $(f^*E)_p = E_{f(p)}$. The total space is

$$f^*E := \{(v, p) \in E \times M | \pi(v) = f(p)\}$$

and projection to the second factor gives $f^*E \to M$. To define a vector bundle we must give an open covering and transition functions. If $\{U_\alpha\}_\alpha, \{h_{\alpha\beta}\}_{\alpha,\beta}$ are the data for E then the covering $\{f^{-1}(U_\alpha)\}_\alpha$ and functions $\{f^*h_{\alpha\beta} = h_{\alpha\beta} \circ f\}_{\alpha,\beta}$ define a vector bundle structure on $f^*E \to M$. (Check this.) There is a pullback map on sections: if s is a section of $E \to N$ then f^*s is the section of f^*E defined by $(f^*s)(p) = s(f(p)) \in E_{f(p)} = (f^*E)_p$. (Check that this defines a smooth section of $f^*E \to M$.)

Example 5.2.1. (i) If $i : W \to M$ is a submanifold and E is a vector bundle on M then i^*E is a vector bundle on W, simply the restriction of E to W.

(ii) If $i : W \to M$ is a submanifold then there is a vector bundle monomorphism $TW \to i^*TM$ and the quotient

$$N_{M/W} = i^*TM/TW$$

is called the **normal bundle** of W in M. If M is given a Riemannian metric then $(N_{M/W})_x$ can be identified with $(T_xW)^{\perp}$ in T_xM and this identifies $N_{M/W}$ with a subbundle of i^*TM. However, this is not canonical and so you should not think of $N_{M/W}$ as a subbundle of i^*TM but only as a quotient.

(iii) There is a natural map $Gr(k, n) \to \mathbb{R}P^{\binom{n}{k}-1}$ by sending $E \subset \mathbb{R}^n$ to $\wedge^k E \subset \wedge^k\mathbb{R}^n$. What is the pullback of the tautological bundle?

5.3. **Presheaves and sheaves.** It will be convenient to use the language of sheaves in what follows.

Let X be a topological space. Let $Op(X)$ be the category with objects the open sets in X, and morphisms inclusions between open sets. A **presheaf of sets** on X is a contravariant functor $F : Op(X) \to Sets$ such that $F(\emptyset)$ is a one-element set and $F(id : U \to U)$ is the identity map of $F(U)$. A **presheaf of abelian groups** on X is a contravariant functor from the category $Op(X)$ to the category of abelian groups which takes the empty set $\emptyset$ to $\{e\}$ and the morphism $U \subset U$ to the identity. Concretely, this means we have, for each open set $U \subset X$, an abelian group $F(U)$ and for each pair of open sets $V \subset U$ we have a homomorphism $\rho^U_V : F(U) \to F(V)$, such that for $W \subset V \subset U$ we have

$$\rho^V_W \circ \rho^U_V = \rho^U_W$$

and moreover $F(\emptyset) = \{e\}$ and $\rho^U_U = id$. The homomorphism ρ^U_V will be called "restriction from U to V". An element $s \in F(U)$ is referred to as a **section** of F over U.

Some examples of presheaves:

(1) For a topological space X, the functor $U \mapsto C(U, \mathbb{R})$ (continuous real-valued functions on U) with ρ^V_U restriction of functions from V to $U \subset V$.
(2) For a smooth manifold M the functor $U \mapsto C^\infty(U)$ with restriction of functions defines the sheaf of smooth functions.
(3) For a complex manifold the functor $U \mapsto \{\text{holomorphic functions } U \to \mathbb{C}\}$ with restriction of functions.
(4) For a smooth manifold M the functor $U \mapsto \mathscr{X}(U) =$ smooth vector fields on U with obvious restriction maps.
(5) Let X be a topological space and $k \geq 0$ an integer. The functor $U \mapsto H^k(U, \mathbb{Z})$ (singular cohomology groups of U) defines a presheaf, where $\rho^V_U : H^k(V, \mathbb{Z}) \to H^k(U, \mathbb{Z})$ is the restriction in cohomology.
(6) For $X = \mathbb{C}$ the functor $U \mapsto \{\text{bounded holomorphic functions on } U\}$ defines a presheaf.

A morphism between presheaves F and G has the obvious meaning, namely a natural transformation of functors from F to G. Concretely this means we are given for each $U \subset X$ a homomorphism $F(U) \to G(U)$ and for every pair $V \subset U$, the diagram

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ {\scriptstyle \rho^U_V}\big\downarrow & & \big\downarrow{\scriptstyle \rho^U_V} \\ F(V) & \longrightarrow & G(V) \end{array}$$

commutes.

A presheaf F on X is a **sheaf** if it satisfies the following two conditions:

(S1) If $s_1, s_2 \in F(U)$ and $\{U_\alpha\}_\alpha$ is an open covering of U such that $\rho^U_{U_\alpha}(s_1) = \rho^U_{U_\alpha}(s_2)$ for all α, then $s_1 = s_2$
(S2) If U is open in X, $\{U_\alpha\}_\alpha$ is an open covering of U and $\{s_\alpha \in F(U_\alpha)\}_\alpha$ a collection of sections satisfying $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$ for all α, β, then there is a section $s \in F(U)$ such that $s_\alpha = \rho^U_{U_\alpha}(s)$ for all α.

Informally, (S2) says that local sections of sheaves glue together to give sections, (S1) says that sections of sheaves are determined by what they are locally.

Among the examples of presheaves given above, (1)–(4) are sheaves. The example (5) for $k \geq 1$ fails (S1) in general (think of a manifold with nontrivial H^1). The example (6) also fails (S2). ([19])

A morphism of sheaves is a morphism between them considered as presheaves, i.e. a natural transformation between the functors.

Given a vector bundle $\pi : E \to M$ on a manifold M, the functor on $Op(M)$

$$U \mapsto \{ \text{ sections of } E \text{ over } U \ \}$$

defines a presheaf of abelian groups on M which is easily seen to be a sheaf. This is called the **sheaf of sections** of $E \to M$. It is a sheaf of modules over the sheaf of smooth functions on M.

• A homomorphism $E \to E'$ is the same as a section of the vector bundle $Hom(E, E')$.

A vector bundle $E \to M$ is trivial if and only if the sheaf of sections is isomorphic, as a module over the sheaf of smooth functions $\mathscr{E}^0$, to the sheaf $\mathscr{E}^0 \oplus \cdots \oplus \mathscr{E}^0$ ($r = rk(E)$ copies).

Let F be a sheaf on X. The **stalk** of F at $x \in X$ is the direct limit

$$F_x := \varinjlim_{x \in U} F(U).$$

(In the case of smooth functions this is the space of germs of smooth functions at x.)

Lemma 5.3.1. *A homomorphism $F \to G$ of sheaves is an isomorphism if and only if the induced map on stalks $F_x \to G_x$ is an isomorphism for all $x \in X$.*

We will say more about sheaves later. For the moment we take as a definition that a sequence of sheaves $E \to F \to G$ is exact if the sequences of stalks $E_x \to F_x \to G_x$ is exact for all $x \in X$.

5.4. **Differentials or 1-forms.**

A differential or **differential 1-form** on a manifold M is a section of the cotangent bundle. (This is the dual notion to the notion of vector field.)

Suppose that we are in $\mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$. The vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ form a basis for the tangent bundle over smooth functions. Let $dx_1, \ldots, dx_n$ be the dual basis of 1-forms defined by

$$dx_j \left(\frac{\partial}{\partial x_i} \right) = \delta_{ij}.$$

Then every 1-form on $\mathbb{R}^n$ can be written as

$$\sum_i f_i \, dx_i$$

for smooth functions f_i. Thus on a smooth manifold a 1-form can be written locally in this form.

Let $f : M \to \mathbb{R}$ be a smooth function. Then $df : TM \to \mathbb{R}$ is a linear form on the tangent bundle, i.e. a section of the cotangent bundle. Let us introduce the notation

$$\mathscr{E}^0 := \text{ sheaf of smooth functions}$$

$$\mathscr{E}^1 := \text{ sheaf of smooth differential 1-forms.}$$

Then $f \mapsto df$ defines a sheaf homomorphism

$$d : \mathscr{E}^0 \to \mathscr{E}^1$$

The kernel is the sheaf of constants, i.e. $ker(d)(U)$ consists of functions that are constant on each connected component of U. The forms in the image are called **exact**.

• Show that in $\mathbb{R}^n$ the 1-form $\omega = \sum_i a_i \, dx_i$ is of the form df if and only if $\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}$ for all i and j.

• Show that the differential $d\theta$ on S^1 is not of the form df for a function f. (Here $d\theta$ means the differential which is $d(\theta)$ in a coordinate patch - the function θ does not make sense globally on S^1.)

• Show that the differential

$$\omega = \frac{-y}{(x^2 + y^2)^{\frac{1}{2}}} \, dx + \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \, dy$$

on $\mathbb{R}^2 - \{0\}$ is not exact. (Hint: Integrate ω over the circle.)

[19] An example of a presheaf which fails (S2) and (S1) is the following: Let $X = \{x, y\}$ with the discrete topology. For an open set $U \subset X$ let $F(U)$= functions from U to $\mathbb{R}$. For $V \subsetneq U$ let $\rho_V^U := 0$.

5.5. **Differential forms.** A **differential k-form** on M is a section of the kth exterior power $\Lambda^k T^*M \to M$ of the cotangent bundle. Concretely, this means we have, for each point $p \in M$, an alternating multilinear form $(T_p^*M)^{\otimes k} \to \mathbb{R}$ which varies smoothly in p. Informally, a k-form is a machine which takes k vector fields and gives back a number, and it is alternating for the action of the symmetric group on k letters on $(T_p^*M)^{\otimes k}$. The assignment

$$\mathscr{E}^k(U) = \{ \text{ differential } k\text{-forms on } U \ \} \qquad (U \subset M \text{ open}).$$

defines a sheaf on M. This is a sheaf of modules over the sheaf of smooth functions $\mathscr{E}^0$.

Suppose we are in $\mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$. The choice of coordinates fixes a trivialization of the tangent bundle as $T\mathbb{R}^n \cong \mathbb{R}^n \times T_0\mathbb{R}^n$ which amounts to taking the n global vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. The dual basis of the cotangent bundle is the set of 1-forms $dx_1, \ldots, dx_n$. Thus a basis (over the smooth functions) of the differential k-forms on an open subset of $\mathbb{R}^n$ is given by the $\binom{n}{k}$ differential k-forms

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $I = \{i_1, \ldots, i_k\}$ runs over all subsets of $\{1, \ldots, n\}$ of size k. Note that dx_I stands for the alternating multilinear form of k variables defined by:

$$dx_I(X_1, \ldots, X_k) = \det\left(dx_{i_r}(X_s)\right)$$

where $X_1, \ldots, X_k$ are sections of TM, i.e. vector fields. So for example $(dx_1 \wedge dx_2)(X_1, X_2) = \begin{vmatrix} dx_1(X_1) & dx_1(X_2) \\ dx_2(X_1) & dx_2(X_2) \end{vmatrix}$ and more generally $dx_I((X_i)_{i \in I}) = \left|(dx_i(X_j))_{i \in I, j \in I}\right|$.

If ω is a k-form on a manifold M and U is the domain of a chart with local coordinates $x_1, \ldots, x_n$ then there is a (unique) local expression

$$\omega = \sum_{|I|=k} \omega_I \, dx_I$$

for smooth functions ω_I. (In words, we are using the basis dx_I to trivialize the kth exterior power of the cotangent bundle and writing a general section of this bundle in terms of the basis, with coefficients in $\mathscr{E}^0$.)

If $f : M \to N$ is a smooth manifold then there is a pullback map:

$$f^* : \mathscr{E}^k(M) \to \mathscr{E}^k(N)$$

Pointwise this is given by the multilinear form $(f^*\omega)_p$ defined by:

$$T_pM \times \cdots \times T_pM \xrightarrow{df_p \times \cdots \times df_p} T_{f(p)}N \times \cdots \times T_{f(p)}N \xrightarrow{\omega_{f(p)}} \mathbb{R}.$$

In terms of sections of vector bundles the pullback comes from the pullback of sections of $\mathscr{E}^k_N$ to sections of $f^*\mathscr{E}^k_N$ followed by the sheaf homomorphism $f^*\mathscr{E}^k_N \to \mathscr{E}^k_M$.

5.6. **Exterior derivative.**

The homomorphism

$$d : \mathscr{E}^0 \to \mathscr{E}^1$$

defined above extends to forms of higher order:

Theorem 5.6.1. *There is a unique $\mathbb{R}$-linear sheaf homomorphism $d : \mathscr{E}^k \to \mathscr{E}^{k+1}$ which agrees with $d : \mathscr{E}^0 \to \mathscr{E}^1$ for $k = 0$ and which satisfies*

(i) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge d\beta$ *(Leibniz rule)*
(ii) $d^2 = 0$ *(flatness).*

The map d commutes with pullback: If $f : M \to N$ is a map of manifolds then $d(f^*(\alpha)) = f^*(d\alpha)$.

Proof. We will first prove the uniqueness of such a map (over any manifold). Then we will define d in coordinate patches by an explicit formula. By uniqueness the definitions must agree on overlaps and hence we have a globally defined $d : \mathscr{E}^k \to \mathscr{E}^{k+1}$. (What we are really using here is the fact that the functor $Hom(\mathscr{E}^k, \mathscr{E}^{k+1})$ defined by $U \mapsto Hom(\mathscr{E}^k|_U, \mathscr{E}^{k+1}|_U)$ is itself a sheaf.) Then we will check (i) and (ii) by calculation in coordinate patches.

Let $U \subset M$ be open and $\omega \in \mathscr{E}^k(U)$. To verify that $d\omega$ is uniquely determined by the (i) and (ii) it is enough to do so on a coordinate patch, i.e .we may assume U is contained in a chart with coordinates

$x_1, \dots, x_n$. Write $\omega = \sum_I \omega_I dx_I$. Then $d\,dx_I = 0$ (reduce to the case $d\,dx_i = 0$ by repeated use of (i) and (ii) and $d\,dx_i = 0$ by (ii)) and hence $d\omega = \sum_I d\omega_I \wedge dx_I$. This proves that $d\omega$ is uniquely determined.

To define d we cover M by charts and just use the formula $d\omega = \sum_I d\omega_I \wedge dx_I$ in each coordinate chart. If we verify the Leibniz rule and flatness conditions then by the uniqueness just proved there is agreement on the overlaps of charts, so $d : \mathscr{E}^k \to \mathscr{E}^{k+1}$ is well-defined. For both it is enough to check on forms of the type $f dx_I$. We check flatness:

$$d(d(f dx_I)) = d(df \wedge dx_I) = d\left(\sum_i \frac{\partial f}{\partial x_i} dx_i \wedge dx_I\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I.$$

Now the equality of mixed partials and the relation $dx_i \wedge dx_j = -dx_j \wedge dx_i$ show this is zero. It only remains to check the Leibniz rule for forms of the form $\alpha = f dx_I$ and $\beta = g dx_J$, which is left as an exercise.

Compatibility with pullback is immediate for exact one-forms (i.e. df for f a smooth function) and for functions. The general case follows using that (1) locally on N every differential form is a linear combination of wedge products of exact one-forms with function coefficients, (2) f^* is a ring homomorphism, and (3) the Leibniz rule holds for d. □

- A 1-form $\omega \in \mathscr{E}^1(\mathbb{R}^n)$ is exact (i.e. $\omega = df$ for $f \in \mathscr{E}^0(\mathbb{R}^n)$) if and only if $d\omega = 0$.

The following lemma gives an expression for the exterior derivative which does not depend on local coordinates:

Lemma 5.6.2. *Let α be a k-form on M. For vector fields $X_0, \dots, X_k$,*

$$\begin{aligned} d\alpha(X_0, \dots, X_k) = &\sum_i (-1)^i X_i \alpha(X_0, \dots, \widehat{X_i}, \dots, X_k) \\ &+ \sum_{i<j} (-1)^{i+j-1} \alpha([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k). \end{aligned}$$

Proof. Check that this definition satisfies the Leibniz rule and flatness and conclude by uniqueness that this is the same. (Alternately, a calculation in local coordinates will suffice.) □

5.7. **Lie derivative.** Let X be a vector field on M. The **contraction** operator $i_X : \mathscr{E}^k \to \mathscr{E}^{k-1}$ is defined as follows: For a k-form ω we have

$$i_X \omega(X_1, \dots, X_{k-1}) := \omega(X, X_1, \dots, X_{k-1}).$$

(When $k = 0$ the contraction of a function is defined to be zero.) Evidently $i_X^2 = 0$. The **Lie derivative** operator $L_X : \mathscr{E}^k \to \mathscr{E}^k$ is defined by:

$$L_X \omega := d i_X \omega + i_X d\omega = (d + i_X)^2 \omega$$

This is **Cartan's formula**. ([20]) If $k = 0$, i.e. we have a function, then we have

$$L_X f = i_X\, df = Xf$$

(the vector field applied to the function). The formal properties of the Lie derivative are easy to verify:

Lemma 5.7.1. *The Lie derivative satisfies:*

(i) $dL_X = L_X d$
(ii) $L_X(\omega_1 \wedge \omega_2) = L_X \omega_1 \wedge \omega_2 + (-1)^{\deg(\omega_1)} \omega_1 \wedge L_X \omega_2$ *(Leibniz rule)*
(iii) *If $\varphi : M \to M'$ is a diffeomorphism then $\varphi^*(L_X \omega) = L_{\varphi^*(X)}(\omega)$.*

Proof. (i) follows from $L_X = d i_X + i_X d$ and $d^2 = 0$.

(ii) follows from the Leibniz rule for i_X and d.

(iii) Exercise. □

There is a more geometric description of the Lie derivative (this is usually given as the definition):

[20] Cartan's formula says that the homomorphism of complexes $L_X : (\mathscr{E}^*(M), d) \to (\mathscr{E}^*(M), d)$ is homotopic to the zero map by the homotopy operator i_X. This has a concrete consequence: If we take $A_X^* \subset \mathscr{E}^*(M)$ to be the subcomplex of forms which satisfy $L_X \omega = 0$ then $A_X^* \subset \mathscr{E}^*(M)$ induces an isomorphism in cohomology. In other words, the complex A_X^* also computes the de Rham cohomology of M.

Lemma 5.7.2. *Suppose X is a vector field with associated local flow α. Then*

$$L_X\omega := \frac{d}{dt}\Big|_{t=0} \alpha_t^*(\omega).$$

The proof is left as an exercise. (Hint: Since $L_X = di_X + i_X d$ is a derivation (i.e. satisfies the Leibniz rule) you can reduce to checking the equality locally on functions and exact 1-forms.)

Here is an example showing that the Lie derivative generalizes familiar objects: Let α_t be a local flow on an open subset of $\mathbb{R}^n$ with associated vector field X. Then one has the identity $L_X(dx_1 \wedge \cdots \wedge dx_n) = div(X)\, dx_1 \wedge \cdots \wedge dx_n$ where $div(X)$ is the classical divergence of a vector field, defined by $div(\sum_i f_i \frac{\partial}{\partial x_i}) = \sum_i \frac{\partial f_i}{\partial x_i}$.

5.8. **Frobenius theorem revisited.** Suppose F is a rank k integrable subbundle of TM. Let

$$\mathscr{I}^1(F) := \{\omega \in \mathscr{E}^1(M) : \omega(p) \text{ vanishes on } F_p \subset T_pM \text{ for all } p \in M\}.$$

Let $\mathscr{I}(F)$ be the ideal in $\mathscr{E}^*(M)$ generated by $\mathscr{I}^1(F)$. Then we have the following slightly more general version of Frobenius theorem (which we will not prove or use here):

Theorem 5.8.1. *A subbundle $F \subset TM$ has integral manifolds through every point if and only if $d\mathscr{I}(F) \subset \mathscr{I}(F)$.*

Proof. Exercise. (Use Lemma 5.6.2 and the Frobenius theorem.) □

5.9. **Orientations.** Recall that an orientation for a real vector space is an equivalence class of bases, under the relation that two bases are equivalent if the matrix relating them has positive determinant. There are thus two possible orientations for a real vector space.

The top exterior power $\Lambda^{n=dimV}V$ is a one-dimensional $\mathbb{R}$-vector space. A choice of generator for $\Lambda^n V$ specifies an orientation uniquely: any generator of $\Lambda^n V$ is of the form $v_1 \wedge \cdots \wedge v_n$ and it gives the basis $\{v_1, \ldots, v_n\}$. Conversely an orientation gives a generator of $\Lambda^n V$ (nonuniquely): If $\{v_1, \ldots, v_n\}$ is any basis in the equivalence class then $v_1 \wedge \cdots \wedge v_n$ spans $\Lambda^n V$. The fact that for $g \in GL(V)$ the operator $\Lambda^n g : \Lambda^n V \to \Lambda^n V$ is multiplication by $det(g)$ makes this well-defined.

An **oriented atlas** for a manifold M is a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ in which the transition functions satisfy the condition

$$det(Jac(\varphi_\beta \circ \varphi_\alpha^{-1})) > 0 \qquad \text{on } \varphi(U_\alpha \cap U_\beta) \subset \mathbb{R}^n.$$

If M admits an oriented atlas it is called **orientable**. Two oriented atlases which are compatible (in the sense that their union is a smooth atlas) are compatibly oriented if their union is oriented. This is an equivalence relation on oriented atlases; an equivalence class under this relation is an **orientation** on M. If we fix an orientation we call M **oriented**.

Here is an equivalent way to define an orientation: An orientation is the choice of an orientation for each $(T_xM)^*$ (or T_xM) which is locally constant in x. Here by locally constant we mean the following: given x there is a chart around U with coordinates $x_1, \ldots, x_n$ such that at any point in U the orientation of $(T_xM)^*$ is the same as $dx_1 \wedge \cdots \wedge dx_n$. It is an easy exercise to see that this is equivalent to the notion defined above.

Just as it is convenient to think of orientations of a vector space in terms of the top exterior power it is convenient to think of orientations in terms of sections of the bundle $\Lambda^n T^*M$. We have:

Lemma 5.9.1. *An n-manifold M is orientable if and only if $\Lambda^n T^*M$ has a nowhere-vanishing section.*

Proof. Suppose M is orientable. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an (locally finite) oriented atlas. Choose a partition of unity $\{f_\alpha\}$ subordinate to the covering. Define

$$\omega := \sum_\alpha f_\alpha \varphi_\alpha^*(dx_1 \wedge \cdots \wedge dx_n).$$

Let us check that ω is nowhere zero. Let $x \in M$ and let $U_{\alpha_1}, \ldots, U_{\alpha_r}$ be the charts containing x. Then

$$\omega(x) = \sum_{i=1}^r f_{\alpha_i}(x)\varphi_{\alpha_i}^*(dx_1^i \wedge \cdots \wedge dx_n^i)$$

where $x_1^i, \dots, x_n^i$ are the coordinates on $\varphi_{\alpha_i}(U_{\alpha_i}) \subset \mathbb{R}^n$. Now the coefficients $f_{\alpha_i}(x)$ are all positive, so it is enough to check that all the $\varphi_{\alpha_i}^*(dx_1^i \wedge \cdots \wedge dx_n^i)$ lie in the same component of $\Lambda^n T_x^* M - \{0\}$. But the change of coordinates from $x_1^i, \dots, x_n^i$ to $x_1^j, \dots, x_n^j$ is $\varphi_{\alpha_j} \circ \varphi_{\alpha_i}^{-1}$ and hence on $U_{\alpha_i} \cap U_{\alpha_j}$ we have

$$\varphi_{\alpha_i}^*(dx_1^i \wedge \cdots \wedge dx_n^i) = det(Jac(\varphi_{\alpha_i} \circ \varphi_{\alpha_j}^{-1}))\, \varphi_{\alpha_j}^*(dx_1^j \wedge \cdots \wedge dx_n^j).$$

Since the atlas is oriented we know $det(Jac(\varphi_{\alpha_i} \circ \varphi_{\alpha_j}^{-1})) > 0$ and we are done.

