

GEOMETRY/TOPOLOGY II

ARVIND NAIR

These are notes for the second semester of the first-year course taught at TIFR in the academic year 2025-26. Sections marked (*) were not covered.

CONTENTS

1. Homology and cohomology of manifolds; Poincaré duality	4
1.1. (Topological) manifolds	4
1.2. Mayer-Vietoris sequences for pairs	5
1.3. Homology in degrees $\geq n$	6
1.4. Orientations	7
1.5. Fundamental class	8
1.6. Compactly supported cohomology	8
1.7. Poincaré duality	9
1.8. Duality and the cup product pairing in cohomology	11
1.9. Duality between cohomology and Borel-Moore homology (*)	11
1.10. Euler characteristics	12
1.11. Finite generation of homology	12
1.12. Manifolds with boundary	13
1.13. Lefschetz duality	14
1.14. Gysin homomorphisms and Poincaré dual classes	15
1.15. Cup product and intersection product	16
1.16. Alexander duality (*)	17
2. Smooth manifolds	18
2.1. Smooth manifolds	18
2.2. Real analytic manifolds	19
2.3. Complex manifolds	19
2.4. Examples	19
2.5. Partitions of unity	20
2.6. Tangent and cotangent spaces	24
2.7. Functoriality	26
2.8. Tangent and cotangent bundles	26
2.9. C^k -manifolds	27
2.10. Inverse and implicit function theorems	27
2.11. Example: Hypersurfaces in \mathbb{R}^{m+1} .	29
2.12. Immersions and submersions	29
2.13. Submanifolds	30
2.14. Normal forms for immersions and submersions	30
2.15. Critical point, critical value, regular value	31
2.16. Transversality, fibre products	32
2.17. Whitney's embedding theorem	33
2.18. Vector bundles	37
2.19. Tubular neighbourhoods	37
2.20. Ehresmann's fibration theorem	39
2.21. Triangulation of smooth manifolds (*)	41

2.22.	Good covers and finite generation of homology	41
2.23.	Orientations	42
3.	Vector fields and the Frobenius theorem	43
3.1.	Vector fields and local flows	43
3.2.	A useful lemma	46
3.3.	Lie bracket	46
3.4.	Local flows and brackets	47
3.5.	Local Frobenius theorem	50
3.6.	Examples	53
3.7.	Integral manifolds and the global Frobenius theorem	54
3.8.	Ehresmann's theorem revisited	58
4.	Basics of complex manifolds (to be finished)	60
4.1.	Orientations	60
4.2.	Holomorphic tangent and cotangent bundles	60
4.3.	Example: Hypersurfaces in projective space.	62
4.4.	Plane curves: Degree-genus formula	63
4.5.	Plane curves: Bézout's theorem	64
4.6.	Embeddings	65
4.7.	Integrability of almost-complex structures (*)	65
5.	Basics of Lie groups	66
5.1.	Lie groups, Lie algebras	66
5.2.	Lie algebra of a Lie group	66
5.3.	Homomorphisms of Lie groups	67
5.4.	Examples	68
5.5.	Lie subgroups and Lie subalgebras, Lie's first theorem	69
5.6.	Example: $GL(n, \mathbb{R})$	70
5.7.	Linear algebraic groups	70
5.8.	Classical groups	73
5.9.	Coverings of Lie groups	74
5.10.	Example: $SU(2), SO(3), Sp(1)$, quaternions	75
5.11.	Exponential map	76
5.12.	Simply connected groups, Lie's third theorem, Cartan's theorem	79
5.13.	Closed subgroups (*)	81
5.14.	Group actions and homogeneous manifolds	81
5.15.	π_0 and π_1 of some classical groups, spin groups (*)	83
6.	Differential forms	86
6.1.	Vector bundles	86
6.2.	Pullback	89
6.3.	Presheaves and sheaves	89
6.4.	Differentials or differential 1-forms	91
6.5.	Differential forms	91
6.6.	Exterior derivative	92
6.7.	Frobenius theorem revisited	93
7.	De Rham cohomology	94
7.1.	Orientations	94
7.2.	Integration on oriented manifolds	95
7.3.	Stokes's theorem	96
7.4.	De Rham cohomology	98
7.5.	Poincaré lemma	100
7.6.	Homotopy invariance	100

7.7.	Mayer-Vietoris sequence	101
7.8.	De Rham theorem	102
7.9.	Relative de Rham cohomology (*)	102
8.	More differential forms	103
8.1.	Lie derivative	103
8.2.	Explicit homotopy operators	104
8.3.	Isotopies and time-dependent vector fields	104
8.4.	Moser's trick and Moser's theorem	106
9.	Basics of symplectic manifolds	109
9.1.	Symplectic linear algebra	109
9.2.	Hermitian, symplectic, orthogonal	112
9.3.	Compatible complex structures	113
9.4.	Lagrangian Grassmannian	115
9.5.	Symplectic vector bundles	115
9.6.	Symplectic manifolds	115
9.7.	Darboux's theorem	117
9.8.	Weinstein's theorem	118
9.9.	Submanifolds of symplectic manifolds	119
9.10.	Miscellaneous remarks	123
10.	Sheaves and cohomology I	125
10.1.	Presheaves and sheaves	125
10.2.	Stalks, sheafification	126
10.3.	Kernels, cokernels, images	128
10.4.	Global sections functor	128
10.5.	Pullback, pushforward, internal Hom, tensor product	129
10.6.	Injective sheaves and injective resolutions	130
10.7.	Derived functors, cohomology of sheaves	132
10.8.	Acyclic resolutions	134
10.9.	Flasque sheaves, canonical flasque resolution	135
10.10.	Singular cohomology and sheaf cohomology	136
10.11.	Fine sheaves and de Rham's theorem	139
10.12.	De Rham's theorem and integration over smooth chains	140
11.	Sheaves and cohomology II	142
11.1.	Cohomology objects, quasi-isomorphisms	142
11.2.	Complexes of injectives	143
11.3.	Derived functors and hypercohomology	145
11.4.	Pullback maps in (hyper)cohomology	146
11.5.	Composition of functors	146
11.6.	Proofs	147
11.7.	Application: Functoriality of the de Rham isomorphism	149
11.8.	Application: Leray-Hirsch theorem	150
11.9.	Example: Homogeneous manifolds for compact Lie groups	152
11.10.	Example: Flag manifolds	153
11.11.	Example: Grassmannians	153
11.12.	Application: Thom isomorphism theorem (*)	155

1. HOMOLOGY AND COHOMOLOGY OF MANIFOLDS; POINCARÉ DUALITY

We assume the following: the basic properties of cohomology (as in the Eilenberg-Steenrod axioms), Mayer-Vietoris sequences, the universal coefficient theorems for homology and cohomology, cross products and the Künneth formula, properties of the cup and cap products. All coefficient rings are assumed to be PIDs. ⁽¹⁾

1.1. (Topological) manifolds.

Definition 1.1.1. A **topological manifold**, or just **manifold** for short, is a second countable Hausdorff topological space M which is locally Euclidean, i.e. every point has a neighbourhood homeomorphic to an open subset of \mathbb{R}^n .

By invariance of domain (i.e. \mathbb{R}^m homeomorphic to \mathbb{R}^n iff $n = m$) the dimension of a manifold is constant on each connected component. We will usually only consider manifolds all of whose connected components are of the same dimension.

Example 1.1.2. An open subset of a manifold is a manifold. A finite product of manifolds is a manifold. Examples of manifolds include Euclidean spaces, spheres, projective spaces (real, complex, quaternionic), Grassmannians (real and complex), and many other spaces which occur naturally.

Example 1.1.3. The universal cover of a manifold (which exists) is a manifold. (It is obvious that \tilde{M} is Hausdorff and locally Euclidean; the only non-obvious part is that it is second-countable. This follows from the fact that $\pi_1(M)$ is countable. ⁽²⁾)

Example 1.1.4. Here are two examples of the kind of pathologies we want to avoid:

(1) Let M be the real line with two origins, i.e. $\mathbb{R} \times \{0, 1\}$ modulo the identification of $(\lambda, 0)$ and $(\lambda, 1)$ for $\lambda \neq 0$. This is clearly locally Euclidean but it is not a manifold as it is not Hausdorff. It cannot be embedded in \mathbb{R}^N . ⁽³⁾

(2) Let $M = (0, 1)^2$ with the topology that a set $U \subset M$ is open iff $U \cap (\{a\} \times (0, 1))$ is open in $\{a\} \times (0, 1)$ for all a . This is locally Euclidean (every point has a neighbourhood homeomorphic to an interval in \mathbb{R}) but is not second countable (e.g. because it has uncountably many connected components).

The following also holds for noncompact manifolds; we only prove the compact case:

Lemma 1.1.5. A compact manifold can be embedded as a closed subset of \mathbb{R}^N .

¹The material here is treated in Hatcher (Ch. 3) and in Appendix A of Milnor's *Characteristic Classes*, which is the reference we follow for the proof of Poincaré duality, adding a few details. Hatcher's presentation is also modeled on Milnor's. The treatment in Spanier (Ch. 6) is different and perhaps less clear.

²Here is a proof that $\pi_1(M)$ is countable for a second countable topological manifold: Take a countable cover $\{U_n\}_n$ of M by coordinate charts homeomorphic to \mathbb{R}^n . For each m, n the intersection $U_m \cap U_n$ has at most countably many components; choose a point in each component and let C be the set of all such points (over all m, n). For each n and $x, y \in C$ such that $x, y \in U_n$ choose a path $\gamma_{n,x,y}$ from x to y in U_n . For a fixed basepoint $p \in C$ there are countably many loops which are finite products of paths of the form $\gamma_{n,x,y}$. It suffices to show that every element of $\pi_1(M, p)$ is of this form.

For a loop $\gamma : [0, 1] \rightarrow M$ based at p there is an N such that when $[0, 1]$ is subdivided into N equal intervals, each subinterval $[(k-1)/N, k/N]$ has image in some U_n . Reparametrizing $\gamma|_{[(k-1)/N, k/N]}$ we get paths γ_k such that $\gamma = \gamma_1 \cdots \gamma_N$. For each k , the point $\gamma(k/N)$ lies in some component $U_n \cap U_m$, in which we have a chosen point, call it $x_k \in C$. Now choose a path δ_k in $U_m \cap U_n$ from x_k to $\gamma(k/N)$ and let $\tilde{\gamma}_k = \delta_{k-1} \gamma \delta_k$. Then $[\gamma] = [\tilde{\gamma}_1] \cdots [\tilde{\gamma}_N]$. But for each k , $\tilde{\gamma}_k$ is a path in U_m from x_{k-1} to x_k , hence homotopic to the reference path γ_{n,x_{k-1},x_k} . Thus γ is homotopic to a product of paths of the form $\gamma_{n,x,y}$.

³Another description of this example: Take the quotient of the action of \mathbb{R}^* on $\mathbb{R}^2 - \{(0, 0)\}$ by $\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y)$.

Proof. Let $\{(U_i, \varphi_i)\}_{i=1, \dots, r}$ be a covering of M by open subsets with $\varphi_i : U_i \rightarrow \mathbb{R}^n$ a homeomorphism onto an open subset of \mathbb{R}^n . Let $\{f_i\}$ be a partition of unity subordinate to it. ⁽⁴⁾ Then define a mapping $F : M \rightarrow \mathbb{R}^{n+n}$ by

$$F := (f_1\varphi_1, \dots, f_r\varphi_r, f_1, \dots, f_r)$$

where $f_i\varphi_i$ has been extended by zero outside U_i (this is continuous). It is easy to check that F is continuous and bijective, hence a homeomorphism onto its image. \square

Remark 1.1.6. The intuitive idea is simply that we cover M by subsets homeomorphic to (open) balls and each ball gives a map $M \rightarrow S^n$ by simply mapping the complement to a point. This would give an embedding in a product of spheres, hence in \mathbb{R}^N . This naive idea fails (as it must: think of the line with doubled origin) and making it work requires a partition of unity as in the proof of the lemma, and so we must assume our manifolds are paracompact Hausdorff.

This can be used to show that any compact manifold is a Euclidean neighbourhood retract and also that a compact manifold has the homotopy type of a CW complex (see the Appendix of Hatcher).

1.2. Mayer-Vietoris sequences for pairs. Let X be a topological space and $U, V \subset X$ open subsets. There is a long exact M-V sequence

$$\rightarrow H_i(X, U \cap V, R) \rightarrow H_i(X, U, R) \oplus H_i(X, V, R) \rightarrow H_i(X, U \cup V, R) \rightarrow H_{i-1}(X, U \cap V, R) \rightarrow \quad (1.2.1)$$

This is proved by the usual argument using “small” chains. Let $C_*^{\{U, V\}}(X)$ be the complex of singular chains spanned by singular simplices which lie in either U or V . By the “small chains argument” we know that $C_*^{\{U, V\}}(U \cup V) \hookrightarrow C_*(U \cup V)$ induces isomorphisms in homology. By the 5-lemma we get that $C_*(X)/C_*^{\{U, V\}}(U \cup V) \rightarrow C_*(X)/C_*(U \cup V)$ induces isomorphisms in homology. Now the short exact sequence of complexes of abelian groups

$$0 \rightarrow C_*(X, U \cap V) \rightarrow C_*(X, U) \oplus C_*(X, V) \rightarrow C_*(X)/C_*^{\{U, V\}}(X) \rightarrow 0$$

is termwise split and so $\otimes_{\mathbb{Z}} R$ gives a short exact sequence of complexes of R -modules, which gives the long exact sequence (1.2.1).

Similarly, in the same setup there is a cohomology M-V sequence

$$\rightarrow H^i(X, U \cup V, R) \rightarrow H^i(X, U, R) \oplus H^i(X, V, R) \rightarrow H^i(X, U \cap V, R) \rightarrow H^{i+1}(X, U \cup V, R) \rightarrow \quad (1.2.2)$$

This comes as follows: $C^*(X, U) \cap C^*(X, V)$ is the complex of singular cochains on X which vanish on chains contained in U or V . There is a short exact sequence

$$0 \rightarrow C^*(X, U \cup V) \rightarrow C^*(X, U) \cap C^*(X, V) \rightarrow C^*(U \cup V, U) \cap C^*(U \cup V, V) \rightarrow 0.$$

But the right term is acyclic by the usual small chains type argument. So $C^*(X, U) \cap C^*(X, V)$ has cohomology $H^*(X, U \cup V)$. The short exact sequence of cochain complexes

$$0 \rightarrow C^*(X, U) \cap C^*(X, V) \rightarrow C^*(X, U) \oplus C^*(X, V) \rightarrow C^*(X, U \cap V) \rightarrow 0.$$

then gives (1.2.2) for $R = \mathbb{Z}$. For general R applying $\otimes_{\mathbb{Z}} R$ gives a short exact sequence of complexes giving (1.2.2) (because this short exact sequence is termwise split).

⁴Recall that in a compact Hausdorff space any open cover has partitions of unity subordinate to it. In fact, this holds for paracompact Hausdorff spaces, and our manifolds are necessarily paracompact as a second countable locally compact and Hausdorff space is always paracompact.

1.3. Homology in degrees $\geq n$. For a compact set K and $x \in K$ the inclusion of pairs $(M, M - K) \subset (M, M - x)$ induces a homomorphism

$$\rho_x : H_n(M, M - K, R) \rightarrow H_n(M, M - x, R).$$

Note that if B is a ball centred at x (i.e. choose a neighbourhood V of x homeomorphic to \mathbb{R}^n by a homeomorphism taking x to 0 and then take B to be the preimage of a ball centred at 0) then excision and deformation retraction give $H_n(M, M - x, R) = H_n(B, B - x, R) = H_n(B, \partial B, R) \cong R$ is a free cyclic R -module of rank one.

Lemma 1.3.1. *If K is a compact set in an n -manifold M then*

- (i) $H_i(M, M - K, R) = 0$ for $i > n$
- (ii) $\alpha \in H_n(M, M - K, R)$ is zero if and only if $\rho_x(\alpha) = 0$ for all $x \in K$.

Proof. We will drop the coefficients R from the notation. A basic fact used repeatedly in the proof is the following: If (i) and (ii) hold for compact sets K_1, K_2 and for their intersection $K_1 \cap K_2$ then (i) and (ii) hold for $K_1 \cup K_2$. This follows easily from the Mayer-Vietoris sequence (1.2.1) by taking $X = M, U = M - K_1, V = M - K_2$:

$$\begin{aligned} \rightarrow H_{i+1}(M, M - K_1 \cap K_2) &\rightarrow H_i(M, M - K_1 \cup K_2) \\ &\rightarrow H_i(M, M - K_1) \oplus H_i(M, M - K_2) \rightarrow H_i(M, M - K_1 \cap K_2) \rightarrow \end{aligned}$$

The vanishing for $i > n$ is obvious and property (ii) is also straightforward since ρ_x for $x \in K_i$ factors through $H_n(M, M - K_i)$. Given this we proceed in several steps:

(1) $M = \mathbb{R}^n, K$ a compact convex set. In this case for an interior point $x \in K$ there is a deformation retraction of both $(\mathbb{R}^n, \mathbb{R}^n - x)$ and $(\mathbb{R}^n, \mathbb{R}^n - K)$ to a pair $(\mathbb{R}^n, \mathbb{R}^n - B)$ for some large ball $B \supset K$. So we have isomorphisms

$$H_i(\mathbb{R}^n, \mathbb{R}^n - K) = H_i(\mathbb{R}^n, \mathbb{R}^n - x)$$

so that both (i) and (ii) are obvious. Notice that in this case we have a stronger version of (ii), namely $\alpha \in H_n(\mathbb{R}^n, \mathbb{R}^n - K)$ is zero if $\rho_x(\alpha) = 0$ for some $x \in K$, i.e. it is enough to check the image under ρ_x for any one point $x \in K$.

(2) $M = \mathbb{R}^n, K = K_1 \cup \dots \cup K_r$ is a finite union of compact convex sets. In this case use the Mayer-Vietoris sequence and induction on r , using the fact that $(\cup_{i < r} K_i) \cap K_r = \cup_{i < r} K_i \cap K_r$ is a union of $r - 1$ compact convex sets. Note that we get a slightly stronger version of (ii), namely if we choose one point $x_i \in K_i$ for each i , then α is nonzero if and only if $\rho_{x_i}(\alpha) \neq 0$ for all i .

(3) $M = \mathbb{R}^n$ and K is an arbitrary compact set. This case requires some argument. Let $\alpha \in H_i(M, M - K)$. Then there exist a compact neighbourhood Ω of K and a class $\tilde{\alpha} \in H_i(M, M - \Omega)$ such that $\tilde{\alpha} \mapsto \alpha$ under the natural map. (To prove this, represent α by a relative cycle, i.e. a chain $\xi \in C_i(M)$ such that $\partial\xi$ lies in $M - K$. Thus $\partial\xi$ is a linear combination of simplices contained in $M - K$, i.e. $\text{supp}(\partial\xi) \subset M - K$. Since $\text{supp}(\partial\xi)$ is compact we can choose a neighbourhood Ω of the compact set K which is disjoint from $\text{supp}(\partial\xi)$. Then ξ also gives a relative cycle in $C_i(M, M - \Omega)$ and the class $\tilde{\alpha}$ of ξ in $H_i(M, M - \Omega)$ has the required property.) Let B_1, \dots, B_r be a family of closed balls covering K with $B_i \subset \Omega$ and $B_i \cap K \neq \emptyset$ for all i . The map $H_i(M, M - \Omega) \rightarrow H_i(M, M - K)$ factors as

$$H_i(M, M - \Omega) \rightarrow H_i(M, M - \cup B_i) \rightarrow H_i(M, M - K).$$

For $i > n$ the middle group is zero by case (2), so $\alpha = 0$. For $i = n$ suppose that $\rho_x(\alpha) = 0$ for all $x \in K$. Let α_B be the image of $\tilde{\alpha}$ under the first homomorphism. Then we know that $\rho_x(\alpha_B) = \rho_x(\alpha) = 0$ for $x \in K$, and by the stronger version of (ii) in case (2), we conclude that $\alpha_B = 0$. Then $\alpha = 0$.

(4) M arbitrary, K is contained in a neighbourhood U homeomorphic to \mathbb{R}^n . In this case the excision $H_i(M, M - K) = H_i(U, U - K)$ reduces it to case (3).

(5) M arbitrary, K arbitrary. In this case write K as a union of compact sets as in (4) and use the same Mayer-Vietoris induction procedure as in (2), using (4) as the input. \square

If M is compact then taking $K = M$ we see that $H_i(M, R) = 0$ for $i > n$. For noncompact manifolds we have a stronger statement:

Proposition 1.3.2. *Let M be a noncompact manifold. Then $H_i(M, R) = 0$ for $i \geq n$.*

Proof. We drop the coefficient ring from the notation. We also assume that M is connected without loss of generality.

An element $\alpha \in H_i(M)$ can be represented by a cycle with support in some open set U with compact closure \bar{U} . Let $K = \bar{U} - U$. The triple $(M, M - K, M - \bar{U})$ has a long exact sequence which is the upper row of the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{i+1}(M, M - K) & \longrightarrow & H_i(M - K, M - \bar{U}) & \longrightarrow & H_i(M, M - \bar{U}) & \longrightarrow & \cdots \\ & & & & \parallel & & \uparrow & & \\ & & & & H_i(U) & \longrightarrow & H_i(M) & & \end{array} \quad (1.3.1)$$

The first vertical arrow is an isomorphism/equality since $M - K = U \sqcup (M - \bar{U})$ as a topological disjoint union.

$i > n$: The outer two groups in the top row vanish for $i > n$ by the previous lemma, so it follows that $H_i(U) = 0$ for $i > n$. By assumption α is in the image of $H_i(U) \rightarrow H_i(M)$, so $\alpha = 0$. This proves $H_i(M) = 0$ for $i > n$.

$i = n$: For $\alpha \in H_n(M)$, we represent it as coming from $\alpha \in H_n(U)$ as above and use the diagram for $i = n$, which is now

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(M - K, M - \bar{U}) & \longrightarrow & H_n(M, M - \bar{U}) & & \\ & & \parallel & & \uparrow & & \\ & & H_n(U) & \longrightarrow & H_n(M) & & \end{array} \quad (1.3.2)$$

by the previous lemma. Consider the image $\bar{\alpha}$ of α in $H_n(M, M - \bar{U})$. Evidently $\rho_x(\bar{\alpha})$ is either nonzero for all $x \in \bar{U}$ or zero for all $x \in \bar{U}$. (Given two points take a path between them in M and cover it by small open balls. Now use the stronger version of (ii) in case (1) of the previous proof to say that $\rho_x(\bar{\alpha})$ is either zero at every point of the path or nonzero at every point of the path.) Since the cycle representing α has compact support contained in U , it must be the case that $\rho_x(\bar{\alpha}) = 0$ for all $x \in \bar{U}$, as it is certainly so for x outside the support. By the previous lemma $\bar{\alpha}$ is zero in $H_n(M, M - \bar{U})$. By the exact sequence $\alpha = 0$ in $H_n(U)$, and hence in $H_n(M)$. \square

1.4. Orientations. Let M be an n -manifold.

A **local R -orientation at $x \in M$** is a choice of generator of the rank one free R -module $H_n(M, M - x, R)$.

An **R -orientation** for M is a choice of local R -orientations at all points, written $x \mapsto \mu_x$, satisfying a continuity condition: For each x , there is a neighbourhood B of x in M such that for $y \in B$, the local orientations μ_x and μ_y are related by the homomorphisms

$$H_n(M, M - x, R) \xleftarrow{\rho_x} H_n(M, M - B, R) \xrightarrow{\rho_y} H_n(M, M - y, R),$$

i.e. there is a $\mu_B \in H_n(M, M - B, R)$ with $\rho_x(\mu_B) = \mu_x$ and $\rho_y(\mu_B) = \mu_y$.

An R -orientation of M may or may not exist. If one exists we say that M is **R -orientable**, and if a specific R -orientation has been chosen we say that M is **R -oriented**.

The **R -orientation manifold** of M is $\tilde{M}_R = \{(x, \mu_x) \mid \mu_x \text{ is an } R\text{-orientation at } x\}$. It is naturally a manifold and the mapping $(x, \mu_x) \mapsto x$ defines a map $p : \tilde{M}_R \rightarrow M$ which is a covering map. (The fibre over any point is (noncanonically) identified with the units R^\times in R .) An R -orientation is simply a section of this covering map. Note that the R -orientation manifold \tilde{M}_R is itself R -oriented, i.e. it comes with an orientation.

If $R = \mathbb{Z}$ (in which case we drop the ring from the notation) then the orientation manifold is a double covering of M . The manifold M is orientable if and only if this is a trivial cover, i.e. $\tilde{M} = M \sqcup M$, and an orientation is a choice of component. (So if M is orientable then it has precisely $2^{|\pi_0(M)|}$ orientations). Since a connected M can have a nontrivial double cover only if $\pi_1(M)$ has an index 2 subgroup, we see that if $\pi_1(M)$ has no such subgroup (e.g. if M is simply connected) then M is orientable. Thus spheres, tori, odd-dimensional real projective spaces, all complex and quaternionic projective spaces, complex Grassmannians etc. are orientable.

If M is oriented (i.e. \mathbb{Z} -oriented) then it is R -oriented for any ring R since $H_n(M, M - x, R) = H_n(M, M - x, \mathbb{Z}) \otimes_{\mathbb{Z}} R$ by the universal coefficient theorem.

If $R = \mathbb{Z}/2$ then $\tilde{M}_{\mathbb{Z}/2} = M$ and any manifold is $\mathbb{Z}/2$ -orientable.

1.5. Fundamental class.

Proposition 1.5.1. *Suppose that M is R -oriented, with local orientations $\mu_x \in H_n(M, M - x, R)$ for each $x \in M$. Then for any compact set $K \subset M$ there is a unique class $\mu_K \in H_n(M, M - K, R)$ such that $\rho_x(\mu_K) = \mu_x$ for all $x \in K$.*

Proof. We drop the ring R from the notation for simplicity. Notice that the previous lemma shows that the class μ_K is unique if it exists, so we have only to prove existence. For K sufficiently small this is contained in the definition of orientation, so we need to patch these together.

Suppose that $K = K_1 \cup K_2$ where μ_{K_1} and μ_{K_2} are known to exist. The M-V sequence is

$$0 \rightarrow H_n(M, M - K) \rightarrow H_n(M, M - K_1) \oplus H_n(M, M - K_2) \rightarrow H_n(M, M - K_1 \cap K_2) \rightarrow$$

By the uniqueness (previous lemma), μ_{K_1} and μ_{K_2} have the same image in $H_n(M, M - K_1 \cap K_2)$, so $(\mu_{K_1}, \mu_{K_2}) \mapsto 0$ in the M-V sequence. Therefore there is a unique $\mu_K \in H_n(M, M - K)$ mapping to it. It is immediate that $\rho_x(\mu_K) = \mu_x$ for all $x \in K$.

For a general K write $K = K_1 \cup \dots \cup K_r$ with each K_i small enough. Induction on r using the Mayer-Vietoris argument gives the existence of μ_K . \square

In the case where M is compact and R -oriented, taking $K = M$ we get a class $\mu_M \in H_n(M, R)$ such that $\rho_x(\mu_M)$ is the local R -orientation at each point $x \in M$.

Definition 1.5.2. *The fundamental class of a compact R -oriented n -manifold is the class $[M] := \mu_M \in H_n(M, R)$ given by the previous proposition.*

Note that if M is compact, R -orientable and connected then necessarily $H_n(M, R)$ is a free rank one R -module with any fundamental class as a generator. (Different choices of R -orientations give different R -fundamental classes which are multiples of each other by units of R .) For example, this shows that even-dimensional projective spaces can only be R -oriented if R is a $\mathbb{Z}/2$ -algebra.

Another consequence of the existence of a fundamental class (at least with $\mathbb{Z}/2$ coefficients) for compact manifolds is that compact manifolds of different dimensions cannot be homotopic to each other. (Intuitively this seems quite plausible, but it is not easy to prove without some machinery.)

1.6. Compactly supported cohomology. Define the **compactly supported cohomology** of a topological space X as the direct limit

$$H_c^i(X, R) := \varinjlim_K H^i(X, X - K, R)$$

where the direct limit is taken over compact subsets $K \subset X$ with respect to the pullback maps induced by $(X, X - K) \subset (X, X - K')$ if $K' \subset K$. This can also be defined as the cohomology of the direct limit complex $\varinjlim_K C^*(X, X - K, R)$ since taking direct limits commutes with cohomology. The obvious map $H_c^*(X, R) \rightarrow H^*(X, R)$ to cohomology is an isomorphism if X is compact. Compactly supported cohomology is not a homotopy invariant: For example,

$H_c^i(\mathbb{R}^n, R)$ is zero for $i \neq n$ and R for $i = n$ (take the limit as K runs over closed balls, which are cofinal among all compacts).

Exercise 1.6.1. Check the following properties of compactly supported cohomology:

- (1) If $U \subset X$ is an open set then there is a map $H_c^*(U, R) \rightarrow H_c^*(X, R)$.
- (2) If $f : X \rightarrow Y$ is a proper map then there is a pullback $f^* : H_c^i(Y, R) \rightarrow H_c^i(X, R)$.
- (3) If $X = U \sqcup Z$ with Z closed then there is a long exact sequence

$$\cdots \rightarrow H_c^i(U, R) \rightarrow H_c^i(X, R) \rightarrow H_c^i(Z, R) \rightarrow \cdots$$

in compactly supported cohomology.

Let $X = U \cup V$ where U, V are open. There is a Mayer-Vietoris exact sequence

$$\rightarrow H_c^i(U \cap V, R) \rightarrow H_c^i(U, R) \oplus H_c^i(V, R) \rightarrow H_c^i(X, R) \rightarrow . \quad (1.6.1)$$

Indeed, this follows from the cohomology Mayer-Vietoris sequence for pairs (1.2.2) as follows (we drop the coefficients R for notational simplicity): For $K \subset U$ and $L \subset V$ compact sets we have a long exact sequence

$$\rightarrow H^i(M, M - K \cap L) \rightarrow H^i(M, M - K) \oplus H^i(M, M - L) \rightarrow H^i(M, M - K \cup L) \rightarrow$$

Using excisions this is the same as

$$\rightarrow H^i(U \cap V, U \cap V - K \cap L) \rightarrow H^i(U, U - K) \oplus H^i(V, V - L) \rightarrow H^i(M, M - K \cup L) \rightarrow$$

Now the set of compact subsets of $U \cap V$ which are of the form $K \cap L$ is cofinal among all compact subsets, so taking limits over K and L gives the desired sequence (1.6.1) since direct limits preserve exactness.

1.7. Poincaré duality. Recall the cap product

$$H^i(X, A, R) \times H_j(X, A \cup B, R) \rightarrow H_{j-i}(X, B, R)$$

which exists if A, B are open in $A \cup B$. (We will generally use it for open sets A, B in X or for sets A, B which can obviously be replaced by open sets without changing the (co)homology groups involved.) It has a simple relationship with cup product, namely

$$\langle a \smile b, c \rangle = \langle b, a \frown c \rangle$$

where $\langle \cdot, \cdot \rangle$ is the pairing of cohomology and homology. (On the level of chains and cochains this is the definition of \frown .)

Now suppose that $\mu_K \in H_n(M, M - K, R)$ where M is an n -manifold. Then $a \mapsto a \frown \mu_K$ defines a map

$$\frown \mu_K : H^i(M, M - K, R) \rightarrow H_{n-i}(M, R).$$

If M is an R -oriented manifold and μ_K is the class given by the previous proposition, then the maps $\frown \mu_K$ are compatible for varying K (by uniqueness of μ_K and naturality of the cap product with respect to inclusions) and so we get a map from the direct limit

$$PD_M : H_c^i(M, R) \rightarrow H_{n-i}(M, R).$$

In the compact case $PD_M(\cdot) = (\cdot) \frown [M]$ is capping with the fundamental class.

Theorem 1.7.1. *Let M be an R -oriented manifold of dimension n . Then $PD_M : H_c^i(M, R) \rightarrow H_{n-i}(M, R)$ is an isomorphism for all i .*

Proof. The proof is again by a Mayer-Vietoris patching argument from small cases.

(1) $M = \mathbb{R}^n$. Let B be a ball and $\mu_B \in H_n(\mathbb{R}^n, \mathbb{R}^n - B, R)$ a generator. By the universal coefficient theorem $H^n(\mathbb{R}^n, \mathbb{R}^n - B, R)$ is the dual of $H_n(\mathbb{R}^n, \mathbb{R}^n - B, R)$ (because $H_{n-1}(\mathbb{R}^n, \mathbb{R}^n - B, R) = 0$), so it is free of rank one over the coefficient ring with a generator e and $\langle \mu_B, e \rangle = 1$. Now

$$1 = \langle 1 \smile e, \mu_B \rangle = \langle 1, e \frown \mu_B \rangle$$

so that $e \frown \mu_B$ is a generator of $H_0(\mathbb{R}^n, R) = R$. Thus $\frown \mu_B$ maps $H^n(\mathbb{R}^n, \mathbb{R}^n - B, R)$ isomorphically to $H_0(\mathbb{R}^n, R)$. Taking the limit over larger balls B gives the statement.

(2) (M-V principle) Suppose that $M = U \cup V$ and $PD_U : H_c^*(U) \rightarrow H_{n-*}(U)$, PD_V and $PD_{U \cap V}$ are known to be isomorphisms. (We drop the coefficients from now on for notational simplicity.) We have the diagram

$$\begin{array}{ccccccccc}
\longrightarrow & H_c^i(U \cap V) & \longrightarrow & H_c^i(U) \oplus H_c^i(V) & \longrightarrow & H_c^i(M) & \longrightarrow & H_c^{i+1}(U \cap V) & \longrightarrow \\
& \downarrow PD_{U \cap V} & & \downarrow PD_U \oplus (-PD_V) & & \downarrow PD_M & & \downarrow PD_{U \cap V} & \\
\longrightarrow & H_{n-i}(U \cap V) & \longrightarrow & H_{n-i}(U) \oplus H_{n-i}(V) & \longrightarrow & H_{n-i}(M) & \longrightarrow & H_{n-i-1}(U \cap V) & \longrightarrow
\end{array} \tag{1.7.1}$$

The naturality of cap product implies that the first two squares commute. For the third square let $K \subset U$ and $L \subset V$ be compact sets and consider

$$\begin{array}{ccc}
H^i(M, M - K \cup L) & \longrightarrow & H^{i+1}(U \cap V, U \cap V - K \cap L) \\
\downarrow \sim \mu_{K \cup L} & & \downarrow \sim \mu_{K \cap L} \\
H_{n-i}(M) & \longrightarrow & H_{n-i-1}(U \cap V).
\end{array}$$

The excision $H^{i+1}(M, M - K \cap L) = H^{i+1}(U \cap V, U \cap V - K \cap L)$ shows that the third square of the previous diagram is the limit of these squares over K and L . That these squares commute up to a sign requires an argument which we skip, referring to Hatcher, p. 246f. The 5-lemma then implies that PD_M is an isomorphism.

(3) If M is homeomorphic to an open ball then (1) applies. If M is the union of finitely many open balls then (2) and an induction on the number of balls implies that PD_M is an isomorphism.

(4) (direct limits) Suppose M is the nested union of subspaces, i.e. $M = \cup_{\alpha \in A} U_\alpha$ where A is directed and $U_\alpha \subset U_\beta$ if $\alpha \leq \beta$ and assume further that every compact set in M is contained in some U_α . Then PD_M is an isomorphism if PD_{U_α} is an isomorphism for all $\alpha \in A$. Indeed, this is because $H_c^i(M) = \varinjlim H_c^i(U_\alpha)$ and $H_{n-i}(M) = \varinjlim H_{n-i}(U_\alpha)$ and the direct limit of isomorphisms is an isomorphism.

(5) If M is an arbitrary open subset of \mathbb{R}^n then (since \mathbb{R}^n is second countable) it can be written as the union of countably many open balls B_1, B_2, \dots . Let $U_i = \cup_{j \leq i} B_j$; then U_i is a nested family of open subsets of M and PD_{U_i} is an isomorphism for all i by (3). By (4) PD_M is an isomorphism.

(6) If M is second countable then cover M by countably many open subsets $\{U_i\}_{i \geq 1}$ homeomorphic to open subsets of \mathbb{R}^n . Let $U'_i = \cup_{j \leq i} U_j$. By (2) and induction we know $PD_{U'_i}$ is an isomorphism for all i . By (4) we get that PD_M is an isomorphism.

(7) If M is arbitrary then we use a Zorn's lemma argument. Let

$$\mathcal{U} = \{U \subset M \text{ open} \mid PD_U \text{ is an isomorphism}\}.$$

Since every chain in \mathcal{U} has an upper bound (the union is one, by (4)), Zorn's lemma says that there is a maximal element U . If $U \neq M$ choose $x \in M - U$. Let V be a neighbourhood of x homeomorphic to \mathbb{R}^n . Then by (1), (2), and (5) we know that $PD_{U \cup V}$ is an isomorphism, contradicting the maximality of U . \square

(Note that the theorem actually holds even without assuming M is second countable.)

The special case when M is compact is of particular importance:

Corollary 1.7.2. *If M is a compact R -oriented n -manifold then $\cap [M] : H^i(M, R) \rightarrow H_{n-i}(M, R)$ is an isomorphism for all i .*

It follows that for a compact connected R -oriented n -manifold $H^n(M, R)$ is a free rank one R -module with canonical generator ω_M where $\omega_M \cap [M] = 1 \in H_0(M, R)$.

Note that if $R = \mathbb{Z}/2$ then any manifold is $\mathbb{Z}/2$ -orientable, so that the corollary applies. In particular, this "explains" why the homology of $\mathbb{R}P^n$ (n even) looks nice with $\mathbb{Z}/2$ coefficients.

1.8. Duality and the cup product pairing in cohomology. Let M be an R -oriented n -manifold. There is a canonical trace homomorphism $Tr : H_c^n(M, R) \rightarrow R$ defined by

$$H_c^n(M, R) \xrightarrow{PD} H_0(M, R) \rightarrow R$$

where $H_0(M, R) = \bigoplus_{\pi_0(M)} R \rightarrow R$ is given by summing components. (The trace should be thought of as integration over the manifold. This will be made precise in the context of de Rham cohomology later.)

Corollary 1.8.1. *Let M be an R -oriented manifold. The pairing*

$$H_c^i(M, R) \times H^{n-i}(M, R) \rightarrow R \quad \text{by } (a, b) \mapsto Tr(a \smile b)$$

is perfect modulo torsion.

Proof. Since we are working modulo torsion the pairing between homology and cohomology is perfect.

$$\begin{array}{ccc} H_c^i(M) \times H^{n-i}(M) & \xrightarrow{PD \times id} & H_{n-i}(M) \times H^{n-i}(M) \\ \downarrow & & \downarrow \\ H_c^n(M) & \longrightarrow & R \end{array}$$

The diagram commutes since $\langle PD(\alpha), \beta \rangle = Tr(\beta \smile \alpha)$, as one checks by working with $H^i(M, M/K)$ instead of $H_c^i(M)$ for a compact set K . \square

Now consider the case where M is compact and oriented and $R = \mathbb{Z}$. If the dimension of M is $n = 2m$ then this defines a perfect bilinear form on $H^2(M)/tors$ with values in \mathbb{Z} . By the graded-commutivity of cup product this is symmetric if m is even and alternating (=skew-symmetric) if m is odd. We get as a corollary that if m is odd then $H^m(M)/tors$ must be even dimensional. (There cannot be a nondegenerate alternating form on an odd dimensional free abelian group or vector space over a field of characteristic zero.) The case m odd is not so interesting because any two perfect alternating forms on \mathbb{Z}^r are equivalent, so assume that $n = 2m = 4l$. Then the pairing above is a perfect symmetric pairing on $H^{2l}(M, \mathbb{Z})/tors$, i.e. a perfect symmetric bilinear form. Perfect symmetric bilinear forms have many useful invariants, all of which give invariants of the manifold. For example, the signature of the real bilinear form on $H^{2l}(M, \mathbb{R}) = H^{2l}(M, \mathbb{Z}) \otimes \mathbb{R}$ is then an invariant of the manifold.

Now suppose that M is simply-connected and $l = 1$, i.e. M is a compact simply-connected (and hence orientable) 4-manifold. Then $H_1(M) = 0$ and hence $H^1(M) = 0$ (universal coefficients) and $H^3(M) = 0$ (duality). So the only interesting cohomology is $H^2(M)$, which is free (by the universal coefficient theorem), and it has the symmetric bilinear form given by the intersection form. An amazing theorem of M. Freedman (1982) says that there are (up to homeomorphism) at most two simply-connected compact four manifolds (without boundary) with a given $H^2(M, \mathbb{Z})$ and intersection form, and only one of them has a smooth structure.

1.9. Duality between cohomology and Borel-Moore homology (*). For noncompact manifolds, there is another Poincaré duality statement relating cohomology to something called Borel-Moore homology. For now, we will just give the statement:

Theorem 1.9.1. *Let M be an R -oriented n -manifold. Then there is a fundamental class $[M] \in H_n^{BM}(M, R)$ and cap product with $[M]$ gives an isomorphism $PD : H^i(M, R) \rightarrow H_{n-i}^{BM}(M, R)$ for all i .*

The definition and properties of Borel-Moore homology and the proof of this theorem will be given in a problem set.

1.10. Euler characteristics. The Euler characteristic of a space X is defined as

$$\chi(X) = \sum_i (-1)^i \text{rank}(H_i(X, \mathbb{Z})).$$

We assume here that the homology groups have finite rank. The mod 2 Euler characteristic is defined as

$$\chi_2(X) = \sum_i (-1)^i \dim H_i(X, \mathbb{Z}/2).$$

Lemma 1.10.1. *If $H_*(X)$ is finitely generated then $\chi(X) = \chi_2(X)$.*

Proof. First note that $\sum_i (-1)^i \dim H^i(X, \mathbb{Z}/2) = \sum_i (-1)^i \dim H_i(X, \mathbb{Z}/2)$ since $\mathbb{Z}/2$ is a field and $H_i(X, \mathbb{Z}/2)^* = H^i(X, \mathbb{Z}/2)$. The universal coefficient theorem for cohomology gives short exact sequences

$$0 \rightarrow \text{Ext}(H_{i-1}(X, \mathbb{Z}), \mathbb{Z}/2) \rightarrow H^i(X, \mathbb{Z}/2) \rightarrow \text{Hom}(H_i(X, \mathbb{Z}), \mathbb{Z}/2) \rightarrow 0.$$

Write $H_i(X, \mathbb{Z})$ as a sum of a free summand \mathbb{Z}^r and its torsion subgroup. Each \mathbb{Z} summand of $H_i(X, \mathbb{Z})$ contributes $\mathbb{Z}/2$ to $H^i(X, \mathbb{Z}/2)$ and $\text{Ext}(\mathbb{Z}, \mathbb{Z}/2) = 0$ to $H^{i+1}(X, \mathbb{Z})$. So each \mathbb{Z} summand contributes $(-1)^i$ to both $\chi(X)$ and $\chi_2(X)$. Each torsion summand \mathbb{Z}/m contributes

$$\text{Hom}(\mathbb{Z}/m, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases} \text{ to } H^i(X, \mathbb{Z}/2) \text{ and also } \text{Ext}(\mathbb{Z}/m, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

to $H^{i+1}(X, \mathbb{Z})$. In either case the contribution of a torsion summand of $H_i(X, \mathbb{Z})$ to $H^*(X, \mathbb{Z}/2)$ is zero, proving the lemma. \square

Corollary 1.10.2. *The Euler characteristic of a compact odd-dimensional manifold is zero.*

Proof. We will assume the fact (to be proved later) that $H_*(M, \mathbb{Z})$ is finitely generated for a compact manifold. By the lemma, $\chi(M) = \chi_2(M)$. Poincaré duality with $\mathbb{Z}/2$ coefficients implies that

$$\dim H_i(M, \mathbb{Z}/2) = \dim H^i(M, \mathbb{Z}/2) = \dim H_{n-i}(M, \mathbb{Z}/2).$$

Since n is odd, these dimensions appear with opposite signs in χ_2 . \square

1.11. Finite generation of homology. Consider the following trivial remark: If Poincaré duality holds for a zero-dimensional manifold X then X must be finite (because $H^0 \cong H_0$ is required to be finitely generated). This suggests that Poincaré duality should force the homology of a compact manifold to be finitely generated. In fact:

Theorem 1.11.1. *If M is a compact R -oriented manifold then $H^*(M, R)$ and $H_*(M, R)$ are finitely generated R -modules.*

Proof. (from Spanier, Algebraic Topology) With a space X , coefficient ring R , and a coefficient R -module M , one can associate two cochain complexes. The first is the usual cochain complex

$$C^*(X, M) = \text{Hom}_R(C_*(X, R), M) = \text{Hom}(C_*(X), M)$$

with cohomology $H^*(X, M)$. There is also

$$C^*(X, R) \otimes_R M = \text{Hom}(C_*(X), R) \otimes_R M = \text{Hom}_R(C_*(X, R), R) \otimes_R M$$

with differential $\delta \otimes 1$. There is a map from the second complex to the first:

$$\mu : \text{Hom}_R(C_*(X, R), R) \otimes_R M \rightarrow \text{Hom}_R(C_*(X, R), M) \quad \text{by } \mu \left(\sum_i \varphi_i \otimes m_i \right) (\xi) = \sum_i \langle \varphi_i, \xi \rangle m_i.$$

This will not usually induce an isomorphism in cohomology, but for a compact manifold X Poincaré duality implies that it does. To see this, recall that the universal coefficient theorem for homology takes the form

$$0 \rightarrow H_k(X, R) \otimes_R M \rightarrow H_k(X, M) \rightarrow \text{Tor}_1(H_{k-1}(X, R), M) \rightarrow 0.$$

Using Poincaré duality isomorphisms $H_k(X) \cong H^{n-k}(X)$ this gives

$$0 \rightarrow H^k(X, R) \otimes_R M \rightarrow H^k(X, M) \rightarrow \text{Tor}_1(H^{k+1}(X, R), M) \rightarrow 0.$$

Now consider the second complex above. Let $K^* = \text{Hom}_R(C_*(X), R)$. The definition of Tor_1 gives a short exact sequence

$$0 \rightarrow H^k(K^*) \otimes_R M \rightarrow H^k(K^* \otimes_R M) \rightarrow \text{Tor}_1(H^{k+1}(K^*), M) \rightarrow 0.$$

Since $H^k(K^*) = H^k(X, R)$ we see that the two sequences are the same, and the natural mapping μ induces isomorphisms in cohomology.

Now we see that this property implies finite generation of the homology. Let C_* be a chain complex of R -modules such that

(*) for every R -module M the natural map of cochain complexes

$$\mu : \text{Hom}_R(C_*, R) \otimes_R M \rightarrow \text{Hom}_R(C_*, M)$$

induces isomorphisms in cohomology.

We will show that this implies that $H_*(C_*)$ is finitely generated. The cohomology universal coefficient theorem gives the sequence

$$0 \rightarrow \text{Ext}^1(H_{k-1}(C_*), M) \rightarrow H^k(\text{Hom}_R(C_*, M)) \xrightarrow{h} \text{Hom}_R(H^k(C_*), M) \rightarrow 0.$$

Taking $M = H_k(C_*)$ the surjectivity of h and the property (*) imply that we can write

$$h\mu \left(\sum_i a_i \otimes b_i \right) = id \quad \in \text{Hom}_R(H_k(C_*), H_k(C_*))$$

for some $a_i \in H^k(\text{Hom}_R(C_*, R))$, $b_i \in H_k(C_*)$. Now apply both sides to an element $b \in H_k(C_*)$:

$$b = h\mu \left(\sum a_i \otimes b_i \right) (b) = \left\langle \mu \left(\sum a_i \otimes b_i \right), b \right\rangle = \sum \langle a_i, b \rangle b_i.$$

Thus the b_i generate $H_k(C_*)$.

This proves that $H_*(X, R)$ is finitely generated; it follows from the universal coefficient theorem (or Poincaré duality) that $H^*(X, R)$ is finitely generated. \square

In fact the homology of a compact manifold is always finitely generated, with no assumption about orientability. To generalize the proof above requires introducing Čech cohomology (cf. Spanier) so we will leave it aside for now. A different, more topological way to prove finite generation is by showing that a compact manifold is homotopy equivalent to a finite CW complex. (See the Appendix of Hatcher for a proof of this.) Note that the theorem would follow immediately if one could show that (a) a compact manifold has a CW structure or (b) a compact manifold has a simplicial structure. (Of course, (b) implies (a).) But (a) is unknown in general and (b) is false in general. A theorem of Whitehead says that (b) (and hence (a)) is true for smooth manifolds, as we shall see later.

1.12. Manifolds with boundary. A **manifold with boundary** is a second-countable Hausdorff topological space M such that every point has a neighbourhood homeomorphic to an open subset in the half-space $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$. If in such a neighbourhood x is carried to a point with $x_n = 0$ then we say that x is in the boundary of M , denoted ∂M . It is easy to check that ∂M is a manifold of dimension $n - 1$. It is also easy to check that the product of manifolds with boundary is naturally a manifold with boundary.

A **collar** for ∂M is a neighbourhood of ∂M in M which is homeomorphic to $\partial M \times [0, 1]$ under a homeomorphism taking ∂M to $\partial M \times \{0\}$ by the identity.

Lemma 1.12.1. *If M is a manifold with compact boundary ∂M then ∂M admits a collar in M .*

Proof. (cf. Hatcher p. 253) Let M' be the manifold one gets by attaching a collar $\partial M \times [0, 1]$ to ∂M , with $\partial M \times \{0\}$ being glued to ∂M via the identity. It is enough to show that M' is homeomorphic to M .

Choose a partition of unity $\varphi_i : \partial M \rightarrow [0, 1]$ such that each $\text{supp}(\varphi_i)$ has closure contained in an open set U_i in M which is homeomorphic to \mathbb{R}_+^n .

$$M_k := M \cup \{(x, t) \in \partial M \times [0, 1] : t \leq (\varphi_1 + \cdots + \varphi_k)(x)\}.$$

For k large enough, $M_k = M'$. So it suffices to construct homeomorphisms $h_k : M_{k-1} \cong M_k$ for each k . Now since $U_k \cong \mathbb{R}_+^n$ we can find a homeomorphism of $\partial U_k \times [-1, 1]$ onto a neighbourhood of ∂U_k in M' . Define h_k to be the identity outside $\partial U_k \times [-1, 1]$, and inside $\partial U_k \times [-1, 1]$ we choose a homeomorphism which maps $\{x\} \times [-1, \sum_{i \leq k-1} \varphi_i(x)]$ onto $\{x\} \times [-1, \sum_{i \leq k} \varphi_i(x)]$ linearly. This gives the desired h_k . \square

(The proof works if ∂M is paracompact.)

We will say that a manifold-with-boundary is R -oriented (or R -orientable) if $M - \partial M$ is R -oriented (or R -orientable).

Lemma 1.12.2. *If M is R -orientable then ∂M is R -orientable.*

Proof. Exercise. \square

Now if M is a compact R -oriented manifold with boundary then using the collar above we can write $H_n(M, \partial M, R) = H_n(M - \partial M, \partial M \times (0, \epsilon))$, which has a generator coming from the R -orientation. Thus there is a fundamental class $[M] \in H_n(M, \partial M, R)$.

1.13. Lefschetz duality. Let M be an R -oriented n -manifold with boundary. The following theorem can be generalized to the noncompact case but we will be content to state the compact version:

Theorem 1.13.1. *Let M be an R -oriented compact n -manifold with boundary ∂M . Then $\frown [M] : H^i(M, \partial M, R) \rightarrow H_{n-i}(M, R)$ and $\frown [M] : H^i(M, R) \rightarrow H_{n-i}(M, \partial M, R)$ are isomorphisms.*

Proof. Both homomorphisms exist because $[M] \in H_n(M, \partial M)$. (In the cap product $H^i(X, A) \times H_j(X, A \cup B) \rightarrow H_{j-i}(X, B)$ they correspond to the choices $A = \partial M, B = \emptyset$ and $A = \emptyset, B = \partial M$.)

The existence of a collar for ∂M easily implies that $H_c^k(M - \partial M) = H^k(M, \partial M)$ and $H_{n-i}(M) = H_{n-i}(M - \partial M)$. So the first isomorphism is just Poincaré duality for the noncompact manifold $M - \partial M$. For the second, look at the long exact sequences of the pair $(M, \partial M)$

$$\begin{array}{ccccccc} \longrightarrow & H^i(M, \partial M) & \longrightarrow & H^i(M) & \longrightarrow & H^i(\partial M) & \longrightarrow & H^{i+1}(M, \partial M) & \longrightarrow \\ & \frown [M] \downarrow & & \frown [M] \downarrow & & \frown [\partial M] \downarrow & & \frown [M] \downarrow & \\ \longrightarrow & H_{n-i}(M) & \longrightarrow & H_{n-i}(M, \partial M) & \longrightarrow & H_{n-i-1}(\partial M) & \longrightarrow & H_{n-i-1}(M) & \longrightarrow \end{array}$$

The diagram commutes because the image of $[M]$ under the connecting homomorphism $H_{n-i}(M, \partial M) \rightarrow H_{n-i-1}(\partial M)$ is a fundamental class, which we have denoted $[\partial M]$. Thus Poincaré duality for the boundary and the first part of the theorem give the second part of the theorem. \square

Corollary 1.13.2. *Let M be a compact R -oriented n -manifold with boundary ∂M . Then*

$$H_*(M, R), H_*(M, \partial M, R) \quad \text{and} \quad H^*(M, R), H^*(M, \partial M, R)$$

are finitely generated R -modules.

Proof. Substitute Lefschetz duality in the proof of finite generation above. \square

Corollary 1.13.3. *If M is a compact $n+1$ -dimensional manifold with boundary then $\chi(\partial M) = (1 + (-1)^n)\chi(M)$. (So $\chi(\partial M)$ is even.)*

Proof. If (X, A) has finitely generated homology then $\chi(X, A) = \chi_2(X, A)$ (this was proved earlier for $A = \emptyset$ and the same proof works). Now for $A \subset X$ we have $\chi_2(X) = \chi_2(A) + \chi_2(X, A)$ by the relative sequence of the pair, assuming finite generation. In our case take $X = M, A = \partial M$ and use the equality $\dim H_i(M, \partial M, \mathbb{Z}/2) = \dim H^i(M, \partial M, \mathbb{Z}/2)$ and duality $H^*(M, \partial M, \mathbb{Z}/2) \cong H_{n+1-*}(M, \partial M, \mathbb{Z}/2)$ to get

$$\chi_2(M) = \chi_2(\partial M) + \chi_2(M, \partial M) = \chi_2(\partial M) + (-1)^{n+1} \chi_2(M) \quad (1.13.1)$$

which gives the corollary. \square

(Strictly speaking we have only proved this for M oriented as we have not proved finite generation of $H_*(M, \partial M)$ in the nonorientable case, but assuming that the corollary is proved.)

In particular a manifold of odd Euler characteristic (for example, $\mathbb{C}P^{2k}$) cannot be the boundary of a compact manifold. The oriented topological cobordism ring Ω_*^{top} is defined as follows: Ω_n^{top} consists of classes of oriented manifolds (without boundary) modulo the equivalence relation of cobordism: N_1 and N_2 are oriented cobordant if $N_1 \sqcup (-N_2) = \partial M$ where M is an oriented $n+1$ -manifold with boundary. (Here $-N_2$ is N_2 with orientation reversed. Evidently this relation is weaker than homeomorphism.) This is a group under taking disjoint unions (the identity is the empty set) and $\Omega_*^{top} = \bigoplus_n \Omega_n^{top}$ is a ring under taking direct products. The corollary tells us that $[N] \neq 0$ if $\chi(N)$ is odd; in particular $[\mathbb{C}P^{2k}] \in \Omega_{4k}^{top}$ is nonzero. A theorem of Sullivan says that $\Omega_{4*}^{top} \otimes \mathbb{Z}[1/2] = \bigoplus_{k \geq 0} \Omega_{4k}^{top} \otimes \mathbb{Z}[1/2]$ is a polynomial ring generated by the classes of even-dimensional complex projective spaces. (The analogous theorem in the context of smooth manifolds and the corresponding ring Ω_* is a famous theorem of Thom.) The complete structure of the ring Ω_*^{top} is quite subtle.

1.14. Gysin homomorphisms and Poincaré dual classes. Let $f : M \rightarrow N$ be a continuous map of R -oriented manifolds of dimension m and n respectively. Then using Poincaré duality we get a map

$$f_! : H_c^k(M, R) \rightarrow H_c^{k+n-m}(N, R),$$

defined by $f_! := PD_N^{-1} \circ f_* \circ PD_M$. which is called the **Gysin homomorphism**. In the case where M and N are compact the relation to the usual pullback f^* is given by

$$f_!(f^*(\alpha) \smile \beta) = \alpha \smile f_!(\beta) \quad (\alpha \in H^k(N, R), \beta \in H^*(M, R))$$

(Check this formula.) Thus $f_!$ is a $H^*(N, R)$ -module homomorphism if $H^*(M, R)$ is given the $H^*(N, R)$ -module structure via f^* .

Now consider the case where $i : M \hookrightarrow N$ is injective (and M and N are compact). Thus i is a continuous bijection onto its image, i.e. a homeomorphism of M with a closed subspace of N . We will refer to M as an embedded submanifold. Then there is a defined a class

$$\xi_M := i_!(1) \in H^{n-m}(M, R)$$

which we call the **Poincaré dual class to M** . It has the property that if $\alpha \in H^m(N, R)$ and $i^*(\alpha) = \omega_M$ then $\alpha \smile \xi_M = \omega_N$. (Note that ξ_M may be zero. For example if $M = S^1$ is the equator of $N = S^2$ then $\xi_M \in H^1(S^2, R) = 0$.)

There is also a naturally defined homology class $i_*([M]) \in H^m(N, R)$. (Intuitively, M defines a cycle in N (for example, if one had a simplicial complex structure on N in which M is a subcomplex) and this is the class of that cycle.) These two are dual in the sense that

$$\xi_M \smile [N] = i_*([M]).$$

(Note that $[M], [N]$, and ξ_M depend on the choices of orientations. But for example if $R = \mathbb{Z}$ then each of these is independent of the choice up to ± 1 .)

1.15. Cup product and intersection product. Let M be compact and R -oriented. Then Poincaré duality can be used to define the **intersection product**

$$H_i(M, R) \times H_j(M, R) \rightarrow H_{i+j-n}(M, R)$$

by inverting the vertical maps in the following diagram:

$$\begin{array}{ccc} H_i(M, R) \times H_j(M, R) & \longrightarrow & H_{i+j-n}(M, R) \\ PD \times PD \uparrow & & \uparrow PD \\ H^{n-i}(M, R) \times H^{n-j}(M, R) & \xrightarrow{\smile} & H^{2n-i+j}(M, R) \end{array}$$

Intuitively it is given by intersecting homology cycles (think of them, e.g. as simplicial cycles) but it is very difficult to make this rigorous. Note also that this works only for oriented compact manifolds, whereas the cup product on cohomology is defined for arbitrary spaces.

In some cases it is possible to give a more geometric description of the intersection product, which is very useful in many applications. Let M be a compact oriented n -manifold. Suppose that $i_A : A \hookrightarrow M, i_B : B \hookrightarrow M$ are closed submanifolds of dimension a, b respectively, each of which is oriented. Suppose also that $A \cap B$ is a connected manifold of dimension $a + b - n$ and moreover, for each $x \in A \cap B$ there is a neighbourhood U of x in M and a homeomorphism $U \cong \mathbb{R}^n$ such that

$$(U \cap A, U \cap B, U \cap A \cap B) \cong (\mathbb{R}^a, \mathbb{R}^b, \mathbb{R}^a \cap \mathbb{R}^b \cong \mathbb{R}^{a+b-n}).$$

We say that A and B **intersect transversely**. The orientations of A and B and the orientation of M together determine an orientation on $A \cap B$. (*Exercise:* Check this.) We will not prove the following theorem which gives a very geometric description of some cup products. (A proof of this version can be found in Dold, *Lectures on Algebraic Topology*, VIII.11.13. We will prove a version later in the context of smooth manifolds.)

Theorem 1.15.1. *Let M be a compact oriented manifold and let A, B be oriented compact submanifolds that intersect transversely (as defined above). Then*

$$\xi_A \smile \xi_B = \sum_C \varepsilon_C \xi_C$$

where C runs over connected components of $A \cap B$ and $\varepsilon_C \in \{\pm 1\}$. The intersection product of the homology classes is

$$i_{A*}([A]) \cdot i_{B*}([B]) = \sum_C \varepsilon_C i_{C*}([C])$$

where C, ε_C are as before.

Here the sign $\varepsilon_C \in \{\pm 1\}$ is determined by the orientations of A and B and M . For a single C we can arrange $\varepsilon_C = +1$ by choosing orientations correctly, but we cannot necessarily make all $\varepsilon_C = +1$ simultaneously.

For example if $\dim A + \dim B = n$ then a transverse intersection means $A \cap B$ is a finite set of points, so that $[M] \frown \xi_{A \cap B} \in H_0(M)$ is a scalar. This is a sum $\sum_{x \in A \cap B} \varepsilon_x$ where ε_x is plus or minus one according to whether the orientations of A and B at x give the orientation of M or its negative. (A trivial example showing that signs are necessary is the following: Let $M = S^3$, $A = S^2$, and B be any S^1 which meets A transversely. The typical B meets A in two points, but the signs are opposite, which they must be as both classes $i_{A*}([A]), i_{B*}([B])$ are zero as $H^1(S^3) = H^2(S^3) = 0$.)

This theorem is very useful in practice because many cohomology classes (but not all!) of a manifold M can be written as linear combinations of Poincaré dual classes of submanifolds. (*Exercise:* Use this theorem to determine the cohomology ring of $\mathbb{C}P^n$.)

The theorem can also sometimes be used to prove nonvanishing of cohomology of a complicated manifold M if one knows the existence of some nice submanifolds A, B of complementary dimension for which we can understand the intersection.

1.16. **Alexander duality (*)**. This is a duality for a closed subset $X \subset M$ of a “background” manifold, which is taken to be a sphere or Euclidean space in the classical Alexander duality. (The case of manifolds-with-boundary can be reduced to this using a doubling construction.)

Since we are now allowing the subset X to be complicated (e.g. not a manifold), we will need Čech cohomology. We will just state the results for now.

Let $X \subset M$ be a closed subset of a compact n -manifold. Then there are isomorphisms

$$\check{H}^i(M, X) \cong H_{n-i}(M - X)$$

and

$$\check{H}^i(X) \cong H_{n-i}(M, M - X)$$

where \check{H}^* denotes Čech cohomology (to be defined later).

So for example, if X is a manifold of dimension m , embedded as a closed subset of M , then

$$H_i(M, M - X) \cong H^{n-i}(X) \cong H_{n-m+i}(X)$$

The dual version $H^i(M, M - X) \cong H^{i-(n-m)}(X)$ is usually referred to as the Gysin isomorphism.

If X is assumed to be a locally contractible subset of a manifold then $\check{H}^i = H^i$, so that we get simpler statements which do not involve Čech cohomology. If the background manifold M is S^n , then this reduces to a statement called Alexander duality:

$$\tilde{H}_i(S^n - X) \cong \tilde{H}^{n-1-i}(X).$$

Here we have used reduced homology. Note that the homology of $S^n - X$ depends only on X and not on how it sits in S^n .

Remark 1.16.1 (Verdier duality). Wouldn't it be nice if there were a single statement encapsulating all the statements above? There is, namely Verdier duality, which was formulated by Grothendieck and proved by Verdier in the 1960s. The formulation requires more sophisticated homological algebra tools, so maybe later.

2. SMOOTH MANIFOLDS

From now on we will only deal with C^∞ things, and the words “ C^∞ ”, “smooth” and “differentiable” will be used interchangeably. A reference for most of the material here is Warner’s *Foundations of Differentiable Manifolds and Lie Groups* or any other book on manifolds.

2.1. Smooth manifolds. Let M be a paracompact second-countable Hausdorff space. ⁽⁵⁾ ⁽⁶⁾

A **chart** (or **coordinate neighbourhood**) on M is a triple (U, V, φ) consisting of an open subset $U \subset M$, an open subset V of \mathbb{R}^n (for some n), and a homeomorphism $\varphi : U \rightarrow V$. Two charts (U, V, φ) and (U', V', φ') are **compatible** if the **transition functions** $\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \rightarrow \varphi'(U \cap U')$ and $\varphi \circ (\varphi')^{-1} : \varphi'(U \cap U') \rightarrow \varphi(U \cap U')$ are smooth functions between open subsets of \mathbb{R}^n .

An **atlas** for M is a collection of charts such that the open sets cover M . Notice that a space M which has an atlas of charts is necessarily locally compact.

A **smooth atlas** for M is a collection of compatible charts which cover M , i.e. a collection of charts $\mathcal{A} = \{(U_\alpha, V_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ which are pairwise compatible and such that $\cup_\alpha U_\alpha = M$. Any smooth atlas \mathcal{A} is contained in a maximal (with respect to inclusion) smooth atlas: simply include in the collection any chart which is compatible with all the charts in \mathcal{A} .

A **smooth (or differentiable or C^∞) manifold** is a pair (M, \mathcal{A}) consisting of a paracompact second-countable Hausdorff space M and a maximal smooth atlas \mathcal{A} on M . We will usually drop the atlas from the notation. (Note that a smooth atlas \mathcal{A} determines a (unique) maximal smooth atlas (as above take the collection of all charts compatible with the atlas). The word maximal is just put in so that things are unique, in practice in all arguments with any atlas contained in the maximal atlas will do.)

Note that by our second countability assumption a manifold has at most countably many connected components, and each connected component is second countable. ⁽⁷⁾ ⁽⁸⁾

A continuous function $f : M \rightarrow \mathbb{R}$ is called **smooth (or differentiable or C^∞)** if, for each open set $\varphi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$ of an atlas, the composition $f \circ \varphi_\alpha^{-1} : V_\alpha \rightarrow \mathbb{R}$ is a smooth function. (Clearly this holds for one atlas contained in the given maximal atlas if and only if it holds for all atlases contained in the given maximal atlas.) There is an obvious definition for a function defined on an open subset $U \subset M$ to be smooth. The assignment $U \mapsto \{\text{smooth functions on } U\}$ is a sheaf (of \mathbb{R} -algebras), the sheaf of smooth functions on M .