Conversely suppose ω is a nowhere-vanishing section of $\Lambda^n T^* M$. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas. For each α write ω on U_α as

$$\omega = g_\alpha\, \varphi_\alpha^*\,(dx_1 \wedge \cdots \wedge dx_n)\,.$$

Then $g_\alpha \in C^\infty(U_\alpha)$ is either always > 0 or always < 0. If, at $x \in M$, the $g_\alpha(x)$ for $U_\alpha \ni x$ all have the same sign then the transition functions $\varphi_{\alpha_i} \circ \varphi_{\alpha_j}^{-1}$ are all orientation preserving at x. Therefore it is enough to modify the atlas so that this holds. Indeed, this is easy to do: If $g_\alpha < 0$ on U_α then replace the chart $(U_\alpha, \varphi_\alpha)$ by the chart $(U_\alpha, s \circ \varphi_\alpha)$ where $s(x_1, x_2, \dots, x_n) = (x_2, x_1, \dots, x_n)$. Now all the functions g_α are > 0 and hence the new atlas is oriented. □

A nowhere-vanishing top form on a manifold is called a **volume form**.

- Show that S^n is orientable and $\mathbb{R}P^n$ is orientable if and only if n is odd.

Remark 5.9.2. Suppose M is a complex manifold. Then its underlying real manifold is always orientable and in fact comes with a preferred choice of orientation. Indeed, consider first the case of $\mathbb{C}^n$ or any open subset of it with coordinates $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$. The nowhere-vanishing $2n$-form

$$(-1)^{\frac{n(n-1)}{2}} \frac{i^n}{2^n} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

gives an orientation on $\mathbb{C}^n$. Now any holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ or between open subsets of $\mathbb{C}^n$ automatically satisfies $det(Jac(f)) \geq 0$. (For $n = 1$ the Jacobian of a holomorphic function $f(z) = u(x, y) + iv(x, hy)$ is $Jac(f) = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$ which by the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ is equal to $\begin{pmatrix} u_x & v_x \\ -v_x & u_x \end{pmatrix}$ so $det(Jac(f)) = u_x^2 + v_x^2 \geq 0$. The determinant is strictly positive everywhere since one cannot have $u_x = v_x = 0$ unless f is constant. For $n > 1$ one has to work with the generalization of the CR conditions.) So a holomorphic atlas of a complex manifold is automatically oriented, and the choice of the orientation above locally fixes a choice of orientation.

5.10. **Integration on oriented manifolds.** On $\mathbb{R}^n$ there is a canonical measure, namely the Lebesgue measure, which is the unique (up to scalar multiple) translation-invariant measure, and with respect to which you can integrate functions. On a general manifold there is no canonical measure (obviously you cannot patch together the Lebesgue measures on charts), so we don't know how to integrate functions. The correct thing to integrate are top forms (on an oriented manifold).

First let us consider the case of an open subset $U \subset \mathbb{R}^n$. Then $\omega \in \mathscr{E}_c^n(U)$ can be written $\omega = f dx_1 \wedge \cdots \wedge dx_n$ and we make the definition

$$\int_U \omega := \int_{x_n=-\infty}^{\infty} \cdots \int_{x_2=-\infty}^{\infty} \int_{x_1=-\infty}^{\infty} f\, dx_1 dx_2 \cdots dx_n \tag{5.10.1}$$

where on the right-hand side one has the usual Lebesgue (or, in this case, Riemann) integral of a function. (f is extended by zero outside U.)

Now if $\varphi : V \to U$ is an orientation-preserving diffeomorphism, then, by assumption, the Jacobian determinant $\det\left(\frac{\partial \varphi_i}{\partial y_j}\right)$ is positive on V. Then by the usual change-of-variables formula for integration in $\mathbb{R}^n$ we have

$$\int_V \varphi^*(\omega) = \int_U \omega. \tag{5.10.2}$$

If φ is orientation-reversing then this is true with a minus sign.

We will use the following simple lemma. (We could have used the earlier lemmas about partitions of unity, but this is even easier.)

Lemma 5.10.1. *Let $C \subset M$ be a compact subset of a manifold and let $\{U_\alpha\}_\alpha$ be a family of open subsets such that $C \subset \cup_\alpha U_\alpha$. Then there exist finitely many smooth functions $f_i : M \to [0, 1]$ such that each f_i has compact support contained in some U_{α_i} and $\sum_i f_i = 1$ on C.*

Proof. For each $x \in C$ choose $\alpha(x)$ with $x \in U_{\alpha(x)}$. Let $g_x : M \to [0,1]$ be a smooth function with compact support $supp(g_x) \subset U_{\alpha(x)}$ and with g_x identically 1 on some open neighbourhood V_x of x. (If $\varphi : W \to \mathbb{R}^n$ is a chart neighbourhood with $W \subset U_{\alpha(x)}$ then take a bump function b which is constant 1 in a neighbourhood of $\varphi(x)$ and with $supp(b) \subset \varphi(W)$. Then take $g_x = \varphi^* b$ and extend it by zero outside W.)

Now $\{V_x\}_{x\in C}$ is an open cover of C hence has a finite subcover $V_{x_1}, \dots, V_{x_r}$. The product $\prod_i (1-g_{x_i})$ is zero on $V_{x_1} \cup \cdots \cup V_{x_r}$ and hence $h = \sum_i g_{x_i} + \prod_i (1-g_{x_i})$ is > 0 everywhere. Then the collection $f_i := g_{x_i}/h$ is as required. □

Now let M be an oriented n-manifold, i.e. a manifold with a chosen orientation, and let $\omega \in \mathscr{E}^n_c(M)$ be a compactly supported n-form. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas of charts. The lemma gives a finite collection of functions f_i partitioning unity on C with $supp(f_i)$ compact, and for each i there is a α_i with $supp(f_i) \subset U_{\alpha_i}$. Define the **integral of** ω **over** M to be the finite sum:

$$\int_M \omega := \sum_i \int_{\varphi_{\alpha_i}(U_{\alpha_i})} (\varphi_{\alpha_i}^{-1})^*(f_i\omega). \tag{5.10.3}$$

The integrals on the right are defined by (5.10.1).

Lemma 5.10.2. *(i) The integral $\int_M \omega$ is well-defined, i.e. does not depend on the choice of oriented chart giving the fixed orientation or on the choice of partition of unity.*

(ii) It depends on the choice of orientation only up to a sign, i.e. if the orientation on M is reversed then the integral changes sign (because each term in the sum (5.10.3) changes sign).

(iii) If the orientation of M is defined by the nowhere-vanishing form $\omega_0 \in \Lambda^n T^*M$ and $\omega = f\omega_0$ for $f \geq 0$, compactly supported, and not identically zero, then $\int_M \omega \neq 0$.

Proof. Suppose $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are two oriented atlases giving the same orientation. Let $\omega \in \mathscr{E}^n_c(M)$ be a top form with compact support. Using the lemma we can find functions $\{f_i\}_{i=1}^r$ and $\{g_i\}_{i=1}^s$. Then we have

$$\sum_i \int_{U_{\alpha_i}} (\varphi_{\alpha_i}^{-1})^*(f_i\omega) = \sum_i \int_{U_{\alpha_i}} (\varphi_{\alpha_i}^{-1})^* \left(\sum_j g_j f_i \omega\right) = \sum_{i,j} \int_{\varphi_{\alpha_i}(U_{\alpha_i}\cap V_{\beta_j})} (\varphi_{\alpha_i}^{-1})^*(g_j f_i \omega)$$

while

$$\sum_j \int_{V_{\beta_j}} (\psi_{\beta_j}^{-1})^*(f_i\omega) = \sum_i \int_{V_{\beta_j}} (\psi_{\beta_i}^{-1})^* \left(\sum_i f_i g_j \omega\right) = \sum_{i,j} \int_{\psi_{\beta_j}(U_{\alpha_i}\cap V_{\beta_j})} (\psi_{\beta_j}^{-1})^*(g_j f_i \omega).$$

Now the diffeomorphism

$$\varphi_{\alpha_i} \circ \psi_{\beta_j}^{-1} : \psi_{\beta_j}(U_{\alpha_i} \cap V_{\beta_j}) \to \varphi_{\alpha_i}(U_{\alpha_i} \cap V_{\beta_j})$$

is orientation-preserving (by the assumption that the two atlases define the same orientation) and hence by (5.10.2) the i, j terms in each expression above are the equal. This proves the first statement.

The other statements of the lemma are easily checked. □

5.11. **Stokes's theorem.**

This theorem is usually stated in the context of manifolds with boundary. To avoid the trouble of introducing this concept we will deal with a slightly simpler situation. A **domain** in an n-manifold M is an open subset $\Omega \subset M$ such that

(i) the (topological) boundary $\partial\Omega = \overline{\Omega} - \Omega$ is an embedded submanifold of M of dimension $n-1$ and
(ii) there is an atlas of M such that for each $(U, x_1, \dots, x_n)$ in the atlas for which $U \cap \partial\Omega \neq \emptyset$, the intersection $\Omega \cap U$ is given by $x_n \leq 0$ (and $U \cap \partial\Omega$ is given by $x_n = 0$).

(This terminology is not entirely standard.) So for example the unit ball B^n in $\mathbb{R}^n$ is a domain; its boundary is S^{n-1}.

Note that the integral defined above can easily be extended to the case of a domain in an oriented manifold. For $\omega \in \mathscr{E}^n_c(M)$ and a domain $\Omega \subset M$, choose an oriented atlas with property (ii) above. Next choose functions f_i and charts U_{α_i} as before. If $U_{\alpha_i} \cap \partial\Omega \neq \phi$ the contribution to the sum (5.10.3) is simply defined to be the Riemann integral

$$\int_{\varphi_{\alpha_i}(U_{\alpha_i}\cap\Omega)} (\varphi_{\alpha_i}^{-1})^*(f_i\omega) = \int_{x_n=-\infty}^{x_n=0} \cdots \int_{x_1=-\infty}^{\infty} g\; dx_1 \dots dx_n.$$

where g has been extended by zero outside $\varphi_{\alpha_i}(U_{\alpha_i} \cap \Omega)$.

The following lemma will allow us to consistently orient the boundary of a domain in an oriented manifold. The proof is left as an exercise.

Lemma 5.11.1. *If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism which carries $\mathbb{H}^n := \{x_n \leq 0\}$ to itself then $det(Jac(\varphi)) > 0$ implies $det(Jac(\varphi|_{\partial\mathbb{H}^n})) > 0$.*

If M is an oriented manifold and $\Omega \subset M$ is a domain then there is an **induced orientation** on $\partial\Omega$. To define this, first consider the special case $(M, \Omega) = (\mathbb{R}^n, \mathbb{H}^n)$. In this case if the orientation of $\mathbb{R}^n$ is defined by the n-form $dx_1 \wedge \cdots \wedge dx_n$ then the induced orientation of $\partial\mathbb{H}^n = \mathbb{R}^{n-1}$ is defined by $(-1)^{n-1}dx_1 \wedge \cdots \wedge dx_{n-1}$. In the general case suppose that M is oriented, i.e. we have fixed an oriented atlas $\{(U_\alpha, \varphi_\alpha)\}$. We may further assume that for a chart U_α such that $U_\alpha \cap \partial\Omega \neq \emptyset$ in the coordinates given by φ_α the domain $U_\alpha \cap \Omega$ is given by $x_n \leq 0$. Then by the previous lemma $\{(U_\alpha \cap \partial\Omega, \varphi_\alpha)\}$ is an oriented atlas for $\partial\Omega$. Locally in a coordinate chart U the orientation on $\partial\Omega \cap U$ is either the induced one defined in the previous paragraph (special case $(M, \Omega) = (\mathbb{R}^n, \mathbb{H}^n)$ or its reverse. The induced orientation is the one which locally looks like the one defined above for the special case. ([21])

Theorem 5.11.2. *(Stokes) Let M be an oriented smooth manifold of dimension n and let Ω be a domain. Let $\partial\Omega$ be the boundary of Ω, with the induced orientation. Let ω be a compactly supported $n-1$-form on M. Then*

$$\int_\Omega d\omega = \int_{\partial\Omega} i^*\omega.$$

In $\mathbb{R}$ this is the fundamental theorem of calculus; in $\mathbb{R}^2$ it is Green's theorem; in $\mathbb{R}^3$ it contains the divergence theorem (Gauss's theorem) and the classical Stokes theorem. Taking $\Omega = M$ gives:

Corollary 5.11.3. *If $\partial M = 0$ then $\int_M d\omega = 0$.*

Proof. First consider the essential special case: $M = \mathbb{R}^n$ and $\Omega = \mathbb{H}^n = \{x_n \leq 0\}$. We assume $\mathbb{R}^n$ is given the orientation $dx_1 \wedge \cdots \wedge dx_n$. Write

$$\omega = \sum_i g_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Then

$$d\omega = \sum_i (-1)^{i-1} \frac{\partial g_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_n$$

and

$$\int_\Omega d\omega = \sum_i (-1)^{i-1} \int_{x_n=-\infty}^{x_n=0} \int_{x_{n-1}=-\infty}^{x_{n-1}=\infty} \cdots \int_{x_1=-\infty}^{x_1=\infty} \frac{\partial g_i}{\partial x_i}\, dx_1 \dots dx_n.$$

Because each g_i vanishes outside a compact set all terms except the $i = n$ term are zero, giving:

$$\begin{aligned}\int_\Omega d\omega &= (-1)^{n-1} \int_{-\infty}^{x_{n-1}=\infty} \cdots \int_{x_1=-\infty}^{x_1=\infty} g_n(x_1, \dots, \dots, x_{n-1}, 0)\, dx_1 \dots dx_{n-1} \\ &= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} i^* g_n\, dx_1 \dots dx_{n-1}.\end{aligned}$$

On the other hand, $i^*\omega = i^* g_n\, dx_1 \wedge \cdots \wedge dx_{n-1}$ since $i^* dx_n = 0$ and $i^* dx_j = dx_j$ for $j < n$. So the equality $\int_\Omega \omega = \int_{\partial\Omega} i^*\omega$ holds if we take into account the definition of induced orientation (i.e. the fact that $dx_1 \wedge \cdots \wedge dx_{n-1}$ is $(-1)^{n-1}$-oriented for the induced orientation on $\mathbb{R}^{n-1}$). The same argument shows that if $\Omega = M = \mathbb{R}^n$ then $\int_{\mathbb{R}^n} d\omega = 0$ for compactly supported ω.

Now consider a general oriented manifold M and a domain $\Omega \subset M$. Choose an oriented atlas $\{U_\alpha\}$ in which each U_α is diffeomorphic to $\mathbb{R}^n$ and such that if $U_\alpha \cap \partial\Omega \neq \emptyset$ then there are coordinates on U_α such that $U_\alpha \cap \partial\Omega$ is given by $x_n \leq 0$. (This can be done since Ω is a domain.) Let $\omega \in \mathscr{E}_c^{n-1}(M)$. Choose

[21] The purpose of the sign in the induced orientation is to avoid a sign in the statement of Stokes's theorem. You will see a sign in books which have a different convention about the induced orientation.

a finite collection of functions $\{f_i\}$ which partition unity on $supp(\omega)$ and with each $supp(f_i) \subset U_{\alpha_i}$ compact. Write $\omega = \sum_i f_i\omega$ so that

$$d\omega = \sum_i d(f_i\omega).$$

Then

$$\begin{aligned}\int_M d\omega = \sum_i \int_M d(f_i\omega) &= \sum_i \int_{U_{\alpha_i}} d(f_i\omega)\\ &= \sum_i \int_{\varphi_{\alpha_i}(U_{\alpha_i}\cap\Omega)} (\varphi_{\alpha_i}^{-1})^*(d(f_i\omega))\\ &= \sum_i \int_{\varphi_{\alpha_i}(U_{\alpha_i}\cap\Omega)} d\left((\varphi_{\alpha_i}^{-1})^*(f_i\omega)\right).\end{aligned}$$

The charts with $U_{\alpha_i} \subset \Omega$ contribute nothing by the special case above for $\Omega = \mathbb{R}^n$. The remaining charts (which intersect $\partial\Omega$) have coordinates in which Ω is given by $x_n \leq 0$. By the special case proved above we have, for such i,

$$\begin{aligned}\int_{\varphi_{\alpha_i}(U_{\alpha_i}\cap\Omega)} d\left((\varphi_{\alpha_i}^{-1})^*(f_i\omega)\right) &= \int_{\varphi_{\alpha_i}(U_{\alpha_i}\cap\partial\Omega)} j^*(\varphi_{\alpha_i}^{-1})^*(f_i\omega)\\ &= \int_{\varphi_{\alpha_i}(U_{\alpha_i}\cap\partial\Omega)} (\varphi_{\alpha_i}^{-1})^* i^*(f_i\omega)\\ &= \int_{U_{\alpha_i}\cap\partial\Omega} i^*(f_i) i^*\omega.\end{aligned}$$

(Here j is the inclusion of $\varphi_{\alpha_i}(U_{\alpha_i} \cap \partial\Omega)$ in $\varphi_{\alpha_i}(U_{\alpha_i} \cap \Omega)$, i.e. $j = \varphi_{\alpha_i} \circ i \circ \varphi_{\alpha_i}^{-1}$.) Since $\{i^* f_i\}$ is a partition of unity for the oriented atlas $\{U_{\alpha_i} \cap \partial\Omega\}$, summing over i gives $\int_{\partial\Omega} i^*\omega$. □

5.12. **De Rham cohomology.** The smooth differential forms on M with the exterior derivative d form a complex of sheaves of $\mathbb{R}$-vector spaces, the **de Rham complex**:

$$0 \to \mathscr{E}^0 \to \mathscr{E}^1 \to \cdots \to \mathscr{E}^n \to 0.$$

The homology groups of the complex of global sections

$$0 \to \mathscr{E}^0(M) \to \mathscr{E}^1(M) \to \cdots \to \mathscr{E}^n(M) \to 0$$

are the **de Rham cohomology** groups of M:

$$H^i_{dR}(M) := \frac{ker(d : \mathscr{E}^i(M) \to \mathscr{E}^{i+1}(M))}{im(d : \mathscr{E}^{i-1}(M) \to \mathscr{E}^i(M))}.$$

These are real vector spaces (not obviously finite-dimensional). Note that in degree zero we get just the locally constant functions, so that if M is connected

$$H^0_{dR}(M) = \mathbb{R}.$$

The formal properties of de Rham cohomology are summarized in the following proposition:

Proposition 5.12.1. *The de Rham cohomology groups have the following properties:*

(i) *The sum $H^*_{dR}(M) = \oplus_{k\geq 0} H^k_{dR}(M)$ is a graded-commutative algebra with the product defined by exterior product of differential forms: $[\omega] \wedge [\tau] = [\omega \wedge \tau]$.*
(ii) *They are contravariantly functorial for smooth maps: For $f : M \to N$ there is a ring homomorphism*

$$f^* : H^k_{dR}(N) \to H^k_{dR}(M) \qquad \text{by } [\omega] \mapsto [f^*\omega].$$

(iii) *They are diffeomorphism invariant, i.e. if $f : M \to N$ is a diffeomorphism then f^* is an isomorphism.*

Proof. (i) We check that the product is well-defined: Suppose ω and τ are closed forms. If ω is replaced by $\omega + d\eta$ then

$$[(\omega + d\eta) \wedge \tau] = [\omega \wedge \tau] + [d\eta \wedge \tau] = [\omega \wedge \tau] + [d(\eta \wedge \tau) + (-1)^{deg(\eta)+1}\eta \wedge d\tau] = [\omega \wedge \tau].$$

Graded-commutativity then follows from graded-commutativity of the wedge product of forms.

(ii) follows immediately from $f^*(d\omega) = d(f^*\omega)$.

(iii) $f^* : \mathscr{E}^*(N) \to \mathscr{E}^*(M)$ is an isomorphism of complexes. □

One can also consider the complex of sheaves of k-forms with compact support:

$$0 \to \mathscr{E}_c^0 \to \mathscr{E}_c^1 \to \cdots \to \mathscr{E}_c^n \to 0$$

where $\mathscr{E}_c^k(U)$ is the space of k-forms with support in a compact set. The homology groups of the global sections are the **de Rham cohomology groups with compact support**:

$$H^i_{dR,c}(M) = \frac{ker(d : \mathscr{E}_c^i(M) \to \mathscr{E}_c^{i+1}(M))}{im(d : \mathscr{E}_c^{i-1}(M) \to \mathscr{E}_c^i(M))}.$$

Of course $H^i_{dR,c}(M) = H^i_{dR}(M)$ if M is compact. If M is noncompact then $H^0_{dR,c}(M) = 0$.

Proposition 5.12.2. *The groups $H^*_{dR,c}(M)$ have the following properties:*

(i) *The sum $H^*_{dR,c}(M) = \oplus_{k\geq 0} H^k_{dR,c}(M)$ is a ring under wedge of differential forms and also a module over $H^*_{dR}(M)$.*
(ii) *The groups $H^*_{dR,c}(M)$ are contravariantly functorial for proper maps of manifolds.*
(iii) *The groups $H^*_{dR,c}(M)$ are covariantly functorial for inclusions of open subsets.*
(iv) *For oriented M, integration defines a nonzero map $\int_M : H^n_{dR,c}(M) \to \mathbb{R}$.*
(v) *If an orientation on M is fixed then*

$$(\omega, \tau) \mapsto \int_M \omega \wedge \tau$$

defines a nonzero pairing

$$H^k_{dR}(M) \times H^{n-k}_{dR,c}(M) \to \mathbb{R}.$$

Proof. (i) is straightforward from the Leibniz rule for d.

(ii) comes from the fact that if $\omega \in \mathscr{E}_c^k(N)$ and $f : M \to N$ is proper then $f^*(\omega) \in \mathscr{E}_c^k(M)$.

(iii) comes as follows: if $U \subset M$ is open and $\omega \in \mathscr{E}_c^k(U)$ then extending ω by zero outside U defines a form $i_*(\omega) \in \mathscr{E}_c^k(M)$. Clearly $d(i_*\omega) = i_*(d\omega)$ so that this gives a map in cohomology $H^k_{dR,c}(U) \to H^k_{dR,c}(M)$.

(iv) follows from the corollary to Stokes's theorem.

(v) check that the pairing descends to cohomology (Stokes's theorem again). This is the same at the product $H^k_{dR}(M) \otimes H^{n-k}_{dR,c}(M) \to H^n_{dR,c}(M)$ composed with integration $\int_M$. □

For a connected oriented manifold the integration map in (iv) is an isomorphism. It is a version of Poincaré duality that for a connected oriented manifold the pairing in (v) is nondegenerate. (We will not prove these facts here; see Bott-Tu for a proof.)

5.13. **Poincaré lemma.**

Lemma 5.13.1. *For any smooth manifold M the projection $\pi : M \times \mathbb{R} \to M$ induces an isomorphism $\pi^* : H^k_{dR}(M) \to H^k_{dR}(M \times \mathbb{R})$.*

Proof. Let $i_t : M \to M \times \mathbb{R}$ be $i_t(x) = (x, t)$. Since $\pi \circ i_t = \mathrm{id}$ we know that π^* is injective. It remains to prove it is surjective. A k-form on $M \times \mathbb{R}$ can be written as $\omega = \omega_1 + dt \wedge \omega_2$ where ω_1 and ω_2 involve no dt. (Both ω_1 and ω_2 may depend on t.) Consider the form

$$\sigma := \omega - d\left(\int_0^t \omega_2(s)ds\right).$$

This means the following: In local coordinates on any coordinate patch of $M \times \mathbb{R}$, which we assume to be of the form $(x_1, \ldots, x_n, t)$ where x_i are local coordinates on a patch in M, we can write ω_2 as a sum of terms like $\sum_I f_I(x_1, \ldots, x_n, t)dx_I$ running over subsets $I \subset \{1, \ldots, n\}$ with $|I| = k$. Then the above integral means: In each such coordinate patch take $\sum_I \left(\int_0^t f(x_1, \ldots, x_n, s)ds\right) dx_I$ where the integral here is an ordinary Riemann integral. These expressions obviously agree on overlaps (as one varies the patch in M) and hence define a differential k-form, which is what is meant by $\int_0^t \omega_2(s)ds$.