Let $f : M \rightarrow N$ be a continuous map between two smooth manifolds of dimension m and n respectively. It is called a **smooth map** if the following holds at all points p : For each point $p \in M$, using charts around p and $f(p) \in N$, we can get a map between open subsets of \mathbb{R}^m and \mathbb{R}^n . This map should be smooth. Again, it is clear that it suffices to check this for any one pair of atlases contained in the maximal atlases defining M and N respectively. Clearly the composition of smooth maps is smooth.

A smooth map $f : M \rightarrow N$ is called a **diffeomorphism** if it has a smooth inverse, i.e. there exists a smooth map $g : N \rightarrow M$ such that $f \circ g = g \circ f = id$. (A smooth 1-1 map need not have smooth inverse, for example look at $\mathbb{R} \rightarrow \mathbb{R}$ by $x \mapsto x^3$.)

⁵A topological space is paracompact if every open covering admits a locally finite open refinement. (A covering is locally finite if every point has an open neighbourhood which meets only finitely many members of the covering. A refinement of $\{U_\alpha\}$ is a covering $\{V_\beta\}$ such that for each V_β there exists α with $V_\beta \subset U_\alpha$.)

⁶A topological space is second-countable if the topology admits a countable base (i.e. there is a countable collection of open sets such that every open set is a union of members of the collection).

⁷Here is an example to show that the Hausdorff condition is not redundant: Take two copies of \mathbb{R} and identify them outside the origin. (Equivalently, take $\mathbb{R}^2 - \{0\}$ and quotient by the relation $(x, y) \sim (\lambda x, \lambda^{-1}y)$ for $\lambda \in \mathbb{R}^*$, with the quotient topology.) This gives a space which is locally homeomorphic to \mathbb{R} but is not Hausdorff.

⁸Here is another funny example: Take the square $(0, 1)^2$ and put the topology in which a set U is open if $U \cap \{x = a\}$ is open in $\{x = a\} = (0, 1)$ for every $a \in (0, 1)$. This space has uncountably many connected components, each of which is homeomorphic to \mathbb{R} . However it is not a smooth manifold in our definition because it is not second countable.

The product of two smooth manifolds is a smooth manifold. (• Exercise.) The universal cover of a smooth manifold is a smooth manifold. (Exercise.) At the moment it is not clear that an open subset of a manifold is a manifold because of the paracompactness condition, but we will prove it shortly.

Smooth manifolds with smooth maps between them form a category Man with the obvious forgetful functor to the category $TopMan$ of topological manifolds and continuous maps and further to Top .

2.2. Real analytic manifolds. By replacing “smooth” everywhere by “real analytic” we get the notion of a real analytic manifold. So a real analytic manifold is a nice topological space M with an open covering by subsets homeomorphic to \mathbb{R}^n such that the transition maps are real analytic functions. It makes sense to talk about real analytic maps between such manifolds, real analytic functions on them, and real analytic maps between them. Real analytic manifolds are usually referred to as “ C^ω -manifolds”.

2.3. Complex manifolds. By replacing \mathbb{R}^n by \mathbb{C}^n and replacing “smooth” everywhere in the above by “holomorphic” (i.e. complex analytic) we get the notion of a complex manifold. So a complex manifold is (determined by) a nice topological space M with a covering by open sets which are homeomorphic to open subsets of \mathbb{C}^n such that the transition functions are holomorphic functions. Since holomorphic functions are smooth functions of the underlying real variables M is also a smooth manifold of (real) dimension $2n$.

(Note that a complex function is differentiable once if and only if it is analytic so there is only one notion of complex manifold, in contrast to the real case, where we have C^k for $k \geq 0$, C^∞ , and C^ω .)

Much, but not all, of what is said below about smooth manifolds also holds for real-analytic and complex manifolds. (*Exercise:* Decide which statements and their proofs are valid in these categories.)

2.4. Examples. Here are some examples of smooth manifolds.

Spheres. Let S^n be the n -sphere, with north pole N and south pole S . Stereographic projection gives homeomorphisms $S^n - \{N\} \cong \mathbb{R}^n$ and $S^n - \{S\} \cong \mathbb{R}^n$, which is an atlas with two charts. (It can be checked that they are compatible by writing explicit formulae.)

Real projective spaces. Consider the real projective space $\mathbb{R}P^n$. One description of it is as $\mathbb{R}P^n = S^n / \sim$ where $x \sim -x$.

- Use this description to get an atlas.

Another description is $\mathbb{R}P^n = \mathbb{R}^{n+1} - \{0\} / \mathbb{R}^*$ or rather $n+1$ -tuples of numbers $(x_0, \dots, x_n) \neq (0, \dots, 0)$ upto scaling by nonzero reals. Let

$$U_i = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} - \{0\} : x_i \neq 0\} / \mathbb{R}^*.$$

Write $[x_0 : \dots : x_n]$ for the homogeneous coordinates of the point represented by $(x_0, \dots, x_n) \in \mathbb{R}^{n+1} - \{0\}$. Then $\varphi_i : U_i \rightarrow \mathbb{R}^n$ by

$$\varphi_i([x_0 : \dots : x_n]) = \left(\frac{x_0}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_n}{x_i} \right)$$

is a homeomorphism, so it gives a chart, and $\{U_i\}_{i=0}^n$ is an open covering of $\mathbb{R}P^n$. For any $0 \leq i < j \leq n$ the transition function $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$ is defined on the open subset $\{t_j \neq 0\} = \varphi_i(U_i \cap U_j) \subset \mathbb{R}^n = \varphi_i(U_i)$ by

$$(t_1, \dots, t_n) \mapsto [t_1 : \dots : 1 : \dots : t_n] \mapsto \left(\frac{t_1}{t_j}, \dots, \frac{1}{t_j}, \dots, \frac{\widehat{t_j}}{t_j}, \dots, \frac{t_n}{t_j} \right)$$

where the entry “1” and the entry “ $\frac{1}{t_j}$ ” are at the i th position. This is obviously smooth on $\{t_j \neq 0\}$. The case $i > j$ is similar. So the various U_i are compatible and $\{(U_i, \varphi_i)\}_{i=1, \dots, n}$ is a smooth atlas for $\mathbb{R}P^n$.

Complex projective spaces. The complex projective space $\mathbb{C}P^n = \mathbb{P}^n$ is a smooth manifold of dimension $2n$. Since $\mathbb{P}^n = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$ we can describe an atlas of charts by

$$U_i = \{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} - \{0\} : z_i \neq 0\}/\mathbb{C}^*.$$

The isomorphism $U_i \cong \mathbb{C}^n = \mathbb{R}^{2n}$ by $(z_0, \dots, z_n) \mapsto \left(\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i}\right)$ gives a chart. As in the previous example, the transition functions are smooth, so that this is a smooth atlas. Note that in this case the charts of the atlas are modeled on (open subsets of) \mathbb{C}^n and the transition functions are holomorphic. This is an example of a complex manifold of dimension n and of a real manifold of dimension $2n$.

Hypersurfaces in $\mathbb{C}P^n$. Let f be a homogeneous polynomial in $n+1$ variables. Its zero-set in $\mathbb{C}^{n+1} - \{0\}$ is invariant under dilations by \mathbb{C}^* , so it defines a subset $V(f) \subset \mathbb{C}P^n$ which is in fact closed. So we can ask: When is $V(f)$ smooth manifold? (If f has real coefficients then we can also ask the same question for the subset of $\mathbb{R}P^n$ defined by f .) We can also ask the question for several homogeneous polynomials f_1, \dots, f_k . These questions will be answered in the next section.

Grassmannians. For $F = \mathbb{R}$ or \mathbb{C} (or \mathbb{H}) let $Gr(k, F^n)$ be the set of k -dimensional F -subspaces (left F -submodules in the case $F = \mathbb{H}$) of F^n . This is a smooth manifold, which can be shown as follows: For $E \in Gr(k, F^n)$ and choose a complement E^\perp to E , i.e. $F^n = E \oplus E^\perp$, and let π_E, π_{E^\perp} be the projections. Consider the subset

$$U_E := \{E' \in Gr(k, n) : E' \cap E^\perp = \{0\}\}.$$

For any $E' \in U_E$ the restriction $\pi_E|_{E'}$ is an isomorphism of E' with E . Let

$$\varphi_{E'} := \pi_{E^\perp} \circ (\pi_E|_{E'})^{-1} \in Hom(E, E^\perp).$$

It is easy to check that $E' \mapsto \varphi_{E'}$ gives an identification of U_E with $Hom(E, E^\perp) \cong Mat(k \times n - k, F) = F^{k(n-k)}$, so this is a chart at $E \in Gr(k, F^n)$.

Exercise: Check compatibility, i.e. that transition maps between charts are smooth.

Thus the collection $\{U_E\}_{E \in Gr(k, F^n)}$ are the charts of a smooth atlas making $Gr(k, F^n)$ into a smooth manifold of dimension $k(n-k)$ when $F = \mathbb{R}$ and a complex manifold of dimension $k(n-k)$, and hence also a smooth real manifold of dimension $2k(n-k)$ when $F = \mathbb{C}$.

Exercise: Calculate the cohomology of the complex Grassmannian $Gr(2, 4)$.

A configuration space. Let $|\cdot|$ denote the standard Euclidean norm on \mathbb{R}^3 . By a *ring* C in \mathbb{R}^3 we will mean a subset which (1) is contained in a plane P and (2) C is a Euclidean circle in P , i.e. there is a point $x_0 \in P$ and real number $r > 0$ such that $C = \{x \in P : |x - x_0| = r\}$. Two disjoint rings C_1, C_2 in \mathbb{R}^3 are either linked or unlinked. (They are linked if and only if $\pi_1(\mathbb{R}^3 - C_1 \cup C_2) = \mathbb{Z} \times \mathbb{Z}$, unlinked if and only if $\pi_1(\mathbb{R}^3 - C_1 \cup C_2) = \mathbb{Z} * \mathbb{Z}$ (see Hatcher, Algebraic Topology, p. 46).) Fix an integer $n \geq 1$. Let X_n be the set of all configurations of n pairwise unlinked rings in \mathbb{R}^3 . So an element of X_n is a set $\{C_1, \dots, C_n\}$ where each C_i is a ring, $C_i \cap C_j = \emptyset$ and C_i and C_j are unlinked for all i, j . *Exercise:* Show that X_n has a natural structure of a noncompact manifold of dimension $6n$. * Show that X_n is path connected.

2.5. Partitions of unity. Recall that if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of a topological space X then a **partition of unity subordinate to** $\{U_\alpha\}_{\alpha \in A}$ is a collection of continuous functions $\{f_\alpha : X \rightarrow [0, 1]\}$ with $supp(f_\alpha) \subset U_\alpha$ for all $\alpha \in A$ such that:

- (1) $\{supp(f_\alpha)\}_{\alpha \in A}$ is locally finite
- (2) $\sum_{\alpha \in A} f_\alpha \equiv 1$.

It follows that $\{\{x \in X : f_\alpha(x) \neq 0\}\}_{\alpha \in A}$ is a locally finite open cover refining $\{U_\alpha\}_{\alpha \in A}$ and with the same indexing set.

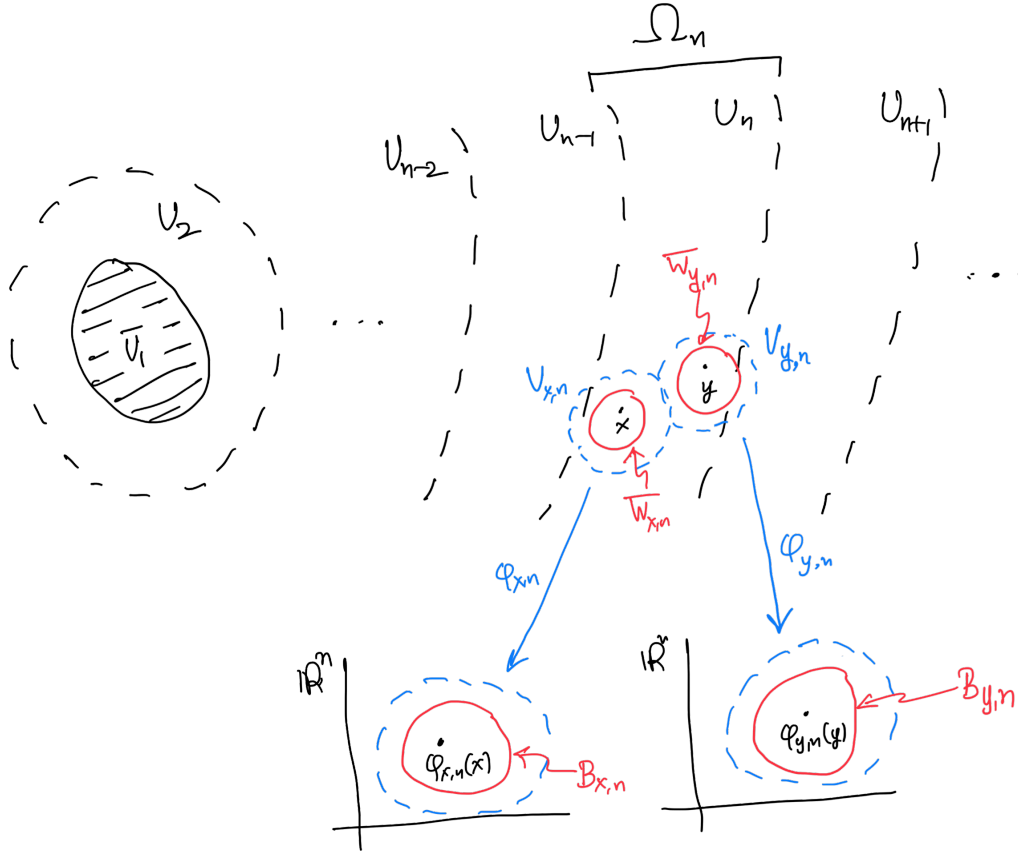
The following fact is usually proved using Urysohn's lemma: *In a paracompact Hausdorff space any open covering has a continuous partition of unity subordinate to it.* For smooth manifolds we will need a variant which allows us to construct a smooth partition of unity subordinate to any covering (Proposition 2.5.4 below). We start with some lemmas.

Lemma 2.5.1. *If $\{U_\alpha\}_\alpha$ is an open covering of a smooth manifold then one can find a countable locally finite refinement by coordinate charts $\{(V_\beta, \varphi_\beta : V_\beta \rightarrow \mathbb{R}^n)\}_{\beta \in B}$ and an open covering $\{W_\beta\}_{\beta \in B}$ refining $\{V_\beta\}$ such that $\overline{W}_\beta \subset V_\beta$ and \overline{W}_β is compact for all β .*

Proof. Since M is second countable it has a countable base consisting of charts. Since M is also locally compact it has a countable base of open sets with compact closure. (Suppose $\{U_n\}$ is a countable base. Then for any point x there is a neighbourhood V_x of x with compact closure, and hence a $U_{n=n(x)} \subset V_x$ with $x \in U_n$. Then U_n has compact closure (since $\overline{U}_n \subset \overline{V}_x$ is closed, and we are in a Hausdorff space) and contains x . Thus the U_i with compact closure already form a basis at each point, hence a base for the topology.) Thus we can assume that the original covering $\{U_\alpha\}$ in the lemma is a countable covering $\{U_n\}$ by charts with compact closure.

So let $\{U_n\}_{n \geq 1}$ be a countable covering of M consisting of charts with compact closure. Replacing U_n by $\cup_{j \leq n} U_j$ we can assume that $\{U_n\}$ is a nested collection of open sets with compact closure. Since \overline{U}_n is compact and is covered by the collection we must have that $\overline{U}_n \subset U_m$ for some $m > n$. Replacing U_{n+1} by U_m (and forgetting the U_i for $n+1 < i < m$), we can arrange that we have a nested collection of open sets with compact closures, and $\overline{U}_n \subset U_{n+1}$ for all n . The sets $\Omega_n = \overline{U}_n - U_{n-1}$ (with $\Omega_1 = \overline{U}_1$) are compact and $U_n \cap \Omega_m = \emptyset$ for $m > n$.⁽⁹⁾ The picture is as follows:

⁹Since M is the union of the compact sets Ω_i we see that M is σ -compact.



We finish the proof of the lemma as follows: For each $x \in \Omega_n$ choose a coordinate neighbourhood $(V_{x,n}, \varphi_{x,n})$ containing x and contained in $\bar{U}_{n+1} - U_{n-2}$. Let $B_{x,n}$ be a compact subset of $\varphi_{x,n}(V_{x,n})$ with nonempty interior and $\varphi_{x,n}(x) \in \text{int}(B_{x,n})$ (e.g. a ball in \mathbb{R}^n centred at $\varphi_{x,n}(x)$). The sets $\varphi_{x,n}^{-1}(\text{int}(B_{x,n}))$ cover Ω_n and hence we can choose a finite subcovering, which we call $\{W_\beta\}_{\beta \in B(n)}$ (here $B(n)$ is the finite index set). Then the corresponding $V_{x,n}$ also form a finite covering of Ω_n , also indexed by $B(n)$. Taking the union over $n \geq 1$ defines the locally finite refinement $\{V_\beta\}_{\beta \in B}$ indexed by $B = \cup_n B(n)$ and its refinement $\{W_\beta\}_{\beta \in B}$ as in the lemma. \square

Remark 2.5.2. The proof of the lemma shows the following: A second-countable and locally compact Hausdorff space is paracompact. (Indeed, in the proof just drop the requirement that V_β is a chart and drop the W_β entirely.) In particular, an open subset of a smooth manifold is again a smooth manifold (since this remark ensures paracompactness).

Lemma 2.5.3. *If B is a closed ball in \mathbb{R}^n then there is a smooth bump function supported exactly on B , i.e. a smooth function $f : \mathbb{R}^n \rightarrow [0, 1]$ with $\{f \neq 0\} = \text{int}(B)$.*

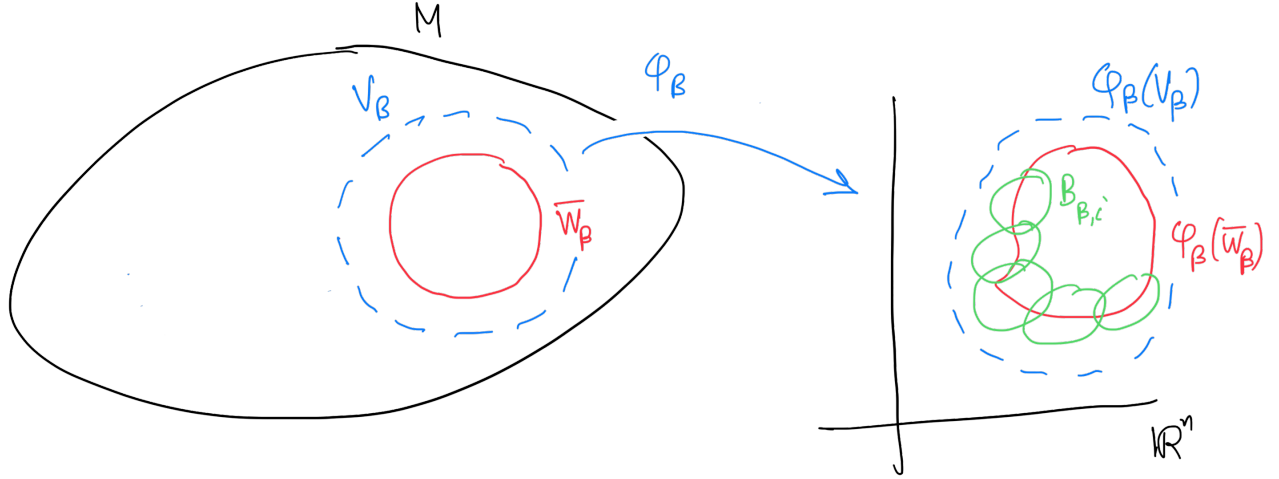
Proof. Exercise. \square

Proposition 2.5.4. *A smooth manifold has a smooth partition of unity subordinate to any open cover. In other words, if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of M then there exist smooth functions $f_\alpha : M \rightarrow [0, 1]$ such that*

- (i) $\text{supp}(f_\alpha) = \text{closure of } \{f_\alpha \neq 0\}$ is contained in U_α
- (ii) $\{\text{supp}(f_\alpha)\}_{\alpha \in A}$ is locally finite
- (iii) $\sum_\alpha f_\alpha \equiv 1$

(Note that (iii) ensures that $\{\{f_\alpha \neq 0\}\}_\alpha$ is an open cover and (i) implies that it is a locally finite refinement of $\{U_\alpha\}_\alpha$ (with the same indexing set).)

Proof. Note that if a refinement of an open covering has a partition of unity subordinate to it then so does the original covering. Indeed, if $\{U_\alpha\}_{\alpha \in A}$ is refined by $\{V_\beta\}_{\beta \in B}$ and f_β is the partition of unity subordinate to $\{V_\beta\}_{\beta \in B}$ then for any function $I : B \rightarrow A$ such that $V_\beta \subset U_{I(\beta)}$, $g_\alpha = \sum_{\beta \in I^{-1}(\alpha)} f_\beta$ defines a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in A}$. So it is enough to construct a partition of unity for a refinement.



Starting with an arbitrary open cover $\{U_\alpha\}_\alpha$ we can take a locally finite refinement by charts $\{(V_\beta, \varphi_\beta)\}_\beta$ and a refinement $\{W_\beta\}$ with $\overline{W}_\beta \subset V_\beta$ compact as in the earlier lemma. For each $\beta \in B$ cover the compact subset $\varphi_\beta(\overline{W}_\beta)$ of \mathbb{R}^n by finitely many balls $B_{\beta,1}, \dots, B_{\beta,r}$ which are contained in $\varphi_\beta(V_\beta)$. Choose C^∞ bump functions $f_{\beta,i}$ for $i = 1, \dots, r$ as in the previous lemma, i.e. with $f_{\beta,i}(x) \neq 0 \Leftrightarrow x \in \text{int}(B_{\beta,i})$. Let $f_\beta = \sum_i f_{\beta,i}$. This is a function from \mathbb{R}^n to $[0, \infty)$ which is > 0 on $\varphi_\beta(\overline{W}_\beta)$ and which vanishes outside $\cup_i B_{\beta,i}$. Thus we can consider it as a smooth function on M which is zero outside V_β and is given by $f_\beta(\varphi_\beta(x))$ for $x \in V_\beta$ (and so is > 0 on W_β). Then $\text{supp}(f_\beta) \subset V_\beta$ and $0 < \sum_\beta f_\beta < \infty$ on the whole of M (> 0 because W_β cover M and $< \infty$ because $\{V_\beta\}_\beta$ is locally finite) and then dividing each one by $\sum_\beta f_\beta$ gives the partition of unity subordinate to $\{V_\beta\}$. By the remark of the previous paragraph we are finished. \square

We will also need to use the following later:

Lemma 2.5.5. *Let M be a smooth manifold, $C \subset M$ a closed subset, and $g : C \rightarrow \mathbb{R}$ a function. Suppose that g satisfies:*

- (*) *for each $p \in C$, there is a neighbourhood U of p in M and a smooth function \tilde{g} on U such that $\tilde{g}|_C = g$.*

Then there exists a smooth function $\tilde{g} : M \rightarrow \mathbb{R}$ such that $\tilde{g}|_C = g$.

Proof. Let U_α be a covering by charts such that for each point p , the neighbourhood U in property (*) contains a U_α containing p . Then we have a smooth function g_α on each U_α which restricts to g on $C \cap U_\alpha$. Let f_α be a partition of unity subordinate to the U_α . The function $\tilde{g} = \sum_\alpha g_\alpha f_\alpha$ then agrees with g on C and is smooth on M . \square

Another application of partitions of unity:

Proposition 2.5.6. *If $C \subset M$ is a closed subset of a smooth manifold then there exists a smooth function $f : M \rightarrow \mathbb{R}$ such that $C = f^{-1}(0)$.*

Proof. First consider the case where $M = U$ is an open subset of \mathbb{R}^n . The complement $U - C$ is a countable union of open balls B_i . Choose smooth functions $f_i : U \rightarrow [0, \infty)$ such that (1) $f_i(x) > 0 \Leftrightarrow x \in B_i$ and (2) all derivatives of f_i of order $\leq i$ are bounded on B_i (uniformly) by $1/2^i$. (The second condition can always be achieved by multiplying by a scalar.) Now $f = \sum_i f_i$ converges uniformly on U , so that the function f is smooth. Obviously $f(x) = 0$ iff $x \in C$.

Now consider the general case. Take a locally finite covering by charts $\{U_\alpha\}$ and a partition of unity $\{g_\alpha\}$ subordinate to it. For each α the intersection $C \cap U_\alpha$ is a closed subset so using the previous case let f_α be a smooth function on U_α with $f_\alpha^{-1}(0) = C \cap U_\alpha$. Consider the sum $f = \sum_\alpha g_\alpha f_\alpha$, which makes sense as a smooth function on M . Then $f^{-1}(0) = C$. \square

This shows that the preimage of a point under a smooth map can be arbitrarily bad. We will soon see a condition ensuring that the preimage is good, e.g. a manifold.

(Nothing in this section holds for real-analytic or complex manifolds as there cannot be real-analytic or holomorphic partitions of unity on them, for obvious reasons.)

2.6. Tangent and cotangent spaces. Let M be a smooth manifold and $p \in M$.

A **germ of a smooth function at p** is an equivalence class of pairs (U, f) consisting of an open neighbourhood U of p and a smooth function f on U , modulo the equivalence relation that $(U, f) \sim (V, g)$ if there is an open neighbourhood $W \subset U \cap V$ on which f and g agree. Let G_p be space of germs of smooth functions at p . This is an \mathbb{R} -algebra under multiplication of functions. The ideal \mathfrak{m}_p of germs vanishing at p has codimension one, i.e. $G_p/\mathfrak{m}_p = \mathbb{R}$ (by evaluation at p). The subspace of germs with vanishing first partial derivatives (in some local coordinate system) at p is denote S_p and called the space of **stationary germs**. (It does not depend on the coordinate system because under a local change of coordinates the first partial derivatives change by the Jacobian matrix of the coordinate change, so they vanish in any other coordinate system).

The **cotangent space** to M at p is the vector space

$$T_p^*M := G_p/S_p = \mathfrak{m}_p/\mathfrak{m}_p \cap S_p$$

of germs modulo stationary germs (or germs vanishing at p modulo stationary germs vanishing at p). The **tangent space** to M at p is the dual vector space

$$T_pM := (T_p^*M)^*.$$

Lemma 2.6.1. *Let x_1, \dots, x_n be local coordinates centred at $p \in M$. Any germ $f \in \mathfrak{m}_p$ can be written as $f = \sum_{i=1}^n x_i g_i$ for some smooth germs g_i with $\frac{\partial f}{\partial x_i}(0) = g_i(0)$. If f is also stationary, i.e. $f \in \mathfrak{m}_p \cap S_p$ then the g_i can be chosen to vanish at 0, i.e. $g_i \in \mathfrak{m}_p$, so that $\mathfrak{m}_p \cap S_p = \mathfrak{m}_p^2$.*

Proof. Since we are dealing with germs we can always restrict to neighbourhoods of p which, in the local coordinates, are starlike in \mathbb{R}^n . ($U \subset \mathbb{R}^n$ is starlike if it is invariant under dilation by elements of $[0, 1]$.) Then one can write, for $(x_1, \dots, x_n) \in U$

$$\begin{aligned} f(x_1, \dots, x_n) &= \int_0^1 \frac{d}{dt} f(tx_1, \dots, tx_n) dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n) \cdot x_i dt \\ &= \sum_{i=1}^n x_i \cdot \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n) dt \end{aligned}$$

Both assertions of the lemma follow. \square

This lemma implies that the images of the x_i give a basis of T_p^*M . (Indeed, for $f \in G_p$ write $f - f(p) = \sum_i x_i g_i = \sum_i x_i (g_i - g_i(p)) + \sum_i x_i g_i(p)$. The germ $f(p) + \sum_i x_i (g_i - g_i(p))$ is stationary. They are obviously linearly independent.) The dual basis of T_pM consists of the linear functionals on germs $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ defined by:

$$\frac{\partial}{\partial x_i}(f) = \frac{\partial f}{\partial x_i}(0).$$

Summarizing:

Lemma 2.6.2. *The tangent space T_pM is a vector space of dimension n . If x_1, \dots, x_n are local coordinates centred at p then it is spanned by the tangent vectors $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ which are defined by*

$$\frac{\partial}{\partial x_i}(f) = \frac{\partial f}{\partial x_i}(0).$$

*The dimension of the cotangent space T_p^*M is n and the dual basis to $\left\{ \frac{\partial}{\partial x_i} \right\}$ is given by the germs of the coordinate functions x_i .*

Remark 2.6.3. It follows that an equivalent description of the tangent space is

$$T_pM = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*.$$

This definition (called the Zariski tangent space) is carried over in algebraic geometry for algebraic varieties.

A **derivation** on M at p is a linear map $D : G_p \rightarrow \mathbb{R}$ which satisfies the Leibniz rule: for two germs f and g , $D(fg) = f(p)D(g) + D(f)g(p)$. It is clear that D must vanish on the constant germs, and also on \mathfrak{m}_p^2 . So it is determined by its restriction to \mathfrak{m}_p and gives an element of the tangent space. Let Der_p be the space of derivations at p . We have defined an injective map

$$Der_p \rightarrow T_pM.$$

It follows also that $D = \sum_i a_i \frac{\partial}{\partial x_i}$ in local coordinates, where $a_i = D(x_i)$. This is an isomorphism because $\left\{ \frac{\partial}{\partial x_i} \right\}$ is exactly the dual basis to the basis $\{x_i\}$ of $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Another description of the tangent space is the most intuitive. A smooth map $\gamma : \mathbb{R} \rightarrow M$ is called a **smooth curve in M** . Suppose that $\gamma(0) = p$. Then we can define a derivation D_γ as follows: For any smooth germ f at p , let

$$D_\gamma(f) = \left. \frac{d}{dt} \right|_{t=0} f \circ \gamma.$$

(i.e. pullback using the map γ and then take the derivative at 0.) We will call this derivation the **tangent vector to γ at p** . If we look at smooth curves in \mathbb{R}^n this corresponds to the usual notion of “directional derivative along the curve γ at p ”. By choosing local coordinates at p we know that all derivations are linear combinations of the $\frac{\partial}{\partial x_i}$, and $\frac{\partial}{\partial x_i} = D_{\gamma_i}$ where γ_i is the curve given by $\gamma_i(t) = (0, \dots, 0, t, 0, \dots, 0)$ in the local coordinates. So we have:

Lemma 2.6.4. *Any derivation at p is of the form D_γ for some curve γ , so that every element of the tangent space at p is the tangent to a smooth curve.*

Remark 2.6.5. It is useful to keep in mind the following fact: If $V \subset \mathbb{R}^n$ is an open set, then choosing coordinates x_1, \dots, x_n , gives tangent vectors $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ form a basis for the tangent space T_pV at any point $p \in V$, and give an isomorphism $T_pV \cong \mathbb{R}^n$. So all tangent vectors at different points in U can be thought of as belonging to the same \mathbb{R}^n . (You are already used to this, you often compare vectors at different points in \mathbb{R}^n by “translation to the origin”.) A more invariant way of saying this is the following: If V is a finite-dimensional real vector space, considered as a manifold via any linear isomorphism $V \cong \mathbb{R}^n$, then there are canonical identifications $T_vV = V$ for any $v \in V$.

2.7. Functoriality. Given a smooth map $f : M \rightarrow N$ between smooth manifolds, there is a linear map of vector spaces $df_p : T_pM \rightarrow T_{f(p)}N$ for each $p \in M$. This can be seen in any of the equivalent descriptions above, but let us see it using derivations. Given $D \in \text{Der}_p$, define a derivation $df(D) \in \text{Der}_{f(p)}$ by $a \mapsto D(a \circ f)$ for a germ a at $f(p)$. Check that this is a derivation.

What is the map df_p in more concrete terms? If you write it out in local coordinates x_1, \dots, x_m near p and y_1, \dots, y_n near $f(p)$, and use the bases $\frac{\partial}{\partial x_i}$ of T_pM and $\frac{\partial}{\partial y_i}$ of $T_{f(p)}N$, then df_p is just the Jacobian matrix of partial derivatives, i.e.

$$(df_p)_{ij} = \frac{\partial f_i}{\partial x_j}$$

where $f = (f_1, \dots, f_n)$ are the component functions of f , i.e. $f_i = y_i \circ f$.

From this local calculation and the chain rule it is clear that if $g : N \rightarrow X$ is another smooth map, then $d(g \circ f)_p = dg_{f(p)} \circ df_p$.

Note that a diffeomorphism $f : M \rightarrow N$ induces an isomorphism $df_p : T_pM \rightarrow T_{f(p)}N$, so they must be of the same dimension. In particular, \mathbb{R}^n and \mathbb{R}^m are not diffeomorphic if $m \neq n$, so that the dimension of a manifold is a diffeomorphism invariant. (Note that to prove that \mathbb{R}^n and \mathbb{R}^m are nonhomeomorphic for $n \neq m$ (“invariance of domain”) requires nontrivial algebraic topology arguments.)

Note that if $\gamma : \mathbb{R} \rightarrow M$ is a smooth curve in M with $\gamma(0) = p$ then in local coordinates x_i around p ,

$$d\gamma_0 \left(\frac{\partial}{\partial t} \right) = D_\gamma = \sum_i \frac{d\gamma_i}{dt} \Big|_{t=0} \frac{\partial}{\partial x_i}$$

where $\gamma_i(t) = x_i(\gamma(t))$.

A geometric way to think about the map $df_p : T_pM \rightarrow T_{f(p)}N$ is using tangents to smooth curves. The map df_p simply sends the tangent vector to $\gamma : (-\epsilon, \epsilon) \rightarrow M$ at $\gamma(0) = p$ to the tangent vector to $f \circ \gamma : (-\epsilon, \epsilon) \rightarrow N$ at $f(p)$.

2.8. Tangent and cotangent bundles. Let TM be the set of all pairs (p, v) where $p \in M$ and $v \in T_pM$ is a tangent vector at p . There is a map $\pi : TM \rightarrow M$ by $(p, v) \mapsto p$. The space TM is called the **tangent bundle** of M ; let us see that it is itself a smooth manifold of dimension $2n$ ($n = \dim(M)$). For this we need to put a topology (second countable, Hausdorff and paracompact) and provide an smooth atlas of charts. Choose an atlas for M and let $\varphi : U \rightarrow V \subset \mathbb{R}^n$ be a chart of the atlas giving local coordinates x_1, \dots, x_n . A tangent vector at any point of U is then of the form $v = \sum_i q_i(v) \frac{\partial}{\partial x_i}$. This gives a homeomorphism $\pi^{-1}(U) \cong U \times \mathbb{R}^n \cong V \times \mathbb{R}^n$ by sending (p, v) to $(x_1(p), \dots, x_n(p), q_1(v), \dots, q_n(v))$. Put on TM the topology generated by the open sets of the form $W \times W'$ where $W \subset \pi^{-1}(U)$ and $W' \subset \mathbb{R}^n$ are open. (You must check that finite intersections of such sets are again of this form.) It is easy to see that TM is second countable, Hausdorff and locally compact (and hence also paracompact). Moreover there is an obvious atlas and the transition maps are smooth, so that the sets $\pi^{-1}(U)$ give a smooth atlas for TM , making it into a smooth manifold of dimension $2n$.

Given a smooth map $f : M \rightarrow N$ there is an induced map on the tangent bundle $Tf : TM \rightarrow TN$ defined by $Tf(p, v) = (f(p), df_p(v))$. It is smooth. If $g : N \rightarrow W$ is another smooth map then it satisfies $T(gf) = Tg \circ Tf$. So T is a functor.

Exercise: Check that $T_{(p,q)}(M \times N) = T_pM \times T_qN$ (natural isomorphism of vector spaces) and $T(M \times N) = TM \times TN$ (natural diffeomorphism of manifolds).

The tangent bundle of \mathbb{R}^n is identified with $\mathbb{R}^n \times T_0\mathbb{R}^n \cong \mathbb{R}^{2n}$ by the remark earlier.

Similar remarks apply to the **cotangent bundle**, denoted T^*M .

2.9. C^k -manifolds. Instead of requiring the transition functions of an atlas to be smooth, one can require them to be only C^k for some k . The formal definition is the following: A C^k -**atlas** on a space M is a covering of M by charts $\{(U_\alpha, \varphi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n)\}_\alpha$ such that if $U_\alpha \cap U_\beta \neq \emptyset$ then the transition function $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is a C^k map between open subsets of \mathbb{R}^n . (Recall that C^k means that all derivatives of order $\leq k$ exist and are continuous.) A C^k -**manifold** is a pair of a space M and a maximal C^k -atlas on M (as before, the maximal atlas is unique). The atlas allows us to talk about the C^k functions on a C^k manifold, and also C^k maps between C^k manifolds.

In general, if $k \geq 1$ things work quite similarly to the C^∞ case. (In particular, it is a theorem that every C^k ($k \geq 1$) atlas on a C^k manifold contains at least one C^∞ atlas. It is also a theorem that if two C^∞ manifolds are diffeomorphic as C^k manifolds for $k \geq 1$ then they are actually smoothly diffeomorphic.)

If $k = 0$ a C^0 manifold is usually called a **topological manifold** and a C^0 map is simply a continuous map. The world of topological manifolds is quite different from the smooth case. (For example, it is known that there are topological manifolds which admit no smooth atlas at all! Also, two smooth manifolds may be homeomorphic without being smoothly diffeomorphic. Unfortunately the simplest examples are already nonelementary.)

Note that for $k \geq 1$ it makes sense to talk about the tangent space and tangent bundle to a C^k manifold: It is defined as before as germs of C^k functions modulo stationary germs of C^k functions. The same proof as above shows that the tangent space $T_p M$ to a n -dimensional C^k manifold has dimension n . A C^k map $f : M \rightarrow N$ induces a linear map $df : T_p M \rightarrow T_{f(p)} N$ which, in local coordinates, is given by the Jacobian.

From now on we will only work in the smooth category, so that the words “manifold, function, mapping” etc. will always refer to smooth things, unless indicated otherwise.

Remark 2.9.1. A useful fact which we will not prove is the following: Let $f : M \rightarrow N$ be a continuous map between C^k manifolds for $k \geq 1$ or $k = \infty$. If C is a closed subset such that f is C^k on some open neighbourhood of C then there exists a continuous map $F : [0, 1] \times M \rightarrow M$ such that $F|_{\{0\} \times M} = f$, $F|_{\{1\} \times M}$ is C^k , and $F|_{\{t\} \times C} = f|_C$ for all $0 \leq t \leq 1$. In particular, taking C to be empty, we see that any continuous map between smooth manifolds is homotopic to a smooth map. In particular, every homotopy class of loops in a smooth manifold contains a smooth loop.

Remark 2.9.2 (Tangent bundle of a complex manifold). Let X be a complex manifold and $x \in X$. The tangent space $T_x X$ is then a complex vector space, i.e. it is a real vector space of dimension $2n$ which has an \mathbb{R} -linear mapping $J_x : T_x X \rightarrow T_x X$ such that $J^2 = -Id$. (The operator J_x is multiplication by $\sqrt{-1} \in \mathbb{C}$.) The operator J_x varies smoothly in x in an obvious sense.

The complexified tangent space

2.10. Inverse and implicit function theorems. In addition to the chain rule we will need to use the inverse and implicit function theorems of multivariable calculus. These show that the linear approximation to a map (i.e. its derivative) controls the behaviour of the map up to change of coordinates, under an assumption on the derivative (basically that it should have the maximal possible rank). (For proofs of these see e.g. the book *Ordinary Differential Equations* by Arnol’d.)

Inverse function theorem: Let $f : U \rightarrow V$ be a map between open subsets U, V of \mathbb{R}^n which is smooth, i.e. all derivatives exist and are continuous. f is given by n differentiable functions f_1, \dots, f_n , each of n variables. Let $p \in U$ be a point at which the Jacobian matrix

$$df_p = \left(\frac{\partial f_i}{\partial x_j} \Big|_p \right)$$

is nonsingular. Then there is a neighbourhood $U' \subset U$ of p such that $f : U' \rightarrow V = f(U')$ is an isomorphism, i.e. the inverse function $g : V \rightarrow U'$ defined by $g(y) = x$ if $f(x) = y$ is smooth and has derivative $dg_y = (df_{g(y)})^{-1}$ for $y \in V$.

If you are given n functions f_1, \dots, f_n of $n + m$ variables $x_1, \dots, x_n, y_1, \dots, y_m$ you can ask whether in the system of n equations

$$\begin{aligned} f_1(x_1, \dots, x_n, y_1, \dots, y_m) &= 0 \\ &\vdots \\ &\vdots \\ f_n(x_1, \dots, x_n, y_1, \dots, y_m) &= 0 \end{aligned}$$

it is possible to solve for the variables x_1, \dots, x_n in terms of the y_1, \dots, y_m . The implicit function theorem gives a sufficient condition to do this:

Implicit function theorem: Suppose we are given n differentiable functions f_1, \dots, f_n of the $n + m$ variables $x_1, \dots, x_n, y_1, \dots, y_m$. If $(p, q) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $f(p, q) = 0$ and if the matrix

$$d(f|_{\mathbb{R}^n \times \{q\}})_{(p,q)} = \left(\frac{\partial f_i}{\partial x_j} \Big|_{(p,q)} \right)_{i=1, \dots, n; j=1, \dots, n}$$

is nonsingular at (p, q) then there exist open neighbourhoods U of q in \mathbb{R}^m and V of p in \mathbb{R}^n and a (unique, differentiable) map $g : U \rightarrow V$ such that $g(q) = p$ and $f(g(y), y) = 0$ for all $y \in U$. The derivative of g is given by

$$dg_y = - \left(\frac{\partial f_i}{\partial x_j} \Big|_{(g(y), y)} \right)^{-1} \left(\frac{\partial f_i}{\partial y_j} \Big|_{(g(y), y)} \right) = - (d(f|_{\mathbb{R}^n \times \{y\}})_{(g(y), y)})^{-1} (d(f_{\{g(y)\} \times \mathbb{R}^n})_{(g(y), y)})$$

for $y \in U$. Moreover, there is a neighbourhood W of (p, q) such that

$$W \cap \{(x, y) : f(x, y) = 0\} = \{(g(y), y) : y \in U\},$$

i.e. every solution of $f(x, y) = 0$ near (p, q) is of the form $(g(y), y)$.

These two theorems are equivalent: To show that the implicit function theorem implies the inverse function theorem, suppose $f : U \rightarrow V$ is a smooth map between open subsets of \mathbb{R}^n with df_p (=the Jacobian matrix at p) nonsingular. Consider the mapping $F : U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $F(x, y) = y - f(x)$. The n functions $F_i(x_1, \dots, x_n, y_1, \dots, y_n) = y_i - f_i(x_1, \dots, x_n, y_1, \dots, y_n)$ of $2n$ variables satisfy the hypotheses of the implicit function theorem. Therefore there is a mapping $g = (g_1, \dots, g_n)$ from an open neighbourhood of $f(p) \in \mathbb{R}^n$ to the open subset U such that $F(g(y), y) = 0$, i.e. $f(g(y)) = y$, i.e. $f \circ g$ is the identity.

Conversely, given a smooth mapping $f : U \rightarrow V$ with $U \subset \mathbb{R}^{n+m}$ and $V \subset \mathbb{R}^n$, we will apply the inverse function theorem to $F : U \rightarrow \mathbb{R}^{n+m}$ given by $F(x, y) = f(x, y) + y$ where $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and $(x, y) \in U$. The differential $dF_{(x,y)}$ is block upper-triangular:

$$dF_{(x,y)} = \left(\begin{array}{cc} \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i, j \leq n} & * \\ 0 & I_m \end{array} \right)$$

where I_m denotes the $m \times m$ identity. So under the hypothesis of the implicit function theorem it is invertible at (p, q) , and so by the inverse function theorem F has a local inverse, i.e. there is a neighbourhood W of (p, q) in which it has a smooth inverse G . Now let $g(y) = G(0, y)$, defined for y in the open neighbourhood $U = \{y \in \mathbb{R}^m : (0, y) \in W\}$ of q . The identities $FG = id$ and $GF = id$ lead to the conclusions of the theorem.

The following immediate consequence of the inverse function theorem is useful: A bijective smooth map $f : M \rightarrow N$ such that df_p is an isomorphism everywhere is a diffeomorphism.

Remark 2.10.1. The real-analytic version of the inverse and implicit function theorems hold, e.g. a real-analytic map with nonsingular differential at a point has real-analytic inverse in a neighbourhood of the point. Similarly, the complex-analytic also holds, i.e. a holomorphic map between open subsets of \mathbb{C}^n with nonsingular differential has a holomorphic inverse. Also the version with “smooth” replaced by C^r for any $r \geq 1$.

2.11. Example: Hypersurfaces in \mathbb{R}^{m+1} . Let $f : \mathbb{R}^{1+m} \rightarrow \mathbb{R}$ be a smooth function. When is the zero-set $M := \{f(x_0, \dots, x_m) = 0\}$ a smooth manifold? (This is a special case of a more general situation discussed below.) It might be useful to keep in mind a situation like the function $f(x, y, z) = x^2 + y^2 - z^2$ in \mathbb{R}^3 . For each point $p \in M$ we must produce a chart containing p , so that the collection of charts form a smooth atlas. If the partial derivative

$$\frac{\partial f}{\partial x_0}(p) \neq 0$$

then by the implicit function theorem there is a mapping g from an open neighbourhood U of (p_1, \dots, p_m) to an open neighbourhood of $p_0 \in \mathbb{R}$, i.e. a smooth function $g(x_1, \dots, x_m)$ with $f(g(x_1, \dots, x_m), x_1, \dots, x_m) = 0$. Now the map $U \rightarrow M$ by

$$(x_1, \dots, x_m) \mapsto (g(x_1, \dots, x_m), x_1, \dots, x_m)$$

is a chart at p . On the other hand, if $\frac{\partial f}{\partial x_0}(p) = 0$ but $\frac{\partial f}{\partial x_i}(p) \neq 0$ for $i > 0$ then we can do the same argument with x_i playing the role of x_0 to produce a chart at p . This means that for every $p \in M$ such that

$$\nabla f(p) := \left(\frac{\partial f}{\partial x_0}(p), \frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_m}(p) \right) \neq (0, \dots, 0)$$

we have produced a chart at p . I leave it to you to check that this is a smooth atlas. So M is a manifold if $\nabla f(p) \neq 0$ at every point of M .

- Show that the tangent space map $df_p : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ is given by $v \mapsto \nabla f(p) \cdot v$.

Now we can use this to show that several standard examples are smooth manifolds: The sphere S^n is the zero-set of $f(x_0, \dots, x_n) = x_0^2 + x_1^2 + \dots + x_n^2 - 1$. Since $\nabla f = 2(x_0, \dots, x_n)$ is nonzero when $f = 0$, this is a smooth manifold.

2.12. Immersions and submersions. Consider a smooth map $f : M \rightarrow N$ of smooth manifolds of dimension m and n respectively. We say:

- f is an **immersion** at $p \in M$ if $df_p : T_p M \rightarrow T_{f(p)} N$ is injective
- f is a **submersion** at $p \in M$ if $df_p : T_p M \rightarrow T_{f(p)} N$ is surjective.

These are both open conditions, i.e. if they hold at a point p then they must hold in a neighbourhood of the point. (To see this, if f is an immersion at p , then df_p , considered as a linear transformation from $T_p M \cong \mathbb{R}^m$ to $T_{f(p)} N \cong \mathbb{R}^n$ is an $n \times m$ matrix with rank m . An $n \times m$ matrix has rank m if and only if it has a nonsingular $m \times m$ submatrix. Since determinant is continuous in the entries, that same submatrix is nonsingular at points close to p , which means that the rank of df_q is still m if q is close to p . The same reasoning applies to submersions, because f is a submersion at p means df_p has rank n .)

Example 2.12.1. Consider the example of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by $f(x_1, \dots, x_n) = -x_1^2 + x_2^2 + \dots + x_n^2$. The map df_p at $p = (x_1, \dots, x_n)$ is the $1 \times n$ matrix $(-2x_1, 2x_2, \dots, 2x_n)$, so f is a submersion at all points except $0 \in \mathbb{R}^n$. Note also that $f^{-1}(t)$ is a manifold except at $t = 0$. We will see below that this happens exactly because f is submersive at all points of $f^{-1}(t)$ if $t \neq 0$.

We say that $f : M \rightarrow N$ is an **immersion** (resp. a **submersion**) if it is an immersion (resp. submersion) at all $p \in M$.

2.13. Submanifolds. Let M be a smooth manifold. There are various possible notions of submanifold which need to be distinguished.

A **submanifold** is a pair (W, i) consisting of a smooth manifold W and a smooth map $i : W \rightarrow M$ which is a 1-1 immersion. (This is sometimes also called an immersed submanifold.)

A submanifold $i : W \rightarrow M$ which induces a homeomorphism of M with the subspace $i(W) \subset N$ (with the induced topology) an **embedded submanifold** and we say that i is an **embedding**.

If $i : W \rightarrow M$ is a 1-1 immersion which is proper then $i : W \rightarrow M$ will be called a **closed submanifold**. (The terminology makes sense because a proper map between manifolds is necessarily closed, so that $i(W)$ is closed in N .) If a 1-1 immersion i is proper then $i : W \rightarrow i(W)$ is a homeomorphism, so that a closed submanifold is embedded. In particular, if W is compact then this holds, so a 1-1 immersion $i : W \rightarrow M$ with W compact is an embedding.

Remark 2.13.1. It is important to remember that a submanifold of M is not simply a subset of M . Here are some examples: For an irrational $\alpha \in \mathbb{R} - \mathbb{Q}$ consider the map $i : \mathbb{R} \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ by $t \mapsto (t, \alpha t) \bmod \mathbb{Z}^2$. This is an immersion with dense image (by Kronecker's theorem). It is evidently not an embedded submanifold since the induced topology on $i(W)$ is not the usual one. An example of an embedded submanifold which is not closed is $(0, 1) \subset \mathbb{R}$.

2.14. Normal forms for immersions and submersions. The inverse and implicit function theorems imply some properties of immersions and submersions, namely that they look standard, locally on the source of the map.

The first one says that an immersed submanifold can be "straightened out" by a suitable choice of local coordinates.

Proposition 2.14.1 (Local normal form for immersions). *Let $f : M^m \rightarrow N^n$ be an immersion at $p \in M$. Then there exist charts U at p and U' at $f(p)$ and systems of coordinates x_1, \dots, x_m on U and y_1, \dots, y_n on U' such that $f : U \rightarrow U'$ is given by $f(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$, i.e. f looks locally like the inclusion $\mathbb{R}^m \subset \mathbb{R}^n$.*

Proof. Choose local charts $\phi : U \rightarrow V \subset \mathbb{R}^m$ and $\psi : U' \rightarrow V' \subset \mathbb{R}^n$ at p and $f(p)$ respectively, so that $\phi(p) = 0$ and $\psi(f(p)) = 0$. We want to show that by changing the local coordinates on U' we can make $\psi \circ f \circ \phi^{-1}$ the map induced by the inclusion $i : \mathbb{R}^m \rightarrow \mathbb{R}^n$ as the first m coordinates. So we are in the following situation: $f : V \rightarrow V'$ is a smooth map between open neighbourhoods of 0 in \mathbb{R}^m and \mathbb{R}^n respectively. It is enough to find an open $W' \subset V'$ and a diffeomorphism $T : W' \rightarrow T(W')$ such that $T^{-1} \circ f' = i$ on W' .

First of all we can clearly assume that $df_0 : T_0V \rightarrow T_0V'$ is the inclusion $i : \mathbb{R}^m \rightarrow \mathbb{R}^n$. (Indeed, by a linear change of coordinates $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we can arrange this by replacing f by $S^{-1} \circ f$.) Now define a smooth mapping $T : V \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^n$ by

$$T(x, y) := f(x) + (0, y)$$

where $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ are coordinates on \mathbb{R}^n . At $y = 0$ we get the map f , so that $f = T \circ i$. Then

$$dT_{(0,0)}(x, y) = df_0(x) + (0, y) = (x, y)$$

so $dT_{(0,0)}$ is the identity $T_{(0,0)}(V \times \mathbb{R}^{n-m}) \rightarrow T_{(0,0)}\mathbb{R}^n = T_0V'$. By the inverse function theorem, there is a small neighbourhood $W' \subset V'$ of $(0, 0)$ on which there is a smooth inverse $T^{-1} : W' \rightarrow V \times \mathbb{R}^{n-m}$ and $T^{-1} \circ f = i$ on W' . \square

This implies (and is essentially the same as) the statement that any smooth function on M extends, locally on M , to a smooth function on N . Note that this is only true locally on M (consider the immersion $\mathbb{R} \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ given by a line of irrational slope, which is not an embedding). If M is an embedded submanifold of N , then it says that the smooth functions on M are exactly the restrictions of smooth functions from N .

Proposition 2.14.2 (Local normal form for submersions). *Let $f : M^m \rightarrow N^n$ be a submersion at $p \in M$. Then there is a chart U at p and a chart U' of N at $f(p)$ such that $f(U) \subset U'$ and $f|_U$ looks like the projection $\mathbb{R}^m = \mathbb{R}^{m-n} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.*

Proof. Since the question is local we may assume that f maps an open subset of \mathbb{R}^m to an open subset of \mathbb{R}^n . Write $\mathbb{R}^m = \mathbb{R}^{m-n} \times \mathbb{R}^n$ and write (x, y) for the coordinates in each factor. Let p be a point at which f is submersive. We may arrange by a change of coordinates in \mathbb{R}^m that $df_p|_{\{0\} \times \mathbb{R}^n}$ is nonsingular. Define $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by $F(x, y) = (x, f(x, y))$. The differential dF_p is nonsingular. By the inverse function theorem we know that F is invertible in a neighbourhood W of p . Let $W' \times V'$ be a product neighbourhood (w.r.t. $\mathbb{R}^m = \mathbb{R}^{m-n} \times \mathbb{R}^m$) in $F(W)$. Then $F^{-1} : W' \times V' \rightarrow W$ is given by $F^{-1}(x, y) = (x, g(x, y))$ for some smooth function g from m variables to n variables, and we have

$$(x, y) = F \circ F^{-1}(x, y) = F(x, g(x, y)) = (x, f(x, g(x, y)))$$

so that $f(x, g(x, y)) = y$, i.e. $fF^{-1}(x, y) = y$ and for $y \in V'$, $f^{-1}(y) \cap W$ is the graph of $W' \ni x \mapsto g(x, y)$. In particular, in the coordinates given by F^{-1} , f is the projection. \square

(Remark: Show that this proposition implies the implicit function theorem. This again shows that the inverse function theorem implies the implicit function theorem.)

Remark 2.14.3. A C^k map $f : M \rightarrow N$ induces a linear map $df : T_pM \rightarrow T_{f(p)}N$ which, in local coordinates, is given by the Jacobian. The notions of C^k immersion, C^k submersion are defined in the obvious way and they have normal forms as above (the same proofs given above work because for $k \geq 1$ one can replace the word “smooth” by “ C^k ” everywhere in the inverse and implicit function theorems and they remain true).

2.15. Critical point, critical value, regular value. Let $f : M \rightarrow N$ be a smooth map.

A point $p \in M$ is called a **critical point** for f if df_p has rank $< n = \dim N$, i.e. f is not a submersion at p .

A point $q \in N$ is called a **critical value** if $f^{-1}(q)$ contains a critical point.

The complement of the set of critical values is the set of **regular values**. Thus if $q \in N$ is a regular value and $p \in f^{-1}(q)$ then f is a submersion at p . (The fibre over a regular value is allowed to be empty.) For example, if $m < n$ then any point is critical and hence every point of $f(M)$ is a critical value.

The normal form for a submersion implies the following (fill in the necessary details):

Corollary 2.15.1. *If $f^{-1}(q) \neq \emptyset$ and $f : M^m \rightarrow N^n$ is a submersion at all points in $f^{-1}(q)$ (i.e., q is a regular value of f), then $f^{-1}(q)$ is an embedded submanifold of M of dimension $m - n$. The tangent space to $f^{-1}(q)$ at p is identified with the subspace $\ker(df_p : T_pM \rightarrow T_qN)$.*

Examples 2.15.2. Let us use the corollary to get some interesting examples of manifolds as “level sets” of smooth maps.

- (i) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by $f(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2$. Then f is submersive at all points of $f^{-1}(\lambda)$ for $\lambda > 0$, so that the fibres $f^{-1}(\lambda)$ are smooth manifolds of dimension $n - 1$, which are all diffeomorphic to the sphere S^{n-1} .
- (ii) For $f : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $f(x_1, \dots, x_n) = -x_1^2 + x_2^2 + \dots + x_n^2$ we showed that f is submersive at all points of $f^{-1}(t)$ for any $t \neq 0$. So $f^{-1}(t)$ is a manifold for $t \neq 0$.
- (iii) (Hypersurfaces) Let $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a smooth function. The induced map $\mathbb{R}^{n+1} \rightarrow \mathbb{R}$ on tangent spaces is given by $v \mapsto \nabla f(p) \cdot v$. (Check this.) So a level set $\{f = \lambda\}$ is a manifold if $\nabla f(p)$ is nonzero at all points $p \in \{f = \lambda\}$. This is exactly what we saw earlier. Note also that since M is a closed submanifold the tangent space T_pM is a subspace of $T_p\mathbb{R}^{m+1} = \mathbb{R}^{m+1}$. What is this subspace? By the corollary it is $T_pM = \{v \in \mathbb{R}^{m+1} : v \cdot \nabla f(p) = 0\}$.

- (iv) Let $GL(n, \mathbb{R})$ be the group of nonsingular $n \times n$ matrices with real entries. It is the complement in $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ of the closed set $\{\det = 0\}$. So it is a manifold. To show that

$$SL(n, \mathbb{R}) = \det^{-1}(1)$$

is a manifold we will compute the tangent map $d \det$ and show that $\det : GL(n, \mathbb{R}) \rightarrow \mathbb{R}^*$ has 1 as a regular value, which will also give us the tangent space as a subspace of $M(n, \mathbb{R})$.

Since $GL(n, \mathbb{R})$ is identified with an open subset of the real vector space $M(n, \mathbb{R})$ of all $n \times n$ matrices with real entries, its tangent space at the identity I is naturally $M(n, \mathbb{R})$. Then $T_I SL(n, \mathbb{R}) \subset T_I GL(n, \mathbb{R}) = M(n, \mathbb{R})$ is $\ker(d \det)$, so we must compute the differential $d \det : M(n, \mathbb{R}) \rightarrow \mathbb{R}$. To do so, for $A \in M(n, \mathbb{R})$ choose a path through I which will have as tangent vector a given matrix $A \in M(n, \mathbb{R})$. There are many ways to choose one; the easiest is $\gamma(t) = I + tA$. For small enough t this is invertible and the tangent vector to γ at I is A . To compute the image of A under $d \det$ we compute

$$\left. \frac{d}{dt} \right|_{t=0} \det(I + tA) = \operatorname{tr}(A).$$

So $d \det_I = \operatorname{tr} : M(n, \mathbb{R}) \rightarrow \mathbb{R}$ is surjective and $T_I SL(n, \mathbb{R}) = \ker(\operatorname{tr})$ is the space of trace zero matrices.

For $g \in SL(n, \mathbb{R})$ one can make a similar computation of the map $d \det_g : T_g GL(n, \mathbb{R}) \rightarrow T_1 \mathbb{R}^* = \mathbb{R}$ by using the curve $\gamma(t) = g + tA$. Another way to do this is to use the diffeomorphism left translation $L_{g^{-1}} : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ by $h \mapsto g^{-1}h$. Then $\det = \det \circ L_{g^{-1}}$ and hence $d \det_g = d \det_I \circ (dL_{g^{-1}})_g$ must also be surjective. (This argument also computes $T_g SL(n, \mathbb{R})$ as a subspace of $M(n, \mathbb{R})$.)

- (iv) Consider the map $f : GL(n, \mathbb{R}) \rightarrow M(n, \mathbb{R})$ given by $f(x) = {}^T x x$. It is surjective onto the space of symmetric positive definite matrices, and it is submersive at all points. (Exercise.) In particular, the identity matrix $I \in M(n, \mathbb{R})$ is a regular value, so the preimage $f^{-1}(I) = O(n)$ is a smooth manifold.
- (v) *Exercise:* Show that $SU(n)$ has a manifold structure by doing something similar to (iv).

2.16. Transversality, fibre products. Let $f : M \rightarrow N$ be a smooth map, and suppose that $W \subset N$ is an embedded submanifold. We say that f is **transverse along** W if

$$T_{f(p)}N = df_p(T_p M) + T_{f(p)}W \quad \text{for all } p \in f^{-1}(W).$$

(The sum need not be direct.) Roughly speaking, this says that the image of f and W are transverse (of course, the image of f need not be a manifold). Two special cases of this definition are:

- (1) if f is an embedding of M in N then we say M and W **meet transversely** if f is transverse along W . This is compatible with the definition of transverse intersection of topological manifolds given earlier.
- (2) if f is submersive at all points of $f^{-1}(q)$ for $q \in N$, then f is transverse along $\{q\}$. (In particular, a submersion is transverse along any submanifold of the target.)

The appropriate generalization of the previous corollary is

Proposition 2.16.1. *If $f : M^m \rightarrow N^n$ is a smooth map transverse along an embedded submanifold $W^w \subset N^n$ then $f^{-1}(W)$ is an embedded submanifold of M of dimension $m - n + w$.*

Exercise: Prove this by generalizing the proof of the previous proposition.

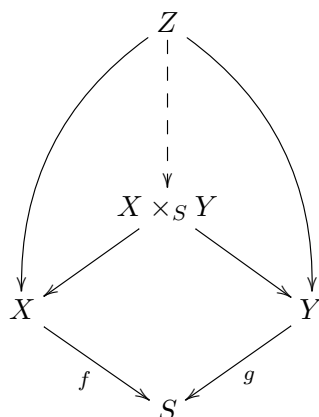
Example 2.16.2. Here are two simple examples illustrating the proposition: First consider the intersection in \mathbb{R}^3 of the surface $M = \{z = xy\}$ and the plane $W = \{z = \lambda\}$. It is easy to check that for $\lambda \neq 0$ the intersection is transverse and $M \cap W$ is a manifold. For $\lambda = 0$ the intersection fails to be transverse at $(0, 0, 0)$ (the two tangent spaces are the same plane) and the intersection is $\{xy = 0 = z\}$, which is a pair of crossing lines. This is not a manifold. Second consider the intersection in \mathbb{R}^3 of the manifolds given by equations $x^2 + y^2 + z^2 = 1$ (2-sphere)

and $x^2 + y^2 - z^2 = \lambda$ ($\lambda \neq 0$). For $\lambda \neq 1$ the intersection is transverse and consists of two circles. For $\lambda = 1$ the intersection is not transverse, but nonetheless it is a manifold, though not of the dimension given in the theorem.

Example 2.16.3 (Whitney's umbrella). This is the zero set of $x^2 - y^2z$ in \mathbb{R}^3 .

Example 2.16.4 (Whitney's cusp). This is the zero set $X \subset \mathbb{C}^3$ of $x^3 + y^2 = x^2z^2$.

The following categorical notion is important: Given a category \mathcal{C} and morphisms $X \rightarrow S, Y \rightarrow S$ in \mathcal{C} , the **fibre product** is an object $X \times_S Y$ with a morphism $X \times_S Y \rightarrow S$ in \mathcal{C} with the following universal property: Given Z in \mathcal{C} and maps $Z \rightarrow X$ and $Z \rightarrow Y$ such that $Z \rightarrow X \rightarrow S$ and $Z \rightarrow Y \rightarrow S$ coincide, there exists a unique map $Z \rightarrow X \times_S Y$ making the following diagram commute:



In the categorical language the fibre product is simply a limit, namely take the category I with three objects and morphisms $\bullet \rightarrow \bullet \leftarrow \bullet$ and consider functor $I \rightarrow \mathcal{C}$ which takes this to $X \rightarrow S \leftarrow Y$ and take the limit. The fibre product may or may not exist, but if it exists it is unique (as it satisfies a universal property).

Example 2.16.5. The category *Sets* has arbitrary fibre products, namely the set-theoretic fibre product $X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$. The category of subsets of a given set has arbitrary fibre products, which are just intersections.

What about the category *Man* of smooth manifolds? An easy consequence of the previous proposition is that the fibre product of $f : X \rightarrow S$ and $g : Y \rightarrow S$ in *Man* exists if either of f or g is a submersion. Indeed, look at $(f, g) : X \times Y \rightarrow S \times S$. This is transverse along the diagonal $\Delta_S \subset S \times S$ if either map is a submersion. By the previous proposition the preimage $(f, g)^{-1}(\Delta_S)$ is an embedded submanifold.

Exercise: Show that this is indeed a fibre product in *Man* by verifying the universal property.

Exercise: If $X \rightarrow S$ is a submersion and $T \rightarrow S$ is arbitrary map of smooth manifolds then $X \times_S T \rightarrow T$ is a submersion.

2.17. Whitney's embedding theorem. With our abstract definition of manifold it is not obvious whether every manifold can be smoothly embedded as a submanifold of a Euclidean space.

Theorem 2.17.1 (Whitney). *A compact smooth manifold can be embedded in \mathbb{R}^N for some N .*

Proof. We will use a partition of unity. From Lemma 2.5.1 we can assume the following: We have a finite (because M is compact) covering by charts $(V_i, \phi_i : V_i \rightarrow \mathbb{R}^n)$ and an open refinement W_i with $\overline{W}_i \subset V_i$ (here $i = 1, \dots, r$). Let $\{f_i\}_i$ be a partition of unity subordinate to the V_i with $f_i|_{\overline{W}_i} \equiv 1$ (as was constructed in the proof of existence of partitions of unity). Define a map $\Phi : M \rightarrow \mathbb{R}^{N=nr+r}$ by

$$\Phi = (f_1\phi_1, \dots, f_r\phi_r, f_1, \dots, f_r).$$

Then $d\Phi = (d(f_1\phi_1), \dots, d(f_r\phi_r), df_1, \dots, df_r)$. Since, for any $p \in M$, some f_i is identically one near p , and ϕ_i is a chart near p , we see that Φ is an immersion at p . So it is an immersion. Let us see that it is 1-1. If $\Phi(p) = \Phi(q)$ then $f_i(p) = f_i(q)$ for all i . In particular, there is some i for which these are both equal to one (since the W_i also cover M), and then p and q belong to the closure of W_i . But $\overline{W_i} \subset V_i$, so $p, q \in V_i$. But then $\phi_i(p) = \phi_i(q)$ and hence $p = q$.