Now $d\sigma = d\omega = 0$ and clearly σ and ω represent the same element of $H^k_{dR}(M \times \mathbb{R})$. The form dt does not appear in σ, so if, in local coordinates, $\sigma = \sum_{|I|=k} \sigma_I dx_I$, then $d\sigma = 0$ implies that $\frac{\partial}{\partial t}(\sigma_I) = 0$ for all I. Hence σ is actually constant in t, i.e. $\sigma = \pi^*(\tau)$ for τ a closed form on M. This proves the lemma. □

Corollary 5.13.2 (Poincaré lemma). *$H^k_{dR}(\mathbb{R}^n) = 0$ for $k > 0$ and $H^0_{dR}(\mathbb{R}^n) = \mathbb{R}$.*

Since any point on a manifold has neighbourhoods diffeomorphic to $\mathbb{R}^n$, the Poincaré lemma says that the following is an exact sequence of sheaves on M:

$$0 \to \mathbb{R}_M \to \mathscr{E}^0 \to \mathscr{E}^1 \to \cdots \to \mathscr{E}^n \to 0$$

where $\mathbb{R}_M$ is the constant sheaf on M (i.e. the sheaf of locally constant functions). In other words, the **de Rham complex** $(\mathscr{E}^\bullet, d)$ is a resolution of the constant sheaf.

5.14. **Homotopy invariance.** Two smooth maps $f, g : M \to N$ are **smoothly homotopic** if there exists a smooth map $F : \mathbb{R} \times M \to N$ such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$. (Here $\mathbb{R}$ can be replaced by any open interval containing $0, 1$.)

Lemma 5.14.1. *If f and g are smoothly homotopic then $f^* = g^*$.*

Proof. Let $i_t : M \to M \times \mathbb{R}$ be the inclusion $i_t(x) = (x, t)$ and let $\pi : M \times \mathbb{R} \to M$ be the projection. Then $\pi \circ i_t = id$ and hence $i_t^* \pi^* = id$. We saw earlier that π^* is an isomorphism, hence $i_t^* = (\pi^*)^{-1}$ is independent of t. In particular $i_0^* = i_1^*$ and hence $f^* = i_0^* F^* = i_1^* F^* = g^*$. □

It is a fact that in any (continuous) homotopy class of maps between two smooth manifolds there is a smooth map, and two smooth maps in the class are smoothly homotopic. Thus we see that a (continuous) homotopy class of maps between manifolds induces a well-defined map in de Rham cohomology.

5.15. **Mayer-Vietoris sequence.** Let $U, V \subset M$ be open subsets (hence manifolds themselves) such that $U \cup V = M$. Let $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ be the inclusions. For each k there is a short-exact sequence

$$0 \to \mathscr{E}^k(M) \to \mathscr{E}^k(U) \oplus \mathscr{E}^k(V) \xrightarrow{i^* - j^*} \mathscr{E}^k(U \cap V) \to 0.$$

Exactness at the first two places is obvious, so we must only check surjectivity at the end. Let $\omega \in \mathscr{E}^k(U \cap V)$. Let $\rho_U + \rho_V = 1$ be a partition of unity subordinate to $\{U, V\}$. Then $\rho_U \omega$ defines an element of $\mathscr{E}^k(V)$ by extending it by zero on $V - U \cap V$. Similarly $\rho_V \omega$ defines an element of $\mathscr{E}^k(U)$. Then

$$\sigma := (-\rho_V \omega, \rho_U \omega) \in \mathscr{E}^k(U \sqcup V)$$

goes to ω under $i^* - j^*$.

Since the maps in the short exact sequence are compatible with differentials we have a short-exact sequence of complexes

$$0 \to \mathscr{E}^*(M) \to \mathscr{E}^*(U) \oplus \mathscr{E}^*(V) \xrightarrow{i^* - j^*} \mathscr{E}^*(U \cap V) \to 0.$$

and hence a long exact sequence in de Rham cohomology

$$\cdots \to H^k_{dR}(M) \to H^k_{dR}(U \sqcup V) \to H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(M) \to \cdots.$$

A **good cover** of a manifold is a covering $\{U_\alpha\}$ such that all finite intersections $\cap_{i=1}^k U_{\alpha_i}$ are diffeomorphic to $\mathbb{R}^n$. The Mayer-Vietoris sequence, induction, and the Poincaré lemma give:

Theorem 5.15.1. *The de Rham cohomology groups of a manifold with a finite good cover are finite-dimensional.*

There is a standard argument to show that a compact manifold always has a (finite) good cover, which uses a Riemannian metric and geodesically convex sets. However, we will not give it here.

There is a Mayer-Vietoris sequence for compactly supported de Rham cohomology: Let $\{U, V\}$ be an open covering of M. The Mayer-Vietoris sequence

$$\cdots H^i_{dR,c}(U \cap V) \to H^i_{dR,c}(U \sqcup V) \to H^i_{dR,c}(M) \to \cdots$$

comes from the following short exact sequence of complexes

$$0 \to \mathscr{E}^*_c(U \cap V) \to \mathscr{E}^*_c(U \sqcup V) \xrightarrow{i_* - j_*} \mathscr{E}^*_c(M) \to 0$$

where $\{i : U \hookrightarrow M, j : V \hookrightarrow M\}$ is an open cover of M. (The maps i_*, j_* here come from the covariant functoriality of $\mathscr{E}^k_c(-)$ under open inclusion; verify the exactness of the sequence.)

Example 5.15.2. It follows easily from the M-V sequence that $H^k_{dR}(S^n) = \mathbb{R}$ for $k = 0, n$ and is zero otherwise. We can write down an explicit generator for $H^n_{dR}(S^n)$ as follows: Consider the form $\omega \in \mathscr{E}^n(\mathbb{R}^{n+1})$ defined by

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i \, dx_{\{1,\dots,n+1\}-\{i\}}.$$

Then $d\omega = (n+1)dx_{\{1,\dots,n+1\}}$ is a multiple of the volume form on $\mathbb{R}^{n+1}$. Let $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion. Then

$$\sigma := i^*\omega \in \mathscr{E}^n(S^n)$$

is closed and $[\sigma]$ is a generator of $H^n_{dR}(S^n)$ because:

$$\int_{S^n} \sigma = \int_{B^{n+1}} d\omega \neq 0$$

where B^{n+1} is the ball of radius one.

Example 5.15.3. Let us write down a generator for $H^*_{dR}(\mathbb{C}P^n)$. (At this stage we do not yet know that $H^*_{dR}(\mathbb{C}P^n) = \mathbb{R}[x]/(x^{n+1})$, so assume this.) Consider the map $p : S^{2n+1} \to \mathbb{C}P^n$ which is the restriction of $\mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$. For $i = 0$ to n let $dz_i = dx_i + \sqrt{-1}dy_i$ and $d\bar{z}_i = dx_i - \sqrt{-1}dy_i$ so that $dz_i \wedge d\bar{z}_i = -2\sqrt{-1}dx_i \wedge dy_i$. Restricting $\frac{\sqrt{-1}}{2}\sum_i dz_i \wedge d\bar{z}_i = \sum_i dx_i \wedge dy_i$ to S^{2n+1} gives a closed real 2-form σ.

- Show that there is a closed 2-form ω on $\mathbb{C}P^n$ such that $p^*(\omega) = \sigma$.
- Show that ω is invariant under the natural action of $U(n+1)$ on $\mathbb{C}P^n$.
- Show that ω^k is nonzero for any $k \leq n$.

5.16. **De Rham theorem.** The de Rham cohomology groups of a smooth manifold depend, *a priori*, on the smooth structure of the manifold. In fact, they depend only on the underlying topological space and they compute the singular cohomology groups, tensored with $\mathbb{R}$:

Theorem 5.16.1. *There are natural isomorphisms $H^*_{dR}(M) \cong H^*(M, \mathbb{R})$ and $H^*_{dR,c}(M) \cong H^*_c(M, \mathbb{R})$.*

Proof. For the general case this will be proved in the next section using sheaf-theoretic methods. Here let's prove it for compact manifolds using the existence of a finite good cover. Indeed, if $\{U_i\}_{i=1}^n$ is a finite good cover, then a straightforward induction using the Mayer-Vietoris sequence and the Poincaré lemma proves the theorem. □

The isomorphism of the theorem is functorial with respect to smooth maps, but we will not prove this now.

Note that for an oriented manifold M, the isomorphism $H^n_{dR,c}(M) \cong H^n_c(M, \mathbb{R})$ relates the integration map $\int_M : H^n_{dR,c}(M) \to \mathbb{R}$ to the map $PD_M : H^n_c(M, \mathbb{R}) \to H_0(M, \mathbb{R}) = \mathbb{R}$ used in the Poincaré duality theorem. (In the case M is compact PD_M is cap product with the fundamental class $[M]$ given by the orientation). Thus we can conclude that if M is connected then $H^n_{dR,c}(M) \cong \mathbb{R}$. Also, we can apply the Poincaré duality theorem to conclude that the pairing

$$H^k_{dR}(M) \times H^{n-k}_{dR,c}(M) \to \mathbb{R}.$$

by $(\omega, \tau) \mapsto \int_M \omega \wedge \tau$ is a nondegenerate pairing. (For a purely de Rham cohomology proof of this duality see Bott&Tu.)

5.17. **Relative de Rham cohomology (∗).** Let $f : M \to N$ be a smooth map. We will define de Rham cohomology groups $H^*(f : M \to N)$. When f is an embedding this will give back the relative cohomology $H^*(M, N, \mathbb{R})$ under a relative version of the de Rham theorem.

THIS IS STILL TO BE WRITTEN

6. Sheaves and cohomology

In this section we will discuss sheaves on topological spaces and their cohomology and apply them to manifolds. Good references for the material in this section are Voisin's *Hodge Theory and Complex Geometry*, vol. I or Ramanan's *Global Calculus*.

(Note that the definitions we use, which are the ones in these books, are sometimes different from those of Warner's book and some other books.)

6.1. **Presheaves.** Let X be a topological space. Let $Op(X)$ be the category with objects the open sets in X, and morphisms inclusions between open sets. A **presheaf of sets** on X is a contravariant functor $F : Op(X) \to Sets$ such that $F(\emptyset)$ is a one-element set and $F(id : U \to U)$ is the identity map of $F(U)$. A **presheaf of abelian groups** on X is a contravariant functor from the category $Op(X)$ to the category of abelian groups which takes the empty set $\emptyset$ to $\{e\}$ and the morphism $U \subset U$ to the identity. Concretely, this means we have, for each open set $U \subset X$, an abelian group $F(U)$ and for each pair of open sets $V \subset U$ we have a homomorphism $\rho^U_V : F(U) \to F(V)$, such that for $W \subset V \subset U$ we have

$$\rho^V_W \circ \rho^U_V = \rho^U_W$$

and moreover $F(\emptyset) = \{e\}$ and $\rho^U_U = id$. The homomorphism ρ^U_V will be called "restriction from U to V". An element $s \in F(U)$ is referred to as a **section** of F over U.

A morphism between presheaves F and G has the obvious meaning, namely a natural transformation of functors from F to G. Concretely this means we are given for each $U \subset X$ a homomorphism $F(U) \to G(U)$ and the diagram below commutes for every pair $V \subset U$:

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ {\scriptstyle \rho^U_V}\downarrow & & \downarrow{\scriptstyle \rho^U_V} \\ F(V) & \longrightarrow & G(V). \end{array}$$

A presheaf of rings (or algebras) has the obvious meaning: a covariant functor from $Op(X)$ to rings (or algebras) such that $F(\emptyset) = \{0\}$ and $F(U \subset U) = id$.

Examples 6.1.1. Here are some examples of presheaves:

(1) The assignment $U \mapsto C^0(U, \mathbb{R})$ (continuous functions from U to $\mathbb{R}$ with restriction of functions defines a presheaf of abelian groups. Here $\mathbb{R}$ may be replaced by any topological space Y to get a presheaf of sets.

(2) Fix an abelian group A. The assignment $U \mapsto A$ defines the constant presheaf, where all $\rho^U_V = id$.

(3) Let $X = \mathbb{C}$, or more generally any complex manifold. For an open subset $U \subset X$ define $\mathscr{O}(U)$ to be the ring of holomorphic functions on U, and let $\rho^U_V : \mathscr{O}(U) \to \mathscr{O}(V)$ be the restriction of functions.

(4) Let M be a smooth manifold and let $\mathscr{E}(U)$ be the smooth functions from U to $\mathbb{R}$, and ρ^U_V is restriction of functions. This defines a presheaf $\mathscr{E}$ on M.

(5) For $X = \mathbb{C}$ let $\mathscr{B}$ be the presheaf defined by $\mathscr{B}(U)$= bounded holomorphic functions on U, ρ^U_V= restriction of functions.

(6) Fix a point $x \in X$ and an abelian group A. Define F by $F(U) = A$ if $x \in U$ and $F(U) = \{0\}$ if $x \notin U$. The restriction maps are the obvious ones (identity or zero). This presheaf is called the **skyscraper presheaf** at x with value A.

(7) Let X be a smooth manifold. The functor $U \mapsto \mathscr{X}(U)$ = smooth vector fields on U defines a presheaf, as does the functor of sections of any smooth vector bundle on X.

(8) Let X be a topological space and $k \geq 0$ an integer. The functor $U \mapsto H^k(U, \mathbb{Z})$ (singular cohomology groups of U) defines a presheaf. Restriction maps are restriction maps in cohomology.

6.2. **Sheaves.** A **sheaf** on X is a presheaf F on X which further satisfies the following two conditions:

(S1) If $s_1, s_2 \in F(U)$ and $\{U_\alpha\}_\alpha$ is an open covering of U such that $\rho^U_{U_\alpha}(s_1) = \rho^U_{U_\alpha}(s_2)$ for all α, then $s_1 = s_2$

(S2) If U is open in X, $\{U_\alpha\}_\alpha$ is an open covering of U and $\{s_\alpha \in F(U_\alpha)\}_\alpha$ a collection of sections satisfying $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$ for all α, β, then there is a section $s \in F(U)$ such that $s_\alpha = \rho^U_{U_\alpha}(s)$ for all α.

Informally, (S2) says that local sections of sheaves glue together to give sections, (S1) says that sections of sheaves are determined by what they are locally.

Examples 6.2.1. Among the examples of presheaves given above, (1), (3), (4), (6), (7) are sheaves and (8) is a sheaf in the case $k = 0$. (2) and (5) are not sheaves because they fail to satisfy (S2). (An example of a presheaf which fails (S2) and (S1) is the following: Let $X = \{x, y\}$ with the discrete topology. For an open set $U \subset X$ let $F(U)=$ functions from U to $\mathbb{R}$. For $V \subsetneq U$ let $\rho_V^U := 0$.) The presheaf (8) for $k \geq 1$ fails (S1) in general, e.g. if X is a manifold or locally contractible.

• For any topological space Y, the presheaf $U \mapsto C^0(U, Y)$ defines a sheaf of sets on X. If Y is a(n abelian) topological group then this defines a sheaf of (abelian) groups.

• For any smooth manifolds M and N the presheaf $U \mapsto C^\infty(U, N)$ defines a sheaf of sets on M. If N is a(n abelian) Lie group then this defines a sheaf of (abelian) groups.

6.3. **Stalks, sheafification.** Sheaves are (as we shall see) much more convenient to work with than presheaves, although it is usually presheaves which appear "in nature". So it is useful to have a canonical (i.e. functorial) way to associate a sheaf to a presheaf.

Let F be a presheaf on X. The **stalk** of F at $x \in X$ is the direct limit:

$$F_x := \varinjlim_{U \ni x} F(U).$$

For $U \ni x$ denote the natural map to the stalk by

$$\rho_x^U : F(U) \to F_x.$$

Now if F satisfies (S1) then the collection $(\rho_x^U(s))_{x \in U}$ determines the section $s \in F(U)$, in the sense that the map

$$F(U) \to \prod_{x \in U} F_x$$

is injective. In fact this is equivalent to (S1). This suggests how, given a presheaf, we should make a sheaf out of it.

A **continuous section** of F over U is an element $(s_x)_{x \in U} \in \prod_{x \in U} F_x$ such that for each $x \in U$ there exists a V and $\sigma \in F(V)$ such that for all $y \in V$ we have $s_y = \rho_y^U(\sigma)$. ([22]) Define the presheaf $\tilde{F}$ by

$$\tilde{F}(U) = \{ \text{ continuous sections of } F \text{ over } U \ \}.$$

The restriction maps $\tilde{F}(U) \to \tilde{F}(V)$ are given by $(s_x)_{s \in U} \mapsto (s_x)_{x \in V}$. There is an obvious presheaf homomorphisms

$$\theta : F \to \tilde{F}$$

(since an actual section of F over U gives a continuous section).

Let us see that $\tilde{F}$ defined like this is a sheaf:

Lemma 6.3.1. *The sheafification $F \mapsto \tilde{F}$ has the following properties:*

(1) *The construction $F \mapsto \tilde{F}$ is functorial.*
(2) *The map θ induces an isomorphism $F_x \cong \tilde{F}_x$ of stalks.*
(3) *If F is a sheaf then $\theta : F \to \tilde{F}$ is an isomorphism.*
(4) *The construction has a universal property: If G is a sheaf and $\phi : F \to G$ is a presheaf homomorphism then there is a unique sheaf homomorphism $\tilde{\phi} : \tilde{F} \to G$ such that $\phi = \tilde{\phi} \circ \theta$.*
(5) *$Hom(F, G) = Hom(\tilde{F}, \tilde{G})$, i.e. sheafification is a faithful functor.*

Example 6.3.2. (i) Consider the examples (2) and (5) above which were not sheaves. It is easy to see that the sheafification of the constant presheaf A is the sheaf which assigns to U the group of locally constant functions $U \to A$, i.e. $A^{\pi_0(U)}$. This is called the constant sheaf with value A. The sheafification of (5) is the sheaf of all holomorphic functions.

(ii) Note that a presheaf can have nonzero global sections even if all its stalks are zero. (This cannot happen for a sheaf by (S1).) An example is the presheaf $U \mapsto H^k(U)$ of de Rham or singular cohomology

[22] Here "continuous" does not refer to a topology; it is simply a terminology for us. However there is a natural topology on the coproduct $\coprod_{x \in X} F_x$ such that these are exactly the continuous sections over U of the natural map $\coprod_{x \in X} F_x \to X$ which takes the stalk F_x to x. (Exercise) The space $\coprod_{x \in X} F_x$ is called the *espace etalé* of F, cf. e.g. Hartshorne, Chapter II, Exs.

groups on a manifold for $k > 0$. By the Poincaré lemma (for de Rham cohomology) or the contractibility of balls (for singular cohomology) this has zero stalk at all points, but it has global sections e.g. if $H^k(M) \neq 0$. This presheaf fails to satisfy (S1) because any two classes in $H^k(M)$ have the same (=zero) restriction to small enough coordinate neighbourhoods. In this case the associated sheaf is zero.

Example 6.3.3. Here is an important example of a sheaf arising from differential equations. Let $X = \mathbb{C}^*$ and fix $\alpha \in \mathbb{C}$. Let F be the sheaf of holomorphic solutions of the differential equation

$$z\frac{d}{dz}f = \alpha f \tag{6.3.1}$$

i.e. over an open set $U \subset \mathbb{C}^*$ the group $F(U)$ is the holomorphic functions on U satisfying the differential equation. The restriction maps are given by restricting functions. This defines a presheaf and it is easy to see that it is a sheaf, a subsheaf of the sheaf of all holomorphic functions on $\mathbb{C}$.

(1) Suppose first that $\alpha \in \mathbb{Z}$. In this case $f(z) = z^\alpha$ is a holomorphic solution to (6.3.1) which makes sense in all of $\mathbb{C}^*$. Moreover it is clear that every solution is a multiple of this one. Therefore the sheaf F is isomorphic to the constant sheaf $\mathbb{C}$.

(2) Suppose now that $\alpha \notin \mathbb{Z}$. Then $z^\alpha = exp(\alpha\, log(z))$ defines a solution of (6.3.1) in any domain U in which we can choose a branch of the logarithm function, i.e. any domain not containing a real half-line. Moreover in such a domain any solution is a multiple of z^α, so $F(U) \cong \mathbb{C}$ with a basis given by z^α. On the other hand $F(U) = \{0\}$ if U contains a punctured disk. Thus F is not the constant sheaf, but it is locally constant in the sense that any point has a neighbourhood in which F is isomorphic to the constant sheaf $\mathbb{C}$. It follows that the stalks are all isomorphic to $\mathbb{C}$.

This example can be generalized by looking at solutions to

$$\left(z\frac{d}{dz} - \alpha\right)^n f = 0$$

for $n \geq 1$. For example if $\alpha \in \mathbb{Z}$ then there are n linearly independent solutions defined on all of $\mathbb{C}^*$ and the sheaf is isomorphic to the constant sheaf $\mathbb{C}^n$. For $\alpha \notin \mathbb{Z}$ the sheaf of solutions is locally constant and locally isomorphic to the constant sheaf $\mathbb{C}^n$.

Example 6.3.4. Consider the previous example with $\alpha \notin \mathbb{Z}$ but with domain $X = \mathbb{C}$, i.e. let G be the sheaf of local solutions of (6.3.1) on $\mathbb{C}$. Now $G(U) = \{0\}$ if $0 \in U$. Thus the stalk at 0 is zero. The stalks at other points are the same since $i^{-1}G = F$ where $i : \mathbb{C}^* \to \mathbb{C}$ is the inclusion and F is as earlier.

6.4. **Kernels, cokernels, images.** Let $F \to G$ be a homomorphism of sheaves. The presheaf kernel, presheaf cokernel, and presheaf image are the functors

$$U \mapsto ker(F(U) \to G(U)),$$
$$U \mapsto im(F(U) \to G(U)),$$
$$U \mapsto coker(F(U) \to G(U)).$$

It is easy to see that the first is already a sheaf, called the kernel sheaf and denoted $ker(F \to G)$. However, simple examples show that the image and cokernel presheaves are not necessarily sheaves. ([23]) So we define the **image sheaf** and the **cokernel sheaf** to be the sheaves associated with the image presheaf and cokernel presheaf respectively. We will use the notation $im(F \to G)$ and $coker(F \to G)$ for these objects as we will never need to use the presheaf image and presheaf cokernel.

In general, if $F \subset G$ is a subsheaf then we will define the **quotient sheaf** to be the sheaf associated with the presheaf $U \mapsto G(U)/F(U)$. In this way the kernel and cokernel become the sheaves associated with the presheaf kernel and cokernel. With this definition, we have

Lemma 6.4.1. *(1) A homomorphism of sheaves $F \to G$ is an isomorphism if and only if it is an isomorphism on all stalks.*

(2) Two subsheaves F' and F'' of F are equal if and only if $F'_x = F''_x$ in F_x.

(3) A sequence $F_1 \to F_2 \to F_3$ of sheaves is exact if and only if the corresponding sequence of stalks $F_{1,x} \to F_{2,x} \to F_{3,x}$ is exact for every $x \in X$.

[23]Here is an example for cokernel: Let $F = G = \mathscr{O}$ be the sheaf of holomorphic functions on $X = \mathbb{C}^*$ and let $\mathscr{O} \to \mathscr{O}$ be the operator $\frac{d}{dz}$. Then every point $x \in \mathbb{C}^*$ has a neighbourhood U for which the presheaf cokernel has sections over U equal to $\{0\}$ (any small enough neighbourhood of x will do). But the presheaf cokernel has nonzero sections over $X = \mathbb{C}^*$ since e.g. $\frac{1}{z}$ is not in the image of $\frac{d}{dz}$ on $\mathbb{C}^*$. Thus the presheaf cokernel is not a sheaf.

6.5. **Example: Exponential sequence.** Here is a basic example of a short exact sequence of sheaves on any smooth manifold X. For any abelian Lie group G let $\underline{G}_X$ be the sheaf on X defined by $U \mapsto C^\infty(U, G)$. Then there is a short exact sequence

$$0 \to \underline{\mathbb{Z}}_X \to \underline{\mathbb{C}}_X \to \underline{\mathbb{C}^*}_X \to 0$$

where the second map is defined by $f \mapsto exp(2\pi i f)$. That this map is surjective on stalks is easy to check and comes down to the fact that for small enough open neighbourhoods of any point in $\mathbb{C}^*$ the exponential map can be inverted. This also gives an example of a short exact sequence of sheaves such that taking sections over an open set does not necessarily give a short exact sequence of abelian groups.

6.6. **Exact functors, global sections functor.** Recall the notions of exact functor and left-exact functor. A functor $T : Sh(X) \to AbGps$ or $T : Sh(X) \to Sh(Y)$ is called **exact** if it takes short-exact sequences to short-exact sequences. It is **left-exact** if $0 \to T(A) \to T(B) \to T(C)$ is exact whenever $0 \to A \to B \to C$ is exact. It is **right-exact** if $T(A) \to T(B) \to T(C) \to 0$ is exact whenever $A \to B \to C \to 0$ is exact.