This proves that Φ is a 1-1 immersion of M in \mathbb{R}^N . Since M is compact it is a homeomorphism onto its image, i.e. Φ is an embedding. \square

(We will improve this to $N = 2n + 1$ below using Sard's theorem.)

Whitney's embedding theorem tells us that two ways of defining manifolds – abstractly as we have or as submanifolds of Euclidean space – are equivalent. For example, this tells us that our manifolds are metric spaces. (Of course, this also follows from the Urysohn metrization theorem, which is much more general.)

Recall that $q \in N$ is a critical value of a smooth map $f : M \rightarrow N$ if there is a critical point in $f^{-1}(q)$, i.e. a point $p \in f^{-1}(q)$ such that f is not submersive at p .

Theorem 2.17.2 (Sard's theorem). *The set of critical values of a smooth map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is of (Lebesgue) measure zero.*

In particular, the set of critical values has no interior. The set of regular values is of full measure, so in particular it is nonempty. (There is another version of Sard's theorem which says that the set of critical values is nowhere dense, i.e. its closure has empty interior. This also implies that the set of regular values is nonempty, which is all we need below.)

We will only use the case $m < n$ below so let's only prove that case, which is very easy. If $m < n$ then every point is critical, so we must show that the image of a smooth mapping f is of measure zero.

Lemma 2.17.3. *Let $U \subset \mathbb{R}^n$ be open and let $f : U \rightarrow \mathbb{R}^n$ be smooth (C^1 is enough). If $C \subset U$ has measure zero then $f(C)$ has measure zero.*

Proof. It is enough to consider the case where the closure of C is contained in a compact set. (In general, we can write C as a countable union of subsets with closure contained in compact sets and prove that each has measure zero.) For C with closure contained in compact set $\Omega \subset U$, since df is continuous, there is a constant μ such that

$$|f(x) - f(y)| \leq \mu |x - y| \quad (x, y \in \Omega).$$

(Use the mean value theorem and let μ be the supremum of $\|df_x\|$ for $x \in \Omega$.) For a given $\epsilon > 0$, since C has measure zero, we may cover C by closed balls B_i (contained in Ω) such that $\sum_i \text{vol}(B_i) < \epsilon/\mu^n$. By the previous estimate $f(B_i)$ is contained in a ball B'_i of radius $\mu \cdot \text{radius}(B_i)$, hence $f(C) \subset \cup_i B'_i$ and $\sum_i \text{vol}(B'_i) \leq \mu^n \sum_i \text{vol}(B_i) < \epsilon$. Thus $f(C)$ has measure zero. \square

Proof of Sard's theorem for $m < n$. Given $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ apply the lemma to the composition of f with the projection $\mathbb{R}^n \rightarrow \mathbb{R}^m$, i.e. to the map $\mathbb{R}^m \times \mathbb{R}^{n-m} \ni (x, y) \mapsto f(x)$. Since the fibres of the projection have measure zero we see that the image of f has measure zero, which is what Sard's theorem says in this case. \square

(The general version of Sard's theorem requires a little more work, see any standard text for a proof.)

Now we say that a subset $Z \subset M$ of a smooth manifold has measure zero if there is a countable covering by charts $\{(U_i, \varphi_i : U_i \rightarrow \mathbb{R}^n)\}$ such that $\varphi_i(U_i \cap Z)$ has Lebesgue measure zero in \mathbb{R}^n . This notion is meaningful, i.e. independent of the choice of $\{U_i\}$, by the lemma above. With this definition, we have:

Corollary 2.17.4. *The set of critical values of a smooth map $f : M \rightarrow N$ is of measure zero. In particular, it has no interior.*

We will use the consequence that the set of regular values is nonempty if $m < n$.

Theorem 2.17.5 (Whitney). *A smooth compact n -manifold can be embedded in \mathbb{R}^{2n+1} and immersed in \mathbb{R}^{2n} .*

Proof. We start with an embedding $M \subset \mathbb{R}^N$ for some N . The idea is to use a linear projections $\mathbb{R}^N \rightarrow \mathbb{R}^{N-1}$ to reduce the dimension.

Suppose that there exists a nonzero vector $v \in \mathbb{R}^N$ such that

- (1) v is never a tangent vector to M , i.e. $v \notin T_p M$ for all $p \in M$. (We are identifying the tangent spaces of all points of \mathbb{R}^N with \mathbb{R}^N simultaneously.)
- (2) v is never a multiple of $p - q$ for $p, q \in M$.

Let $\pi : \mathbb{R}^N \rightarrow \mathbb{R}^{N-1}$ be orthogonal projection along v to $v^\perp \cong \mathbb{R}^{N-1}$. It is easy to check that $d\pi_p : T_p \mathbb{R}^N \rightarrow T_{\pi(p)} \mathbb{R}^{N-1}$ is simply the projection $\pi : \mathbb{R}^N \rightarrow \mathbb{R}^{N-1}$ under the identifications $T_p \mathbb{R}^N = \mathbb{R}^N$ and $T_{\pi(p)} \mathbb{R}^{N-1}$. Then (1) ensures that $d\pi_p$ is injective for $p \in M$, so that $\pi|_M : M \rightarrow \mathbb{R}^{N-1}$ is an immersion, while (2) ensures that it is 1-1. Since M is compact it is an embedding, i.e. a homeomorphism onto its image. So it remains to find such a vector v .

Note the following:

- (a) For a pair of points $p, q \in M$, $p \neq q$, let $\ell(p, q)$ denote the line in \mathbb{R}^N spanned by $p - q$. Define a map $f : M \times M - \Delta \rightarrow \mathbb{R}P^{N-1} = \mathbb{R}^N - \{0\}/\mathbb{R}^*$ by $(p, q) \rightarrow \ell(p, q)$. (Here $\Delta \subset M \times M$ denotes the diagonal $\{(p, p) : p \in M\}$.) This is evidently a smooth map. If $2n < N - 1$ then every point of $image(f)$ is critical, and so by Sard's theorem $image(f)$ is of measure zero. For a line $L \notin image(f)$, any $v \in L - \{0\}$ satisfies (1).
- (b) Define a map $g : TM - M \rightarrow \mathbb{R}P^{N-1}$ by $(p, v) \mapsto \mathbb{R}v$. If $2n < N - 1$, then by Sard's theorem $image(g)$ is of measure zero. For a line $L \notin image(f)$, any $v \in L - \{0\}$ satisfies (2).

So if $2n < N - 1$ the union $image(f) \cup image(g) \subset \mathbb{R}P^{N-1}$ is of measure zero, so there is a point in the complement, i.e. a line L in \mathbb{R}^N which is not in either image. Any nonzero vector $v \in L$ has properties (1) and (2) above, and so $\pi|_M : M \rightarrow v^\perp = \mathbb{R}^{N-1}$ is a 1-1 immersion, hence an embedding.

Now we repeat the argument, reducing the dimension N until $2n < N - 1$ fails, i.e. $N = 2n + 1$. This proves the $2n + 1$ embedding theorem.

For the immersion in \mathbb{R}^{2n} we argue a little more carefully with the map g . It is easy to see that g factors through a map $X \rightarrow \mathbb{R}P^{N-1}$ where $X = (TM - M)/\mathbb{R}^*$ is the quotient space. Now this quotient space is naturally a manifold of dimension $2n - 1$ in a way that makes the map g smooth. Thus if $2n - 1 < N - 1$ we know (by Sard's theorem) that the image is of measure zero. Projecting along a direction not in the image will still give an immersion. (Instead of this quotient you can put a Riemannian metric on M and take the unit sphere bundle in the tangent bundle for X and make the same argument.) \square

The argument in the proof proves slightly more: If an n -manifold has a finite cover by open neighbourhoods homeomorphic to \mathbb{R}^n then it has a 1-1 immersion in \mathbb{R}^{2n+1} . (The projection argument nowhere used compactness of M .) This will be used below in the proof of the embedding theorem for noncompact manifolds.

Remark 2.17.6 (Whitney's $2n$ embedding theorem). This argument cannot work to get an embedding in \mathbb{R}^{2n} since we have no control on the initial embedding. For example, if we start with $M = S^1$ we may, in the first step, end up with an embedding like the embedding of the trefoil knot in \mathbb{R}^3 . There is no vector in \mathbb{R}^3 along which we can project and get a 1-1 map to the plane. On the other hand, it is obvious that the circle can be embedded in \mathbb{R}^2 . So Whitney's theorem that M^n can be embedded in \mathbb{R}^{2n} requires a new idea, the famous "Whitney trick" in dimensions ≥ 5 . (For $n = 2$ it fails but it is elementary that any 2-manifold embeds in \mathbb{R}^3 if it is orientable and \mathbb{R}^4 otherwise.) We will not go into this here (see e.g. Milnor's *Notes on the h-cobordism theorem*).

Remark 2.17.7. Whitney proved some much stronger theorems: If $k \geq 2n + 1$ then the embeddings are dense (in a suitable topology) among all smooth maps $M^n \rightarrow \mathbb{R}^k$. Moreover, any two embeddings are isotopic. If $k = 2n$ then the immersions are dense and any two immersions are isotopic. (See R. Narasimhan's book *Analysis on real and complex manifolds* or his TIFR lecture notes *Topics in Analysis* for some of this. See also Whitney's original papers (1944).)

We can now prove the noncompact version of the embedding theorem.

Theorem 2.17.8 (Whitney). *A smooth manifold can be embedded in \mathbb{R}^N for some N .*

Proof. Recall that in the proof of Lemma 2.5.1 we showed that there are open sets U_n with $\bar{U}_n \subset U_{n+1}$ and \bar{U}_n compact such that $M = \cup_n U_n$. Using this we constructed countable open covers $\{V_\beta\}_{\beta \in B}$ and refinement $\{W_\beta\}_{\beta \in B}$ with $B = \cup_n B(n)$ such that $\{W_\beta\}_{\beta \in B(n)}$ is a finite cover of $\bar{U}_n - U_{n-1}$ and $\bar{W}_\beta \subset V_\beta \subset \bar{U}_{n+1} - U_{n-2}$ if $\beta \in B(n)$. For $i = 0, 1, 2$ let

$$B_i = \cup_{k \geq 0} B(i + 3k).$$

Then M has an open covering by three sets $M = M_0 \cup M_1 \cup M_2$ where

$$M_i = \cup_{\beta \in B_i} V_\beta = \cup_{k \geq 0} M_{i,k} \quad \text{where } M_{i,k} = \cup_{\beta \in B(i+3k)} V_\beta.$$

We will first immerse each M_i in a Euclidean space and then use the immersions to immerse M . To immerse M_i note that on each finite union $M_{i,k} = \cup_{\beta \in B(i+3k)} V_\beta$ one can define an immersion $\Phi_{i,k}$ using the construction in the compact Whitney theorem (all that was required was a finite cover as in Lemma 2.5.1 and then a partition of unity as in Proposition 2.5.4). Thus $M_{i,k}$ has an injective immersion $\Phi_{i,k}$ in \mathbb{R}^N and the image, being contained in the union of the finitely many compact sets $\Phi_{i,k}(\bar{W}_\beta)$ for $\beta \in B(i + 3k)$, is contained in some ball. Moreover, the projection argument as in the $2n + 1$ theorem says that we can take $N = 2n + 1$. Translating the immersion of $M_{i,k}$ by a multiple of a suitable fixed vector so that the images of the various $\Phi_{i,k}$ for $k \geq 0$ are disjoint, we get injective immersions $\Phi_i : M_i \rightarrow \mathbb{R}^{N=2n+1}$ for $i = 0, 1, 2$.

Now take a partition of unity $\{f_0, f_1, f_2\}$ subordinate to the covering $\{M_0, M_1, M_2\}$ and consider the map

$$\Phi := (f_0 \Phi_0, f_1 \Phi_1, f_2 \Phi_2, f_0, f_1, f_2).$$

It is easy to see that this is an injective immersion in \mathbb{R}^{3N+3} . It only remains to see that it is an embedding, which would follow if we show that the map Φ is closed. Now any closed subset $C \subset M$ can be written as a union $C = C_0 \cup C_1 \cup C_2$ where $C_i = \cup_{\beta \in B_i} C \cap (\bar{U}_i - U_{i-1})$. This is a disjoint union of compact sets, which maps under Φ to a disjoint union of compact sets which can be seen to be closed. \square

Once we have a closed embedding in $\mathbb{R}^{3(2n+1)+3}$, the same projection idea as in the compact case reduces the dimension to $2n + 1$ and $2n$ for an immersion. Thus we arrive at:

Theorem 2.17.9 (Whitney $2n+1$ embedding). *A smooth n -manifold admits a closed embedding in \mathbb{R}^{2n+1} and a closed immersion in \mathbb{R}^{2n} .*

Proof. The previous proof can be modified to get that any smooth manifold embeds in \mathbb{R}^{2n+1} by using the projection argument in the compact $2n + 1$ theorem. (Exercise: Work out the details of this argument, in particular you have to ensure that the map is proper.) \square

Remark 2.17.10 (Real analytic manifolds). It is natural to ask about a real-analytic version of Whitney's theorem. It is true that a real-analytic manifold has a real-analytic embedding in Euclidean space, but it's difficult (Morrey, Grauert (1958)).

Another theorem of Whitney is the following: A smooth n -manifold admits an embedding in \mathbb{R}^{2n+1} such that the image is a real-analytic submanifold. Thus any smooth manifold has real-analytic structures (but not a canonical one). Nash (*Annals of Math.* 56 (1952)) proved that a connected compact smooth n -manifold can be embedded in \mathbb{R}^N in a way that it is a connected component of the set of real points of a system of polynomial equations.

Remark 2.17.11 (Complex manifolds). For complex manifolds there can be no such theorem, because e.g. a connected compact complex manifold has no nonconstant everywhere-holomorphic functions, so that any map to \mathbb{C}^n has to be constant. A **Stein manifold** is a complex manifold (necessarily noncompact) which admits a closed embedding in \mathbb{C}^N for some N . (This is one of many equivalent definitions.) Bishop and Narasimhan (1958) proved that a Stein manifold of (complex) dimension n admits a holomorphic embedding in \mathbb{C}^{2n+1} . What about compact complex manifolds? It is natural to try to embed in $\mathbb{C}P^n$. We will discuss this later.

2.18. Vector bundles. We will need the notion of vector bundle on a smooth manifold, so let's introduce it here, leaving a detailed treatment for later.

A **(smooth) vector bundle** on a smooth manifold M is a smooth manifold E with a smooth mapping $\pi : E \rightarrow M$ such that there exists an open covering $\{U_\alpha\}_\alpha$ of M and diffeomorphisms $\{\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^r\}_\alpha$ such that π is given on $\pi^{-1}(U_\alpha)$ by the second projection composed with φ_α and if $U_\alpha \cap U_\beta \neq \emptyset$ then the transition map

$$U_\alpha \cap U_\beta \times \mathbb{R}^r \xrightarrow{\varphi_\beta^{-1}} \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \rightarrow \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\alpha} U_\alpha \cap U_\beta \times \mathbb{R}^r$$

is of the form $(id, g_{\alpha\beta})$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(r, \mathbb{R})$ is a smooth function. The collection $\{U_\alpha, \varphi_\alpha\}_\alpha$ is called a **trivialization**. The integer r is called the **rank** of the vector bundle.

Exercise: The tangent bundle with $\pi : TM \rightarrow M$ is a vector bundle of rank $\dim M$ over M : For a trivialization we choose a covering $\{(U_\alpha, \phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n)\}$ by charts and then the identification $\pi^{-1}(U_\alpha) = TU_\alpha \cong \phi_\alpha(U_\alpha) \times \mathbb{R}^n$ by choosing local coordinates and the corresponding vector fields. Show that $g_{\alpha\beta}(x) = Jac(\phi_\alpha \circ \phi_\beta^{-1})(x)$ in this trivialization.

A **section** of a smooth vector bundle $\pi : E \rightarrow M$ is a smooth map $s : M \rightarrow E$ such that $\pi \circ s = id$. Any vector bundle has a canonical section, the **zero section** defined by $s(x) = 0 \in E_x$ for all $x \in M$.

A **vector bundle homomorphism** from a vector bundle $\pi : E \rightarrow M$ to a vector bundle $\pi' : E' \rightarrow M$ is a smooth mapping $\phi : E \rightarrow E'$ such that $\pi' \circ \phi = \pi$ and which restricts to a linear mapping of vector spaces on each fibre $E_p \rightarrow E'_p$ for each $p \in M$.

Any linear algebra construction that can be applied to vector spaces can be applied to smooth vector bundles over a manifold: duals, tensor products, exterior powers, quotients, Homs etc. We will discuss this in more detail later. Vector bundles over M form an abelian category.

If $f : M \rightarrow N$ is a smooth map and E is a vector bundle on N then there is a **pullback bundle** f^*E with fibres $(f^*E)_p = E_{f(p)}$ for $p \in M$. (Exercise: Prove this.)

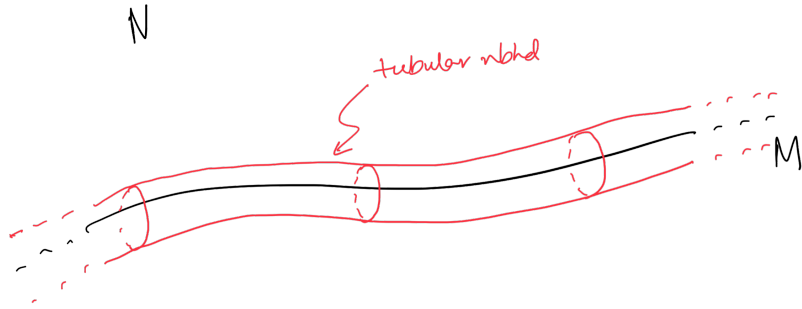
Exercise: For a smooth map $f : M \rightarrow N$ check that $df : TM \rightarrow TN$ gives a homomorphism of vector bundles $TM \rightarrow f^*TN$.

Remark 2.18.1 (Variants). We can also talk of a complex vector bundle on a smooth manifold, which means that the fibres are complex vector spaces, and the transition maps are \mathbb{C} -linear, i.e. given by maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(r, \mathbb{C})$. There is an obvious notion of complexification, i.e. a real vector bundle gives a complex vector bundle by tensoring the fibres with \mathbb{C} .

The notion of vector bundle also makes sense in the other categories above. So for example if M is a real analytic manifold then we can talk of **real analytic vector bundles**. The definition is identical to the above, except that $g_{\alpha\beta}$ is required to be a real-analytic map.

Similarly, if X is a complex manifold, then we can talk of **holomorphic vector bundles**, i.e. complex vector bundles such that the transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(r, \mathbb{C})$ are holomorphic.

2.19. Tubular neighbourhoods. Let $i : M \rightarrow N$ be an embedded submanifold of N . A **tubular neighbourhood** of M in N is an open neighbourhood U which is diffeomorphic to a neighbourhood of the zero section in some vector bundle on M . The picture is as follows:



It is a general theorem that an embedded submanifold of a manifold has a tubular neighbourhood, with the vector bundle being the **normal bundle** $\mathcal{N} = i^*TN/TM$ which has fibre $T_pN/di(T_pM)$ at $p \in M$. (Exercise: Check that this indeed is a vector bundle.) We will prove this when the submanifold is compact.

To simplify the notation we will think of M as a subset of N , which causes no problems since it is an embedded submanifold. We first treat the case $N = \mathbb{R}^n$ (without assuming M is compact).

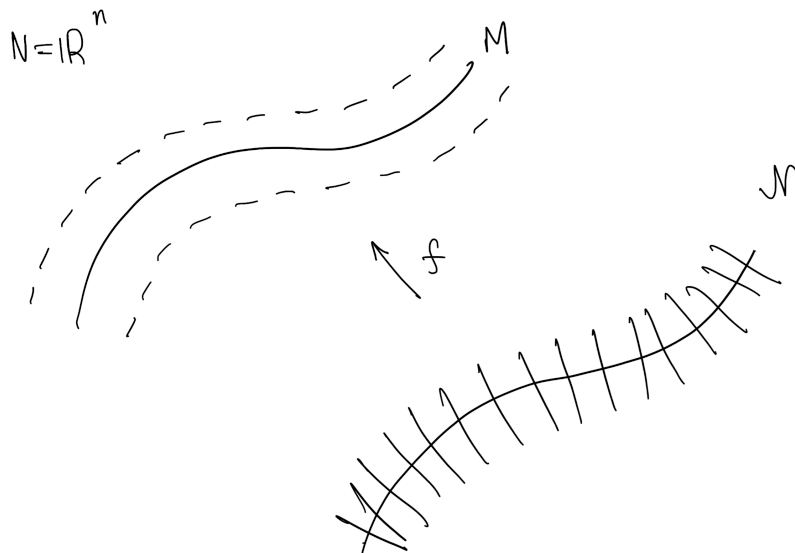
Proposition 2.19.1. *Let M be a smooth manifold embedded in \mathbb{R}^n . Then M admits a tubular neighbourhood in \mathbb{R}^n .*

Proof. Let \mathbb{R}^n be given the standard inner product. Then at each point $p \in M$ there is a decomposition $T_p\mathbb{R}^n = T_pM \oplus (T_pM)^\perp$. It is straightforward to check that

$$\mathcal{N} := \{(p, v) \in M \times \mathbb{R}^n : v \in (T_pM)^\perp\}$$

with the obvious map $\mathcal{N} \rightarrow M$ is a smooth vector bundle of rank $n - \dim M$ and is isomorphic to the normal bundle.

Now define a map $f : \mathcal{N} \rightarrow \mathbb{R}^n$ by $(p, v) \mapsto p + v$. The picture is as follows:



This map is the identity on the zero section ($= M$) and in the identification $T_{(p,v)}\mathcal{N} = T_pM \oplus \mathcal{N}_p$ the map $df_{(p,v)}$ is given by $(v, w) \mapsto v + w$. Thus df is an isomorphism along the zero section, in particular it follows that f is a local diffeomorphism along the zero section $M \subset \mathcal{N}$. We must

show that there is a neighbourhood U of M in \mathcal{N} on which f is a homeomorphism. It is enough to show that there is neighbourhood on which f is injective. Now for each p and $\delta > 0$ let

$$V_{p,\delta} = \{(x, v) : |x - p| + |v| < \delta\}.$$

Since f is a local homeomorphism there exists $V_{p,\delta}$ on which f is injective. Let $\delta(p) = \sup \{\delta : f \text{ is injective on } V_{p,\delta}\}$. Then $x \mapsto \delta(x)$ is a continuous function. (Indeed, since $V_{y,\delta-|x-y|} \subset V_{x,\delta}$ for all $\delta > 0$, we have $\delta(y) \geq \delta(x) - |x - y|$. The same holds with x, y reversed. This shows that $|\delta(x) - \delta(y)| \leq |x - y|$, so that δ is continuous. In fact it is Lipschitz with Lipschitz constant one.)

Now let $U = \{(p, v) \in \mathcal{N} : |v| < \delta(p)/4\}$. It will suffice to show that f is injective on U . Let $(p, v), (q, w) \in U$ and $f(p, v) = p + v = q + w = f(q, w)$ and assume $\delta(p) \leq \delta(q)$. Then $|p - q| = |w - v| \leq |w| + |v| \leq \delta(p)/2$, so $|p - q| + |w| = |w - v| + |w| \leq \delta(p)$. Thus both (p, v) and (q, w) lie in $V_{p,\delta}$ for some $\delta < \delta(p)$. But f is injective on such a neighbourhood so $(p, v) = (q, w)$. \square

Theorem 2.19.2 (Tubular neighbourhood theorem). *Let N be a smooth manifold and $M \subset N$ a smooth embedded compact submanifold. Then there exists a neighbourhood U of the zero section in the normal bundle and a diffeomorphism of U with an open neighbourhood of M in N .*

Proof. We first embed N in \mathbb{R}^n and use the previous proposition to get a neighbourhood W of N which is tubular, or more precisely, as in that proof the mapping $(p, v) \mapsto p + v$ is a diffeomorphism of a neighbourhood of N in its normal bundle onto W . Inverting this gives a map from W to the neighbourhood of N in the normal bundle of N in \mathbb{R}^n and then projecting to N defines a retraction $r : W \rightarrow N$ which is the identity on N . Let

$$\mathcal{N} = \{(p, v) \in M \times \mathbb{R}^n : v \in (T_p M)^\perp \cap T_p N\}.$$

(This is clearly isomorphic to the normal bundle as defined earlier.) Define a map $F : \mathcal{N} \rightarrow N$ by $(p, v) \mapsto r(p + v)$ where $r : W \rightarrow N$ is the retraction. It is easy to check that F is a local diffeomorphism along $M \times \{0\}$ and hence a local homeomorphism, so it remains to check that it is injective on some neighbourhood of $M \times \{0\}$ when M is compact. Suppose not. Then we can find sequences of points (p_n, v_n) and (q_n, w_n) such that $(p_n, v_n) \neq (q_n, w_n)$ and $v_n \rightarrow 0, w_n \rightarrow 0$ but $F(p_n, v_n) = F(q_n, w_n)$ for all n . By compactness of M we have subsequences converging to p and q with $F(p, 0) = F(q, 0)$, i.e. $p = q$. But then for large enough n , (p_n, v_n) and (q_n, w_n) must both lie in a neighbourhood $V_{p,\delta}$ for some $\delta > 0$, on which F is injective, contradicting that $F(p_n, v_n) = F(q_n, w_n)$. \square

Notice that a tubular neighbourhood W of M in N admits a retraction $r : W \rightarrow M$, corresponding to the projection $\mathcal{N} \rightarrow M$ in the normal bundle. (We can clearly also make a deformation retraction of W to M .)

The tubular neighbourhood theorem also holds if the embedded submanifold is noncompact, but requires more work.

2.20. Ehresmann's fibration theorem. We first define the notion of fibre bundle. Let $\pi : E \rightarrow B$ be a continuous mapping of topological spaces. Let F be a topological space. We say that π is a **locally trivial fibre bundle with fibre F** or just **fibre bundle with fibre F** for short, if there is an open covering $\{U_\alpha\}_\alpha$ of B and homeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ (called **local trivializations**) such that

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times F \\ \pi \downarrow & & \downarrow pr \\ U_\alpha & \xlongequal{\quad} & U_\alpha \end{array} \quad (2.20.1)$$

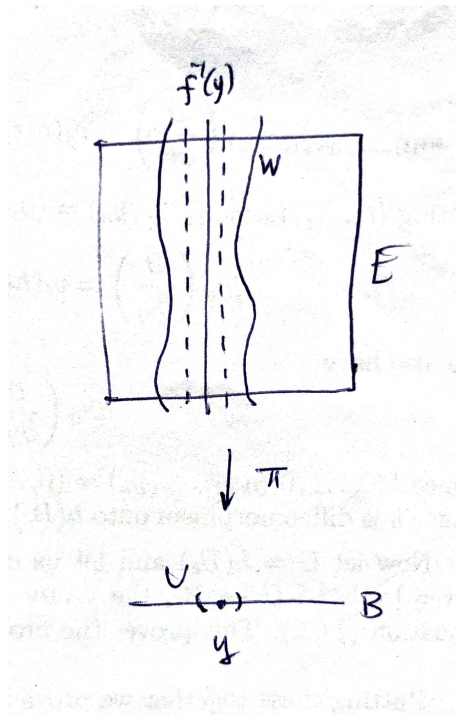
commutes for all α . When E and B and π are smooth we can speak of a **smooth fibre bundle** by requiring the local trivializations to be smooth maps.

For $\pi : E \rightarrow B$ a fibre bundle E and B are referred to as the **total space** and the **base** respectively. The space F is referred to as the **fibre**.

The following important theorem gives a general criterion for a smooth map to be a fibre bundle:

Theorem 2.20.1 (Ehresmann). *A proper surjective submersion of smooth manifolds is a smooth fibre bundle.*

Proof. This will follow easily from the tubular neighbourhood theorem. Let $f : M \rightarrow N$ be a proper submersion. A fibre $f^{-1}(y)$ is a smooth embedded submanifold of M (by the normal form for submersions), and it is compact (as f is proper). Therefore it has a tubular neighbourhood, i.e. there is a neighbourhood W of $f^{-1}(y)$ in M and a smooth retraction $r : W \rightarrow f^{-1}(y)$. The picture is as follows:



Consider the map

$$F = (r, f) : W \rightarrow f^{-1}(y) \times N$$

by $F(w) = (r(w), f(w))$. This is a diffeomorphism along $f^{-1}(y)$ and since this is compact, there is an open neighbourhood $W' \subset W$ of $f^{-1}(y)$ such that $F|_{W'}$ is a diffeomorphism onto its image. Since $F(W')$ is an open neighbourhood of $f^{-1}(y)$ in $f^{-1}(y) \times N$ and $f^{-1}(y)$ is compact, the tube lemma says that W' contains an open set of the form $f^{-1}(U)$ for some open set $U \subset N$ containing y . Then $F = (r, f)$ restricts to a diffeomorphism $f^{-1}(U) \cong f^{-1}(y) \times U$ and this is over U , i.e. the map $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is the second projection. We leave it to be checked that this gives local trivializations showing that $M \rightarrow N$ is a smooth fibre bundle. \square

In particular, if the base N is connected, all the fibres of such a map are diffeomorphic to each other.

Example 2.20.2. A simple example showing that properness is necessary in Ehresmann's theorem: Take $X = S^1 \times \mathbb{R} - \{(1, 0)\}$ and take $X \rightarrow \mathbb{R}$ by projection to the second factor. This is a submersion everywhere but clearly not a fibre bundle. (Where does the proof of the theorem fail in this case?)

Later we will give a different proof of Ehresmann's theorem using flows.

Remark 2.20.3 (Real analytic setting). ⁽¹⁰⁾ It appears that the tubular neighbourhood theorem and the Ehresmann theorem continue to hold in the real analytic category. In fact, starting from a real analytic embedding in \mathbb{R}^N , the proof above works as the only thing used is the inverse function theorem.

Remark 2.20.4 (Complex analytic setting). The tubular neighbourhood theorem has no analogue in the complex setting, i.e. a complex submanifold $M \subset N$ need not have an open neighbourhood W (which is again a complex manifold) with a *holomorphic* retraction $r : W \rightarrow M$. The analogue of Ehresmann's theorem also fails: The fibres of a smooth submersion $f : M \rightarrow N$ of complex manifolds, which are complex submanifolds if f is holomorphic, are not biholomorphic to each other. Put another way, there are interesting families of complex manifold structures on the same underlying smooth manifold. The simplest example is that of complex tori of dimension one, or equivalently, elliptic curves, or equivalently, plane cubics (i.e. hypersurfaces of degree 3 in $\mathbb{C}P^2$), which we will discuss later.

Example 2.20.5. A **complex hypersurface of degree d** is the subset of $\mathbb{C}P^n$ defined by a homogeneous polynomial f in $n + 1$ variables of degree d , i.e.

$$X_f := \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} - \{0\} : f(z_0, \dots, z_n) = 0\} / \mathbb{C}^* \subset \mathbb{C}P^n.$$

The space of homogeneous polynomials in z_0, \dots, z_n is identified with \mathbb{C}^N for some N , and the subset $B_d \subset \mathbb{C}^N$ consisting of f for which X_f is nonsingular is an open set. Let

$$\mathcal{X}_d := \{(f, x) \in B_d \times \mathbb{C}P^n : x \in X_f\}.$$

and let $\pi_d : \mathcal{X}_d \rightarrow B_d$ be the projection; this is a proper map. Ehresmann's theorem can be used to show that any two smooth hypersurfaces of the same degree are diffeomorphic, by showing that (1) π_d is a proper smooth submersion of manifolds and (2) the base B_d is path-connected. The details will be worked out in a problem set.

2.21. Triangulation of smooth manifolds (*). We will not prove the following theorem in this course:

Theorem 2.21.1 (Whitehead). *A smooth compact manifold admits a triangulation (i.e. there is a (finite) simplicial complex with topological realization homeomorphic to the manifold).*

The analogue for topological manifolds is *false*: In every dimension $n \geq 4$ there exists a topological n -manifold which does not admit a triangulation. In dimension 4 this was known in the 1980s (Freedman, Casson) but in (all) dimensions ≥ 5 it was proved only recently (Manolescu, 2013).

As a corollary, we see that the (simplicial or singular) homology and cohomology groups of a compact smooth manifold are finitely generated. We will prove this in a different way in the next subsection.

2.22. Good covers and finite generation of homology. A **good cover** of an n -manifold is a locally finite cover $\{U_i\}_{i \in I}$ such that all nonempty intersections $U_{i_0} \cap \dots \cap U_{i_n}$ for $i_0, \dots, i_n \in I$ are homeomorphic to \mathbb{R}^n .

Proposition 2.22.1. *Let M be a smooth manifold. Then M has a good cover.*

Sketch of proof. ⁽¹¹⁾ Using Whitney's embedding theorem we may embed M in \mathbb{R}^N . For each $x \in M$, orthogonal projection ϕ_x in \mathbb{R}^N to the n -plane $T_x M$, when restricted to a sufficiently small neighbourhood U_x of $x \in M$ homeomorphic to \mathbb{R}^n , is actually a chart of M around x . The collection $\{(U_x, \phi_x)\}_x$ is therefore an atlas for M . The transition function $\phi_y \circ \phi_x^{-1} : \phi_x(U_x) \rightarrow \phi_y(U_y)$ between open subsets of $T_x M$ and $T_y M$ is close to the identity for x and y close enough. (Here we fix a reference linear isomorphism between $T_x M$ and $T_y M$ which is the restriction to $T_x M \subset T_x \mathbb{R}^n \cong \mathbb{R}^n$ of the orthogonal projection $\mathbb{R}^n \rightarrow T_y M$, and "close to the

¹⁰This remark is from a MathOverflow comment. I have not checked the details, but it looks reasonable.

¹¹following Weil, Sur les théorèmes de de Rham, *Commentarii Math. Helvetici* 26 (1952)).

identity” means close to this reference isomorphism (in any norm.) So we may assume that $\phi_y \circ \phi_x^{-1}$ has first (Jacobian) and second (Hessian) derivative matrices close to the identity. But a diffeomorphism of \mathbb{R}^n which has Jacobian and Hessian sufficiently close to the identity takes compact strictly convex domains to compact strictly convex domains. (Here a strictly convex domain in \mathbb{R}^n is defined to be a domain with smooth boundary (i.e. the boundary is a smooth manifold embedded in \mathbb{R}^n) and with the Hessian of any function defining the boundary locally is strictly positive definite at every point of the boundary in its domain.) We may assume the U_x are all interiors of strictly convex domains (i.e. interiors of closed balls). Then the finite intersections are intersections of interiors of strictly convex domains and one can show that such an intersection is homeomorphic to \mathbb{R}^n or empty (as any open convex set in \mathbb{R}^n is homeomorphic to a ball). It follows that the cover $\{U_x\}_x$ is a good cover. \square

The Mayer-Vietoris sequence and an induction on cardinality immediately gives:

Corollary 2.22.2. *The homology and cohomology of a manifold with arbitrary coefficients of a manifold with a finite good cover (e.g. a compact smooth manifold) are finitely generated.*

Recall that in the lectures on Poincaré duality we had assumed this fact for some applications, e.g. to prove $\chi(M) = \chi_2(M)$ etc., so now we have justified all that, at least for smooth manifolds. In fact the corollary is true with the word “smooth” omitted, i.e. for topological manifolds, but this is harder to prove. (I am not sure if any compact topological manifold has a good cover, so one has to prove finite generation of homology in a different way.)

2.23. Orientations. Recall that an orientation for a real vector space is an equivalence class of bases, under the relation that two bases are equivalent if the matrix relating them has positive determinant. There are thus two possible orientations for a real vector space.

The top exterior power $\Lambda^{n=\dim V} V$ of a real vector space V of dimension n is a one-dimensional \mathbb{R} -vector space. A choice of generator for $\Lambda^n V$ specifies an orientation uniquely: any generator of $\Lambda^n V$ is of the form $v_1 \wedge \cdots \wedge v_n$ and it gives the basis $\{v_1, \dots, v_n\}$. Conversely an orientation gives a generator of $\Lambda^n V$ (nonuniquely): If $\{v_1, \dots, v_n\}$ is any basis in the equivalence class then $v_1 \wedge \cdots \wedge v_n$ spans $\Lambda^n V$. The fact that for $g \in GL(V)$ the operator $\Lambda^n g : \Lambda^n V \rightarrow \Lambda^n V$ is multiplication by $\det(g)$ makes this well-defined.

An **oriented atlas** for a manifold M is a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ in which the transition functions satisfy the condition

$$\det(\text{Jac}(\varphi_\beta \circ \varphi_\alpha^{-1})) > 0 \quad \text{on } \varphi(U_\alpha \cap U_\beta) \subset \mathbb{R}^n.$$

If M admits an oriented atlas it is called **orientable**. Two oriented atlases which are compatible (in the sense that their union is a smooth atlas) are compatibly oriented if their union is oriented. This is an equivalence relation on oriented atlases; an equivalence class under this relation is an **orientation** on M . If we fix an orientation we call M **oriented**.

Here is an equivalent way to define an orientation: An orientation is the choice of an orientation for each $(T_x M)^*$ (or $T_x M$) which is locally constant in x , or, equivalently, it is a nowhere-vanishing smooth section of the fibre bundle $\Lambda^n TM / \mathbb{R}_+$ (or of the fibre bundle $\Lambda^n T^*M / \mathbb{R}_+$) where the \mathbb{R}_+ acts by dilations in the each $T_x M$.

Exercise: Check these two definitions are equivalent.

Exercise: Show using this definition that S^n is orientable and $\mathbb{R}P^n$ is orientable if and only if n is odd.

Exercise: Convince yourself that an orientation of a smooth manifold as defined here is the same as an orientation of the underlying topological manifold as we defined it earlier.

3. VECTOR FIELDS AND THE FROBENIUS THEOREM

3.1. Vector fields and local flows. A **vector field** on a smooth manifold M is a smooth section of the tangent bundle, i.e. a smooth map $V : M \rightarrow TM$ such that $\pi \circ V = id$. Formally, $V(p) = (p, v(p))$, i.e. we are given, for each point p , a tangent vector $v(p) \in T_p M$ and it “varies smoothly as we vary p ”. We will usually ignore the first factor and speak of “the vector field v .”

Let $\varphi : M \rightarrow M'$ be a diffeomorphism and $T\varphi : TM \rightarrow TM'$ the induced diffeomorphism of the tangent bundle. Given a vector field v on M and v' on M' we can define new vector fields $\varphi_*(v)$ and $\varphi^*(v')$, called the **pushforward** of v and **pullback** of v' respectively, by

$$\begin{aligned}\varphi_*(v) &:= T\varphi \circ v \circ \varphi^{-1} \\ \varphi^*(v') &:= T\varphi^{-1} \circ v' \circ \varphi.\end{aligned}$$

These are both sections of the respective tangent bundles, i.e. vector fields. (There is a mild abuse of notation here as $T\varphi$ must be applied to $V(p) = (p, v(p))$.) It is immediate that

$$\begin{aligned}\varphi^* \varphi_*(v) &= v \\ \varphi_* \varphi^*(v') &= v'\end{aligned}$$

and $(\varphi^{-1})^* = \varphi_*$, $\varphi^* = (\varphi^{-1})_*$. Concretely,

$$\begin{aligned}\varphi_*(v)(q) &= d\varphi_p(v(p)) & \text{for } \varphi(p) = q \\ \varphi^*(v')(p) &= d\varphi_q^{-1}(v'(q)) & \text{for } \varphi(p) = q.\end{aligned}$$

(Here again we abuse notation by thinking of $v(p)$ as a tangent vector rather than a pair (point, tangent vector at the point).) Thinking of tangent vectors as derivations on (germs of) smooth functions, we have:

$$\begin{aligned}\varphi_*(v)(f')(x') &= \varphi^{-1*}(v(\varphi^*(f')))(x') \\ &= v(f' \circ \varphi)(\varphi^{-1}(x')) & (x \in M', f' \text{ smooth function on } M') \\ \varphi^*(v')(f)(x) &= \varphi^*(v'(\varphi^{-1*}(f)))(x) \\ &= v'(f \circ \varphi^{-1})(\varphi(x)) & (x \in M, f \text{ smooth function on } M)\end{aligned}$$

where $\varphi^*(f') = f' \circ \varphi$ is the pullback of smooth functions from M' to M .

- If f' is a smooth function on M' then $\varphi^*(f'v') = \varphi^*(f')\varphi^*(v')$ (i.e. pullback of vector fields is a module over pullback of functions)

In general, it is only possible to pushforward and pullback vector fields by diffeomorphisms, not by general smooth maps. (There are some special classes of maps for which pullback of vector fields exists.)

Consider what a vector field looks like “locally”. Let $p \in M$ and let $(U, \varphi, U' \subset \mathbb{R}^n)$ be a chart containing p . The smooth map $V : M \rightarrow TM$ restricts to a smooth map $V : U \rightarrow TU$ which is a section of $\pi : TU \rightarrow U$. Now using the diffeomorphism $\varphi : U \rightarrow U'$ we can get a vector field $\varphi_*(v)$ on $U' \subset \mathbb{R}^n$, i.e. a section of $TU' \rightarrow U'$. The basis $\left\{ \frac{\partial}{\partial x_i} \right\}$ gives a trivialization $TU' = U' \times T_{\varphi(p)}U'$, in terms of which we can write

$$\varphi_*(v) = \sum_i a_i \frac{\partial}{\partial x_i} \quad \text{or} \quad v = \varphi^* \left(\sum_i a_i \frac{\partial}{\partial x_i} \right) \quad (3.1.1)$$

for smooth functions $a_i = a_i(x_1, \dots, x_n) \in C^\infty(U')$. This is what a vector field looks like locally, i.e. in a coordinate chart. In the sequel we will frequently write $v = \sum_i a_i \frac{\partial}{\partial x_i}$ as a local expression for the vector field v ; it is to be understood as meaning (3.1.1). (In other words we will usually drop the diffeomorphism φ from the notation.)

Note that a vector field is determined by its action as a derivation on smooth functions (i.e., if v, w are vector fields and $v(f) = w(f)$ for all smooth functions f then $v = w$, e.g. because of

the local expression). In other words, a vector field is a linear differential operator of order one on smooth functions.

A **flow** on M is an action $\alpha : \mathbb{R} \times M \rightarrow M$ of the group $(\mathbb{R}, +)$ on M such that the map $\alpha : \mathbb{R} \times M \rightarrow M$ is smooth. A flow gives a vector field: For each point p let $v(p)$ be the tangent vector to the smooth curve $t \mapsto \alpha(t, p)$.

In the other direction, given a vector field v , a smooth curve $\gamma : (a, b) \rightarrow M$ is called an **integral curve** of v if, at points p along the curve, the tangent vector at p is $v(p)$. This means that for any smooth function f ,

$$\left. \frac{d}{dt} \right|_{t=s} f \circ \gamma = v(\gamma(s))(f)$$

for $s \in (a, b)$, or, in different notation, that:

$$d\gamma_s \left(\frac{\partial}{\partial t} \right) = v(\gamma(s)).$$

We shall now see that an integral curve of a vector field through a given point p always exists, at least locally, and it is unique if we require $\gamma(0) = p$. This will be a consequence of the existence-uniqueness theorem for ordinary differential equations.

Since the question is local, we may assume using a coordinate chart that we are in an open subset O of \mathbb{R}^n . Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a smooth curve in our open subset with $\gamma(0) = q \in O$. The condition that γ is an integral curve of v is that

$$d\gamma_s \left(\frac{\partial}{\partial t} \right) = v(\gamma(s))$$

holds for s in a neighbourhood of 0. In the local coordinates x_1, \dots, x_n this is

$$\sum_i \left. \frac{d\gamma_i}{dt} \right|_{t=s} \frac{\partial}{\partial x_i} = \sum_i a_i(\gamma_1(s), \dots, \gamma_n(s)) \frac{\partial}{\partial x_i}$$

which (dropping the dummy variable s), is the same as

$$\sum_i \frac{d\gamma_i}{dt} \frac{\partial}{\partial x_i} = \sum_i a_i(\gamma_1(t), \dots, \gamma_n(t)) \frac{\partial}{\partial x_i}$$

with the initial conditions $\gamma_i(0) = q_i$. (Here q_i are the local coordinates of q .) Solving this equation is equivalent to solving the following system of n ordinary differential equations with initial conditions:

$$\begin{aligned} \frac{d\gamma_i}{dt} &= a_i(\gamma_1(t), \dots, \gamma_n(t)) \\ \gamma_i(0) &= q_i. \end{aligned}$$

Since the a_i are smooth functions, the existence-uniqueness theorem for o.d.e. (Picard's theorem) says that:

- (i) there exists a relatively compact neighbourhood U of p and an ϵ (depending on U) such that for any $q \in U$, the system of differential equations with initial condition has a unique solution $\gamma_q(t)$ which is defined for $t \in (-\epsilon, \epsilon)$
- (ii) the solution is smooth and depends smoothly on $q \in U$, i.e. the mapping

$$\alpha : (-\epsilon, \epsilon) \times U \rightarrow \mathbb{R}^n \quad \alpha(t, q) = \gamma_q(t)$$

(“follow the integral curve starting at q for a time t ”) is smooth.

(For a discussion and proof of the existence-uniqueness theorem see e.g. Arnol'd's book.) Let us see that α satisfies

$$\alpha(t, \alpha(s, p)) = \alpha(s + t, p)$$

whenever both sides make sense. This equation is the same as $\gamma_{\gamma_p(s)}(t) = \gamma_p(s + t)$. Fixing s , we see that both these satisfy the same system of differential equations with the same initial condition, so by uniqueness they are the same.

The map α defined here is not a flow, because $\alpha(t, p)$ is only defined for small t using the integral curve. Such a thing is called a “local flow”.

A **local flow** (Φ, α) on a manifold M is a pair consisting of an open neighbourhood Φ of $\{0\} \times M$ in $\mathbb{R} \times M$ and a map $\alpha : \Phi \rightarrow M$ such that:

- (i) $\Phi \cap \mathbb{R} \times \{p\}$ is an interval containing $0 \times \{p\}$
- (ii) $\alpha(0, x) = x$ for all $x \in M$
- (iii) $\alpha(s, \alpha(t, x)) = \alpha(s + t, x)$ whenever both sides make sense.

(The open set Φ will usually be dropped from the notation.)

The earlier discussion shows that a vector field gives rise to a local flow and vice-versa. This evidently sets up a bijection

$$\{\text{vector fields}\} \longleftrightarrow \{\text{local flows}\} / \sim \quad \text{where } (\alpha, \Phi) \sim (\alpha', \Phi') \text{ if } \alpha|_{\Phi \cap \Phi'} = \alpha'|_{\Phi \cap \Phi'}.$$

The following property of a local flow (α, Φ) associated with the vector field v is key:

$$d\alpha_{(t,x)} \left(\frac{\partial}{\partial t} \right) = v(\alpha(t, x)) \quad (\text{if } (t, x) \in \Phi). \quad (3.1.2)$$

In terms of the action on smooth functions one has:

$$v(f)(x) = \frac{d}{dt} \Big|_{t=0} \alpha_t^*(f)(x) = \frac{d}{dt} \Big|_{t=0} f(\alpha_t(x)) \quad (3.1.3)$$

The right-hand side has the following meaning: $(t, x) \mapsto f(\alpha_t(x))$ is a smooth function of t and x , so we can differentiate in t and set $t = 0$. We will also write this as:

$$v(f) = \frac{d}{dt} \Big|_{t=0} \alpha_t^*(f). \quad (3.1.4)$$

(This expression could have a different meaning, namely consider $t \mapsto \alpha_t^*(f)$ as a function from \mathbb{R} to $C^\infty(M)$ and take the derivative (in the usual limit definition) at 0. This requires putting a topology on $C^\infty(M)$, which can certainly be done (uniform convergence of all derivatives on compact sets), but we will avoid this and simply take (3.1.4) to mean (3.1.3) for all $x \in M$.) In some books you will find this as $v = \frac{d}{dt} \Big|_{t=0} \alpha_t$.

Note that v is a vector field on M and if $V \subset M$ is relatively compact, then there is an ϵ such that $(-\epsilon, \epsilon) \times V$ is contained in the local flow of v , i.e. $\alpha : (-\epsilon, \epsilon) \times V \rightarrow M$ is defined (since any neighbourhood of the closure of V contains a product neighbourhood).

Example 3.1.1. One cannot always get a (global) flow from a vector field: Take $M = \mathbb{R}$, $v(x) = x^2 \frac{\partial}{\partial x}$. The integral curve starting at $a \in \mathbb{R}$ is $\gamma(t) = \frac{a}{1-at}$. If $a < 1$ this is defined for $t \in (-\infty, a^{-1})$. In particular, it is not possible to choose ϵ such that $(-\epsilon, \epsilon) \times \mathbb{R}$ is contained in the local flow. In fact, there is no ϵ which will work for any unbounded interval of starting points.

Lemma 3.1.2. *If v is a vector field vanishing outside a compact set, then its integral curves are defined for all t . In particular, if M is compact then every vector field v arises from a flow $\alpha : \mathbb{R} \times M \rightarrow M$.*

Proof. • Exercise. Do this in two steps: First show that if M is compact there is an ϵ such that the local flow contains $(-\epsilon, \epsilon) \times M$ (e.g. by the tube lemma). Next show that on any manifold M if a local flow contains $(-\epsilon, \epsilon) \times M$ then it extends (uniquely) to a flow $\mathbb{R} \times M \rightarrow M$. □

A vector field is called **complete** if its local flows exist for all t , i.e. its local flow is a flow. The lemma implies that any vector field with compact support is complete.

Example 3.1.3. The vector fields $x^2 \sin^2(x) \frac{\partial}{\partial x}$ and $x^2 \cos^2(x) \frac{\partial}{\partial x}$ are complete, but their sum is not. Thus completeness is not a linear condition in the space of all vector fields.

3.2. A useful lemma. Suppose we have a vector field v on a one-dimensional manifold M . Let $p \in M$. The integral curve of v through p is a map $\gamma : (-\epsilon, \epsilon) \rightarrow M$ with the property that $d\gamma_t \left(\frac{\partial}{\partial t} \right) = v(\gamma(t))$. If we assume that $v(p) \neq 0$ then there is a neighbourhood of $0 \in (-\epsilon, \epsilon)$ on which γ is a diffeomorphism onto a neighbourhood of p , say $\gamma : (-\epsilon', \epsilon') \cong W$. Then in the coordinate t on W given by $\gamma^{-1} : W \rightarrow \mathbb{R}$, the vector field v becomes $d\gamma_{\gamma^{-1}(t)}^{-1} (v(\gamma(t))) = \frac{\partial}{\partial t}$, i.e. $(\gamma^{-1})_*(v) = \gamma^*(v) = \frac{\partial}{\partial t}$.

Given a vector field on a general manifold, locally one can choose coordinates in which it is simple:

Lemma 3.2.1. *Let v be a vector field on M and $p \in M$ a point such that $v(p) \neq 0$. Then we can find a neighbourhood U of p and local coordinates x_1, \dots, x_n on U such that $v = \frac{\partial}{\partial x_1}$ on U . (More precisely, there is a chart $(U, \varphi : U \rightarrow U' \subset \mathbb{R}^n)$ with $p \in U$ such that $v = \varphi^* \left(\frac{\partial}{\partial x_1} \right)$ on U .)*

Proof. Let U be a neighbourhood of p in a chart on which the local flow is defined for $t \in (-\epsilon, \epsilon)$. Let N be any submanifold of dimension $n - 1$ in U such that $p \in N$ and $v(p) \notin T_p N$. (For example, choose any local coordinates in which $v(p) = \frac{\partial}{\partial x_1}$ and take $N = \{x_1 = 0\}$.) The flow map $\alpha : (\epsilon, \epsilon) \times N \rightarrow M$ is a diffeomorphism in some neighbourhood of $(0, p) \in (-\epsilon, \epsilon) \times N$ (by the inverse function theorem). Choose coordinates y_2, \dots, y_n in N near p and set

$$\begin{aligned} x_i &= y_i \circ (\text{projection to } N) \circ \alpha^{-1} & (i \geq 2) \\ x_1 &= (\text{projection to } (-\epsilon, \epsilon)) \circ \alpha^{-1} \end{aligned}$$

on a neighbourhood on which α is invertible. In these coordinates the vector field v is simply $\frac{\partial}{\partial x_1}$, because of the fact (3.1.2) that

$$d\alpha_{(t,x)} \left(\frac{\partial}{\partial t} = \frac{\partial}{\partial x_1} \right) = v(\alpha(t,x))$$

for all t, x in some neighbourhood of $(0, p)$. □

This leads naturally to the next question: if we have k vector fields, all nonvanishing and linearly independent at p , when can we find local coordinates so that they are the $\frac{\partial}{\partial x_i}$ for $i = 1, \dots, k$? (More precisely, we should ask, does there exist a chart (U, φ, U') containing p such that $v_i = \varphi^* \left(\frac{\partial}{\partial x_i} \right)$ on U .) The answer is given by the Lie bracket of vector fields and Frobenius' theorem.

3.3. Lie bracket. Let v and w be vector fields on M . The **Lie bracket** of v and w is $[v, w]$ defined by its action on smooth functions:

$$[v, w](f) := v(w(f)) - w(v(f)).$$

Defined in this way it is not obvious that this is a vector field. We claim that this is actually a derivation (linearity is clear, one must check the Leibniz rule) on germs of smooth functions at p , and that it varies smoothly with p , so that it defines a vector field. To check these statements it will suffice to check in local coordinates that it can be written as $\sum_i c_i \frac{\partial}{\partial x_i}$ for smooth functions $c_i(x_1, \dots, x_n)$ ($i = 1, \dots, n$). Fix a point of M and fix local coordinates x_1, \dots, x_n in a neighbourhood of the point. Write the vector fields as $v = \sum_i a_i \frac{\partial}{\partial x_i}$ and $w = \sum_i b_i \frac{\partial}{\partial x_i}$. Then

$$(vw - wv) = \sum_i c_i \frac{\partial}{\partial x_i} \quad \text{for} \quad c_i = \sum_j \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right)$$

by calculation.

Remark 3.3.1 (Lie bracket of C^k vector fields). On a C^k (or better) manifold one has the notion of C^k vector field. The bracket of two C^k vector fields is only C^{k-1} , as the previous local formula suggests.

Remark 3.3.2 (Lie bracket of real analytic vector fields). On a real analytic manifold there is a notion of real analytic vector field (i.e. real analytic section of the tangent bundle), and the bracket of two real analytic vector fields is again real analytic.

Remark 3.3.3 (Lie bracket of holomorphic vector fields). On a complex manifold a holomorphic vector field is (by definition) a holomorphic section of the holomorphic tangent bundle $T_X = TX^{1,0}$. It can be written in local holomorphic coordinates as $\sum_i f_i \frac{\partial}{\partial z_i}$ where f_i are holomorphic functions. The bracket of two holomorphic vector fields is holomorphic.

The Lie bracket has the following properties, which follow from the definition:

- (i) $[v, v] = 0$ (antisymmetry)
- (ii) $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$ (Jacobi identity)

These two conditions mean that the space of vector fields on M with the Lie bracket is a Lie algebra over \mathbb{R} (of infinite dimension if $\dim M > 0$). ⁽¹²⁾ We also have:

- (iii) for functions f, g and vector fields v, w , we have

$$[fv, gw] = fg[v, w] + fv(g)w - gw(f)v$$

There is also the following very important naturality property:

- (iv) If $\varphi : M \rightarrow M$ is a diffeomorphism then

$$\begin{aligned}\varphi_*[v, w] &= [\varphi_*v, \varphi_*w] \\ \varphi^*[v, w] &= [\varphi^*v, \varphi^*w]\end{aligned}$$

This is easy from the fact that $\varphi_*(u)(f) = \varphi^{-1*}(u(\varphi^*f))$. Indeed if $u = [v, w]$ we have

$$\varphi_*([v, w])(f) = \varphi^{-1*}([v, w](\varphi^*f)) = \varphi^{-1*}(vw(\varphi^*f) - wv(\varphi^*f)).$$

Now

$$\varphi_*v((\varphi_*w)(f)) = \varphi^{-1*}(v(\varphi^*((\varphi_*w)(f)))) = \varphi^{-1*}(v(\varphi^*(\varphi^{-1*}w(\varphi^*f)))) = \varphi^{-1*}(vw(\varphi^*f))$$

(the last equality comes from $\varphi^{-1*}\varphi^*f = f$), so that we have $\varphi_*[v, w](f) = [\varphi_*v, \varphi_*w](f)$ for all smooth functions. The case of φ^* follows since $\varphi^*\varphi_* = id$.

More generally, if $f : M \rightarrow N$ is a smooth map then we say vector fields v on M and v' on N are **f -related** if we have $df_p(v(p)) = v'(f(p))$ for all $p \in M$. The same proofs show that

- (v) If $f : M \rightarrow N$ is a smooth mapping and v, w are vector fields on M which are f -related to vector fields v', w' on N respectively, then $[v, w]$ is f -related to $[v', w']$.

3.4. Local flows and brackets. Let (α, Φ) be a local flow on M . For $p \in M$ let U be a relatively compact neighbourhood of p and let $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times U \subset \Phi$, i.e. $\alpha(t, x)$ makes sense for $t \in (-\epsilon, \epsilon)$ and $x \in U$. For such t , $\alpha_t(-) := \alpha(t, -)$ is a diffeomorphism of U onto an open set in M (indeed, it has a smooth inverse, namely α_{-t}). So for $|t| < \epsilon$, the pullback $\alpha_t^*(v)$ and the pushforward $(\alpha_t)_*(v)$ make sense as vector fields on neighbourhoods of p . Writing things out in local coordinates shows that these depend smoothly on t . We also have

$$\alpha_t^*(v) = (\alpha_{-t})_*(v)$$

(from the definitions).

A more geometric picture of the bracket can be given using the local flow of one of the vector fields. The idea is to pullback one vector field using the flow of the other, and measure how it changes, infinitesimally. We will use the following: If v_t is a vector field depending smoothly on a parameter t , then $\frac{dv_t}{dt}$ is the vector field which acts on smooth functions by $\frac{dv_t}{dt}(f) = \frac{d}{dt}(v_t(f))$. (That this is actually a vector field can be checked by a local calculation: In local coordinates if we write $v_t = \sum_i a_i(x, t) \frac{\partial}{\partial x_i}$ then $\frac{dv_t}{dt} = \sum_i \frac{d}{dt} a_i(x, t) \frac{\partial}{\partial x_i}$.) The following alternative description of the bracket is often given as the definition:

¹²By definition, a **Lie algebra** over a field k is a k -vector space V with a k -bilinear operation $[\cdot, \cdot] : V \times V \rightarrow V$ which is antisymmetric and satisfies the Jacobi identity.

Lemma 3.4.1. *Let α_t be the local flow associated with v . Then:*

$$[v, w] = \left. \frac{d}{dt} \right|_{t=0} \alpha_t^*(w) = - \left. \frac{d}{dt} \right|_{t=0} (\alpha_t)_*(w).$$

Proof. The identity is local, so it is enough to check this in local coordinates centred at p . We will look separately at the cases $v(p) \neq 0$ and $v(p) = 0$.

If $v(p) \neq 0$ then by the previous lemma, we may assume that coordinates have been chosen so that $v = \frac{\partial}{\partial x_1}$. Then the definition of the bracket gives

$$(vw - wv)(p) = \sum_i \frac{\partial b_i}{\partial x_1} \frac{\partial}{\partial x_i}.$$

On the other hand, the integral curve of v through (p_1, \dots, p_n) is simply $t \mapsto (p_1 + t, p_2, \dots, p_n)$, i.e. $x_1(t) = p_1 + t, x_i(t) = p_i$ for $i \neq 1$. The local flow is given by $\alpha_t(x_1, x_2, \dots, x_n) = (x_1 + t, x_2, \dots, x_n)$. Write $w = \sum_i b_i \frac{\partial}{\partial x_i}$. Using the fact that $\alpha_t^* \left(b_i \frac{\partial}{\partial x_i} \right) = \alpha_t^*(b_i) \frac{\partial}{\partial x_i}$ (exercise), we get:

$$\alpha_t^*(w) = \sum_i \alpha_t^* \left(b_i \frac{\partial}{\partial x_i} \right) = \sum_i b_i(x_1 + t, x_2, \dots, x_n) \frac{\partial}{\partial x_i}.$$

So then $[v, w]$ is

$$[v, w] := \left. \frac{d}{dt} \right|_{t=0} \alpha_t^*(w) = \sum_i \left. \frac{\partial b_i}{\partial x_1} (x_1 + t, x_2, \dots, x_n) \right|_{t=0} \frac{\partial}{\partial x_i} = \sum_i \frac{\partial b_i}{\partial x_1} \frac{\partial}{\partial x_i}$$

and we have proved the lemma.

If $v(p) = 0$ then the flow α_t fixes p , and hence gives a one-parameter subgroup of linear transformations of $T_p \mathbb{R}^n$, namely $d\alpha_t$. Since any one-parameter subgroup of transformations of a vector space is e^{tA} for a (complex) matrix A ⁽¹³⁾, we have that $d\alpha_t$ acts by e^{tA} on $T_p \mathbb{R}^n$. Thus we can compute the right-hand side at p by:

$$\left(\left. \frac{d}{dt} \right|_{t=0} \alpha_t^*(w) \right) (p) = \lim_{t \rightarrow 0} \frac{(e^{tA})^{-1} w(p) - w(p)}{t} = -Aw(p).$$

On the other hand, by our assumptions $v(p) = 0$, so that in coordinates centred at p with $v = \sum_i a_i \frac{\partial}{\partial x_i}$ and $w = \sum_i b_i \frac{\partial}{\partial x_i}$ we have

$$[v, w](p) = \sum_i \left(\sum_j a_j(p) \frac{\partial b_i}{\partial x_j} - b_j(p) \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i} = - \sum_i \left(\sum_j b_j(p) \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

So we must prove that for $w = \sum_j b_j \frac{\partial}{\partial x_j}$ we have, for each i , that

$$A \sum_j b_j(p) \frac{\partial}{\partial x_j} (x_i) = Aw(p)(x_i) = \sum_j b_j(p) \frac{\partial a_i}{\partial x_j}.$$

For this it is enough to check that $A \frac{\partial}{\partial x_j} = \sum_i \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i}$, which follows from the fact that e^{tA} is the local flow of v (exercise). \square

Remark 3.4.2. It is possible to give a more direct proof by computation, see e.g. standard textbooks.

There is a nice formula for the bracket in terms of both the associated local flows:

¹³Here we are using that any one-parameter subgroup of transformations can be written as e^{tA} . Here “one-parameter subgroup” means a smooth map $\phi : \mathbb{R} \rightarrow GL(n, \mathbb{C})$ which is also a group homomorphism. Let us prove this: Suppose $\phi : \mathbb{R} \rightarrow GL(n, \mathbb{C})$ is a one-parameter subgroup. Let $A := d\phi_0 \left(\frac{\partial}{\partial t} \right) \in T_{\phi(0)=I} GL(n, \mathbb{C}) = M(n, \mathbb{C})$. Then $t \mapsto e^{tA}$ and ϕ both satisfy the same ODE with the same initial condition, so they are the same.

Lemma 3.4.3. *Let v, w be vector fields with local flows α_t, β_s respectively. Then*

$$[v, w] = \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} \alpha_t^* \beta_s^* - \beta_s^* \alpha_t^*$$

(as operators on smooth functions).

Proof. By definition,

$$(\beta_s^* v(f))(x) = v(f)(\beta_s(x)) = \frac{\partial}{\partial t} \Big|_{t=0} (\alpha_t^* f)(\beta_s(x)) = \frac{\partial}{\partial t} \Big|_{t=0} f(\alpha_t \beta_s(x)).$$

Taking $\frac{\partial}{\partial s} \Big|_{s=0}$ and using $w(g)(x) = \frac{\partial}{\partial s} \Big|_{s=0} (\beta_s^* g)(x)$ for $g = v(f)$ gives

$$w(v(f))(x) = \frac{\partial}{\partial s} \Big|_{s=0} \frac{\partial}{\partial t} \Big|_{t=0} f(\alpha_t \beta_s(x)) = \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} (\beta_s^* \alpha_t^* f)(x).$$

This implies the lemma using the similar expression for $v(w(f))$. \square

Let α and β be two local flows on M . Let $U \subset M$. We say that α and β **commute** on U , if, for $p \in U$,

$$\alpha(t, \beta(s, p)) = \beta(s, \alpha(t, p)) \quad (\text{equivalently, } \alpha_t \circ \beta_s = \beta_s \circ \alpha_t)$$

whenever s, t are small enough so that both sides make sense. Note that if this equation holds for s, t in some neighbourhood of $(0, 0)$, then it holds for all s, t where it makes sense.

The previous lemma shows that if the local flows of v and w commute then $[v, w] = 0$. The converse is also true, i.e. we will show that if $[v, w] = 0$ then the local flows of v and w commute.

Let M be a manifold. Let V, W be relatively compact subsets of M and $\varphi : V \rightarrow W$ a diffeomorphism. If v is a vector field on M we say that v **is invariant under** φ if $(\varphi^{-1})^*(v) = v$. Concretely, this means that for all $q \in W$,

$$(d\varphi^{-1})_q(v(q)) = v(\varphi^{-1}(q))$$

Similarly, we can consider the condition $\varphi_*(v) = v$ which concretely amounts to

$$d\varphi_q(v(p)) = v(\varphi(p))$$

for $p \in V$ and is easily seen to be equivalent.

Let v, w be vector fields on M , with (α, Φ) a local flow of v . We say that w **is invariant under the local flow of v** if, for any $(-\epsilon, \epsilon) \times V \subset \Phi$, we have $\alpha_{t*}(w) = w$ for $|t| < \epsilon$ (equivalently, $\alpha_t^*(w) = w$ for $|t| < \epsilon$).