The **global sections functor** is the functor

$$\Gamma(X, -) : Sh(X) \to AbGps$$

defined by taking sections over X. So in the earlier notation, $\Gamma(X, F) = F(X)$.

Lemma 6.6.1. *The global sections functor is left-exact.*

Proof. This is straightforward. □

Examples 6.6.2. Here are examples to show that $\Gamma(X, -)$ is not an exact functor:

(1) Let $X = \mathbb{C}^*$ and consider the exponential sequence on X. The map $\Gamma(X, \underline{\mathbb{C}}_X) \to \Gamma(X, \underline{\mathbb{C}^*}_X)$ is not since e.g. the function z has no logarithm in X.

(2) Let $X = \mathbb{C}P^1$ and $\mathscr{O}$ the sheaf of holomorphic functions on X. Let $x, y \in X$ be two distinct points. Then there is a short exact sequence

$$0 \to \mathscr{I}_{\{x,y\}} \to \mathscr{O} \to i_{x*}\mathbb{C} \oplus i_{y*}\mathbb{C} \to 0.$$

(Here the second map is $f \mapsto (f(x), f(y))$ and $\mathscr{I}_{\{x,y\}}$ is the kernel, i.e. the ideal of functions vanishing at x and y.) The only global sections of $\mathscr{O}$ are constant functions, so the map $\mathbb{C} = \Gamma(X, \mathscr{O}) \to \Gamma(X, i_{x*}\mathbb{C} \oplus i_{y*}\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$ is not surjective.

6.7. **Pullback, pushforward.** Let $f : X \to Y$ be a continuous map of topological spaces. Given a sheaf F on X we can define a presheaf on Y by

$$U \mapsto F(f^{-1}(U)).$$

This is already a sheaf, called the **pushforward** or **direct image** of F by f, and denoted f_*F. (Exercise: Check that this defines a sheaf.) Note that a special case of direct image is $\Gamma(X, -)$, which is direct image by the map $X \to pt$.

Given a sheaf G on Y we can defind a presheaf on X by

$$U \mapsto \varinjlim_{V \supset f(U)} G(V).$$

This is not a sheaf in general, so we define the **pullback** or **inverse image** sheaf $f^{-1}G$ to be the sheaf associated with this presheaf. The stalks are related by $(f^{-1}G)_x = G_{f(x)}$ for $x \in X$. For any abelian group A we have $f^{-1}\underline{A}_Y = \underline{A}_X$.

(Remark: Note that if $i_x : \{x\} \to X$ is the inclusion of a point then for an abelian group A the skyscraper sheaf at x with value A is $i_{x*}A$. For a sheaf F we have $i_x^{-1}F = F_x$ (the stalk at x).)

Lemma 6.7.1. *The inverse image functor $f^{-1} : Sh(Y) \to Sh(X)$ is exact. The direct image functor $f_* : Sh(X) \to Sh(Y)$ is left-exact.*

Proof. The first assertion follows from the fact that $(f^{-1}F)_x = F_y$ and the fact that a short-exact sequence of sheaves is exact if and only if it is exact on every stalk. The second assertion is left as an exercise. (Note that if $f : X \to pt$ is the map to a point then $f_*F = \Gamma(X, F)$.) □

There are natural transformations of functors $id \to f_*f^{-1}$ and $f^{-1}f_* \to id$ which define homomorphisms $Hom(f^{-1}F, G) \leftrightarrow Hom(F, f_*G)$. The pair (f^{-1}, f_*) is an **adjoint pair** of functors, which means (by definition) that there is a natural isomorphism

$$Hom(f^{-1}G, F) = Hom(G, f_*F)$$

for $F \in Sh(X), G \in Sh(Y)$. (Exercise: Verify this.) We say f^{-1} is **left adjoint** to f_* or f_* is **right adjoint** to f^{-1}.

Lemma 6.7.2. *(1) If L is an exact functor and R is right adjoint to L then R takes injective objects to injective objects.*

(2) f_* takes injective sheaves to injective sheaves.

(3) f_* takes flasque sheaves to flasque sheaves.

6.8. **Internal Hom and tensor product.** Given two sheaves F, G of abelian groups on a space X we have the **internal Hom** presheaf defined by

$$U \mapsto \underline{Hom}(F, G)(U) := Hom(F|_U, G|_U)$$

where $F|_U$ means the pullback by the inclusion $U \hookrightarrow X$. ([24])

Lemma 6.8.1. *If F and G are sheaves then $\underline{Hom}(F, G)$ is a sheaf.*

Proof. Exercise. □

Something to keep in mind: In general, the natural map $\underline{Hom}(F, G)_x \to Hom(F_x, G_x)$ is neither injective nor surjective. ([25])

The adjointness relation between f_* and f^{-1} then has a sheaf version, which gives the previous one after taking global sections:

Lemma 6.8.2. *If $f : X \to Y$ is a continuous map of topological spaces and $F \in Sh(X)$ and $G \in Sh(Y)$ then $f_*\underline{Hom}(f^{-1}G, F) \cong \underline{Hom}(F, f_*G)$.*

Proof. Exercise. □

For $F, G \in Sh(X)$ the **tensor product** $F \otimes G$ is the sheaf associated with the presheaf

$$U \mapsto F(U) \otimes G(U).$$

(The presheaf is not in general a sheaf, so we must sheafify.) The stalks of the tensor product are given by $(F \otimes G)_x = F_x \otimes G_x$. (Easy exercise.)

A **complex of sheaves** on a topological space X is a sequence of sheaves $(F^i)_{i\in\mathbb{Z}}$ on X and morphisms $F^i \to F^{i+1} (i \in \mathbb{Z})$ such that the composition of any two successive morphisms is zero. We write:

$$\cdots \to F^{-1} \to F^0 \to F^1 \to F^1 \to \cdots$$

We say "F^i appears in degree i". Taking sections over any open set $U \subset X$ gives a complex of abelian groups $\cdots \to \Gamma(U, F^i) \to \Gamma(U, F^{i+1}) \to \cdots$.

Let F be a sheaf on X. A **resolution** of F is a complex of sheaves $0 \to K^0 \to K^1 \to \cdots$ such that the following sequence is exact:

$$0 \to F \to K^0 \to K^1 \to \cdots .$$

[24] One might think of defining it by $\underline{Hom}(F, G)(U) = Hom(F(U), G(U))$ but this does not make any sense: There are no natural restriction maps for $V \subset U$.

[25] Here is an example: Let $X = \mathbb{C}$ and $i : \{0\} \hookrightarrow \mathbb{C}$ the inclusion. There is an injection $i_*\mathbb{Z} \hookrightarrow \mathbb{Z}_X$; let Q be the cokernel. Then $\mathbb{Z}_X \to Q$ gives a nonzero element of $\underline{Hom}(\mathbb{Z}_X, Q)_0$ while $Hom(\mathbb{Z}, Q_0) = 0$ since $Q_0 = 0$.

6.9. **Injective sheaves and injective resolutions.** An **injective sheaf** is an injective object in the abelian category of sheaves, i.e. a sheaf $I \in Sh(X)$ for which the functor $Hom(-, I)$ is exact. It is easy to see that this is equivalent to the following property: In the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & \big\downarrow & & \\ & & I & & \end{array}$$

if $A \to B$ is a sheaf monomorphism then there exists a homomorphism $B \to I$ making the diagram commute. A short exact sequence $0 \to I \to G \to H \to 0$ with I injective has a splitting. (Look at the identity $I \to I$ to get a morphism $G \to I$ which splits the sequence.)

Example 6.9.1. (i) Let I be an injective abelian group. Then the skyscraper sheaf $i_{x*}I$ is injective. Indeed, this follows from the equality $Hom(F, i_{x*}I) = Hom(F_x, I)$ (which is a special case of the adjointness of i_x^{-1} and i_{x*}).

(ii) The product of injective sheaves is injective. (Indeed, this follows from $Hom(-, \prod_\alpha F_\alpha) = \prod_\alpha Hom(-, F_\alpha)$.)

The following lemma is usually quoted as saying "$Sh(X)$ has enough injectives":

Lemma 6.9.2. *Any sheaf admits a monomorphism to an injective sheaf.*

Proof. Recall that any abelian group admits a monomorphism to an injective (=divisible) abelian group. Thus for each $x \in X$ the stalk F_x admits an injection $j_x : F_x \hookrightarrow I_x$ for some injective I_x. Now define $I = \prod_{x \in X} i_{x*} I_x$. This is an injective sheaf. Consider the morphism

$$F \hookrightarrow \prod_{x \in X} i_{x*}F_x \hookrightarrow \prod_{x \in X} i_{x*}I_x = I.$$

defined on U by composing $\prod_{x \in U} \rho_x^U$ with the product $\prod_{x \in U} j_x$. This is injective on stalks and hence is injective. □

A resolution

$$0 \to F \to I^0 \to I^1 \to \cdots$$

where each I^k is injective is called an **injective resolution**.

Lemma 6.9.3. *(i) Every sheaf of abelian groups has an injective resolution.*

(ii) Given two sheaves F and G and resolutions $F \to I^\bullet$ and $G \to J^\bullet$ such that $J^\bullet$ is injective, any homomorphism $\alpha : F \to G$ lifts to a homomorphism $I^\bullet \to J^\bullet$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\ & & \alpha\big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots \end{array} \tag{6.9.1}$$

commutes. Any two such liftings are homotopy equivalent, i.e. if $\varphi = (\varphi^i : I^i \to J^i)_i$ and $\psi = (\psi^i : I^i \to J^i)_i$ are two liftings then there is a homomorphism $H = (H^i : I^i \to J^{i-1})_i$ from $I^\bullet$ to $J^{\bullet-1}$ such that $d_J H + H d_I = \varphi - \psi$. (Here $d_I = (d_I : I^i \to I^{i+1})_i$ and $d_J = (d_J^i)$ are the morphisms in the resolutions.)

(iii) An injective resolution of a sheaf is unique up to homotopy, i.e. given two injective resolutions $F \to I^\bullet$ and $F \to J^\bullet$ there are maps $\varphi : I^\bullet \to J^\bullet$ and $\psi : J^\bullet \to I^\bullet$ such that $\varphi\psi$ and $\psi\varphi$ are both homotopic to the identity.

(iv) Given an injective homomorphism $\alpha : F \hookrightarrow G$ we can find injective resolutions $F \to I^\bullet$ and $F \to J^\bullet$ and injective homomorphisms $I^k \hookrightarrow J^k$ such that the diagram (6.9.1) commutes.

Proof. (i) Existence is proved in the usual way: First let

$$F \hookrightarrow I^0$$

be a monomorphism to an injective sheaf. The cokernel of $F \to I^0$ can be embedded in injective sheaf I^1. The morphism

$$I^0 \to I^1$$

is the composition $I^0 \to coker(F \to I^0) \to I^1$. If $I^0 \to I^1$ is not surjective embed its cokernel in I^2 and continue.

(ii) is straightforward, (iii) follows from (ii).

(iv) We will first prove the following: Given $F \hookrightarrow G$ one can find $F \hookrightarrow I$ and $G \hookrightarrow J$ (injections into injectives) such that $I \hookrightarrow J$ and the natural map $I/F \to J/G$ is injective. Indeed, look at the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{\phi} & G & \xrightarrow{\psi} & Q & \longrightarrow & 0 \\ & & \downarrow u & & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & I & \longrightarrow & J := I \oplus K & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

Here $u : F \hookrightarrow I$ and $w : Q \hookrightarrow K$ with I, K injective have been chosen. The map $u : F \to I$ extends to a map $\alpha : G \to I$ with $\alpha\phi = u$ (by injectivity of I). Define $J = I \oplus K$ and $v : G \hookrightarrow J$ by $v = (w\psi, \alpha)$. The maps in the bottom row are the obvious ones.

Now suppose $F \hookrightarrow G$ is injective. Applying the above we can find injectives I^0 and J^0 and maps such that all rows and columns in the diagram below are exact:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & I^0 & \longrightarrow & coker(F \to I^0) & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & G & \longrightarrow & J^0 & \longrightarrow & coker(G \to J^0) & \longrightarrow & 0 \end{array}$$

Now apply the above observation to the injection $coker(F \to I^0) \hookrightarrow coker(G \to J^0)$ we find $I^1 \hookrightarrow J^1$ and continue as before. □

6.10. **Derived functors, cohomology of sheaves.** Let $T : Sh(X) \to AbGps$ be a left-exact functor to abelian groups (or any abelian category). The **derived functors** of T are defined as follows: Let F be a sheaf. Take an injective resolution $F \to I^\bullet$. Apply the functor T to get a complex of abelian groups

$$0 \to T(I^0) \to T(I^1) \to \cdots$$

and take cohomology:

$$R^iT(F) := H^i(T(I^\bullet)) = \frac{ker\left(T(I^i) \to T(I^{i+1})\right)}{im\left(T(I^{i-1}) \to T(I^i)\right)}.$$

(We set $T(I^{-1}) = 0$.) Note that since T is left-exact we have $R^0T(F) = T(F)$.

The main example of a left-exact functor is the global sections functor $\Gamma(X, -) : Sh(X) \to AbGps$. The cohomology groups $H^i(X, F)$ are defined as its derived functors, i.e.

$$H^i(X, F) := R^i\Gamma(X, F).$$

In other words, given a sheaf F choose a resolution by injective sheaves

$$0 \to F \to I^0 \to I^1 \to \cdots$$

then apply $\Gamma(X, -)$ to get a complex of abelian groups

$$0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \cdots$$

and then take cohomology to define

$$H^i(X, F) := \frac{ker(\Gamma(X, I^k) \to \Gamma(X, I^{k+1}))}{im(\Gamma(X, I^{k-1}) \to \Gamma(X, I^k))}$$

It follows from Lemma 6.9.3 above that this definition does not depend on the choice of injective resolution.

The following proposition summarizes the properties of derived functors:

Proposition 6.10.1. *The objects $R^iT(F)$ are determined up to canonical isomorphism. The assignment $F \to R^iT(F)$ has the following properties:*

(1) $R^0T(F) = T(F)$
(2) *If I is an injective sheaf then $R^iT(I) = 0$ for $i > 0$.*

(3) *A short exact sequence*

$$0 \to F \to G \to Q \to 0$$

of sheaves gives a long exact sequence

$$0 \to T(F) \to T(G) \to T(Q) \to R^1T(F) \to R^1T(G) \to R^1T(Q) \to \cdots$$

(4) *If $F \to G$ is a homomorphism of sheaves and injective resolutions $F \to I^\bullet$ and $G \to J^\bullet$ are chosen then there is a canonical induced homomorphism $R^iF \to R^iG$.*

Proof. (1), (2), (4) follow from the definitions and Lemma 6.9.3.

(3) is proved using the last part of Lemma 6.9.3: Choose injective resolutions $F \to I^\bullet$ and $G \to J^\bullet$ such that $I^\bullet \hookrightarrow J^\bullet$ compatibly with $F \hookrightarrow G$. Then the quotients

$$K^i := J^i/I^i$$

is a resolution of Q by injective objects. (The quotient of an injective object by an injective subobject is injective: use the splitting property to write $J^i = I^i \oplus K^i$ and then show that $Hom(-, K^i)$ is exact). Moreover the short exact sequences

$$0 \to I^i \to J^i \to K^i \to 0$$

are split (because I^i is injective). Thus applying the functor T gives split short exact sequences

$$0 \to T(I^i) \to T(J^i) \to T(K^i) \to 0$$

Thus we have a short exact sequence of complexes $0 \to T(I^\bullet) \to T(J^\bullet) \to T(K^\bullet) \to 0$ which splits at each i and hence by the usual argument (used, for example, to produce the long exact sequence in cohomology for a pair (X, A) from the short exact sequence of cochain complexes) gives a long exact sequence in cohomology:

$$\cdots \to H^i(T(I^\bullet)) \to H^i(T(J^\bullet)) \to H^i(T(K^\bullet)) \to H^{i+1}(T(I^\bullet)) \to \cdots .$$

This is the desired long exact sequence. □

Remarks 6.10.2. (i) One can ask whether it is possible to left-derive right-exact (covariant) functors, for example the functor $Hom(F, -) : A \mapsto Hom(F, A)$, or (in the way one does for modules over a ring in order to define Ext^i or Tor_i). One problem is that the category $Sh(X)$ does not usually have enough projectives, i.e. not every object admits a surjection $P \to F$ from a projective object. (A projective object is one for which $Hom(P, -)$ is exact.) ([26]) So we will usually derive such bifunctors by resolving the second variable by injectives. (Another option, when working with sheaves of modules over a fixed sheaf of rings R, is to use resolutions by sheaves of R-flat modules.)

(ii) Although we call the R^iT derived functors, they are not really functors because the objects $R^iT(F)$ depend on the choice of injective resolution, thus they have only been defined up to canonical isomorphism. Nevertheless, the property (4) means they behave essentially like functors.

We note the following fact for later use:

Lemma 6.10.3. *If $i : Z \hookrightarrow X$ is the inclusion of a closed subspace then i_* is an exact functor and*

$$H^k(Z, F) = H^k(X, i_*F)$$

for $F \in Sh(X)$.

Proof. ([27]) The exactness of i_* is easily checked on stalks. Now if $F \to I^\bullet$ is an injective resolution of F then $i_*I^\bullet$ is a resolution of i_*F and each i_*I^k is injective (because i_* is the right adjoint to the exact functor i^{-1} it takes injectives to injectives). Thus $i_*I^\bullet$ is an injective resolution of i_*F and so one has

$$H^k(Z, F) = H^k(\Gamma(Z, I^\bullet)) = H^k(\Gamma(X, i_*I^\bullet)) = H^k(X, i_*F)$$

because $\Gamma(Z, I^k) = \Gamma(X, i_*I^k)$ for each k. □

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[26] Here is an example: Suppose X is Hausdorff and F is a nonzero sheaf on X. Choose $x \in X$ such that $F_x \neq 0$. Then $i_{x*}F_x$ admits no epimorphism from a projective sheaf. Indeed, suppose $P \to i_{x*}F_x$ is such an epimorphism. Then there is an open set U for which $P(U) \to F_x$ is nonzero. Let $y \in U$ be a point distinct from x and let $j : U - \{y\} \hookrightarrow X$ be the inclusion. Then $j_!j^{-1}F \to i_{x*}F_x$ is surjective (it is surjective on stalks). Thus the map $P \to i_{x*}F_x$ lifts to $P \to j_!j^{-1}F$. Applying $\Gamma(U, -)$ gives that $P(U) \to F_x$ is factors as $P(U) \to \Gamma(U, j_!j^{-1}F) \to F_x$. But $\Gamma(U, j_!j^{-1}F) = 0$ so this is a contradiction.

This shows that on any reasonable space there are no projectives in the category of sheaves. Note however that there are often plenty of projectives in the category of presheaves.

[27] The proof here shows the following: If $\mathfrak{A} \xrightarrow{T} \mathfrak{B} \xrightarrow{U} \mathfrak{C}$ are left-exact functors between abelian categories and T is exact and takes injectives to injectives then $R^k(U \circ T) = R^kU \circ T$.

6.11. **Acyclic resolutions.** Let T be a left-exact functor from $Sh(X)$ to $AbGps$ or $Sh(Y)$ as before. A sheaf F is called T**-acyclic** if $R^iT(F) = 0$ for $i > 0$.

Proposition 6.11.1. *Let $0 \to F \to A^\bullet$ be a resolution of F by T-acyclic objects. Then*

$$R^iT(F) = H^i(T(A^\bullet)) = \frac{ker(T(A^i) \to T(A^{i+1}))}{im(T(A^{i-1}) \to T(A^i))},$$

i.e. the derived functors of T can be computed using T-acyclic resolutions. More precisely, if $F \hookrightarrow A^\bullet$ is a resolution and $F \hookrightarrow I^\bullet$ is an injective resolution then the (canonical) induced map

$$H^i(T(A^\bullet)) \to H^i(T(I^\bullet)) = R^iT(F)$$

is an isomorphism when $A^\bullet$ consists of T-acyclic objects.

Proof. The proof is by induction on degree, i.e. we assume that R^kT for $k \leq i-1$ is computed by acyclic resolutions. We want to prove the same for R^iT. Let $F \in Sh(X)$ and let $F \to A^\bullet$ be an acyclic resolution of F. Look at the short exact sequence:

$$0 \to F \to A^0 \to coker(d_0) \to 0.$$

The long exact sequence of R^iT is:

$$0 \to T(F) \to T(A^0) \to T(coker(d_0)) \to R^1T(F) \to R^1T(A^0) \to \cdots$$
$$\cdots R^{i-1}T(A^0) \to R^{i-1}T(coker(d_0)) \to R^iT(F) \to R^iT(A^0) \to \cdots$$

By T-acyclicity of A^0 one deduces:

$$R^1T(F) = coker(T(A^0) \to T(coker(d_0))$$
$$R^iT(F) = R^{i-1}T(coker(d_0)) \qquad (i \geq 2)$$

On the other hand applying T to the resolution $F \to A^\bullet$ gives the complex

$$0 \to T(F) \to T(A^0) \to T(A^1) \to T(A^2) \to$$

which is exact at the first two places. Thus $H^1(T(A^\bullet)) = \frac{ker(T(A^1) \to T(A^2))}{im(T(A^0) \to T(A^1))}$. But the sequence

$$0 \to coker(d_0) \to A^1 \to A^2$$

is also exact and hence

$$0 \to T(coker(d_0)) \to T(A^1) \to T(A^2)$$

is exact and hence $ker(T(A^1) \to T(A^2)) = im(T(coker(d_0) \hookrightarrow T(A^1))$ and so

$$H^1(T(A^\bullet)) = \frac{im(T(coker(d_0) \hookrightarrow T(A^1))}{im(T(A^0) \to T(A^1))} = R^1T(F).$$

Now $coker(d_0)$ has the resolution by T-acyclic objects

$$0 \to coker(d_0) \to A^1 \to A^2 \to \cdots.$$

Then the induction hypothesis gives

$$R^iT(F) = R^{i-1}T(coker(d_0)) = H^i(T(A^1) \to T(A^2) \to \cdots)$$

and we are done. □

6.12. **Flasque sheaves, canonical flasque resolution.** A sheaf is called **flasque** (or sometimes **flabby**) if the restriction maps $F(X) \to F(U)$ are surjective for all U (equivalently, all restrictions $F(V) \to F(U)$ for $U \subset V$ are surjective).

Proposition 6.12.1. *A flasque sheaf is Γ-acyclic.*

Proof. Let F be flasque. We can find a short-exact sequence

$$0 \to F \xrightarrow{\alpha} I \xrightarrow{\beta} G \to 0$$

where I is both flasque and injective. (Indeed, the proof of Lemma 6.9.2 showed this: the sheaf $\prod_x i_{x*}I_x$ in that proof is automatically flasque.) Suppose we show that

$$I(U) \xrightarrow{\beta} G(U) \text{ is surjective for all } U \subset X. \qquad (6.12.1)$$

It would then follow that G is also flasque and we could use the long exact sequence

$$0 \to H^0(X,F) \to H^0(X,I) \to H^0(X,G) \to H^1(X,F) \to H^1(X,I) \to$$
$$\cdots \to H^{i-1}(X,I) \to H^{i-1}(X,G) \to H^i(X,F) \to H^i(X,I) \to \cdots$$

The surjectivity of $I(X) \to G(X)$ would show that $H^1(X,F) = 0$ and then the dimension shift $H^i(X,F) \cong H^{i-1}(X,G)$ plus flasqueness of G would allow an induction to show that $H^i(X,F) = 0$ for all $i > 1$.