Lemma 3.4.4. *Let v and w be vector fields on M with local flows α and β .*

- (i) $\alpha_t^*(v) = v$, i.e. a vector field is invariant under its own local flow.
- (ii) w is invariant under α if and only if α and β commute.
- (iii) For t in some interval around 0 we have:

$$\frac{d}{dt} \alpha_t^*(w) = [v, \alpha_t^*(w)].$$

(At $t = 0$ this recovers the geometric definition of $[v, w]$.)

Proof. (i) We must show that $(d\alpha_t)_p(v(p)) = v(\alpha_t(p))$ for any $p \in V$ if $(t, p) \in (-\epsilon, \epsilon) \times V \subset \Phi$. Since α_t is the local flow of v we have

$$\begin{aligned} (d\alpha_t)_p(v(p)) &= (d\alpha_t)_p \left(d(s \mapsto \alpha_s(p))_{s=0} \left(\frac{\partial}{\partial s} \right) \right) \\ &= d(s \mapsto \alpha_t(\alpha_s(p)))_{s=0} \left(\frac{\partial}{\partial s} \right) \\ &= d(s \mapsto \alpha_{s+t}(p))_{s=0} \left(\frac{\partial}{\partial s} \right) \\ &= v(\alpha_t(p)). \end{aligned}$$

(ii) We show that if $\varphi : V \rightarrow M$ is a diffeomorphism onto a relatively compact open subset of M then $\varphi_*(w) = w$ if and only if $\varphi \circ \beta_t = \beta_t \circ \varphi$ for t in an interval around zero. (Applying this to $\varphi = \alpha_s$ for small enough s gives (ii).) To see this note that $\varphi \circ \beta_t \circ \varphi^{-1}$ is again a local flow on $V' = \varphi(V)$: For a smooth function f on V' ,

$$\begin{aligned} \varphi_*(w)(f)(x) &= \varphi^{-1*}(w(\varphi^*f))(x) \\ &= w(f \circ \varphi)(\varphi^{-1}(x)) \\ &= \left. \frac{d}{dt} \right|_{t=0} \beta_t^*(f \circ \varphi)(\varphi^{-1}(x)) \\ &= \left. \frac{d}{dt} \right|_{t=0} (f \circ \varphi \circ \beta_t \circ \varphi^{-1})(x) \end{aligned}$$

where in the third equality we have used (3.1.4). By (3.1.4) again, $\varphi \circ \beta_t \circ \varphi^{-1}$ induces the vector field $\varphi_*(w)$ on V' . So if $\varphi_*(w) = w$ then the flows $\varphi \circ \beta_t \circ \varphi^{-1}$ and β_t must agree for t small enough. This proves (ii).

(iii) Writing out the derivative, we have

$$\begin{aligned} \frac{d}{dt} \alpha_t^*(w) &= \lim_{h \rightarrow 0} \frac{\alpha_{t+h}^*(w) - \alpha_t^*(w)}{h} \\ &= \alpha_t^* \left(\lim_{h \rightarrow 0} \frac{\alpha_h^*(w) - w}{h} \right) \\ &= \alpha_t^*([v, w]) \\ &= [\alpha_t^*(v), \alpha_t^*(w)] \\ &= [v, \alpha_t^*(w)] \end{aligned} \tag{3.4.1}$$

where we have used the geometric description of the bracket in Lemma 3.4.1 in the third equality and the previous lemma in the last equality. \square

Proposition 3.4.5. *Let v, w be vector fields on M with associated local flows α and β . Then $[v, w] = 0$ in a neighbourhood U of p if and only if the local flows commute on U .*

Proof. (\Leftarrow) This implication follows from the formula $[v, w] = \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=0, t=0} \alpha_t^* \beta_s^* - \beta_s^* \alpha_t^*$ given above, but let us prove it using the previous lemma instead. If α and β commute then by (i) w is invariant under α , i.e. $\alpha_t^*(w) = w$ for $|t|$ small enough. By (iii), for small enough $|t|$ we have:

$$0 = \frac{d}{dt} \alpha_t^*(w) = [v, \alpha_t^*(w)] = [v, w].$$

(\Rightarrow) By (i) and (iii) of the previous lemma and properties of the bracket, for $|t|$ small enough we have:

$$\frac{d}{dt} \alpha_t^*(w) = [v, \alpha_t^*(w)] = [\alpha_t^*(v), \alpha_t^*(w)] = \alpha_t^*[v, w].$$

So if $[v, w] = 0$ then $\alpha_t^*(w)$ is constant in t (for t near zero), hence is equal to w (since $\alpha_0 = id$), i.e. w is invariant under α . By (ii) above α and β must commute. \square

3.5. Local Frobenius theorem. Let M be a manifold. A (smooth) **subbundle** of rank k of the tangent bundle TM is a union $E = \cup_{p \in M} E_p$, where $E_p \subset T_p M$ is a k -dimensional subspace which “varies smoothly in p ”: More precisely, we require that for any $p \in M$ there is an open set $U \subset M$ containing p and k vector fields v_1, \dots, v_k such that $v_1(q), \dots, v_k(q)$ form a basis of E_q for all $q \in U$.¹⁴ It is easy to check that E is itself a smooth manifold of dimension $n + k$, in fact an embedded submanifold of the tangent bundle.

Example 3.5.1. A smooth nonvanishing vector field v on M gives the rank one subbundle $\cup_{p \in M} \mathbb{R}v(p)$. More generally we could take the subbundle spanned by k vector fields, provided they are linearly independent everywhere. (Not every subbundle is of this form.)

¹⁴In older books, the assignment $p \mapsto E_p$ is called a smooth **k -distribution** on M . Since distribution has a well-established meaning in analysis it’s best to avoid this terminology.

A k -dimensional smooth subbundle $E \subset TM$ of the tangent bundle is called **integrable** (or **involutive**) on an open set $U \subset M$ if, for any vector fields v, w such that $v(p), w(p) \in E_p$ for all $p \in U$, the bracket has the same property, i.e. $[v, w](p) \in E_p$ for all $p \in U$. We say that $E \subset TM$ is integrable (or involutive) if each point has a neighbourhood on which E is integrable.

Exercise. Prove that E is integrable if it is integrable on $U = M$. (Use a partition of unity. This is false in the real analytic or complex analytic setting.)

Let $E \subset TM$ be a smooth subbundle. A submanifold $i : I \rightarrow M$ is called an **integral manifold** of E if

$$di_p(T_p I) = E_{i(p)} \quad \text{for all } p \in I.$$

Sometimes we will suppress the notation di_p and simply identify $T_p I$ with its image in $T_{i(p)} M$; we will also suppress the inclusion i and think of points of I as points of M . So we will write the above condition simply as: $T_p I = E_p$ for all $p \in I$.

Remark 3.5.2. An integral manifold need not be a closed or embedded submanifold. Here is an example: Fix $\alpha \in \mathbb{R} - \mathbb{Q}$. The integral manifolds (=integral curves=orbits) of the vector field associated with the flow on $S^1 \times S^1$ defined by $\alpha(t, (z_1, z_2)) = (e^{it} z_1, e^{i\alpha t} z_2)$ are dense in $S^1 \times S^1$, hence are not embedded submanifolds.

Theorem 3.5.3 (Local Frobenius theorem). *A subbundle $E \subset TM$ of the tangent bundle of a manifold M is integrable if and only if it admits an integral manifold through any point of M .*

This will be a consequence of two lemmas.

Lemma 3.5.4. *Any integrable subbundle $E \subset TM$ can be locally spanned by k commuting vector fields, i.e. given p , there is a neighbourhood U of p and there exist vector fields w_1, \dots, w_k on U such that*

- (1) E_x is the span of $w_1(x), \dots, w_k(x)$ for all $x \in U$ and
- (2) $[w_i, w_j](x) = 0$ for all i, j and $x \in U$.

Proof. Let U be a neighbourhood of p with local coordinates x_1, \dots, x_n and let v_1, \dots, v_k be k vector fields which span E_x at each point of U . Write $v_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j}$ for smooth functions $a_{ij} = a_{ij}(x_1, \dots, x_n)$. The matrix $A = (a_{ij})$ has rank k near p , so some $k \times k$ submatrix of it is nonsingular. By renumbering we may assume that $A_0 = (a_{ij})_{i=1, \dots, k, j=1, \dots, k}$ is the nonsingular submatrix, and by shrinking U we may assume that A_0 is nonsingular on all of U . Define

$$\begin{pmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_k \end{pmatrix} = A_0^{-1} \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_k \end{pmatrix} = A_0^{-1} A \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial}{\partial x_n} \end{pmatrix}.$$

Then w_1, \dots, w_k are still a basis for E on U (since they are related to the basis v_1, \dots, v_k by the nonsingular matrix A_0). Since A is of the form $(A_0 | *)$, it follows that $A_0^{-1} A$ is of the form $(I | *)$ and so we have

$$w_i = \frac{\partial}{\partial x_i} + \sum_{\ell > k} c_{i\ell} \frac{\partial}{\partial x_\ell}.$$

Now $\left[\frac{\partial}{\partial x_i}, f \frac{\partial}{\partial x_\ell} \right] = \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_\ell}$, so $[w_i, w_j]$ involves only the $\frac{\partial}{\partial x_\ell}$ for $\ell > k$. As E is integrable this must be a linear combination (with smooth function coefficients) of the w_i , so it must vanish. So we have found vector fields satisfying (1) and (2) as in the lemma. \square

The next idea is that the local flows of these commuting vector fields, which commute among themselves, provide coordinates.

Lemma 3.5.5. *Let v_1, \dots, v_k be linearly independent vector fields on $V \subset \mathbb{R}^n$ satisfying $[v_i, v_j] = 0$ for all i, j . For any $p \in V$, there exists a neighbourhood U of p and coordinates x_1, \dots, x_n on U such that $v_i = \frac{\partial}{\partial x_i}$ for $1 \leq i \leq k$ (on U).*

Proof. Choose a chart W containing p and local coordinates z_1, \dots, z_n centred at p . Using a linear change of coordinates, we can arrange that

$$v_1(p), v_2(p), \dots, v_k(p), \frac{\partial}{\partial z_{k+1}}, \dots, \frac{\partial}{\partial z_n} \text{ are linearly independent at } p. \quad (3.5.1)$$

Let $\alpha_i : (-\epsilon, \epsilon) \times W \rightarrow \mathbb{R}^n$ be the local flows associated with the v_i (we shrink W so that one ϵ works for all α_i).

Let B_δ denote the open ball of radius $\delta < \epsilon$ in \mathbb{R}^n with coordinates $t_1, \dots, t_k; y_{k+1}, \dots, y_n$. Define a map

$$h : B_\delta \rightarrow \mathbb{R}^n$$

by

$$h(t_1, \dots, t_k; y_{k+1}, \dots, y_n) = \alpha_1(t_1, \alpha_2(t_2, \dots, \alpha_k(t_k, (0, \dots, 0, y_{k+1}, \dots, y_n)) \dots)).$$

We will show that this is a coordinate system at p for sufficiently small δ . For this it suffices (by the inverse function theorem) to show that the differential $dh_0 : T_0 B_\delta \rightarrow T_p V$ has rank n . By definition, since α_1 is the flow associated with v_1 , we have (by (3.1.2)):

$$\begin{aligned} dh_{(t_1, \dots, t_k; y_{k+1}, \dots, y_n)} \left(\frac{\partial}{\partial t_1} \right) &= (d\alpha_1)_{(t_1, h(0, t_2, \dots, t_k; y_{k+1}, \dots, y_n))} \left(\frac{\partial}{\partial t_1} \right) \\ &= v_1(\alpha_1(t_1, h(0, t_2, \dots, t_k; y_{k+1}, \dots, y_n))) \\ &= v_1(\alpha_1(t_1, \alpha_2(t_2, \dots, \alpha_k(t_k, (0, \dots, 0, y_{k+1}, \dots, y_n)) \dots))) \end{aligned}$$

or

$$dh_{(t_1, \dots, t_k; y_{k+1}, \dots, y_n)} \left(\frac{\partial}{\partial t_1} \right) = v_1(h(t_1, \dots, t_k; y_{k+1}, \dots, y_n)).$$

Since the vector fields v_i commute among themselves, the flows α_i also do so, so we could change the order of the α_i s in the definition of h and get that for any $j \leq k$:

$$dh_{(t_1, \dots, t_k; y_{k+1}, \dots, y_n)} \left(\frac{\partial}{\partial t_j} \right) = v_j(\alpha_1(t_1, \alpha_2(t_2, \dots, \alpha_k(t_k, (0, \dots, 0, y_{k+1}, \dots, y_n)) \dots)))$$

i.e.

$$dh_{(t_1, \dots, t_k; y_{k+1}, \dots, y_n)} \left(\frac{\partial}{\partial t_j} \right) = v_j(h(t_1, \dots, t_k; y_{k+1}, \dots, y_n)) \quad \text{for } 1 \leq j \leq k. \quad (3.5.2)$$

Putting $(t_1, \dots, t_k; y_{k+1}, \dots, y_n) = (0, \dots, 0) = 0$ we have

$$dh_0 \left(\frac{\partial}{\partial t_j} \right) = v_j(h(0, \dots, 0)) = v_j(p) \quad \text{for } j \leq k.$$

We also have

$$dh_0 \left(\frac{\partial}{\partial y_j} \right) = \frac{\partial}{\partial z_j} \quad \text{for } j > k$$

(since $h(0, \dots, 0; y_{k+1}, \dots, y_n) = (0, \dots, 0, y_{k+1}, \dots, y_n)$). By (3.5.1), dh_0 has rank n and hence by the inverse function theorem h is a diffeomorphism onto $h(B_\delta)$ for small δ .

Now let $U = h(B_\delta)$ and let us check that in the coordinates $t_1, \dots, t_k, y_{k+1}, \dots, y_n$ given by $h^{-1} : U \rightarrow B_\delta$, the vector fields v_j are as we want. But this is precisely what the equation (3.5.2) says. This proves the proposition. \square

Putting these two lemmas together we prove the local Frobenius theorem.

Proof. Suppose first that $E \subset TM$ admits integral manifolds through any point. We must show that it is integrable. Let U be an open set and v, w vector fields with $v(p), w(p) \in E_p$ for all $p \in U$. Let $p \in U$ and let $i : I \rightarrow M$ be an integral manifold through p . Now there are vector fields \tilde{v} and \tilde{w} on I that are i -related to v and w . Indeed, they are defined by

$$di_x(\tilde{v}(x)) = v(i(x)) \quad (x \in I)$$

and similarly for \tilde{w} . It is easy to see that they are smooth vector fields on I e.g. using the normal form for immersions. Now the fact that the bracket of i -related vector fields is i -related shows that

$$di_x([\tilde{v}, \tilde{w}](x)) = [v, w](i(x)) \quad (x \in I).$$

In particular for x such that $i(x) = p$ we get that $[v, w](p) \in di_x(T_x I) = E_p$. Thus E is integrable.

Conversely, suppose $E \subset TM$ is integrable and $p \in M$. There is a neighbourhood U of p on which E is spanned by k commuting vector fields (by the first lemma above). By the second lemma, we may assume that there are local coordinates on U centred at p in which E is the span of the vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$. Clearly for small enough $c = (c_{k+1}, \dots, c_n)$ the “slices”

$$U_c := \{(x_1, \dots, x_n) \in U : x_{k+1} = c_{k+1}, \dots, x_n = c_n\}$$

are integral manifolds of E , and the choice $c = (0, \dots, 0)$ gives an integral manifold through p . \square

(This proof of the local Frobenius theorem is from R. Narasimhan’s book *Analysis on Real and Complex Manifolds* (or see his TIFR lectures *Topics in Analysis*). For a slightly different proof see Warner’s book.)

3.6. Examples. (i) Consider the subbundle in $\mathbb{R}^3 - \{0\}$ given by

$$E_{x=(x_1, x_2, x_3)} = \{v \in T_x \mathbb{R}^3 : v \perp x\}.$$

(Here we are using identifications $T_x(\mathbb{R}^3 - \{0\}) = T_x \mathbb{R}^3 = \mathbb{R}^3$.) This subbundle is integrable and the integral manifolds are spheres centred at the origin.

(ii) Consider the subbundle in $\mathbb{R}^3 - \{0\}$ given by

$$E_{x=(x_1, x_2, x_3)} = \{v \in T_x \mathbb{R}^3 : v \perp (x_2, x_3, x_1)\}.$$

Check that this is not integrable.

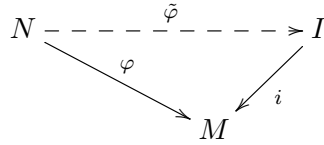
(iii) Consider the subbundle of $T\mathbb{R}^3$ spanned at the point (x, y, z) by $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$. Check that this is not integrable. (Note that $(x(t), y(t), z(t)) = (x_0, y_0 + t, z_0)$ is an integral curve for all (x_0, y_0, z_0) .)

Remark 3.6.1. A **completely nonintegrable hyperplane field** or **contact structure** on an n -manifold M is a subbundle of TM of rank $n - 1$ such that for every pair of vector fields $v, w \in F$ which are nonzero and linearly independent near a point x , the bracket $[v, w]$ does not lie in F .

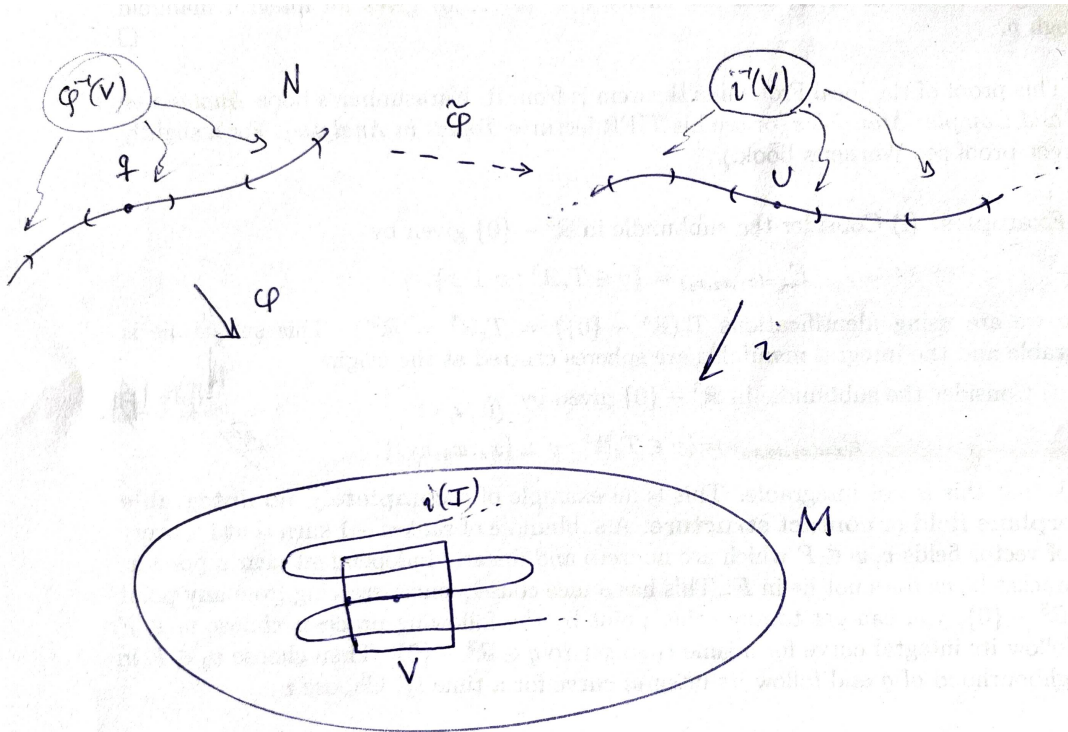
If you have a contact structure F on a manifold M , then you can do the following: Start from any point p_1 , choose a vector field $v_1 \in F$ and follow its integral curve for a time t_1 to get to p_2 . Then choose another vector field $v_2 \in F$ and follow its integral curve for times t_2 . Then choose $v_3 \in F$ and If F is contact then you can reach an open set in M (which will sometimes be the whole of M) by such operations, whereas if F is integrable then you would be confined to an integral manifold in M . (Typically the background manifold M in such considerations arises as a state space, i.e. the set of possible configurations of some physical system, and following the integral curves which pointwise lie in a contact structure allows you to reach a “large” set of configurations.)

3.7. Integral manifolds and the global Frobenius theorem. Integral manifolds of integrable subbundles $E \subset TM$ need not be embedded but they have the following “factoring” property which holds if $i : I \hookrightarrow M$ is embedded:

Theorem 3.7.1 (Factoring property of integral manifolds). *Suppose that $i : I \hookrightarrow M$ is an integral manifold of an integrable subbundle $E \subset TM$ and suppose that $\varphi : N \rightarrow M$ is a smooth mapping such that $\varphi(N) \subset i(I)$. Then there is a smooth mapping $\tilde{\varphi} : N \rightarrow I$ such that $i \circ \tilde{\varphi} = \varphi$:*



Proof. The mapping $\tilde{\varphi}$ is already given as a set map, since i is injective, and it is given by $\tilde{\varphi}(n) = i^{-1}(\varphi(n))$ for $n \in N$. So we have only to prove that it is smooth. The main point is actually to prove continuity. The picture is as follows:



Let $p \in I$ and let U be a neighbourhood of p in I . For $q \in N$ with $\tilde{\varphi}(q) = p$, i.e. with $\varphi(q) = i(p)$ we must find a neighbourhood W of q such that $\tilde{\varphi}(W) \subset U$, i.e. $\varphi(W) \subset i(U)$. Choose a neighbourhood V of $i(p)$ in M such that $i(U) \subset V$ and V has local coordinates $x_1, \dots, x_m = \dim M$ such that the slices

$$V_c = \{x_{k+1} = c_{k+1}, \dots, x_m = c_m\} \quad (\text{for } c = (c_{k+1}, \dots, c_m) \in (-\epsilon, \epsilon)^{m-k})$$

are integral manifolds of E . We further assume the coordinates are centred at $i(p)$, so that $i(U) \cap V$ is contained in the slice $V_0 = \{x_{k+1} = \dots = x_m = 0\}$ of V . By possibly shrinking V further we can arrange that $i(U) \cap V = V_0$.

Let us remark that each connected component of $i^{-1}(V)$ is mapped by i into in a single slice of V . Indeed, this is because the functions $\tilde{x}_i = x_i \circ i$ for $i \geq k + 1$ are locally constant on $i^{-1}(V)$. (If $i > k$ then $d\tilde{x}_i|_{T_p I} = 0$ for $j \leq k$ and $p \in i^{-1}(V)$ because $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$ span $di_p(T_p I)$ and $dx_j \left(\frac{\partial}{\partial x_i} \right) = 0$ if $j > k$ and $i \leq k$).

Now we remark that the connected components of $i^{-1}(V)$ map (under i) onto connected components of $i(I) \cap V$ in the topology of V (i.e. of M). To see this note that since I is second countable the set $i^{-1}(V)$ has countably many connected components, each mapping into a single slice, hence $i(I) \cap V$ maps under the projection $V \rightarrow \mathbb{R}^{m-k}$ to a countable collection of points. The connected components of a countable set in \mathbb{R}^{m-k} are points, hence the V -connected components of $i(I) \cap V$ are contained in slices. Now on a single slice the two topologies agree because the slices are embedded submanifolds, so the assertion follows.

Let $q \in \varphi^{-1}(i(p))$. Let

$$W := \text{connected component of } \varphi^{-1}(V) \text{ containing } q.$$

This is an open neighbourhood of q in N . To prove continuity of $\tilde{\varphi}$, it will suffice to show that $\tilde{\varphi}(W) \subset U$, i.e. that $\varphi(W) \subset i(U)$. By continuity of φ , $\varphi(W)$ is a connected set containing $i(p)$, so to prove that $\varphi(W) \subset i(U)$ it will suffice to show that $\varphi(W)$ is contained in the single slice $V_0 = \{x_{k+1} = \dots = x_m = 0\}$, since we have arranged that $i(U) \cap V = V_0$. But $\varphi(W) \subset i(I) \cap V$, and the components of $i(I) \cap V$ are contained in slices (as remarked earlier), so $\varphi(W)$ is contained in a single slice, which must be the slice V_0 since $i(p) \in \varphi(W)$. So $\varphi(W) \subset V_0 = i(U) \cap V$ and hence $\tilde{\varphi}(W) \subset U$.

The smoothness of $\tilde{\varphi}$ is left as an exercise. □

Note that the integrability of the subbundle is used essentially in the proof to have information on all the integral manifolds in the neighbourhood V , not just on I . Also, the second countability of I played a key role in the proof.

The following example is useful to think about: You can construct a 1-1 immersion $i : \mathbb{R} \rightarrow \mathbb{R}^2$ with image the figure eight in the plane and an immersion $\varphi : (0, 1) \rightarrow \mathbb{R}^2$ with image inside the figure eight but which does not factor through i even continuously. (This shows that the immersion i cannot be an integral manifold of an integrable subbundle of $T\mathbb{R}^2$.)

Integral manifolds of an integral subbundle through a given point are obviously not unique: an open subset of an integral manifold is again an integral manifold. Taking the maximal (with respect to inclusion) integral manifold through a point fixes this:

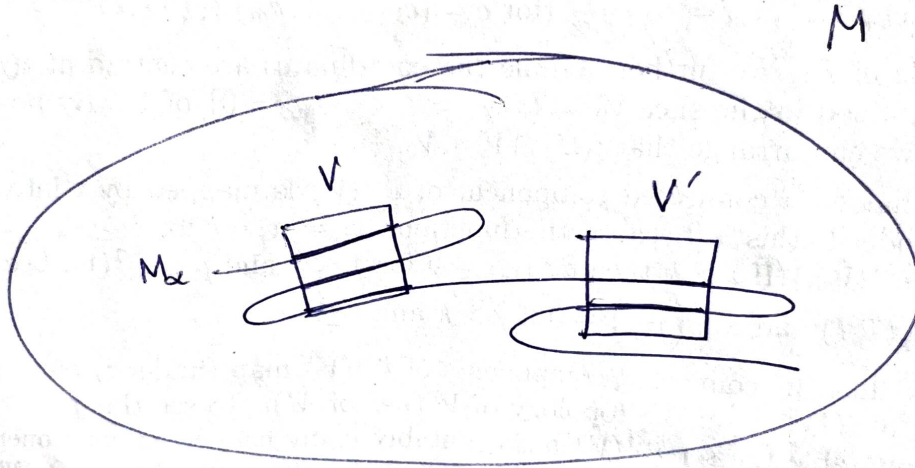
Theorem 3.7.2 (Global Frobenius theorem). *Let M be a smooth manifold and let E be an integrable subbundle of TM . Then for each $p \in M$ there exists a unique maximal connected integral manifold of E through p .*

Proof. Maximality means that every other connected integral manifold of E through p is contained in this one. Uniqueness means that if $i : I \hookrightarrow M$ and $j : J \hookrightarrow M$ are two maximal connected integral manifolds of E through p then there exists a diffeomorphism $\varphi : I \rightarrow J$ such that $i = j \circ \varphi$. It is easy to check that maximality implies uniqueness by the previous theorem (factorization property), so it is left as an exercise. Let us prove existence.

Let \mathcal{S} be the collection of all subsets of M which are unions of integral manifolds of E . This defines a topology on M , which it is easy to check is finer (i.e. has more open sets) than the topology on M . Write $M_{\mathcal{S}}$ for M with this topology. Then $M_{\mathcal{S}}$ is Hausdorff and locally connected, and also locally Euclidean (of dimension the rank of E). Write

$$M_{\mathcal{S}} = \bigsqcup_{\alpha \in A} M_{\alpha}$$

as a union of connected components, which are both open and closed (because $M_{\mathcal{S}}$ is locally connected). Then any connected integral manifold $i : I \rightarrow M$ has $i(I) \subset M_{\alpha}$ for some α . The picture is as follows:



We will show that the M_α are integral manifolds; this will prove the existence part of the theorem. The key point is to prove second countability of each M_α , since the locally Euclidean property (i.e. existence of charts) is clear.

To prove that the M_α are second countable, cover M by countably many neighbourhoods V with coordinates x_1, \dots, x_m taking values in $(-\epsilon, \epsilon)$ such that the slices V_c for $c = (c_{k+1}, \dots, c_n) \in (-\epsilon, \epsilon)^{n-k}$ are integral manifolds of E . (We can do this because of the local Frobenius theorem and second countability of M .) Since M_α and V_c are both open in the topology \mathcal{S} , we know that $M_\alpha \cap V_c$ is open in V_c for all $c = (c_{k+1}, \dots, c_n)$. Now $M_\alpha \cap V$ is open in M_α and

$$M_\alpha \cap V = \bigsqcup_c M_\alpha \cap V_c$$

is a disjoint union (in M_α). Each of the $M_\alpha \cap V_c$ is both open and closed, so a connected component of $M_\alpha \cap V$ is contained in some open in some $M_\alpha \cap V_c$ and hence is second countable since V_c is an open subset of \mathbb{R}^{n-k} . Thus we can write the connected space M_α as a countable union

$$M_\alpha = \bigcup_V M_\alpha \cap V$$

with each connected component of each $M_\alpha \cap V$ being second countable. By the lemma below we conclude that M_α is second countable.

Now there is clearly a unique smooth manifold structure on M_α such that each integral manifold of E contained in M_α is an open submanifold. \square

Let us prove the lemma used above:

Lemma 3.7.3. *Let X be a connected, locally connected Hausdorff space. Suppose that $X = \bigcup_{n \geq 1} X_n$ where $X_n \subset X$ are open and each connected component of X_n is second countable. Then X is second countable.*

Proof. The idea is to write X as a countable union of second countable open sets by “adding one connected component of some X_n at a time”.

Let \mathcal{C}_n be the collection of connected components of X_n and let $\mathcal{C} = \bigsqcup_{n \geq 1} \mathcal{C}_n$. In other words \mathcal{C} indexes all connected components of all X_n s, which are all second countable subsets of X . Now if $C \in \mathcal{C}$, then $\{C' \in \mathcal{C} : C \cap C' \neq \emptyset\}$ is countable. (Indeed, if not then there would exist an n for which $\{C' \in \mathcal{C}_n : C' \cap C \neq \emptyset\}$ is uncountable. But then $\{C \cap C' : C' \in \mathcal{C}_n, C \cap C' \neq \emptyset\}$ gives an uncountable collection of pairwise disjoint open sets in C . Since C is assumed 2nd countable this is impossible.)

Let $C_0 \in \mathcal{C}$ and set $J_0 = \{C_0\}$. Inductively define countable sets J_k by

$$J_k = \{C \in \mathcal{C} : C \text{ meets an element of } J_{k-1}\}.$$

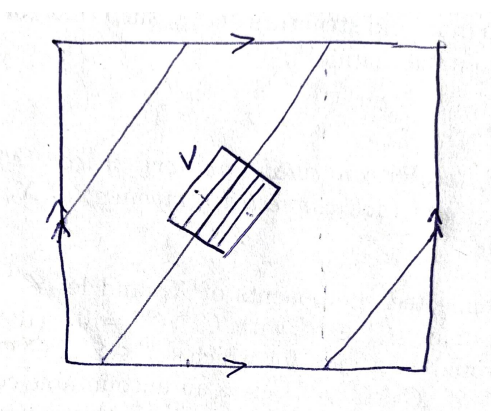
Let $Y = \bigcup_{k \geq 0} \bigcup_{C \in J_k} C$. This is a countable union of second countable open sets, hence second countable and open in X . We show that Y is closed in X . Indeed, if $x \in \bar{Y}$ there is a $C \in \mathcal{C}$ such that $x \in C$. Since $C \cap Y \neq \emptyset$ there is a k and $C' \in J_k$ such that $C \cap C' \neq \emptyset$. But then $C \in J_{k+1}$, and hence $x \in Y$.

Thus Y is open and closed in X , so $Y = X$ as X is connected (and locally connected). Thus $X = Y$ is second countable. \square

(The proof of the global Frobenius theorem here is originally from Chevalley's *Theory of Lie Groups*, as modified slightly in Varadarajan's *Lie Groups and Lie Algebras*. Warner has a (very) slightly different version.)

Here is a different description of the maximal connected integral manifold I through $p \in M$: A point $x \in M$ belongs to I if there exists a piecewise-smooth path joining p and x , each smooth segment of which is an integral curve of E (in the sense that the tangent vector belongs to E pointwise). (Exercise: Show that this is the same as the definition given above.)

Example 3.7.4. It is useful to keep in mind the example of the irrational slope flow on the torus $M = \mathbb{R}^2/\mathbb{Z}^2$. The corresponding vector field defines a one-dimensional integrable subbundle of the tangent bundle, and the integral manifolds are unions of segments. The picture is as follows:



If we take a small square neighbourhood V of a point then the topology induced on V by $M_{\mathcal{G}}$ is homeomorphic to the following: $(0, 1)^2$ with the topology that a set is open if its intersection with $(0, 1) \times \{y_0\}$ is open for every y_0 . In this topology $(0, 1)^2$ has uncountably many connected components, each homeomorphic to $(0, 1)$. So it satisfies all the conditions to be a manifold (in fact, an integral manifold) except 2nd countability. The second countability of maximal connected integral manifolds in the theorem tells us that the intersection $I \cap V$ must have countably many connected components.

Belonging to the same maximal connected integral manifold is an equivalence relation on M . The decomposition of M into equivalence classes is called a **foliation** and the maximal integral manifolds are the **leaves** of the foliation.