Let us prove (6.12.1). Let $\sigma \in G(U)$. Suppose that $V, W \subset U$ are open and $\tau_V \in I(V)$ and $\tau_W \in I(W)$ map (under β) to $\rho^U_V(\sigma)$ and $\rho^U_W(\sigma)$ respectively. On the overlap $V \cap W$ we have

$$\rho^V_{V\cap W}(\tau_V) - \rho^W_{V\cap W}(\tau_W) \quad \overset{\beta}{\mapsto} \quad \rho^V_{V\cap W}\rho^U_V(\sigma) - \rho^W_{V\cap W}\rho^U_W(\sigma) = 0$$

and hence there is $\eta_{V\cap W} \in F(V \cap W)$ such that

$$\eta_{V\cap W} \quad \overset{\alpha}{\mapsto} \quad \rho^V_{V\cap W}(\tau_V) - \rho^W_{V\cap W}(\tau_W) \in I(V \cap W).$$

Since F is flasque there is an $\eta \in F(V)$ restricting to $\eta_{V\cap W}$ (i.e. with $\rho^V_{V\cap W}(\eta) = \eta_{V\cap W}$). Define

$$\tau'_V = \tau_V - \eta.$$

Then $\tau'_V \in I(V)$ and $\tau_W \in I(W)$ agree on $V \cap W$ and $\tau'_V \mapsto \rho^U_V(\sigma)$ and $\tau_W \mapsto \rho^U_W(\sigma)$. Thus there is a section $\tau \in I(V \cup W)$ which goes to $\rho^U_{V\cup W}(\sigma)$, i.e we have increased the domain on which we can lift a section $\sigma \in G(U)$. Now consider the set of pairs $(W, \tau \in I(W))$ where $\tau \mapsto \rho^U_W(\sigma)$, partially ordered according to

$$(W,\tau) \leqslant (W',\tau') \Leftrightarrow W \subset W' \text{ and } \rho^{W'}_W(\tau') = \tau.$$

Since any increasing chain in this poset has a maximal element (by the sheaf axiom (S2)), the poset has a maximal element (Zorn's lemma). By the previous argument its domain must be all of U. In other words, there is a $\tau \in I(U)$ such that $\tau \mapsto \sigma$. This proves (6.12.1) and the lemma. ([28]) □

As a corollary we see that flasque resolutions compute sheaf cohomology. Looking back at the proof of the existence of injective resolutions we see that the same idea gives the **canonical flasque resolution**, usually called the **Godement resolution**: For a sheaf F let

$$C^0(F) := \prod_{x\in X} i_{x*}F_x$$

This is flasque and functorial in F. The natural morphism of sheaves $F \to C^0(F)$ is injective (look at stalks) and we have the cokernel sheaf $coker(F \to C^0(F))$. Define

$$C^1(F) := C^0(coker(F \to C^0(F))).$$

The map $C^0(F) \to C^1(F)$ is given by the composition

$$C^0(F) \to coker(F \to C^0(F)) \hookrightarrow C^0(coker(F \to C^0(F))) = C^1(F).$$

Now set

$$C^2(F) := C^0(coker(C^0(F) \to C^1(F)))$$

and $C^1(F) \to C^2(F)$ is induced by

$$C^1(F) \to coker(C^0(F) \to C^1(F)) \to C^0(coker(C^0(F) \to C^1(F))) = C^2(F).$$

Proceeding in this way gives a complex $C^\bullet(F)$ which is a resolution of F. Because all the terms in the resolution come by the $C^0(-)$ construction they are all flasque. ([29])

We thus have a canonical resolution computing the cohomology of any sheaf. Moreover, as is obvious from the construction, it is functorial in the sheaf (i.e. in F). From this (and Prop. 6.11.1) we see that $F \mapsto R^i\Gamma(X,F) = H^i(X,F)$ is a functor. ([30])

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[28]Note that we have really proved that if $0 \to F \to G \to Q \to 0$ is exact and F and G are flasque then $Q = G/F$ is flasque.

[29]In the case where we work with sheaves of vector spaces over a fixed field, a flasque sheaf is injective. So in this case the Godement resolution is a canonical (and functorial) injective resolution. Thus all derived functors are actually functors. In general, there is no canonical and functorial injective resolution in an abelian category, and this can be a source of subtle problems.

[30]The existence of a canonical and functorial flasque resolution means that we could have *defined* the cohomology groups using it, i.e. we could have defined $H^i(X,F)$ by $H^i(X,F) := H^i(\Gamma(X, C^\bullet(F)))$. This would have the advantage of obviously being a functor and one can directly prove its properties. (This is the approach taken in some books, e.g. Chapter 3 of Wells, *Differential Analysis on Complex Manifolds*.)

6.13. **Singular cohomology and sheaf cohomology.** Let X be a topological space. For an abelian group A we will use $C^*(-,A)$ to denote the complex of singular cochains with values in A, i.e. $C^*(-,A) = Hom(C_*(-),A)$ where $C_*(-)$ is the singular chain complex. For notational simplicity we restrict to $A=\mathbb{Z}$, but everything below works the same for general A.

For each $i \geq 0$, the assignment

$$U \mapsto C^i(U,\mathbb{Z})$$

defines a presheaf of abelian groups on X. The sheafification is denoted $\mathscr{C}^i$. By functoriality of sheafification the coboundary maps $\delta : C^i(U,\mathbb{Z}) \to C^{i+1}(U,\mathbb{Z})$ give sheaf homomorphisms $\mathscr{C}^i \to \mathscr{C}^{i+1}$. The relation $\delta \circ \delta = 0$ means we have a complex of sheaves

$$0 \to \mathscr{C}^0 \to \mathscr{C}^1 \to \cdots$$

There is an inclusion $\mathbb{Z}_X \hookrightarrow \mathscr{C}^0$ coming from the presheaf inclusion given by $\mathbb{Z} \hookrightarrow C^0(U,\mathbb{Z})$ for any U as the constant cochain, i.e. n goes to the cochain which takes the constant value n on every chain in U.

Lemma 6.13.1. *For a locally contractible space X this gives a resolution of the constant sheaf $\mathbb{Z}_X$, i.e. the sequence*

$$0 \to \mathbb{Z}_X \to \mathscr{C}^0 \to \mathscr{C}^1 \to \cdots$$

is an exact sequence of sheaves.

Proof. Recall that the stalks of a presheaf and of the associated sheaf are the same. So it suffices to prove that each $x \in X$ has a sequence of neighbourhoods U for which the sequence

$$0 \to \mathbb{Z} \to C^0(U,\mathbb{Z}) \to C^1(U,\mathbb{Z}) \to \cdots$$

is exact. This follows immediately from the local contractibility of X since this complex computes the reduced singular cohomology of U. □

The following point-set fact is Proposition 1.14 of Ramanan's *Global Calculus*:

Lemma 6.13.2. *If X is a paracompact Hausdorff space in which every open set is paracompact ([31]) and F is a presheaf on X satisfying (S2), then $F(U) \to F^{\#}(U)$ is surjective for all open $U \subset X$.*

Proof. Let $U \subset X$ be open. Giving an element $s \in F^{\#}(U)$ amounts to giving a cover $\{U_i\}$ of U and sections $s_i \in F(U_i)$ such that for each $x \in U_i$, $s_i \mapsto s_x$ in $F_x = F^{\#}_x$. We would like to show that under the assumptions of the lemma s comes from an element of $F(U)$. We may assume (since U is paracompact by assumption on X) that the cover $\{U_i\}$ is locally finite.

Suppose we could find a covering $\{W_x\}$ of U by open subsets, indexed by $x \in U$, such that

- for all x, W_x is an open neighbourhood of x with the property that $W_x \subset U_i$ for some i
- if $W_x \cap W_y \neq \phi$ then $W_x \cup W_y \subset U_i$ for some i.

Indeed, if so then $s_i|_{W_x}$ defines a collection of sections of F which can evidently be glued to give an element $\sigma \in F(U)$ which has stalks $\sigma_x = s_x$ for all $x \in U$, hence $\sigma \mapsto s \in F^{\#}(U)$. (There is a small point here: Given x, the U_i containing W_x may not be unique, so that if $W_x \subset U_j$ then we do not know whether to take $s_i|_{W_x}$ or $s_j|_{W_x}$. However, by shrinking W_x we can ensure this (since s_i and s_j have the same stalk s_x at x), and this does not affect the other properties of $\{W_x\}_{x\in U}$.)

To construct the covering W_x we proceed as follows. First choose a refinement $\{V_i\}_{i\in I}$ of $\{U_i\}_{i\in I}$ with the same indexing set and such that $\overline{V_i} \subset U_i$. ([32]) Since $\{V_i\}$ is also locally finite, we can choose for each $x \in U$ a neighbourhood W'_x such that $\{i \in I : W'_x \cap V_i \neq \phi\}$ is finite. Now define the following, which is open since it is a finite intersection of opens:

$$W_x := W'_x \cap \left(\bigcap_{W'_x \cap V_i \neq \phi, x \notin \overline{V_i}} U - \overline{V_i} \right) \cap \left(\bigcap_{W'_x \cap V_i \neq \phi} U_i \right).$$

The first intersection makes sure that if $W_x \cap V_i \neq \phi$ then $x \in \overline{V_i}$, while the second makes sure that if $W_x \cap V_i \neq \phi$ then $W_x \subset U_i$. Now if $W_x \cap W_y \neq \phi$, say $z \in W_x \cap W_y$, then there exists i such that $z \in V_i$, whence $z \in W_x \cap V_i \neq \phi$, so that $W_x \subset U_i$. The same argument applies to W_y, so $W_x \cup W_y \subset U_i$.

[31] It is a fact, which we do not use here, that if X is a topological space such that every open subset is paracompact, then in fact every subspace of X is paracompact. Such a space is called *hereditarily paracompact*.

[32] This is a general property of paracompact Hausdorff spaces, which can be proved e.g. using the fact that they are normal, and on normal spaces using Urysohn's lemma we can find such refinements. In the situation of manifolds we actually proved this directly in the course of constructing smooth partitions of unity.

We are now in the situation we wanted. The covering $\{W_x\}_{x\in U}$ allows us to glue the sections $s_i|_{W_x} \in F(W_x)$ (which by the remark made earlier we may assume is independent of i) into a section in $F(U)$, which maps to s under $F(U) \to F^{\#}(U)$. □

The lemma implies that on such a space X, if a presheaf F satisfying (S2) is flasque (i.e. $F(X) \twoheadrightarrow F(U)$ for all U), then the associated sheaf $F^{\#}$ is flasque. This follows immediately from the commutative diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & F^{\#}(X) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F^{\#}(U) \end{array} \tag{6.13.1}$$

In particular, on such a space X, the sheaves $\mathscr{C}^i$ are flasque. In particular, this holds for smooth manifolds (since any open set is again a smooth manifold).

Theorem 6.13.3. *If X is a locally contractible and paracompact Hausdorff space in which any open set is paracompact, then for any abelian group A there are canonical isomorphisms*

$$H^*_{sing}(X, A) \cong H^*(X, A_X)$$

between the singular cohomology groups of X with coefficients in A and the sheaf cohomology groups of the constant sheaf A_X.

Proof. We will assume $A = \mathbb{Z}$, the proof is the same in general. Since $\mathbb{Z}_X \to \mathscr{C}^\bullet$ is a flasque resolution we know that

$$H^*(X, \mathbb{Z}_X) \cong H^*(\Gamma(X, \mathscr{C}^\bullet)).$$

Thus it will suffice to show that

$$C^\bullet(X, \mathbb{Z}) \to \Gamma(X, \mathscr{C}^\bullet)$$

induces an isomorphism in cohomology. Let

$$C^i(X, \mathbb{Z})_0 := \{\sigma \in \Gamma(X, \mathscr{C}^i) : \sigma \mapsto 0 \in \mathscr{C}^i_x \text{ for all } x \in X\}.$$

We first show that the following sequence is short exact:

$$0 \to C^i(X, \mathbb{Z})_0 \to C^i(X, \mathbb{Z}) \to \Gamma(X, \mathscr{C}^i) \to 0$$

The surjectivity on the right comes from the previous lemma. That $C^i(X, \mathbb{Z})_0 \subset ker(C^i(X, \mathbb{Z}) \to \Gamma(X, \mathscr{C}^i))$ is clear. Equality follows from the fact that $\Gamma(X, \mathscr{C}^i) \hookrightarrow \prod_x \mathscr{C}^i_x$ by the sheaf axiom (S1).

Thus it is enough to prove that the complex $C^\bullet(X, \mathbb{Z})_0$ is acyclic, i.e. has zero cohomology. This will be deduced from the theorem of small chains. Recall the statement: If X is a topological space, $\mathscr{U} = \{U_\alpha\}_\alpha$ is an open cover of X and $C^{\mathscr{U}}_\bullet(X)$ denotes the complex of singular chains generated by singular simplices with image contained in some element of $\mathscr{U}$, then

$$C^{\mathscr{U}}_\bullet(X) \hookrightarrow C_\bullet(X)$$

is a chain homotopy equivalence. From the universal coefficient theorem we get short-exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ext^1(H_{i-1}(X,\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^i(X,\mathbb{Z}) & \longrightarrow & Hom(H_i(X,\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \pi^*_{\mathscr{U}}} & & \downarrow & & \\ 0 & \longrightarrow & Ext^1(H^{\mathscr{U}}_{i-1}(X,\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^i_{\mathscr{U}}(X,\mathbb{Z}) & \longrightarrow & Hom(H^{\mathscr{U}}_i(X,\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

where the maps are induced by $C^{\mathscr{U}}_\bullet(X) \subset C_\bullet(X)$ or its dual. (The upper sequence is the usual one; the lower sequence comes from exactly the same homological algebra applied to the complexes $C^{\mathscr{U}}_\bullet(X)$ and $Hom(C^{\mathscr{U}}_\bullet(X), \mathbb{Z})$.) Hence we get the dual assertion that the morphism

$$\pi_{\mathscr{U}} : C^\bullet(X, \mathbb{Z}) \to C^\bullet_{\mathscr{U}}(X, \mathbb{Z})$$

induces an isomorphism in cohomology and hence $ker(\pi_{\mathscr{U}})$ is acyclic. Now $ker(\pi_{\mathscr{U}})$ consists precisely of cochains which map to zero in $C^\bullet(U_\alpha, \mathbb{Z})$ for every α, so

$$C^\bullet(X, \mathbb{Z})_0 = \varinjlim_{\mathscr{U}} ker(\pi_{\mathscr{U}})$$

where the limit is over all coverings of X. Since taking cohomology commutes with direct limits we conclude that $C^\bullet(X, \mathbb{Z})_0$ is acyclic. □

Remark 6.13.4. In fact it is true for any locally contractible space that singular cohomology and sheaf cohomology of the constant sheaf agree, i.e. the assumption that every open in X is paracompact can be dropped. A correct proof of this is surprisingly recent, see Y. Sella, arXiv:1602.06674v3. Sella also gives an example where Lemma 6.13.2 fails without some hypothesis, i.e. the sheaves $\mathscr{C}^k$ are not flasque.

Corollary 6.13.5. *If M is a smooth manifold then $H^*(M, \mathbb{Z}_M) \cong H^*_{sing}(M, \mathbb{Z})$.*

6.14. **Fine sheaves.** We would like to relate de Rham cohomology to sheaf cohomology, just as was done above for singular cohomology. The de Rham sheaves $\mathscr{E}^*_M$ are certainly not flasque. Nevertheless, they are acyclic for the global sections functor. The key property that ensures this is that $\mathscr{E}^0(M)$ contains partitions of unity. Abstracting this we say that a sheaf R of (commutative) rings with unity([33]) on a space X **admits partitions of unity** if for every locally finite covering $\{U_\alpha\}_\alpha$ of X there exists a partition of unity $\{f_\alpha\}$ subordinate to it, i.e. elements $f_\alpha \in R(X)$ with $supp(f_\alpha) \subset U_\alpha$ such that

$$\sum_\alpha f_\alpha = 1.$$

(This equation is to be interpreted as making sense locally: In a neighbourhood of any point it is a finite sum and everything makes sense.) Our main example is the sheaf of smooth functions $\mathscr{E}^0$ on a smooth manifold, which admits partitions of unity. The sheaf of holomorphic functions on any complex manifold is an example of a sheaf of rings with unity which does not admit partitions of unity.

A **fine** sheaf is a sheaf of modules over a sheaf of rings R which admits partitions of unity.([34]) ([35]) ([36]) The main example is $\mathscr{E}^k$ for $k \geq 0$. More generally, the sheaf of sections of any smooth vector bundle is a sheaf of $\mathscr{E}^0$-modules, hence it is fine.

Proposition 6.14.1. *A fine sheaf on a paracompact space is Γ-acyclic.*

Proof. Let F be the fine sheaf, which by definition is a sheaf of modules over a sheaf of rings R. We will use the Godement resolution $F \to C^\bullet = C^\bullet(F)$ (the canonical flasque resolution). Note that the differentials $D : C^i \to C^{i+1}$ are, by construction, R-module homomorphisms. By the Γ-acyclicity of flasque sheaves we have

$$H^i(X, F) = \frac{ker(\Gamma(X, C^i) \to \Gamma(X, C^{i+1}))}{im(\Gamma(X, C^{i-1}) \to \Gamma(X, C^i))}.$$

Suppose $\sigma \in \Gamma(X, C^i)$ is a cocycle with $i > 0$. By exactness of the Godement resolution this means that there is a covering $\{U_\alpha\}$ and elements $\sigma_\alpha \in C^{i-1}$ such that

$$\sigma|_{U_\alpha} = D\sigma_\alpha.$$

By refining the covering we may assume (since X is paracompact) that it is locally finite. Let $\{f_\alpha\}$ be a partition of unity for R subordinate to the covering. Let

$$\tilde{\sigma} := \sum_\alpha f_\alpha \sigma_\alpha.$$

Here $f_\alpha \sigma_\alpha$ is extended by zero outside U_α, so this defines a section in $\Gamma(X, C^{i-1})$. Then

$$D\tilde{\alpha} = \sum_\alpha D(f_\alpha \sigma_\alpha) = \sum_\alpha f_\alpha \sigma|_{U_\alpha} = \sigma.$$

This proves that $H^i(X, F) = 0$ for $i > 0$. □

We have the following important corollary:

Corollary 6.14.2. *The de Rham cohomology groups of a smooth manifold M are naturally isomorphic to the sheaf cohomology of the constant sheaf $\mathbb{R}_M$.*

Combining this with the identification of singular cohomology with sheaf cohomology we get:

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[33] A sheaf of rings has the obvious meaning: a contravariant functor $R : Op(X) \to Rings$ satisfying the sheaf conditions. We assume that there is a section $1 \in R(X)$ which restricts to the unit in $R(U)$ for every U.

[34] A sheaf of modules F over a sheaf of rings R means: Each $\Gamma(U, F)$ is a module over $\Gamma(U, R)$ and the module multiplication maps $\Gamma(U, R) \otimes \Gamma(U, F) \to \Gamma(U, F)$ are compatible with restrictions from U to $V \subset U$.

[35] A more natural definition of fine would be the following: F is fine if the sheaf of rings $Hom(F, F)$ admits partitions of unity. Everything here works with this definition also.

[36] There are several different definitions of fine sheaf in the literature; the one we use here is the easiest for us.

Theorem 6.14.3 (de Rham's theorem)**.** *There is a canonical isomorphism*

$$H^*_{dR}(M) \cong H^*_{sing}(M,\mathbb{R}) = H^*_{sing}(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

between the de Rham cohomology and the singular cohomology with $\mathbb{R}$*-coefficients.*

The isomorphism in the theorem is canonical.([37]) The isomorphism is also functorial and a ring isomorphism when $H^*_{sing}(M,\mathbb{Z})$ is given the cup product and $H^*_{dR}(M)$ is given the product given by exterior product of differential forms. These facts will be proved later.

6.15. **De Rham's theorem and integration over smooth chains (∗).** We will discuss a refinement of the de Rham theorem without giving full details, for which we refer to e.g. Warner's book.

Let M be a smooth manifold. A **smooth k-simplex** is a smooth map from a neighbourhood of the standard k-simplex Δ^k in $\mathbb{R}^n$ to the manifold M. A smooth k-chain is a (finite) formal linear sum with integer coefficients of smooth k-simplices. The boundary of a smooth k-chain is a smooth $k-1$-chain and we have the complex of smooth chains:

$$0 \to C_0^{sm}(M) \to C_1^{sm}(M) \to \cdots$$

Taking $Hom(-,\mathbb{Z})$ we get the complex of smooth cochains:

$$0 \to C^0_{sm}(M,\mathbb{Z}) \to C^1_{sm}(M,\mathbb{Z}) \to C^2_{sm}(M,\mathbb{Z}) \to \cdots$$

The definition can be sheafified: Let $\mathscr{C}^k_{sm}$ be the sheaf associated with the presheaf $U \mapsto C^k_{sm}(U,\mathbb{Z})$. Then we have a complex of sheaves on M

$$0 \to \mathscr{C}^0_{sm} \to \mathscr{C}^1_{sm} \to \cdots$$

and an inclusion $\mathbb{Z}_M \hookrightarrow \mathscr{C}^0_{sm}$ as before. To show that this is a resolution of the constant sheaf we must prove that it is exact on stalks in degrees ≥ 1. This requires some argument which I will not give here, referring instead to pp. 194-195 of Warner's book.

It follows from this that $\mathscr{C}^\bullet_{sm}$ is a resolution of the constant sheaf $\mathbb{Z}_M$ and hence we have an isomorphism

$$H^k(M,\mathbb{Z}_M) \cong H^k(C^\bullet_{sm}(M,\mathbb{Z})).$$

As a corollary we also see that the surjection $C^\bullet(M,\mathbb{Z}) \to C^\bullet_{sm}(M,\mathbb{Z})$ induces an isomorphism in cohomology.

Now suppose σ is a smooth k-simplex in $U \subset M$, i.e. a smooth map $\sigma : nbhd(\Delta^k) \to U$. If $\omega \in \mathscr{E}^k(U)$ then $\sigma^*(\omega)$ can be integrated over Δ^k. There is a minor point here: The k-simplex for $k > 1$ is not quite a domain in $\mathbb{R}^n$ since its boundary is not a manifold at the vertex points. Nevertheless, we can integrate over it, as you can easily check by going over the arguments defining integration over domains. Moreover a slight generalization of Stokes's theorem to this situation says that for a smooth k-chain σ and smooth k-form ω we have:

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega|_{\partial\sigma}.$$

Thus integration defines a homomorphism of complexes of sheaves

$$\mathscr{E}^\bullet \to \mathscr{C}^\bullet_{sm}.$$

It is easy to check that this induces the de Rham isomorphism (indeed, this follows by the footnote after Theorem 6.14.3), implying the following:

Theorem 6.15.1 (Explicit de Rham theorem)**.** *Integration over smooth chains defines a nondegenerate pairing*

$$H^i_{dR}(M) \otimes H_i^{sing}(M,\mathbb{R}) \to \mathbb{R}$$

which induces the de Rham isomorphism $H^i_{dR}(M) \cong H^i_{sing}(M,\mathbb{R})$.

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[37] In general if T is a left-exact functor, given two different T-acyclic resolutions $F \to A^\bullet$ and $F \to B^\bullet$ there is a canonical isomorphism between $H^*(T(A^\bullet))$ and $H^*(T(B^\bullet))$: Choose an injective resolution $F \to I^\bullet$; by Lemma 6.9.3 there are morphisms of resolutions $A^\bullet \to I^\bullet \leftarrow B^\bullet$. By Prop. 6.11.1 these induce canonical isomorphisms $H^i(T(A^\bullet)) \to H^i(T(I^\bullet)) \leftarrow H^i(T(B^\bullet))$. The isomorphism $H^i(T(A^\bullet)) \cong H^i(T(B^\bullet))$ is independent of the choice of $I^\bullet$, again by using Lemma 6.9.3.

Moreover, if $A^\bullet$ and $B^\bullet$ are acyclic resolutions and $A^\bullet \to B^\bullet$ lifts the identity $F \to F$ then it induces the same isomorphism as the previous one.

6.16. **Functoriality (∗).** The explicit version of de Rham's theorem shows that the de Rham isomorphism is functorial in M, i.e.

Let $f : M \to N$ be a smooth map of smooth manifolds. If $\sigma : nbhd(\Delta^k) \to M$ is a smooth k-simplex in M and $\omega \in \mathscr{E}^k(N)$ then

$$\int_\sigma f^*\omega = \int_{f\circ\sigma} \omega.$$

This shows that the isomorphism $H^k_{dR}(M) \to H^k_{sing}(M, \mathbb{R})$ induced by the integration pairing is functorial.

(Since we did not give all the details of the explicit de Rham theorem we will give another proof of the functoriality in the next section.)

7. Sheaves and cohomology II (∗)

In this section we introduce hypercohomology, which generalizes cohomology to complexes of sheaves. The necessary formalism is a little cumbersome but we will use it to establish that the de Rham isomorphism is functorial and one of rings. It is also useful in concrete situations, as we will see later. The presentation mostly follows Voisin's book.

In this section we fix an abelian category $\mathfrak{A}$, which is always either $Sh(X)$ or $AbGps$. The word "object" will usually mean object of A and "homomorphism" will mean homomorphism in $\mathfrak{A}$, unless explicitly stated otherwise. All complexes $K^\bullet$ considered below will be in nonnegative degrees, i.e. $K^i = 0$ for $i < 0$. (The proofs work for all complexes bounded below, i.e. if there is a r such that $K^i = 0$ for $i < r$.)

7.1. **Cohomology objects, quasi-isomorphisms.** Let $F^\bullet$ be a complex of objects of $\mathfrak{A}$. The ith **cohomology object** of $F^\bullet$ is:

$$H^i(F^\bullet) = \frac{ker(F^i \to F^{i+1})}{im(F^{i-1} \to F^i)}.$$

(Here if $\mathfrak{A} = Sh(X)$ then the image and quotient are taken in the sheaf sense.) A homomorphism of complexes of sheaves $F^\bullet \to G^\bullet$ is a **quasi-isomorphism** if the induced homomorphism on cohomology objects $H^i(F^\bullet) \to H^i(G^\bullet)$ is an isomorphism for all i.

For example on a manifold M the inclusion $\mathbb{R}_M \to \mathscr{E}^\bullet_M$ is a quasi-isomorphism. More generally, if $F \hookrightarrow A^\bullet$ is a resolution then $F \to A^\bullet$ is a quasi-isomorphism, where we think of F as the complex $0 \to F \to 0$ with F in degree 0.

A **double complex** $K^{\bullet\bullet}$ (in $\mathfrak{A}$) is a collection of objects $(K^{p,q})_{(p,q)\in\mathbb{Z}^2}$ (in $\mathfrak{A}$) with homomorphisms $D_2 : K^{p,q} \to K^{p,q+1}$ (the vertical differential) and $D_1 : K^{p,q} \to K^{p+1,q}$ (the horizontal differential) such that

$$D_1^2 = 0 = D_2^2 \quad \text{and} \quad D_1D_2 = D_2D_1.$$

One should keep in mind the picture:

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
\longrightarrow & & K^{p,q+1} & \xrightarrow{D_1} & K^{p+1,q+1} & \longrightarrow & \\
 & & {\scriptstyle D_2}\uparrow & & {\scriptstyle D_2}\uparrow & & \\
\longrightarrow & & K^{p,q} & \xrightarrow{D_1} & K^{p+1,q} & \longrightarrow & \\
 & & \uparrow & & \uparrow & &
\end{array}$$

The **total complex** is the associated single complex $Tot(K^{\bullet,\bullet})^\bullet$ with

$$Tot(K^{\bullet,\bullet})^i = \oplus_{p+q=i} K^{p,q}$$

and differential

$$D|_{K^{p,q}} = D_1 + (-1)^p D_2$$

(The sign is put so that $D^2 = 0$.)

An example of this construction is the **tensor product** of complexes: If $K^\bullet$ and $L^\bullet$ are complexes then the tensor product complex is defined to be the total complex

$$Tot(K^\bullet \otimes L^\bullet)^i := \oplus_{p+q=i} K^p \otimes L^q$$

with the total differential $D_K \otimes id + (-1)^p id \otimes D_L$. Note that there is a map

$$H^p(K^\bullet) \otimes H^q(L^\bullet) \longrightarrow H^{p+q}(Tot(K^\bullet \otimes L^\bullet))$$

of cohomology objects. We will frequently drop the "*Tot*" from the notation when we are dealing with tensor products of complexes; this will not usually cause any confusion.

7.2. **"Resolutions" of complexes.**

Lemma 7.2.1. *Let $F^\bullet$ be a complex bounded below (say $F^k = 0$ for $k < 0$). There is a double complex $(I^{\bullet,\bullet}, D_1, D_2)$ consisting of injective objects and a monomorphism of complexes $F^\bullet \hookrightarrow I^{\bullet,0}$*

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,2} & \longrightarrow & I^{1,2} & \longrightarrow & I^{2,2} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,1} & \xrightarrow{D_1} & I^{1,1} & \longrightarrow & I^{2,1} & \longrightarrow & \cdots \\
\uparrow{\scriptstyle D_2} & & \uparrow{\scriptstyle D_2} & & \uparrow & & \\
I^{0,0} & \xrightarrow{D_1} & I^{1,0} & \longrightarrow & I^{2,0} & \longrightarrow & \cdots \\
{\scriptstyle i^0}\uparrow & & {\scriptstyle i^1}\uparrow & & {\scriptstyle i^2}\uparrow & & \\
F^0 & \xrightarrow{d_F} & F^1 & \xrightarrow{d_F} & F^2 & \xrightarrow{d_F} & \cdots
\end{array}$$

such that:

(1) *each column is exact (i.e. $I^{k,\bullet}$ is an injective resolution of F^k)*
(2) *the inclusion $F^\bullet \hookrightarrow I^{\bullet,0}$ is a homomorphism of complexes.*

Proof. It is enough to prove the following: A complex $F^\bullet$ admits a monomorphism to a complex of injectives $J^\bullet$. Indeed, if so then we can construct the first row $I^{\bullet,0}$ of the double complex with the inclusion $F^\bullet \hookrightarrow I^{\bullet,0}$. Then we can embed the cokernel $coker(F^\bullet \to I^{\bullet,0})$ in another complex of injectives which will be the row $I^{\bullet,1}$ and proceed from there.

So let us construct the first row. We start with $i^0 : F^0 \hookrightarrow I^{0,0}$ a monomorphism to an injective object. Next consider

$$(i^0, -d_F) : F^0 \hookrightarrow I^{0,0} \oplus F^1.$$

Choose a monomorphism

$$coker(i^0, -d_F) \hookrightarrow I^{1,0}$$

to an injective object. Then the homomorphisms D_1 and i^1 are given by:

$$\begin{array}{ccc}
I^{0,0} & & \\
\downarrow & & \\
I^{0,0} \oplus F^1 & \longrightarrow & coker(i^0, -d_F) \hookrightarrow I^{1,0} \\
\uparrow & & \\
F^1 & &
\end{array}$$

The identity $D_1 \circ i^0 = i^1 \circ d_F$ holds because $i^0(f)$ and $d_F(f)$ are the same in the module $coker(i^0, -d_F)$. The injectivity of $F^1 \to I^{1,0}$ is easy.

Now to continue we take

$$(\bar{i}^1, -d_F) : F_1 \to coker(D_1) \oplus F^2$$

where $\bar{i}^1$ is the composition $F^1 \to I^{1,0} \to coker(D_1)$ and choose a monomorphism

$$coker(\bar{i}^1, -d_F) \hookrightarrow I^{2,0}.$$

The maps $D_1 : I^{1,0} \to I^{2,0}$ and $i^2 : F^2 \to I^{2,0}$ are given by:

$$\begin{array}{ccc}
I^{1,0} & & \\
\downarrow & & \\
I^{1,0} \oplus F^2 & \longrightarrow & coker(\bar{i}^1, -d_F) \hookrightarrow I^{2,0} \\
\uparrow & & \\
F^2 & &
\end{array}$$

As before we see that $D_1 \circ i^1 = i^2 \circ d_F$ by construction, and i^2 is injective. Also $D_1^2 = 0$ by construction.

Proceeding in this fashion constructs the first row. □

Lemma 7.2.2. *Suppose that $I^{\bullet,\bullet}$ is a double complex of objects in $\mathfrak{A}$ and $F^\bullet \hookrightarrow I^{\bullet,0}$ is a monomorphism of complexes such that $F^k \hookrightarrow I^{k,\bullet}$ is a resolution of F^k for each k. Then $F^\bullet \hookrightarrow Tot(I^{\bullet,\bullet})^\bullet$ is a quasi-isomorphism.*

Proof. We give the proof in $\mathfrak{A} = AbGps$; the same proof works in general with some rephrasing (one cannot work with elements).

First let us see that $H^k(F^\bullet) \twoheadrightarrow H^k(Tot(I)^\bullet)$ for all k. A class in $H^k(Tot(I)^\bullet)$ is represented by $\alpha = \sum_{p+q=k} \alpha_{p,q} \in \oplus_{p+q=k} I^{p,q}$ with $D\alpha = 0$. Then $D_2(\alpha_{0,k}) = 0$ and by exactness of the first column $\alpha_{0,k} = D_2(\beta)$ for $\beta \in I_{0,k-1}$. Then $\alpha' = \alpha - D\beta$ represents the same class as α but $\alpha'_{0,k} = 0$. But then $D_2(\alpha'_{k-1,0}) = 0$ and we can run the same argument to produce α'' in the same class as α and with $\alpha''_{1,k-1} = 0$. Proceeding in this fashion we can assume $\alpha \in I^{k,0}$. Then $D\alpha = 0$ implies $D_2\alpha = D_1\alpha = 0$. Hence $\alpha = i^k(f)$ for $f \in F^k$, and $i^{k+1}(d_F(f)) = D_1 i^k(f) = D_1\alpha = 0$. Then the class of f maps to the class of α. This proves surjectivity.

Next let us see that $H^k(F^\bullet) \to H^k(Tot(I)^\bullet)$ is injective for all k. If $k = 0$ then this follows from the injectivity of i^0. Let $k > 0$ and let $f \in F^k$ with $d_F(f) = 0$ and suppose that $i^k(f) = D\beta$ for some $\beta \in Tot(I)^{k-1}$. Then $\beta \in I^{k-1,0}$ and hence $D\beta = i^k(f)$ implies that $D_2(\beta) = 0$ and $D_1\beta = i^k(f)$. Since the columns are exact there is a $e \in F^{k-1}$ with $i^{k-1}(e) = \beta$. Then

$$i^k d_F(e) = D_1 i^{k-1}(e) = D_1\beta = i^k(f).$$

Since i^k is injective we have $d_F(e) = f$. This proves the injectivity. □

Note that the objects in the double complex here were not required to be injective in this lemma.

Combining the two lemmas we get the first part of the following lemma.

Lemma 7.2.3. *(1) If $F^\bullet$ is a bounded below complex there is a monomorphism $F^\bullet \hookrightarrow I^\bullet$ to a complex of injective objects which is a quasi-isomorphism.*

(2) If $F^\bullet \hookrightarrow I^\bullet$ is as in (1) and $F^\bullet \to J^\bullet$ is any other quasi-isomorphism with $J^\bullet$ a complex of injectives, then there exists a morphism of complexes $\phi : I^\bullet \to J^\bullet$ making the diagram

$$\begin{array}{ccc} F^\bullet & \xrightarrow{id} & F^\bullet \\ \downarrow & & \downarrow \\ I^\bullet & \xrightarrow{\phi} & J^\bullet \end{array}$$

commute.

Proof. (2) Exercise. □

7.3. **Derived functors and hypercohomology.** For a left-exact functor T define the **ith (hyper)derived functor** on $F^\bullet$ as follows: Choose a monomorphic quasi-isomorphism $F^\bullet \hookrightarrow I^\bullet$ to a complex of injectives and set:

$$R^iT(F^\bullet) := H^i(T(I^\bullet)).$$

In the special case where T is the global sections functor these are called the **hypercohomology groups of $F^\bullet$** and are denoted:

$$\mathbb{H}^i(X, F^\bullet) = R^i\Gamma(X, F^\bullet) = H^i(\Gamma(X, I^\bullet)).$$

As in the case of derived functors these depend, a priori, on the choice of $F^\bullet \hookrightarrow I^\bullet$. The second part of the previous lemma can be used to show that they are well-defined up to canonical isomorphism and that they have the same functorial property as derived functors.

Thus if $F^\bullet$ consists of a single sheaf F placed in degree zero, then

$$R^iT(F^\bullet) = R^iT(F) \quad \text{and} \quad \mathbb{H}^i(X, F^\bullet) = H^i(X, F).$$

If $I^\bullet$ is a complex of injectives then

$$R^iT(I^\bullet) = H^i(T(I^\bullet)) \quad \text{and} \quad \mathbb{H}^i(X, I^\bullet) = H^i(\Gamma(X, I^\bullet)).$$

In particular, an exact complex of injectives has zero higher ($i > 0$) derived functors.

The proof of the following proposition will be given later:

Proposition 7.3.1. *If $F^\bullet \to J^\bullet$ is a quasi-isomorphism (but not necessarily a monomorphism) with a complex of injective objects and T is a left-exact functor then it induces canonical isomorphisms*

$$R^iT(F^\bullet) \to R^iT(J^\bullet) = H^i(T(J^\bullet))$$

for all i.

We have the following immediate corollary:

Corollary 7.3.2. *If $F^\bullet \to K^\bullet$ is a quasi-isomorphism and T is a left-exact functor then it induces canonical isomorphisms $R^iT(F^\bullet) \to R^iT(K^\bullet)$ for all i.*

Proof. Suppose $\phi : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Choose a monomorphism $i : L^\bullet \hookrightarrow I^\bullet$ which is a quasi-isomorphism to a complex of injectives. Then $i \circ \phi : K^\bullet \to I^\bullet$ is a quasi-isomorphism to a complex of injectives. By Prop. 7.3.1 there are canonical isomorphisms

$$R^iT(K^\bullet) = H^i(T(I^\bullet)) = R^iT(L^\bullet).$$

This proves the corollary. □

The proof of the following proposition (which generalizes Prop. 6.11.1) will be given later:

Proposition 7.3.3. *If T is a left-exact functor and $F^\bullet \to A^\bullet$ is a quasi-isomorphism to a complex of T-acyclic objects then it induces canonical isomorphisms*

$$R^iT(F^\bullet) \to R^iT(A^\bullet) = H^i(T(A^\bullet))$$

for all i.

For example, applying the corollary to the quasi-isomorphism $\mathbb{R}_M \to \mathscr{E}_M^\bullet$ on a smooth manifold we get:

$$H^i(M, \mathbb{R}_M) \cong \mathbb{H}^i(M, \mathscr{E}_M^\bullet) = H^i(\Gamma(M, \mathscr{E}_M^\bullet))$$

where we have also used the second proposition. This gives the isomorphism between de Rham cohomology and sheaf cohomology of $\mathbb{R}_M$.

Suppose $F^\bullet$ is a complex of sheaves. The Godement resolution is functorial hence we get a double complex $C^\bullet(F^\bullet)$ in which each column $C^\bullet(F^k)$ is a resolution of F^k. By Lemma 7.2.2 we conclude that $F^\bullet \hookrightarrow Tot(C^\bullet(F^\bullet))^\bullet$ is a quasi-isomorphism. Since the terms in the Godement resolutions are flasque we get a canonical isomorphism:

$$\mathbb{H}^i(X, F^\bullet) = \mathbb{H}^i(X, Tot(C^\bullet(F^\bullet))^\bullet).$$

As in the case of sheaf cohomology, we see that the functoriality of Godement resolutions shows that hypercohomology is actually a functor from complexes of sheaves to abelian groups.

7.4. **Pullback maps in (hyper)cohomology.** First let $f : Y \to X$ be a continuous map of topological spaces and F a sheaf on X. Then there is an obvious map ([38])

$$\Gamma(X, F) \to \Gamma(Y, f^{-1}F).$$

For each $i \geq 0$ there is a canonical pullback map

$$H^i(X, F) \to H^i(Y, f^{-1}F) \tag{7.4.1}$$

which is the previous map when $i = 0$. This is defined as follows: Choose injective resolutions $F \hookrightarrow I^\bullet$ and $f^{-1}J^\bullet$. Then $f^{-1}I^\bullet$ is a resolution of $f^{-1}F$ (because f^{-1} is exact) and hence there is a homomorphism of complexes $f^{-1}I^\bullet \to J^\bullet$, unique up to homotopy. Applying Γ we have homomorphisms of complexes

$$\Gamma(X, I^\bullet) \to \Gamma(Y, f^{-1}I^\bullet) \to \Gamma(Y, J^\bullet)$$

which induce the desired homomorphisms (7.4.1) in cohomology. It is straightforward to check that these are independent of the choices made.

Now let $F^\bullet$ be a complex of sheaves on X. Then there is a canonical pullback map

$$\mathbb{H}^i(X, F^\bullet) \to \mathbb{H}^i(Y, f^{-1}F^\bullet) \tag{7.4.2}$$

defined as follows: Choose injective resolutions $F^\bullet \hookrightarrow I^\bullet$ and $f^{-1}F^\bullet \to J^\bullet$ as in (1) of Lemma 7.2.3. Since f^{-1} is exact $f^{-1}F^\bullet \hookrightarrow f^{-1}I^\bullet$ is also a quasi-isomorphism ([39]), so by (2) of the same lemma there

[38] Indeed, a section of $f^{-1}F$ over Y consists of a covering of Y and for each U in the covering an element of $\varinjlim_{V\, open, V \supset f(U)} \Gamma(V, F)$. An element of $\Gamma(X, F)$ gives such a section trivially.

[39] An exact functor takes quasi-isomorphisms to quasi-isomorphisms (see Lemma 7.10.2 below).

exists a homomorphism $f^{-1}I^\bullet \to J^\bullet$, unique up to homotopy. Thus the maps on global sections give maps

$$\Gamma(X, I^\bullet) \to \Gamma(Y, f^{-1}I^\bullet) \to \Gamma(Y, J^\bullet)$$

and hence a canonical map

$$\mathbb{H}^i(X, F^\bullet) = H^i(\Gamma(X, I^\bullet)) \to H^i(\Gamma(Y, J^\bullet)) = \mathbb{H}^i(Y, f^{-1}F^\bullet).$$

It is easy to check that this pullback map is independent of choices and is natural in homomorphisms of complexes of sheaves $F^\bullet \to G^\bullet$. It obviously reduces to (7.4.1) in case $F^\bullet$ and $G^\bullet$ are single sheaves in degree zero.

In some cases this map is a familiar one: For example, if $f : Y \to X$ is a continuous map of topological spaces then $f^{-1}\mathbb{Z}_X \cong \mathbb{Z}_Y$. Thus there is a pullback map

$$H^i(X, \mathbb{Z}_X) \to H^i(Y, \mathbb{Z}_Y).$$

If X and Y are locally contractible then under the isomorphisms $H^i(-, \mathbb{Z}_X) \cong H^i_{sing}(-, \mathbb{Z})$ this is the usual pullback map f^* in singular cohomology. (Exercise.)

7.5. **Functoriality of the de Rham isomorphism.** Let us see how this follows nicely using hypercohomology. In the case $f : M \to N$ is a map of manifolds we shall see that the pullback map in either singular or de Rham cohomology is given by the map

$$H^i(N, \mathbb{R}_N) \to H^i(N, f^{-1}\mathbb{R}_N) \cong H^i(M, \mathbb{R}_M)$$

under the respective isomorphisms $H^i_{sing}(?, \mathbb{R}) \cong H^i(?, \mathbb{R}_?)$ and $H^i_{dR}(?) \cong H^i(?, \mathbb{R}_?)$. This will prove the functoriality of the de Rham isomorphism.

First consider de Rham cohomology. We have a diagram

$$\begin{array}{ccc} f^{-1}\mathbb{R}_N & \longrightarrow & \mathbb{R}_M \\ \downarrow & & \downarrow \\ f^{-1}\mathscr{E}^\bullet_N & \longrightarrow & \mathscr{E}^\bullet_M \end{array}$$

The lower map is given by pullback of differential forms from (open sets of) N to (open sets of) M. Both vertical maps are quasi-isomorphisms (the right by the Poincaré lemma on M, the left by the Poincaré lemma on N plus exactness of f^{-1}). The upper map is an isomorphism. The commutativity of the diagram is obvious. Taking hypercohomology gives the right square of:

$$\begin{array}{ccccc} \mathbb{H}^k(N, \mathbb{R}_N) & \longrightarrow & \mathbb{H}^k(M, f^{-1}\mathbb{R}_N) & = & \mathbb{H}^k(M, \mathbb{R}_M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^k(N, \mathscr{E}^\bullet_N) & \longrightarrow & \mathbb{H}^k(N, f^{-1}\mathscr{E}^\bullet_N) & \xrightarrow{\cong} & \mathbb{H}^k(M, \mathscr{E}^\bullet_M) \end{array}$$

The outer vertical maps give the isomorphism between sheaf cohomology and de Rham cohomology on N and M respectively. All the maps in the right square are isomorphisms. The horizontal maps in the left square are pullback maps. Both squares commute, thus the outer isomorphisms are compatible. Checking that the composition across the bottom row is the same as the map induced by pullback of differential forms from N to M is left as an exercise.

The same argument using the complex of sheaves of continuous cochains proves that the pullback

$$H^i(N, \mathscr{C}^\bullet_N) = \mathbb{H}^i(N, \mathscr{C}^\bullet_N) \to \mathbb{H}^i(M, f^{-1}\mathscr{C}^\bullet_N) \cong \mathbb{H}^i(M, \mathscr{C}^\bullet_M) = H^i(M, \mathscr{C}^\bullet_M)$$

is the sheaf pullback. Now this map is clearly compatible with the usual pullback

$$H^i_{sing}(N, \mathbb{Z}) \to H^i_{sing}(M, \mathbb{Z})$$

Thus we have:

Proposition 7.5.1. *The de Rham isomorphism $H^*_{dR}(M) \cong H^*_{sing}(M, \mathbb{R})$ is functorial for smooth maps of manifolds.*

7.6. **Cup products in sheaf cohomology.** To show that the de Rham isomorphism $H^*_{dR}(M) \cong H^*_{sing}(M,\mathbb{R})$ is compatible with products (i.e. relates the product in $H^*_{sing}(M,\mathbb{R})$ induced by cup product of cochains and the product in $H^*_{dR}(M)$ induced by wedge product of differential forms) we will define a suitable cup product in sheaf cohomology $H^*(M,\mathbb{R}_M)$, and then show that it agrees with each of the de Rham and singular cup products under the respective isomorphisms $H^*_{dR}(M) \cong H^*(M,\mathbb{R}_M)$ and $H^*_{sing}(M,\mathbb{R}) \cong H^*(M,\mathbb{R}_M)$. For this we need to understand how cup products work in hypercohomology. For simplicity the case of cohomology of a single sheaf is discussed first.

Before starting with the discussion, recall (for motivation) one way to define cup products in singular cohomology. ([40]) This is via the cross product (or external cup product), which is a map

$$\times : H^i(X,\mathbb{Z}) \otimes H^j(Y,\mathbb{Z}) \to H^{i+j}(X \times Y,\mathbb{Z}). \tag{7.6.1}$$

When $X = Y$ we can compose this with pullback by the diagonal $\delta : X \hookrightarrow X \times X$ to define the cup product

$$\smile: H^i(X,\mathbb{Z}) \otimes H^j(X,\mathbb{Z}) \to H^{i+j}(X,\mathbb{Z}) \quad \text{by } \alpha \smile \beta = \delta^*(\alpha \times \beta). \tag{7.6.2}$$

Now let us define cup products on sheaf cohomology. For sheaves $F, G \in Sh(X)$ we will define a (cup) product map

$$H^p(X,F) \otimes H^q(X,G) \to H^{p+q}(X,F \otimes G). \tag{7.6.3}$$

Let $\pi_i : X \times X \to X$ be the projections. The map (7.6.3) is the composition

$$\begin{array}{c} H^p(X,F) \otimes H^q(X,G) \\ \big\downarrow {\scriptstyle (1)} \\ H^{p+q}(X \times X, \pi_1^{-1}F \otimes \pi_2^{-1}G) \\ \big\downarrow {\scriptstyle (2)} \\ H^{p+q}(X \times X, \delta_*(F \otimes G)) \\ \big\| \\ H^{p+q}(X, F \otimes G) \end{array}$$

where the maps are as follows:

(1) Let π_1, π_2 be the projections of $X \times Y$ to each factor. Then there is a homomorphism

$$H^p(X,F) \otimes H^q(Y,G) \to H^{p+q}(X \times Y, \pi_1^{-1}F \otimes \pi_2^{-1}G). \tag{7.6.4}$$

(This is the analogue of the cross product in singular cohomology.) To get this let $F \hookrightarrow I^\bullet$ and $G \hookrightarrow J^\bullet$ be the Godement resolutions of F and G. Then $\pi_1^{-1}F \otimes \pi_2^{-1}G \hookrightarrow \pi_1^{-1}I^\bullet \otimes \pi_2^{-1}J^\bullet$ is a resolution on $X \times Y$. ([41]) Thus there is a map ([42])

$$H^i(\Gamma(X \times Y, \pi_1^{-1}I^\bullet \otimes \pi_2^{-1}J^\bullet)) \longrightarrow H^i(X \times Y, \pi_1^{-1}F \otimes \pi_2^{-1}G).$$

Now using the maps of complexes

$$\Gamma(X,I^\bullet) \otimes \Gamma(Y,J^\bullet) \overset{\pi_1^{-1}\otimes\pi_2^{-1}}{\longrightarrow} \Gamma(X \times Y, \pi_1^{-1}I^\bullet) \otimes \Gamma(X \times Y, \pi_2^{-1}J^\bullet) \longrightarrow \\ \longrightarrow \Gamma(X \times Y, \pi_1^{-1}I^\bullet \otimes \pi_2^{-1}J^\bullet)$$

and composing with the previous map we get (7.6.4).

(2) comes from the canonical homomorphism of sheaves on $X \times X$

$$\pi_1^{-1}F \otimes \pi_2^{-1}G \longrightarrow \delta_*(F \otimes G)$$

where $\delta : X \hookrightarrow X \times X$ is the diagonal.

(3) Pushforward by δ is an exact functor and hence $H^i(X \times X, \delta_*K) = H^i(X,K)$ (cf. Lemma 6.10.3).

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[40] This is the approach used in §3.B of Hatcher's book.

[41] To prove this look at stalks: The stalk of $\pi_1^{-1}A \otimes \pi_2^{-1}B$ at (x,y) is $A_x \otimes B_y$.

[42] We are dropping the "Tot" notation in tensor products for this subsection.

In the special case where $\mathscr{R}$ is a sheaf of rings, so that there is a map $\mathscr{R} \otimes \mathscr{R} \to \mathscr{R}$, (7.6.3) can be composed with $H^i(X, \mathscr{R} \otimes \mathscr{R}) \to H^i(X, \mathscr{R})$ to give products

$$H^i(X, \mathscr{R}) \otimes H^j(X, \mathscr{R}) \longrightarrow H^{i+j}(X, \mathscr{R}). \tag{7.6.5}$$

In particular, if R is any coefficient ring and R_X the constant sheaf then we have defined a product

$$H^i(X, R_X) \otimes H^j(X, R_X) \longrightarrow H^{i+j}(X, R_X). \tag{7.6.6}$$

In this context we could have replaced $\otimes = \otimes_{\mathbb{Z}}$ everywhere above by $\otimes_R$ and got a product $H^i(X, R_X) \otimes_R H^j(X, R_X) \longrightarrow H^{i+j}(X, R_X)$.

On the other hand, if $\mathscr{R}$ is a sheaf of rings we may be able to resolve it by a sheaf of $\mathscr{R}$-algebras which are Γ-acyclic. This is the case, for example, if $\mathscr{R} = \mathbb{R}_M$ is the constant sheaf on M and we take the de Rham resolution $\mathbb{R}_M \hookrightarrow \mathscr{E}^\bullet_M$. In that case there is another, potentially different, product on $H^*(X, \mathscr{R})$. Again we abstract the situation a bit to make it clearer.

Suppose we are given a commutative ring R (with unity) and two sheaves of R-modules F and G. Suppose we are given Γ-acyclic resolutions $F \hookrightarrow A^\bullet, G \hookrightarrow B^\bullet$ and $F \otimes_R G \hookrightarrow C^\bullet$ and a homomorphism of complexes of sheaves

$$\mu : A^\bullet \otimes_R B^\bullet \longrightarrow C^\bullet$$

such that the diagram

$$\begin{array}{ccc} F \otimes_R G & = & F \otimes_R G \\ \downarrow & & \downarrow \\ A^\bullet \otimes_R B^\bullet & \xrightarrow{\mu} & C^\bullet. \end{array}$$

commutes. Then we have a product

$$H^p(X, F) \otimes_R H^q(X, G) \longrightarrow H^{p+q}(X, F \otimes_R G) \tag{7.6.7}$$

induced as follows: Taking global sections gives

$$\Gamma(X, A^\bullet) \otimes_R \Gamma(X, B^\bullet) \longrightarrow \Gamma(X, A^\bullet \otimes_R B^\bullet) \longrightarrow \Gamma(X, C^\bullet)$$

and thus in cohomology we get (7.6.7) (using acyclicity of $A^\bullet, B^\bullet, C^\bullet$).

Proposition 7.6.1. *The two products (7.6.5) and (7.6.7) are the same, i.e. under the isomorphisms $H^p(X, F) = H^p(\Gamma(X, A^\bullet))$ and $H^p(X, G) = H^p(\Gamma(X, B^\bullet))$ (coming from Prop. 6.11.1) the two products are the same.*

This will follow from a more general statement about hypercohomology proved below.

Let us apply this proposition to singular cohomology. First consider the case of a locally contractible space and the sheaf $\mathbb{Z}_X$ on it. The Γ-acyclic resolution $\mathbb{Z}_X \hookrightarrow \mathscr{C}^\bullet_X$ given by the (sheafified) singular cochain complexes has a natural product

$$\mathscr{C}^\bullet_X \otimes_{\mathbb{Z}} \mathscr{C}^\bullet_X \longrightarrow \mathscr{C}^\bullet_X$$

which comes from the presheaf homomorphisms

$$C^p(U, \mathbb{Z}) \otimes C^q(U, \mathbb{Z}) \longrightarrow C^{p+q}(U, \mathbb{Z}) \quad \alpha \otimes \beta \mapsto \alpha \smile \beta$$

i.e. cup product of cochains. This lifts $\mathbb{Z}_X \otimes_{\mathbb{Z}} \mathbb{Z}_X \to \mathbb{Z}_X$ (multiplication). By Prop. 7.6.1 the induced map in cohomology (7.6.7) is the same as the sheaf-theoretically defined cup product (7.6.5) (which comes from (7.6.3)).

Next let us apply the proposition to de Rham cohomology. We have a morphism of Γ-acyclic resolutions

$$\mathscr{E}^\bullet_M \otimes_{\mathbb{R}} \mathscr{E}^\bullet_M \longrightarrow \mathscr{E}^\bullet_M$$

given by wedge product of differential forms. This lifts the map $\mathbb{R}_M \otimes_{\mathbb{R}} \mathbb{R}_M \to \mathbb{R}_M$. Applying Prop. 7.6.1 we see that the induced map in cohomology (7.6.7) is the same as the sheaf-theoretically defined cup product (7.6.5) (which comes from (7.6.3)).

Thus we have identified both products under the respective isomorphisms with sheaf cohomology. This proves that the de Rham isomorphism is an isomorphism of rings.

7.7. **Cup products in hypercohomology.** To prove Prop. 7.6.1 it is easier to prove a more general statement about hypercohomology. For this we first generalize the sheaf-theoretic definition of cup products to this context.

For complexes of sheaves $F^\bullet$ and $G^\bullet$ we will define a product map ([43])

$$\mathbb{H}^p(X, F^\bullet) \otimes \mathbb{H}^q(X, G^\bullet) \to \mathbb{H}^{p+q}(X, F^\bullet \otimes G^\bullet). \tag{7.7.1}$$

Let $\pi_i : X \times X \to X$ be the projections. The map (7.7.1) is the composition

$$\begin{array}{c} \mathbb{H}^p(X, F^\bullet) \otimes \mathbb{H}^q(X, G^\bullet) \\ \Big\downarrow {\scriptstyle (1)} \\ \mathbb{H}^{p+q}(X \times X, \pi_1^{-1}F^\bullet \otimes \pi_2^{-1}G^\bullet) \\ \Big\downarrow {\scriptstyle (2)} \\ \mathbb{H}^{p+q}(X \times X, \delta_*(F^\bullet \otimes G^\bullet)) \\ \Big\| \\ \mathbb{H}^{p+q}(X, F^\bullet \otimes G^\bullet) \end{array}$$

where the maps are as follows:

(1) Let π_1, π_2 be the projections of $X \times Y$ to each factor. Then there is a homomorphism

$$\mathbb{H}^p(X, F^\bullet) \otimes \mathbb{H}^q(Y, G^\bullet) \to \mathbb{H}^{p+q}(X \times Y, \pi_1^{-1}F^\bullet \otimes \pi_2^{-1}G^\bullet). \tag{7.7.2}$$

To get this let $F \hookrightarrow I^\bullet$ and $G \hookrightarrow J^\bullet$ be the Godement resolutions of F and G.

Lemma 7.7.1. *Then $\pi_1^{-1}F \otimes \pi_2^{-1}G \hookrightarrow \pi_1^{-1}I^\bullet \otimes \pi_2^{-1}J^\bullet$ is a resolution on $X \times Y$.*

Thus there is a map

$$H^i(\Gamma(X \times Y, \pi_1^{-1}I^\bullet \otimes \pi_2^{-1}J^\bullet)) \longrightarrow H^i(X \times Y, \pi_1^{-1}F \otimes \pi_2^{-1}G).$$

Now using the map of complexes

$$\Gamma(X, I^\bullet) \otimes \Gamma(Y, J^\bullet) \longrightarrow \Gamma(X \times Y, \pi_1^{-1}I^\bullet) \otimes \Gamma(Y, \pi_2^{-1}J^\bullet) \longrightarrow \Gamma(X \times Y, \pi_1^{-1}I^\bullet \otimes \pi_2^{-1}J^\bullet)$$

and composing with the previous map we get (7.7.2).

(2) comes from the canonical homomorphism

$$\pi_1^{-1}F^\bullet \otimes \pi_2^{-1}G^\bullet \longrightarrow \delta_*(F^\bullet \otimes G^\bullet)$$

where $\delta : X \hookrightarrow X \times X$ is the diagonal.

(3) Pushforward by δ is an exact functor and hence $\mathbb{H}^i(X \times X, \delta_*K^\bullet) = \mathbb{H}^i(X, K^\bullet)$. ([44])

This defines the cup product on hypercohomology.

Generalizing the second situation above, suppose that we have complexes $F^\bullet$ and $G^\bullet$ and quasi-isomorphisms to complexes of Γ-acyclic objects $F^\bullet \to A^\bullet, G^\bullet \to B^\bullet$ and $F^\bullet \otimes G^\bullet \to C^\bullet$ and assume that there is a map

$$\mu : A^\bullet \otimes B^\bullet \to C^\bullet$$

such that the following diagram commutes:

$$\begin{array}{ccc} F^\bullet \otimes G^\bullet & = & F^\bullet \otimes G^\bullet \\ \downarrow & & \downarrow \\ A^\bullet \otimes_R B^\bullet & \xrightarrow{\mu} & C^\bullet. \end{array}$$

Applying hypercohomology to this diagram gives a map

$$\begin{array}{ccccc} \mathbb{H}^k(X, F^\bullet) \otimes \mathbb{H}^k(X, G^\bullet) & \longrightarrow & \mathbb{H}^k(X, F^\bullet \otimes G^\bullet) & = & \mathbb{H}^k(X, F^\bullet \otimes G^\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^k(X, A^\bullet) \otimes \mathbb{H}^k(X, B^\bullet) & \longrightarrow & \mathbb{H}^k(X, A^\bullet \otimes B^\bullet) & \longrightarrow & \mathbb{H}^k(X, C^\bullet) \end{array}$$

THIS NEEDS TO BE FINISHED

[43] We are dropping the "*Tot*" notation in tensor products for this subsection, so $F^\bullet \otimes G^\bullet$ is actually the total complex of the double complex $(F^p \otimes G^q)$.

[44] This follows from Corollary 7.3.2 above and Lemma 7.10.2 below.

7.8. **Cup products and the de Rham isomorphism.** The following was proved above:

Proposition 7.8.1. *The de Rham isomorphism $H^*_{dR}(M) \cong H^*_{sing}(M,\mathbb{R})$ is an isomorphism of rings.*

Corollary 7.8.2. *If M is a compact oriented n-manifold the pairing $\alpha \otimes \beta \mapsto \int_M \alpha \wedge \beta$ defines a nondegenerate pairing $H^k_{dR}(M) \otimes H^{n-k}_{dR}(M) \to \mathbb{R}$.*

7.9. **Composition of functors.** Let

$$\mathfrak{A} \xrightarrow{T} \mathfrak{B} \xrightarrow{U} \mathfrak{C}$$

be left-exact functors between abelian categories. Assume any one of three conditions:

(1) T carries injective objects to injective objects
(2) T carries injective objects to T-acyclic objects
(3) T carries $U \circ T$-acyclic objects to T-acyclic objects.

Then for a resolution $F \to A^\bullet$ by either injectives (if (1) or (2) hold) or $U \circ T$-acyclics (if (3) holds) we know that

$$R^k(U \circ T)(F) = H^k(U \circ T(A^\bullet)).$$

On the other hand, $T(A^\bullet)$ is a complex of U-acyclic objects and hence the derived functors R^kU can be computed on $T(A^\bullet)$ using the "resolution" $T(A^\bullet) \stackrel{id}{\to} T(A^\bullet)$, i.e.:

$$R^kU(T(A^\bullet)) = H^k(U(T(A^\bullet)).$$

Thus we have that under (1) or (2) or (3) there is a natural isomorphism: ([45])

$$R^k(U \circ T)(F) = R^kU(T(A^\bullet)).$$

We will use this observation (rather, a similar argument) in the following situation: $f : Y \to X$ is a continuous map and we consider the composition $\Gamma(Y,-) = \Gamma(X,-) \circ f_*$:

$$Sh(Y) \xrightarrow{f_*} Sh(X) \xrightarrow{\Gamma(X,-)} AbGps.$$

For $F \in Sh(Y)$ and $A^\bullet$ a flasque resolution of F, since f_* takes flasque sheaves to flasque sheaves and flasques are acyclic, we have:

$$H^k(Y,F) = \mathbb{H}^k(X, f_*A^\bullet).$$

Thus the cohomology of any sheaf on Y (or any complex of sheaves on Y) can be computed by a complex of sheaves on X, namely the direct image of a flasque or injective resolution. For the latter we may, for instance, take the Godement resolution.

7.10. **Proofs.** We now go back and prove various unproven statements. We will need some extra notation.

If $A^\bullet$ is a complex then the shifted complex $A^\bullet[1]$ is defined by

$$A[1]^i := A^{i-1} \qquad \text{and } d^i_{A[1]} = (-1)^i d^{i-1}_A.$$

It has the property that $H^i(A[1]^\bullet) = H^{i-1}(A^\bullet)$. The functor $A \mapsto A[1]$ from complexes to complexes is called the **shift functor**.

If $\phi : A^\bullet \to B^\bullet$ is a morphism of complexes then the **cone of** ϕ is the complex

$$C^i(\phi) := A^i \oplus B^{i-1} \qquad \text{with } d_C := \begin{pmatrix} d^i_A & (-1)^i\phi^i \\ & d^{i-1}_B \end{pmatrix}.$$

There is a split short-exact sequence of complexes

$$0 \to B^\bullet[1] \to C(\phi)^\bullet \to A^\bullet \to 0.$$

We will also need the following facts:

Lemma 7.10.1. *(1) If $I^\bullet$ is an exact complex of injective objects and T is a left-exact functor then $T(I^\bullet)$ is exact.*

(2) If $\phi : K^\bullet \to L^\bullet$ is a quasi-isomorphism then $C(\phi)^\bullet$ is acyclic, i.e. $H^*(C(\phi)^\bullet) = 0$, and conversely.

(3) If $A^\bullet$ is an exact complex of T-acyclic objects for a left-exact functor T then $T(A^\bullet)$ is an exact complex.

[45] A special case of this was seen earlier: If T is actually exact and satisfies (1) then any object is acyclic for T and hence taking $A^\bullet = F$ (in degree zero) we get $R^k(U \circ T)(F) = R^kU(T(F))$. This was proved in Lemma 6.10.3 (see also the footnote there).

Proof. (1) If $I^\bullet$ is an exact complex of injective objects then there is a homotopy between the identity map of $I^\bullet$ and the zero map, constructed as follows: Look at the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & I^2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

Since $I^0 \to I^1$ is a monomorphism and I^0 is injective there is a map $H^1 : I^1 \to I^0$ extending the identity of I^0, i.e. with $H^1 d^0 = id_{I^0}$. Now $ker(H^1)$ is a complement to $d^0(I^0)$, i.e. $I^1 \to coker(d^0)$ gives an isomorphism $ker(H^1) \cong coker(d^0)$. By the diagram

$$\begin{array}{cccc} 0 & \longrightarrow & coker(d^0) = ker(H^1) & \xrightarrow{d^1} I^2 \\ & & \downarrow & \\ & & I^1 & \end{array}$$

and injectivity of $coker(d^0)$ we get $H^2 : I^2 \to I^1$ satisfying

$$H^2 \circ d^1 = Id_{I^1} - d^0 \circ H^1.$$

Continuing in this way constructs a homotopy $H^k : I^k \to I^{k-1}$ such that $d^{k-1}H^k + H^{k+1}d^k = id^k$.

Now applying T to the homotopy gives a homotopy of the identity of $T(I^\bullet)$ with the zero map. Thus $T(I^\bullet)$ is exact.

(2) The split short exact sequence $0 \to B[1]^\bullet \to C(\phi)^\bullet \to A^\bullet \to 0$ gives the long exact sequence

$$\cdots \longrightarrow H^i(B[1]^\bullet) \longrightarrow H^i(C(\phi)^\bullet) \longrightarrow H^i(A^\bullet) \longrightarrow H^{i+1}(B[1]^\bullet) \longrightarrow \cdots$$

The connecting homomorphism $H^i(A^\bullet) \to H^{i+1}(B[1]^\bullet) = H^i(B^\bullet)$ is easily checked to be the map induced by ϕ. Since this is an isomorphism for all i (because ϕ is a quasi-isomorphism) we have that $H^i(C(\phi)^\bullet) = 0$ for all i, i.e. the cone is acyclic. The converse also clearly holds.

(3) Suppose that

$$0 \longrightarrow A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \longrightarrow \cdots$$

is an exact complex of T-acyclics. The sequences

$$0 \longrightarrow im(d^{i-1}) = ker(d^i) \longrightarrow A^i \longrightarrow ker(d^{i+1}) \longrightarrow 0$$

are then short-exact for all i. Induction and the long exact sequence for $R^iT(-)$ shows that $ker(d^i)$ is T-acyclic for all i. Thus applying T gives short-exact sequences

$$0 \longrightarrow T(ker(d^i)) \longrightarrow T(A^i) \longrightarrow T(ker(d^{i+1})) \longrightarrow 0$$

for all i. It follows that $T(ker(d^i)) = ker(T(d^{i+1}))$ for all i and putting these short exact sequences gives exactness of the complex $T(A^\bullet)$. □

The following statements about exact functors were used above:

Lemma 7.10.2. *(1) If $K^\bullet$ is an exact complex and T is an exact functor then $T(K^\bullet)$ is an exact complex.*

(2) If $K^\bullet \to L^\bullet$ is a quasi-isomorphism and T is an exact functor then $T(K^\bullet) \to T(L^\bullet)$ is a quasi-isomorphism.

Proof. (1) Exercise. (Break the complex up into short exact sequences and apply T, similar to the proof of part (3) of the previous lemma.)

(2) By (2) of the previous lemma a morphism $\phi : K^\bullet \to L^\bullet$ is a quasi-isomorphism if and only if the cone $C(\phi)^\bullet$ is exact, i.e. $H^*(C(\phi)^\bullet) = 0$. By (1), $T(C(\phi)^\bullet) = C(T(\phi))^\bullet$ is exact, and hence (by (2) of the previous lemma) $T(\phi)$ is a quasi-isomorphism. □

Proof. (of Proposition 7.3.1) Suppose that $j : F^\bullet \to J^\bullet$ is a quasi-isomorphism to a complex of injective objects and $i : F^\bullet \to I^\bullet$ is a monomorphism which is a quasi-isomorphism to a complex of injective objects. By (2) of Lemma 7.2.3 there is a map $\phi : I^\bullet \to J^\bullet$ such that $\phi \circ i = j$. Then ϕ is also a quasi-isomorphism. By (2) of the previous lemma the cone $C(\phi)^\bullet$ is acyclic. It also consists of injective objects. By (1) of that lemma $T(C(\phi)^\bullet)$ is again an exact complex. Since the short exact sequence

$$0 \longrightarrow J[1]^\bullet \longrightarrow C(\phi)^\bullet \longrightarrow I^\bullet \longrightarrow 0$$

is split exact, so is the sequence

$$0 \longrightarrow T(J[1]^\bullet) \longrightarrow T(C(\phi)^\bullet) \longrightarrow T(I^\bullet) \longrightarrow 0.$$

Using the long exact cohomology sequence of this sequence and the fact that $T(C(\phi)^\bullet)$ is exact we get isomorphisms

$$H^i(T(I^\bullet)) \cong H^{i+1}(T(J[1]^\bullet)) = H^i(T(J^\bullet)).$$

Since $R^iT(I^\bullet) = H^i(T(I^\bullet))$ (by definition), this proves the proposition. □

Proof. (of Proposition 7.3.3). Let $F^\bullet \to A^\bullet$ be a quasi-isomorphism to a complex of acyclic objects for the left-exact functor T. By Corollary 7.3.2 we know that

$$R^iT(F^\bullet) \cong R^iT(A^\bullet)$$

for all i. Thus we must show that for a complex of acyclic objects we have $R^iT(A^\bullet) = H^i(T(A^\bullet))$. Let $A^\bullet \hookrightarrow I^\bullet$ be a monomorphic quasi-isomorphism to a complex of injective objects. Let $Q^\bullet = I^\bullet/A^\bullet$. The sequence

$$0 \longrightarrow A^\bullet \longrightarrow I^\bullet \longrightarrow Q^\bullet \longrightarrow 0$$

shows (via the long-exact sequence in cohomology) that $Q^\bullet$ is an exact complex. Applying T gives (because each A^i is acyclic) a short-exact sequence of complexes

$$0 \longrightarrow T(A^\bullet) \longrightarrow T(I^\bullet) \longrightarrow T(Q^\bullet) \longrightarrow 0.$$

Each Q^i is T-acyclic (by the long-exact sequence for $R^iT(-)$ because A^i and I^i are T-acyclic). Taking cohomology and applying (4) of the previous lemma to $T(Q^\bullet)$ gives

$$H^i(T(A^\bullet)) = H^i(T(I^\bullet)) = R^iT(F^\bullet)$$

proving the proposition. □

8. Fibre bundles: Leray-Hirsch theorem, Thom isomorphism

In this section we look at the cohomology of locally trivial fibre bundles. These are usually studied using spectral sequences, but (following Voisin's book) we will avoid these for some types of fibre bundles using a sheaf-theoretic argument with hypercohomology. The main result is the Leray-Hirsch theorem, which we then apply in some examples and use to establish the Thom isomorphism.

8.1. **Pushforward.** We will need the following lemma, which describes the pushforward of a complex:

Lemma 8.1.1. *Let $f : Y \to X$ be a continuous map and $K^\bullet$ a complex of sheaves on Y. Then $R^i f_* K^\bullet$ is the sheaf on X associated with the presheaf $U \mapsto \mathbb{H}^i(f^{-1}(U), K^\bullet)$.*

Proof. Let $K^\bullet \hookrightarrow I^\bullet$ be a quasi-isomorphism to a complex of injective sheaves. By definition,

$$R^i f_* K^\bullet = H^i(f_* I^\bullet).$$

But $H^i(f_* I^\bullet)$ is the sheaf associated with the presheaf

$$U \mapsto H^i(\Gamma(U, f_* I^\bullet)) = H^i(\Gamma(f^{-1}(U), I^\bullet)) = \mathbb{H}^i(f^{-1}(U), K^\bullet).$$

This proves the lemma. □

8.2. **Leray-Hirsch theorem.** We will now use the formalism above to prove a useful theorem about the homology of certain manifolds.

A map $f : Y \to X$ is **locally trivial** if for each $x \in X$ there is a neighbourhood U_x of x in X and a homeomorphism $f^{-1}(U_x) \cong U_x \times f^{-1}(x)$ such that $f|_{f^{-1}(U_x)}$ is given by the first projection. An example of a locally trivial map is a fibre bundle: A map $\pi : E \to B$ is a **fibre bundle** if there exists a space F (the fibre) and a covering $\{U_\alpha\}$ of B and homeomorphisms $\pi^{-1}(U_\alpha) \cong U_\alpha \times F$ such that $\pi|_{\pi^{-1}(U_\alpha)}$ is given by the first projection. An example of a fibre bundle is a vector bundle.

Theorem 8.2.1 (Leray-Hirsch)**.** *Suppose that X and Y are locally contractible ([46]) and $f : Y \to X$ is a locally trivial map. Assume that*

1. *for each $x \in X$ the cohomology groups $H^*(f^{-1}(x), \mathbb{Z})$ are torsion-free*
2. *there exist classes $\alpha_1, \dots, \alpha_N \in H^*(Y, \mathbb{Z})$ such that for all $x \in X$ the restrictions $\alpha_i|_{f^{-1}(x)}$ form a basis of $H^*(f^{-1}(x), \mathbb{Z})$.*

Let A^* be the graded subgroup of $H^*(Y, \mathbb{Z})$ generated by $\alpha_1, \dots, \alpha_N$. (Thus $A^* \cong H^*(f^{-1}(x), \mathbb{Z})$ for any x). Then the map

$$A^* \otimes H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z}) \qquad \text{by } \sum_i c_i \alpha_i \otimes \omega \mapsto \sum_i c_i \alpha_i \smile f^*(\omega)$$

is an isomorphism of graded groups. In particular, $H^*(Y, \mathbb{Z})$ is freely generated as a graded $H^*(X, \mathbb{Z})$-module by A^*.

Proof. Let $I^\bullet$ be a flasque resolution of the constant sheaf $\mathbb{Z}_Y$. We know that $H^*(Y, \mathbb{Z}) = H^*(Y, \mathbb{Z}_Y) = H^*(\Gamma(Y, I^\bullet))$. Let $\tilde{\alpha}_1, \dots, \tilde{\alpha}_N \in \Gamma(Y, I^\bullet)$ be cocycles representing the classes $\alpha_1, \dots, \alpha_N$. We may assume that the α_i are all homogeneous of a single degree, and that the same holds for the $\tilde{\alpha}_i$. Then they generate a subgroup of $\Gamma(Y, I^\bullet)$ isomorphic to A^* and they give an inclusion

$$A^*_X \hookrightarrow f_* I^\bullet$$

which is an inclusion of complexes of sheaves. (Here A^k_X is the constant sheaf on X and the complex A^*_X has zero differentials.) The local triviality of f and assumption (2) in the theorem imply that this is a quasi-isomorphism (use Lemma 8.1.1). Thus we have isomorphisms in hypercohomology

$$\mathbb{H}^i(X, A^*_X) \cong \mathbb{H}^i(X, f_* I^\bullet)$$

for all i. The right-hand side is $H^*(Y, \mathbb{Z})$ by the lemma above, while the right-hand side is

$$\mathbb{H}^i(X, A^*_X) = \oplus_k \mathbb{H}^{i-k}(X, \mathbb{Z}_X) \otimes A^k = \oplus_k H^{i-k}(X, \mathbb{Z}) \otimes A^k$$

by the assumption (1) on A^*. Thus we have $H^*(X, \mathbb{Z}) \otimes A^* \cong H^*(Y, \mathbb{Z})$. □

Corollary 8.2.2 (Künneth formula)**.** *Let X and Y be locally contractible spaces and assume that $H^*(X, \mathbb{Z})$ is torsion-free. Then $H^*(X \times Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z})$ (as graded $\mathbb{Z}$-modules).*

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[46] The proof will use the identification of singular cohomology with sheaf cohomology for X and Y. Since this was only proved in these notes for locally contractible spaces under the additional hypothesis that every open subset is paracompact, we should probably add this assumption here. In any case, in the applications all spaces will be manifolds.

Remarks 8.2.3. (1) The proof given here works with any ring R as coefficients.

(2) Note in particular that the map $f^* : H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ is an injective ring homomorphism under the conditions of the theorem. Moreover, the image is a direct summand if the fibres are connected: a complement is given by choosing a basis of $A^{>0}$.

(3) The Leray-Hirsch theorem does not claim that the ring structure on $H^*(Y, \mathbb{Z})$ is the tensor product, in fact that is not true in general. (We will see an example below.) In the product case, it is true that the ring structure is given by the tensor product, but the proof here does not prove this.

(4) If X and Y are manifolds and the fibres are manifolds then everything holds in de Rham cohomology. (In fact, a proper surjective smooth submersion of smooth manifolds with connected fibres always satisfies the local triviality condition in the Leray-Hirsch theorem, by Ehresmann's theorem.)

(5) An easy example to see that condition (2) in the theorem is not always satisfied is the Hopf fibration $S^3 \to S^2$, which is locally trivial with fibre S^1.

8.3. **Example: Homogeneous manifolds for compact Lie groups.** Let G be a compact connected Lie group. The de Rham cohomology of a compact Lie group is computed by the bi-invariant differential forms, i.e.

$$H^*_{dR}(G) = (\Lambda^* Lie(G)^*)^{AdG},$$

where $Ad : G \to GL(Lie(G))$ is the adjoint representation. (This is an exercise on the problem set.) If H is a closed connected (Lie) subgroup then $Lie(G)^* \twoheadrightarrow Lie(H)^*$ but it need not be the case that

$$\Lambda^k(Lie(G)^*)^{Ad(G)} \twoheadrightarrow \Lambda^k(Lie(H)^*)^{Ad(H)} \quad \text{for all } k. \tag{8.3.1}$$

Now consider the fibration $\pi : G \to G/H$. It is locally trivial since the smooth manifold structure on G/H is such that π has local smooth sections. The fibre over eH is identified with H. If (8.3.1) holds then one can choose classes $\alpha_1, \ldots, \alpha_N \in H^*_{dR}(G)$ which map isomorphically to a basis of $H^*_{dR}(H)$. Since the fibre $\pi^{-1}(gH) = gH = L_g(H)$ and the classes $\alpha_1, \ldots, \alpha_N$ may be represented by left-invariant differential forms, we see that $\alpha_1, \ldots, \alpha_N$ restrict to a basis of $H^*_{dR}(\pi^{-1}(gH))$ for any fibre. Thus the Leray-Hirsch theorem applies and tells us that

$$H^*(H, \mathbb{R}) \otimes H^*(G/H, \mathbb{R}) \cong H^*(G, \mathbb{R}). \tag{8.3.2}$$

So if (8.3.1) holds then we can conclude (8.3.2).

(An example where (8.3.1) fails is $G = SU(2) \supset H = U(1)$, with $k = 1$. The corresponding map $S^3 = SU(2) \to SU(2)/U(1) = S^2$ is the Hopf fibration.)

8.4. **Example: Unitary groups.** Assume that (8.3.1) holds for the pair $U(n-1) \subset U(n)$ for all $n \geq 1$. Then applying the isomorphism (8.3.2) to the fibration

$$U(n) \to U(n)/U(n-1) = S^{2n-1}$$

and induction on n shows that

$$H^*(U(n), \mathbb{R}) = \Lambda_{\mathbb{R}}[c_1, c_3, \ldots, c_{2n-1}]$$

is the free exterior algebra on generators in each odd degree $\leq 2n-1$. Note that the ring structure is also determined since all products are zero by graded-commutativity of the cup product. For $SU(n)$ the same argument gives

$$H^*(SU(n), \mathbb{R}) = \Lambda_{\mathbb{R}}[c_3, \ldots, c_{2n-1}]$$

(since $SU(1) = \{e\}$ the induction begins with $SU(2) = S^3$ so the first generator is in degree 3). The same assertions hold with $\mathbb{Q}$ coefficients.

We will verify (8.3.1) for the pair $U(n-1) \subset U(n)$ for $n \geq 3$ later.

Remarks 8.4.1. (1) These results illustrate Hopf's theorem, viz. the cohomology of a compact connected Lie group is an exterior algebra on odd-degree generators.

(2) The same method can be used for the sequence of fibrations involving the compact symplectic group: $Sp(1) \subset Sp(2) \subset \cdots$.

(3) The method does not work for $SO(n)$. For example, the fibration $SO(3) \to S^2 = SO(3)/SO(2)$ is not homologically like a product, i.e. the Leray-Hirsch theorem does not apply. (This has to be the case because according to the Hopf theorem $H^*(SO(n), \mathbb{R})$ is also an exterior algebra on odd degree generators. Computing the actual degrees of the exponents is not as easy as in the $SU(n), Sp(n)$ cases; the answer is different for even n and odd n.)

8.5. **Example: Flag manifolds.** A **k-flag** in $\mathbb{C}^n$ is a chain $V_1 \subset V_2 \subset \cdots \subset V_k$ of subspaces of $\mathbb{C}^n$ with $dim(V_i) = i$. The set $Fl(k,n)$ of k-flags in $\mathbb{C}^n$ is a homogenous manifold: The Lie group $GL(n,\mathbb{C})$ acts transitively on it and the isotropy is a certain closed (hence Lie) subgroup. This implies that it has a manifold structure.

It will be more convenient for us to think of a k-flag as an ordered k-tuple $(\ell_1, \ldots, \ell_k)$ of orthogonal lines in $\mathbb{C}^n$. (By orthogonal we mean with respect to the standard Hermitian inner product. The relation with the previous description is the obvious one: $V_i = \ell_1 + \cdots + \ell_i$.)

For each $i = 1, \ldots, k$ there is a map

$$p_i : Fl(k,n) \to \mathbb{C}P^{n-1} \qquad \text{by } (\ell_1, \ldots, \ell_k) \mapsto \ell_i.$$

Pulling back the generator of $H^2(\mathbb{C}P^{n-1})$ for each $i = 1, \ldots, k$ gives elements $x_1, \ldots, x_k \in H^2(Fl(k,n))$.

Lemma 8.5.1. *The cohomology of $Fl(k,n)$ is given by:*

$$H^*(Fl(k,n),\mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_k]/(x_1^n, x_2^{n-1}, \ldots, x_k^{n-k+1}) = H^*(\mathbb{C}P^{n-1} \times \cdots \times \mathbb{C}P^{n-k}, \mathbb{Z}).$$

Proof. Forgetting the last line defines a map to $Fl(k-1,n)$; the fibre over a $k-1$ tuple $(\ell_1, \ldots, \ell_{k-1})$ consists of lines in $(\ell_1 + \cdots + \ell_{k-1})^\perp$, i.e. it is isomorphic to $\mathbb{C}P^{n-k}$. It is easy to see that

$$\begin{array}{ccc} \mathbb{C}P^{n-k} & \longrightarrow & Fl(k,n) \\ & & \downarrow \\ & & Fl(k-1,n) \end{array}$$

is a fibre bundle with fibre $\mathbb{C}P^{n-k}$. Now the element x_k restricts to any fibre as the generator of H^2. Indeed, the fibre over $(\ell_1, \ldots, \ell_{k-1})$ consists of lines in the space $(\ell_1 + \cdots + \ell_{k-1})^\perp$, which is a copy of $\mathbb{C}P^{n-k}$ sitting in the $\mathbb{C}P^{n-1}$ consisting of choices of ℓ_k, from which the generator x_k was pulled back. Thus the restriction to the fibre is surjective in cohomology and the Leray-Hirsch theorem applies to $Fl(k,n) \to Fl(k-1,n)$.

Let us prove the lemma by induction on k. The induction hypothesis gives that $H^*(Fl(k-1,n),\mathbb{Z})$ is generated by $x_1, \ldots, x_{k-1}$ with relations as in the theorem. The Leray-Hirsch theorem then tells us that $H^*(Fl(k-1,n),\mathbb{Z})$ is freely generated over $H^*(Fl(k-1,n),\mathbb{Z})$ by x_k, with the relation $x_k^{n-k+1} = 0$. This proves the lemma. ☐

For the space of k-flags in an infinite-dimensional space $\mathbb{C}^\infty = \varinjlim_n \mathbb{C}^n$, viz. the space $Fl(k,\infty) = \varinjlim_n Fl(k,n)$ the same proof shows that the cohomology is

$$H^*(Fl(k,\infty),\mathbb{Z}) = \mathbb{Z}[x_1, x_2, \ldots]$$

i.e. a polynomial ring in the generators defined by pulling back the generator of $H^2(\mathbb{C}P^\infty)$ under the k maps $Fl(k,\infty) \to \mathbb{C}P^\infty$. Notice that this is simpler than $H^*(Fl(k,n),\mathbb{Z})$. ([47])

8.6. **Example: Grassmannians.** Consider the Grassmannian $Gr(k,n)$ of k-dimensional subspaces of $\mathbb{C}^n$. There is a fibre bundle

$$\begin{array}{ccc} Fl(k,k) & \longrightarrow & Fl(k,n) \\ & & \downarrow \pi \\ & & Gr(k,n) \end{array}$$

where the vertical map π sends $(\ell_1, \ldots, \ell_k)$ to $\ell_1 + \cdots + \ell_k$. The fibre over a subspace $E \in Gr(k,n)$ consists of full flags in E, which is simply $Fl(k,k)$. We have seen above that the restriction map

$$H^*(Fl(k,n),\mathbb{Z}) \to H^*(Fl(k,k),\mathbb{Z})$$

is surjective and this applies to any fibre. Thus the Leray-Hirsch theorem applies and we deduce that the pullback

$$\pi^* : H^*(Gr(k,n),\mathbb{Z}) \to H^*(Fl(k,n),\mathbb{Z})$$

is injective and in fact the image is a direct summand.

Now we consider the case of $Gr(k,\infty)$, in which case we have $H^*(Gr(k,\infty),\mathbb{Z}) \hookrightarrow \mathbb{Z}[x_1, \ldots, x_k]$.

[47] We are being a little careless here: If $Fl(k,\infty)$ is defined as the direct limit of $Fl(k,n)$ as $n \to \infty$, then it is not clear that the cohomology satisfies $H^i(Fl(k,\infty),\mathbb{Z})$ is the group $H^i(Fl(k,n),\mathbb{Z})$, which is independent of n for n large. This same comment applies to $Gr(k,\infty)$ below. Nevertheless, this is okay, see e.g. Milnor-Stasheff.

Lemma 8.6.1. *The image of* $\pi^* : H^*(Gr(k,\infty),\mathbb{Z}) \to H^*(Fl(k,\infty),\mathbb{Z}) = \mathbb{Z}[x_1,\dots,x_k]$ *is the subalgebra of symmetric polynomials.*

Proof. First we show that the image consists of symmetric polynomials. For a permutation σ of $\{1,\dots,k\}$ there is a map $\sigma : Fl(k,n) \to Fl(k,n)$ by permuting the lines. This map induces the automorphism $x_i \mapsto x_{\sigma(i)}$ on $H^*(Fl(k,n),\mathbb{Z})$. Now $\pi\circ\sigma = \pi$ because $\ell_1+\dots+\ell_k = \ell_{\sigma(1)}+\dots+\ell_{\sigma(k)}$, so that $\sigma^*\pi^* = \pi^*$ and thus the image consists of symmetric polynomials.

To show that the image is the subalgebra of symmetric polynomials we need a definition. For a free graded $\mathbb{Z}$-module $A^* = \oplus_{k\geq 0}A^k$ with $A^i = 0$ for odd i the Poincaré series is the formal power series

$$P_A(t) := \sum_{k\geq 0} rank(A^{2k})t^k \quad \in \mathbb{Z}[[t]].$$

For a graded tensor product $A^*\otimes B^*$ we have:

$$P_{A\otimes B} = P_A\, P_B.$$

Thus $P_{H^*(Fl(k,\infty))} = (1-t)^{-k}$. By the computation of the cohomology of $Fl(k,k)$ we have

$$P_{H^*(Fl(k,k))} = (1+t)(1+t+t^2)\cdots(1+t+\dots+t^{k-1}) = \prod_{k=1}^{k}\frac{1-t^i}{1-t} = (1-t)^{-k}\prod_{i=1}^{k}(1-t^i).$$

Thus by the isomorphism of graded groups

$$H^*(Fl(k,\infty),\mathbb{Z}) = H^*(Fl(k,k),\mathbb{Z})\otimes H^*(Gr(k,\infty),\mathbb{Z})$$

coming from Leray-Hirsch, the Poincaré series $Q = P_{H^*(Gr(k,\infty))}$ satisfies:

$$Q\cdot(1-t)^{-k}\prod_{i=1}^{k}(1-t^i) = (1-t)^{-k}$$

and hence

$$Q = \prod_{i=1}^{k}\frac{1}{1-t^i}.$$

Now we use the classical fact (see e.g. Lang's *Algebra*) that the symmetric polynomials are freely generated, as a $\mathbb{Z}$-algebra, by the elementary symmetric polynomials $c_1,\dots,c_k$ where:

$$\begin{aligned} c_1 &= x_1+\dots+x_k \\ c_2 &= \sum_{1\leq i<j\leq k} x_i x_j \\ &\vdots \\ c_k &= x_1\cdots x_k. \end{aligned} \tag{8.6.1}$$

Thus the Poincaré series of the algebra of invariant polynomials is

$$(1+t+t^2+\cdots)(1+t^2+t^4+\cdots)\cdots(1+t^k+t^{2k}+\cdots) = \prod_{i=1}^{k}\frac{1}{1-t^i}.$$

This is exactly Q and hence we have proved the lemma. □

We have proved:

Theorem 8.6.2. $H^*(Gr(k,\infty),\mathbb{Z})$ *is a polynomial ring over* $\mathbb{Z}$ *in* k *generators* $c_1,\dots,c_k$ *where* $deg(c_i) = 2i$.

Note that the generators are canonical (under π^* they map to the symmetric functions in $x_1,\dots,x_k$, which are themselves canonically defined).

The same arguments as above, using the fact that $H^*(\mathbb{R}P^n,\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[x]/(x^{n+1})$, give the $\mathbb{Z}/2\mathbb{Z}$-cohomology of the infinite real Grassmannian:

Theorem 8.6.3. $H^*(Gr_{\mathbb{R}}(k,\infty),\mathbb{Z}/2\mathbb{Z})$ *is the polynomial ring over* $\mathbb{Z}/2\mathbb{Z}$ *in* k *generators* $w_1,\dots,w_k$ *where* $deg(w_i) = 2i$.

These two theorems are of fundamental importance in the theory of characteristic classes, because $Gr(k,\infty)$ and $Gr_{\mathbb{R}}(k,\infty)$ are the classifying spaces for vector bundles: On any reasonable space X (e.g. a manifold) any rank r complex (resp. real) vector bundle E is the pullback of the tautological rank r bundle on $Gr(k,\infty)$ (resp. $Gr_{\mathbb{R}}(k,\infty)$) by a mapping h_E of X to $Gr(k,\infty)$ (resp. to $Gr_{\mathbb{R}}(k,\infty)$) which is unique up to homotopy. The Chern classes $c_i(E) := h_E^*(c_i) \in H^{2i}(X,\mathbb{Z})$ in the complex case (resp. Stiefel-Whitney classes $w_i(E) := h_E^*(w_i) \in H^{2i}(X,\mathbb{Z}/2\mathbb{Z})$ in the real case) are then important invariants of the vector bundle. For example, a famous theorem of Thom says that a complex manifold X is the boundary of a manifold (i.e. cobordant to the empty set) if and only if all the Chern classes of its tangent bundle are trivial. For more on this see Milnor-Stasheff, *Characteristic Classes.*

8.7. **Thom isomorphism theorem.** A (smooth) vector bundle $E \to M$ of rank r is called **oriented** if we are given a nowhere vanishing section of $\wedge^r E \to M$. Informally, we have an orientation of each fibre vector space $E_x = \pi^{-1}(x)$ varying smoothly in $x \in M$. This also gives an orientation in the topological sense, i.e. a class in $H^r(E_x, E_x - \{x\})$. Notice that together with an orientation of M, this determines an orientation of the manifold E.

Theorem 8.7.1 (Thom isomorphism)**.** *Let $\pi : E \to M$ be an oriented smooth vector bundle of rank r on a (compact, oriented) manifold M. Then*

(1) *There exists a unique class $u \in H^r(E, E-M)$ (called the* **Thom class***) such that for each $x \in M$, the image of u under $H^r(E,E-M) \to H^r(E_x, E_x - \{x\})$ is the orientation class of the fibre E_x.*
(2) *The map*
$$H^{*-r}(M) \longrightarrow H^*(E, E-M) \qquad \alpha \mapsto \pi^*(\alpha) \smile u$$
is an isomorphism of $H^*(M)$-modules.

Proof. Put a metric on the vector bundle, i.e. a norm on the vector spaces $E_x = \pi^{-1}(x)$ which varies smoothly in $x \in M$. (Exercise: This can be constructed using a trivialization and a partition of unity.) Let D_E be the unit disk bundle, i.e. in each fibre of π we take the vectors of norm ≤ 1, and let S_E be the sphere bundle (vectors of norm $= 1$ in each fibre). Then D_E is a smooth manifold of dimension $\dim M + r$ with boundary $\partial D_E = S_E$. By excision and a deformation retraction we have
$$H^*(E, E-M) \cong H^*(D_E, S_E).$$
Then we have isomorphisms
$$H^k(D_E, S_E) \overset{\cap [D_E]}{\longrightarrow} H_{\dim M + r - k}(D_E) \cong H_{\dim M + r - k}(M) \overset{\cap [M]}{\longleftarrow} H^{k-r}(M)$$
by Lefschetz duality for D_E, the fact that $M \hookrightarrow D_E$ is a homotopy equivalence, and Poincaré duality for M.

(1) Now for $k = r$ we see that $H^r(E, E-M) \cong H^0(M) = \mathbb{Z}$ (assuming M is connected, which case is enough to consider). Let $u \mapsto 1$ under this isomorphism. The sequence of isomorphisms used above is compatible with restriction to a single fibre of $D_E \to M$ (use the local triviality to reduce to the case of products), and this shows that u restricts to the orientation class in each fibre.

(2) Form the quotient space Q of D_E in which each fibre of S_E is collapsed to a point. (To show this exists, construct it locally using a trivialization of E and then glue.) This is a fibre bundle over M with fibre the r-sphere $D_{E,x}/S_{E,x} \cong S^r$. The Leray-Hirsch theorem applies – check that the existence of the class u gives the condition needed to apply that theorem. This gives that $H^*(S^r) \otimes H^*(M) \to H^*(Q)$ is an isomorphism, and this restricts to the isomorphism in the theorem on the summand $H^r(S^r) \otimes H^*(M) = H^{*+r}(M)$ since $H^r(S^r) = H^r(D_E, S_E)$. □

Applying this to the case of the normal bundle of a submanifold, we get:

Corollary 8.7.2. *Let $i : M \hookrightarrow N$ be an embedding of oriented smooth manifolds with M compact and let c be the codimension of M in N. Then $H^*(N, N-M) \cong H^{*-c}(M)$.*

Proof. Use the tubular neighbourhood theorem and excision to replace the pair $(N, N-M)$ by the disk bundle of the normal bundle to M in N, and apply the previous theorem. (An orientation of the normal bundle is given by the orientations of M and N.) □

Remark 8.7.3. The Thom isomorphism also holds in the case M is not oriented (but E is an oriented vector bundle). See Milnor-Stasheff. It also holds without any assumption if we use $\mathbb{Z}/2\mathbb{Z}$ coefficients (and the proofs above work).

9. Complex manifolds and the Dolbeault theorem (∗)

9.1. **Tangent bundle.** Let X be a complex n-manifold, and TX its tangent bundle considered as a real $2n$ manifold. We can complexify this vector space and write $(TX)_{\mathbb{C}}$ for the complexified tangent bundle.

9.2. **Almost complex structures and integrability.** The main application of the Frobenius theorem will be to the correspondence between Lie groups and Lie algebras, discussed in the section on Lie groups. Here we give another application of the analytic version of the Frobenius theorem, proving (following Voisin's book) a special case of the famous Newlander-Nirenberg theorem.

An **almost-complex structure** on a real $2n$ manifold X is a section $J \in \Gamma(X, End(TX))$, i.e. an endomorphism of the (real) tangent bundle, which satisfies $J^2 = -I$. Thus it induces in each tangent space a complex structure

Example 9.2.1.

Theorem 9.2.2 (Newlander-Nirenberg)**.** *Let (X, J) be a manifold with an almost-complex structure. Then J comes from a complex structure if and only if the subbundle $T_X^{0,1}$ of the complexified tangent bundle of X defined by J is integrable.*

Proof. We will only prove this assuming that (X, J) are real analytic, following the idea of Weil (as in Voisin's book).

□

9.3. **Differential forms: $\partial, \bar{\partial}$, etc.**

9.4. **The $\bar{\partial}$-Poincaré lemma.**

9.5. **Dolbeault theorem.**