Remark 3.7.5. (Real analytic and complex analytic settings) The local and global Frobenius theorems hold also in the real analytic setting, i.e. a real analytic subbundle of the tangent bundle of a real analytic manifold is integrable (i.e. closed under bracket) if and only if it admits integral manifolds through every point. Similarly, in the complex analytic setting, a holomorphic subbundle of the holomorphic tangent bundle of a complex manifold is integrable (i.e. closed under bracket) if and only if it admits integral manifolds through every point (which are then holomorphic submanifolds). The proofs are the same because all that we really used was the inverse function theorem, which holds in both those settings.

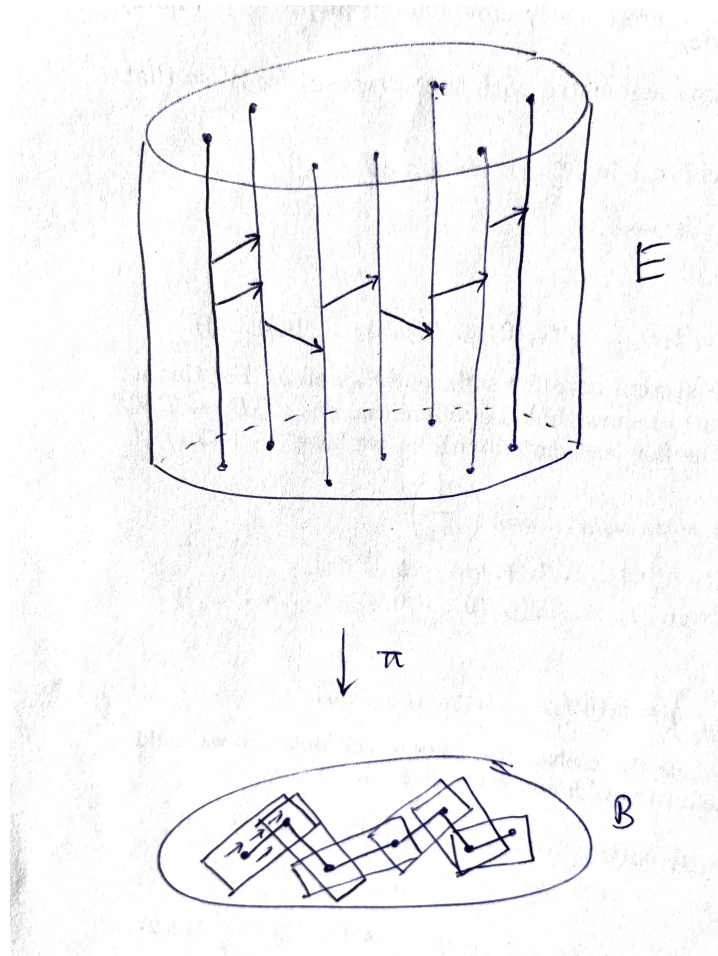
3.8. Ehresmann's theorem revisited. We will use the notion of vector fields and flows to give another proof of a weak version of Ehresmann's theorem, namely the assertion that fibres are diffeomorphic if the base is connected. This will use the following fact:

Proposition 3.8.1. *Let $f : M \rightarrow N$ be a smooth submersion and let v be a smooth vector field on N . Then v can be lifted to a vector field \tilde{v} on M , i.e. there exists a smooth vector field \tilde{v} on M which is f -related to v , i.e. $df_p(v(p)) = \tilde{v}(f(p))$ for all $p \in M$.*

Proof. This is true locally on the source, by the normal form for submersions. Now use a partition of unity. \square

Corollary 3.8.2 (Weak version of Ehresmann theorem). *If $\pi : E \rightarrow B$ is a proper smooth submersion of smooth manifolds and B is connected then any two fibres are diffeomorphic.*

Proof. We may assume B is connected. Let $a, b \in B$. The picture is as follows:



We may choose a piecewise smooth curve in B from a to b such that each smooth segment is within a chart. We may further choose a vector field on each chart such that the two endpoints of the segment are related by the flow of the vector field, for a time t_0 . We may assume that integral curves of v exist for $|t| < \epsilon$ and $t_0 < \epsilon$. By the previous proposition we may lift each vector field to E and the flow of the lifted vector field exists for the same time $|t| < \epsilon$ (by properness of π , see the lemma below). Now compose the flows on E for each segment to get a diffeomorphism of the fibres at a and b . \square

Lemma 3.8.3. *If $f : M \rightarrow N$ is proper and the vector fields \tilde{v} on M and v on N are f -related, then if the flow of v exists on $U \subset N$ exists for $t \in (-\epsilon, \epsilon)$ then the flow of \tilde{v} exists on $f^{-1}(U)$ for $t \in (-\epsilon, \epsilon)$.*

Proof. Exercise. □

Remark 3.8.4. Notice that the proof does not work in the holomorphic or real-analytic category as the proof of the proposition lifting vector fields in submersions fails.

4. BASICS OF COMPLEX MANIFOLDS (TO BE FINISHED)

Standard references are Chapter 1 of Voisin, *Hodge Theory and Complex Algebraic Geometry* or Chapter 0 of Griffiths and Harris, *Principles of Algebraic Geometry*.

4.1. Orientations. The following easy fact of linear algebra will be useful: Suppose V is a complex vector space and $A : V \rightarrow V$ a \mathbb{C} -linear transformation. Let $A_{\mathbb{R}} \in \text{Hom}_{\mathbb{R}}(V_{\mathbb{R}}, V_{\mathbb{R}})$ be the underlying \mathbb{R} -linear map on $V_{\mathbb{R}}$, by which we mean V considered as an \mathbb{R} -vector space. Then $\det(A_{\mathbb{R}}) = \det(A) \overline{\det(A)}$ ⁽¹⁵⁾. In particular, $\det(A) \neq 0 \Leftrightarrow \det(A_{\mathbb{R}}) > 0$. ⁽¹⁶⁾

Recall that an orientation of a real vector space is a choice of generator for the top exterior power, up to scaling by \mathbb{R}_+ . The linear algebra fact just mentioned implies that the underlying real vector space of a complex vector space V has a canonical orientation: If we choose a \mathbb{C} -basis v_1, \dots, v_n of V then

$$v_1, \sqrt{-1}v_1, v_2, \sqrt{-1}v_2, \dots, v_n, \sqrt{-1}v_n$$

is an \mathbb{R} -basis for V , and the class of this \mathbb{R} -basis is independent of the choice. ⁽¹⁷⁾ This is clear: If A is the change of basis matrix for another complex basis then $\det(A_{\mathbb{R}}) = \det(A) \overline{\det(A)} > 0$, so that the \mathbb{R} -bases given like this have the same orientation.

This has the following important consequence: If M is a complex manifold then its underlying real manifold is always orientable, and in fact comes with a preferred choice of orientation. For $M = \mathbb{C}^n$ or any open subset of it with coordinates $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$ this works as for vector spaces. Now any holomorphic map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ or between open subsets of \mathbb{C}^n automatically satisfies $\det(\text{Jac}(f)) \geq 0$, because it is \mathbb{C} -linear. (For $n = 1$ the Jacobian of a holomorphic function $f(z) = u(x, y) + iv(x, y)$ is $\text{Jac}(f) = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$ which by the Cauchy-

Riemann equations $u_x = v_y, u_y = -v_x$ is equal to $\begin{pmatrix} u_x & v_x \\ -v_x & u_x \end{pmatrix}$ so $\det(\text{Jac}(f)) = u_x^2 + v_x^2 \geq 0$.

The determinant is strictly positive everywhere since one cannot have $u_x = v_x = 0$ unless f is constant. For $n > 1$ one has to work with the generalization of the CR conditions.) So a holomorphic atlas of a complex manifold is automatically oriented, and taking the canonical orientation above locally fixes a canonical orientation globally since $\det_{\mathbb{C}}(\text{Jac}(\varphi_{\alpha} \circ \varphi_{\beta}^{-1})) > 0$ for overlapping holomorphic charts.

Remark 4.1.1. The following consequence is very useful: If X is a complex manifold and $A, B \subset X$ are complex submanifolds which meet transversely, then $A, B, A \cap B$ are all canonically oriented and then $\xi_A \cdot \xi_B = \xi_{A \cap B}$ and $[A] \cdot [B] = [A \cap B]$ (no signs!). If, further $X \subset \mathbb{C}P^n$ or, more generally, X is a compact *Kähler manifold* (to be defined later), then the class $[A] \in H_{2 \dim A}(X)$ is also nonzero.

4.2. Holomorphic tangent and cotangent bundles. If X is a complex manifold, then its tangent space (considered as a real manifold) at every point is naturally a \mathbb{C} -vector space: A chart $\varphi_{\alpha} : U \rightarrow \mathbb{C}^n$ at x gives an identification $T_x X \cong T_{\varphi_{\alpha}(x)} \mathbb{C}^n \cong \mathbb{C}^n$ and a different chart at x gives the same complex structure on $T_x X$ as they are, by assumption, holomorphically related. Thus multiplication by $\sqrt{-1}$ gives an operator $J_x : T_x X \rightarrow T_x X$ which varies smoothly in x and

¹⁵*Proof.* Choose a \mathbb{C} -basis v_1, \dots, v_n in which A is upper-triangular. Let $\lambda_1, \dots, \lambda_n$ be the diagonal entries. Then in the \mathbb{R} -basis $v_1, \sqrt{-1}v_1, \dots, v_n, \sqrt{-1}v_n$ the matrix $A_{\mathbb{R}}$ looks block-upper-triangular with the 2×2 blocks $\begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ on the diagonal, where $\lambda_i = a_i + \sqrt{-1}b_i$ with $a_i, b_i \in \mathbb{R}$. Then $\det(A_{\mathbb{R}}) = \prod_i (a_i^2 + b_i^2) = \prod_i \lambda_i \bar{\lambda}_i = \det(A) \overline{\det(A)}$. \square

¹⁶Here is an unnecessarily complicated way to see the same fact: The map $A \mapsto A_{\mathbb{R}}$ defines a map $M(n, \mathbb{C}) \rightarrow M(2n, \mathbb{R})$ by choosing a basis. On invertible elements this defines $GL(n, \mathbb{C}) \rightarrow GL(2n, \mathbb{R})$. Now $GL(n, \mathbb{C})$ is connected (it is the complement in $M(n, \mathbb{C}) = \mathbb{C}^{n^2}$ of the zero set of a single polynomial \det) while $GL(2n, \mathbb{R})$ has two components, namely the matrices with determinant > 0 or < 0 , so that $GL(n, \mathbb{C})$ must map into the identity component $GL(2n, \mathbb{R})$, which has $\det > 0$.

¹⁷It does, however, depend on the choice of square root of -1 , which we assume that we fixed, once and for all, at the beginning of time.

with $J_x^2 = -Id$. In other words, we have a section J of the bundle $End(TX) = Hom(TX, TX)$ which squares to $-Id$.

An **almost complex structure** on a smooth manifold M is an endomorphism J of the tangent bundle TM such that $J^2 = -Id$. By the previous paragraph, a complex manifold gives a real manifold with an almost complex structure. We will come to the question of when an almost complex structure on an even-dimensional real manifold actually comes from a complex manifold structure later.

Now consider the complexification of the tangent space at $x \in X$. The action of J , extended \mathbb{C} -linearly, can be diagonalized and the eigenvalues are $\pm\sqrt{-1}$:

$$T_x X \otimes_{\mathbb{R}} \mathbb{C} = T_x^{1,0} X \oplus T_x^{0,1} X$$

where $T_x^{1,0} X = (T_x X)^{J_x = \sqrt{-1}}$ and $T_x^{0,1} X = (T_x X)^{J_x = -\sqrt{-1}}$ are the **holomorphic tangent space** and the **antiholomorphic tangent space**, respectively.

Example 4.2.1. If $X = \mathbb{C}^n$ then $T_x X \cong \mathbb{C}^n$ and if $z_i = x_i + \sqrt{-1}y_i$ are coordinates then $T_x X$ has the \mathbb{R} -basis $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}_{i=1, \dots, n}$. In this basis the J -operator is given by $J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$. The holomorphic tangent space (resp. antiholomorphic tangent space) is spanned by

$$\frac{\partial}{\partial z_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} - \sqrt{-1} \frac{\partial}{\partial y_i} \right) \quad (\text{resp. } \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} + \sqrt{-1} \frac{\partial}{\partial y_i} \right))$$

for $i = 1, \dots, n$. Similarly, the holomorphic (resp. antiholomorphic) cotangent spaces are spanned by the

$$dz_i = dx_i + \sqrt{-1}dy_i \quad (\text{resp. } d\bar{z}_i = dx_i - \sqrt{-1}dy_i)$$

for $i = 1, \dots, n$. These are the dual bases to $\left\{ \frac{\partial}{\partial z_i} \right\}$ resp. $\left\{ \frac{\partial}{\partial \bar{z}_i} \right\}$.

Of course, the previous example is the local picture on any complex manifold.

If $f : X \rightarrow Y$ is a holomorphic map of complex manifolds, then $df_x \otimes \mathbb{C} : T_x X \otimes \mathbb{C} \rightarrow T_{f(x)} Y \otimes \mathbb{C}$ maps $T_x^{1,0} X$ to $T_{f(x)}^{1,0} Y$ (since df_x commutes with the action of the J -operators). If $\varphi : U \rightarrow U' \subset \mathbb{C}^n$ and $\psi : V \rightarrow V' \subset \mathbb{C}^m$ are local charts at x and $f(x)$, and $f(U) \subset V$, then in these local coordinates the map $df_x : T_x^{1,0} X \rightarrow T_{f(x)}^{1,0} Y$ is given by the Jacobian matrix

$$Jac_{\mathbb{C}}(\psi \circ f \circ \varphi^{-1}) = \left(\frac{\partial(\psi \circ f \circ \varphi^{-1})_i}{\partial z_j} \right)_{i=1, \dots, m; j=1, \dots, n}$$

with respect to the holomorphic coordinates. (Here $((\psi \circ f \circ \varphi^{-1})_i)_{i=1, \dots, m}$ are the m component functions of $\psi \circ f \circ \varphi^{-1} : U' \rightarrow V'$.)

Letting $x \in X$ vary, we get the **holomorphic tangent bundle** $T^{1,0} X = \coprod_{x \in X} T_x^{1,0} X$. This is a complex vector bundle of rank n . Moreover, it is a holomorphic vector bundle, with transition maps given by the Jacobian with respect to the holomorphic coordinates, i.e. the $n \times n$ matrix

$$Jac_{\mathbb{C}}(\varphi_{\alpha} \circ \varphi_{\beta}^{-1}) = \left(\frac{\partial(\varphi_{\alpha} \circ \varphi_{\beta}^{-1})_i}{\partial z_j} \right)$$

of holomorphic functions. Similarly, there is the **antiholomorphic tangent bundle** $T^{0,1} X$, which is not a holomorphic bundle as its transition maps are given by antiholomorphic functions. Note that $T^{1,0} X$ is canonically isomorphic, as a real vector bundle, to the usual tangent bundle TX (by the composition $T^{1,0} X \subset TX \otimes \mathbb{C} \xrightarrow{Re} TX$).

Given $f : X \rightarrow Y$ a holomorphic map of complex manifolds we then have a homomorphism of holomorphic vector bundles $T^{1,0} X \rightarrow f^* T^{1,0} Y$. This is determined by the homomorphism $TX \rightarrow f^* TY$ using the J -operator.

Remark 4.2.2. (Holomorphic immersions and submersions) Let $f : X \rightarrow Y$ be a holomorphic map of complex manifolds and $x \in X$. Then the diagram

$$\begin{array}{ccccc} T_x^{1,0}X & \xrightarrow{\subset} & T_xX \otimes \mathbb{C} & \xrightarrow{Re} & T_xX \\ \downarrow & & \downarrow df_x \otimes \mathbb{C} & & \downarrow df_x \\ T_{f(x)}^{1,0}Y & \xrightarrow{\subset} & T_{f(x)}Y \otimes \mathbb{C} & \xrightarrow{Re} & T_{f(x)}Y \end{array} \quad (4.2.1)$$

commutes. (Exercise: Check this.) Since the horizontal compositions are isomorphisms it follows that the property of df_x being an immersion or submersion at x can be checked using the Jacobian $Jac_{\mathbb{C}}(f)$ with respect to holomorphic coordinates.

Example 4.2.3. Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a holomorphic function. The level set $f^{-1}(c)$ is a complex manifold if the locus

$$\{z \in \mathbb{C}^{n+1} : Jac_{\mathbb{C}}(f)(z) = 0\} \cap f^{-1}(c)$$

is empty. Here $Jac_{\mathbb{C}}(f) = \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)$.

4.3. Example: Hypersurfaces in projective space. Let P be a homogeneous polynomial of degree d in $n+1$ variables with complex coefficients. The closed subset of $\mathbb{C}P^n$ defined by

$$V(P) := \{[z_0 : \dots : z_n] \in \mathbb{C}P^n : P(z_0, \dots, z_n) = 0\}$$

is called a **projective hypersurface of degree d in $\mathbb{C}P^n$** . More generally, for homogeneous polynomials P_1, \dots, P_k we can consider

$$V(P_1, \dots, P_k) := \{[z_0 : \dots : z_n] \in \mathbb{C}P^n : P_i(z) = 0 \text{ for } i = 1, \dots, k\}.$$

which is the same as $V(P_1) \cap V(P_2) \cap \dots \cap V(P_k)$.

When is $V(P)$ smooth? Considering P as a holomorphic function $P : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{C}$ we see that $P^{-1}(0)$ is smooth if $\{z \in \mathbb{C}^{n+1} - \{0\} : Jac_{\mathbb{C}}(P)(z) = 0\} \cap P^{-1}(0)$ is empty. Now for a homogenous polynomial P of degree d ,

$$\sum_{i=0}^n z_i \frac{\partial P}{\partial z_i} = dP$$

so that $Jac_{\mathbb{C}}(P)(z) = 0$ for $z \in \mathbb{C}^{n+1} - \{0\}$ implies that $P(z) = 0$. Thus $P^{-1}(0)$ is smooth if $Jac_{\mathbb{C}}(P) \neq 0$ in $\mathbb{C}^{n+1} - \{0\}$. Now consider what happens in $\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$. In the chart $U_0 = \{z_0 = 1\} \cong \mathbb{C}^n$ we have the function $P_1 := P(1, z_1, \dots, z_n)$, which defines a smooth submanifold unless $\{Jac_{\mathbb{C}}(P_1) = 0\} \cap \{P_1 = 0\}$ is nonempty. Now $Jac_{\mathbb{C}}(P_1)(w)$ is the projection of $Jac_{\mathbb{C}}(P)(1, w)$ to the last n coordinates, so that if w is in the zero locus of $Jac_{\mathbb{C}}(P_1)$ then $(1, w)$ is in the zero locus of $Jac_{\mathbb{C}}(P)$. A similar argument works for the other charts $\{z_i = 1\}$. Thus we have proved the “if” part of:

Lemma 4.3.1 (Jacobian criterion). $V(P)$ is smooth iff $V(Jac_{\mathbb{C}}(P)) = \emptyset$.

We will only use the implication “ $V(Jac_{\mathbb{C}}(P)) = \emptyset \Rightarrow V(P)$ is smooth” here, which was just explained. We will refer to $V(Jac_{\mathbb{C}}(P))$ as the **singular locus** of $V(P)$.

Remark 4.3.2. Notice also that we have identified the tangent space

Example 4.3.3. Suppose $P = z_0^n + z_1 \cdots z_n$. Then

$$Jac_{\mathbb{C}}(P) = (nz_0^{n-1}, z_2 \cdots z_n, \dots, z_1 \cdots z_{n-1}).$$

The singular locus is.

Example 4.3.4 (Fermat hypersurfaces). Let $P = z_0^d + \dots + z_n^d$ for $d \geq 1$. Then $V(P)$ is called the Fermat hypersurface of degree d . The Jacobian criterion implies that it is smooth.

Example 4.3.5. Consider the hypersurface $V(P) \subset \mathbb{C}P^n$ defined by $P(z_0, z_1, z_2, z_3) = \sum_{i=0}^n z_i^{n+1} - z_0 \cdots z_n$. Show that the singular locus is $\{(1 : 0 : \dots : 0), \dots, (0 : \dots : 0 : 1)\}$.

Remark 4.3.6. The Jacobian criterion works for $V(P_1, \dots, P_k)$ as follows:

The space of homogeneous degree d polynomials in $n + 1$ variables with \mathbb{C} -coefficients is identified with \mathbb{C}^N for $N = \binom{n+d}{n}$, which is the number of monomials of degree d in $n + 1$ variables. To see what the condition “ $V(P)$ is smooth” looks like, we need to use a piece of algebra. There is a general algebraic gadget called the **resultant** $Res(Q_0, \dots, Q_n)$ of $n + 1$ homogeneous polynomials Q_0, \dots, Q_n in $n + 1$ variables, which has the property that $Res(Q_0, \dots, Q_n)$ vanishes if and only if the Q_i have a common zero in $\mathbb{C}^{n+1} - \{0\}$. It is a universal polynomial expression in the coefficients of the Q_i which is homogenous of degree $(n + 1)(d - 1)^n$.⁽¹⁸⁾ One can show exists, but it is impossible to write it down explicitly.⁽¹⁹⁾ What matters for us that the **discriminant**, defined by

$$\Delta(P) = Res\left(\frac{\partial P}{\partial z_0}, \dots, \frac{\partial P}{\partial z_n}\right)$$

then satisfies $V(Jac_{\mathbb{C}}(P)) \neq \emptyset \Leftrightarrow \Delta(P) = 0$. In particular, by the Jacobian criterion, the P for which $V(P)$ is smooth is the complement in \mathbb{C}^N of the zero set of a single polynomial Δ . In particular, it is connected.

Let $B_{d,n} = \{P : \Delta(P) \neq 0\} \subset \mathbb{C}^N$ be the subset of P for which $V(P)$ is smooth. Let

$$\mathcal{X}_{d,n} := \{(f, z) \in B_{d,n} \times \mathbb{C}P^n : z \in V(P)\}.$$

Note that $\mathcal{X}_{d,n}$ is a (complex) manifold because

the preimage of $B_{d,n}$ under the Projection to the first factor defines $\pi : \mathcal{X}_{d,n} \rightarrow B_{d,n}$.

Proposition 4.3.7. *The map $\pi : \mathcal{X}_{d,n} \rightarrow B_{d,n}$ is a proper surjective submersion.*

Proof. Properness and surjectivity are clear so we must show it is submersive. □

Since $B_{d,n}$ is connected, Ehresmann’s theorem gives:

Corollary 4.3.8. *All nonsingular hypersurfaces in $\mathbb{C}P^n$ of degree d are diffeomorphic.*

Remark 4.3.9. The base $B_{d,n}$ is

4.4. Plane curves: Degree-genus formula. A (n algebraic) **plane curve** of degree d is a hypersurface of degree d in $\mathbb{C}P^2$, i.e. the subset of the complex projective plane $\mathbb{C}P^2$ defined by

$$C := \{(z_0, z_1, z_2) \in \mathbb{C}^3 - \{0\} : p(z_0, z_1, z_2) = 0\} / \mathbb{C}^* \subset \mathbb{C}P^2$$

where p is a homogeneous polynomial of degree d . It is a smooth complex manifold of dimension one if p satisfies the Jacobian condition, which

Now a nonsingular plane curve is a smooth compact connected real manifold of dimension two, and it is oriented. This means that it is topologically a sphere with g handles attached, i.e. it has a genus $g(C)$.

Theorem 4.4.1 (Degree-genus formula). *Let $C \subset \mathbb{C}P^2$ be a nonsingular plane curve of degree d . Then $g(C) = \binom{d-1}{2}$.*

¹⁸In the case $n = 1$ this is the usual resultant of a pair of polynomials, in the homogeneous version. In the case of $n + 1$ linear forms in $n + 1$ variables the resultant is simply the determinant of the matrix made from their coefficients.

¹⁹E.g. the discriminant of three cubic ($d = 3$) polynomials in $n = 3$ variables is a sum of 2040 monomials in the $(20)^3 = 800$ coefficients.

Proof. We assume $d \geq 2$ as the case $d = 1$ is obvious. Since $\mathcal{X}_{d,2} \rightarrow B_{d,2}$ is a fibre bundle we know that the various plane curves C of a given d are diffeomorphic, so it suffices to show that $g(C) = \binom{d-1}{2}$ for one particular choice of C .

Choose a configuration of d lines L_1, \dots, L_d in $\mathbb{C}P^2$ such that no three lines meet at a point. ⁽²⁰⁾ Thus they intersect at $\binom{d}{2}$ points. Since $B_{d,n}$, being the complement of $\{\Delta(P) = 0\}$, is dense in \mathbb{C}^N , we can choose a one-parameter family of curves P_t , defined for $|t| < \epsilon$ such that $P_0 = L_1 \cdots L_d$ while $P_t \in B_{d,n}$ for $t \neq 0$. ⁽²¹⁾

Now consider $V(P_t)$. When $t = 0$ we have $V(P_0) = L_1 \cup \cdots \cup L_d$.

, which we may assume lie in $U_0 = \{z_0 = 1\}$. Let ℓ_i be a linear form in z_0, z_1, z_2 defining L_i . For $c \in \mathbb{C}$ let

$$P_c = cz_0^d + \ell_1 \cdots \ell_d$$

and consider the curve $V(P_c) \subset \mathbb{C}P^2$. For $c = 0$ we get $V(P_0) = L_1 \cup \cdots \cup L_d$. For $c \neq 0$, applying the Jacobian criterion, we see that $V(P_c)$ is smooth unless a certain polynomial condition holds on the coefficients of the ℓ_i . (If $\ell_i = a_0^i z_0 + a_1^i z_1 + a_2^i z_2$ then

Let us choose a curve C_c which in the chart $U_0 = \{z_0 = 1\} \cong \mathbb{C}^2$ is given by a product of linear forms, i.e. $C_c \cap U_0 \subset \mathbb{C}^2$ is given by the single equation $\ell_1 \cdots \ell_d = c$. (For $d = 2$ we can choose $\ell_i = z_i$, i.e. $C_c \cap U_0 = \{z_1 z_2 = c\}$ but for $d > 2$ we do not want to make this choice. Note that the ℓ_i are inhomogeneous forms, i.e. $\{\ell_i = 0\}$ need not pass through $(0, 0)$.) The example discussed after the Jacobian criterion shows that this is nonsingular for $c \neq 0$. For $c = 0$ it is a collection of d lines. We may assume that ℓ_i have been chosen so that these d lines meet in $\binom{d}{2}$ points, i.e. every line meets every other line and no three lines meet at a single point. ⁽²²⁾

Now consider what the local topological picture is at a meeting point of two lines in $\mathbb{C}P^2$. We may assume by changing coordinates that the lines are the coordinate axes and the point is $(0, 0) \in U_0$, i.e. we consider the topological picture of $z_1 z_2 = c$ at $(0, 0)$. For $c = 0$ it is a pair of complex lines meeting transversely, which topologically is like the wedge of two $\mathbb{C}P^1$ s at a point. For $c \neq 0$ it is topologically like a connected sum of two $\mathbb{C}P^1$ s. In other words, it is again a $\mathbb{C}P^1 \cong S^2$.

Now consider the topological properties of $V(P_c)$. Topologically, it looks like a generic configuration of d lines in which, at each intersection point, the wedge has been replaced by the connected sum. Counting the number of connected sums we have to perform gives the number $\binom{d-1}{2}$ for the genus of this surface (because on ℓ_1 the connected sum has been done d times, which gives $d-1$ handles, then on ℓ_2 we have $d-2$ handles (excluding $\ell_1 \cap \ell_2$), ...). Thus $g(V(P_c)) = \binom{d-1}{2}$, as required. \square

4.5. Plane curves: Bézout's theorem. Let us start in dimension one. A homogeneous polynomial $f(x, y)$ of degree d defines a closed subset of $\mathbb{C}P^1$ which consists of d points if counted with multiplicity (fundamental theorem of algebra).

Now let us move up a dimension to $\mathbb{C}P^2$. Two lines in the projective plane meet in one point unless they coincide. A line and a conic (= degree 2 curve) meet in two points unless the conic is degenerate (i.e. a single line doubled, e.g. given by an equation like z_0^2). The general statement is:

²⁰This is the generic situation: The space of lines in $\mathbb{C}P^2$ is identified with $Gr(2, 3) \cong \mathbb{C}P^1$, so collections of d lines are parametrized by $(\mathbb{C}P^1)^d$. The subset of collections of d lines meeting in $\binom{d}{2}$ points is open dense in $(\mathbb{C}P^2)^d$, and the complement is a closed subset of strictly smaller dimension.

²¹It is not too difficult to write an explicit family P_t .

²²This is the generic situation: The space of lines in $\mathbb{C}P^2$ is identified with $Gr(2, 3) \cong \mathbb{C}P^1$, so collections of d lines are parametrized by $(\mathbb{C}P^1)^d$. The subset of collections of d lines meeting in $\binom{d}{2}$ points is open dense in $(\mathbb{C}P^2)^d$, and the complement is a closed subset of strictly smaller dimension.

Theorem 4.5.1 (Bézout's theorem). *Let $C_1, C_2 \subset \mathbb{C}P^2$ be nonsingular plane curves of degrees d_1, d_2 . Then $[C_1] \cdot [C_2] = d_1 d_2$ in $H_*(\mathbb{C}P^2)$.*

Proof. The idea is to show that $[C] = d[L]$ if C has degree d and L is a line in $\mathbb{C}P^2$. □

The theorem can be extended to arbitrary plane curves, if we keep track of multiplicities properly when we define $[C]$. (If you think of the case, e.g. of the doubled line, e.g. the line defined by z_0^2 , which is of degree 2, it becomes clear you have to do this.)

4.6. Embeddings. As remarked earlier, a Stein manifold (i.e. one which admits a holomorphic embedding in some \mathbb{C}^N) of dimension n can be embedded in \mathbb{C}^{2n+1} (Bishop-Narasimhan). There can be no embedding of a compact complex manifold X of dimension > 0 in $\mathbb{C}P^N$. What about embeddings in $\mathbb{C}P^n$? There are plenty of meromorphic functions on $\mathbb{C}P^n$ (e.g. z_i) and so the same must be true of X . More precisely, there are

4.7. Integrability of almost-complex structures (*). The following is an important theorem but the proof requires methods of PDE. Following Voisin (following Weil) we will give a proof in the real analytic case.

Theorem 4.7.1 (Newlander-Nirenberg). *An almost-complex structure J on a smooth complex manifold X is a complex structure if and only if $T^{1,0}X \subset TX$ is an integrable subbundle.*

Proof (in the real analytic case). □

5. BASICS OF LIE GROUPS

We will use the Frobenius theorems to establish the basic correspondence between Lie groups and Lie algebras, which is the starting point of Lie theory.

5.1. Lie groups, Lie algebras. A **Lie group** is a smooth manifold G with a group structure in which the multiplication map $\mu : G \times G \rightarrow G$ by $\mu(g, h) = gh$ and the inversion map $\iota : G \rightarrow G$ by $\iota(g) = g^{-1}$ are smooth maps. (Obviously it is enough to require that $\alpha(g, h) = gh^{-1}$ is smooth.) ⁽²³⁾

Examples 5.1.1. $\mathbb{R}, \mathbb{R}^*, \mathbb{C}^*, \mathbb{C}, S^1, GL(n, \mathbb{R})$, the group of affine transformations $x \mapsto Ax + b$ of \mathbb{R}^n , $SO(n)$,

If G, H are Lie groups a **Lie group homomorphism** is a smooth map $\varphi : H \rightarrow G$ which is also a group homomorphism. Lie groups with homomorphisms form a category. It has a faithful functor to smooth manifolds.

A **Lie algebra** over a field k is a k -vector space \mathfrak{g} with a bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which is antisymmetric:

$$[x, x] = 0 \quad \text{for all } x \in \mathfrak{g}$$

and satisfies the Jacobi identity:

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

We will mainly take $k = \mathbb{R}$ or $k = \mathbb{C}$ here.

Examples 5.1.2. Here are some examples:

\mathbb{R}^n with the trivial bracket ($[x, y] = 0$ for all x, y). More generally, any vector space with the trivial bracket.

$M(n, \mathbb{R})$ with the bracket $[A, B] = AB - BA$. More generally, if A is an associative algebra over k we can define $[a, b] := ab - ba$ for $a, b \in A$ and this is a Lie algebra.

\mathbb{R}^3 with the bracket $[v, w] = v \times w$ (cross product of vectors) where we set $v \times v = 0$.

Remark 5.1.3. Obviously \mathbb{R} is the unique real Lie algebra of dimension one. There are two real Lie algebras of dimension two (up to isomorphism), namely \mathbb{R}^2 with the trivial bracket and the Lie algebra $\mathfrak{g} = \mathbb{R}x + \mathbb{R}y$ with bracket $[x, y] = y$.

A **Lie algebra homomorphism** is a linear map $\psi : \mathfrak{h} \rightarrow \mathfrak{g}$ which respects the bracket, i.e. $\psi([x, y]) = [\psi(x), \psi(y)]$ for all $x, y \in \mathfrak{h}$. Lie algebras with homomorphisms between them form a category.

5.2. Lie algebra of a Lie group. Let G be a Lie group. It has diffeomorphisms $L_g : G \rightarrow G$ and $R_g : G \rightarrow G$ defined by $L_g(x) = gx$ and $R_g(x) = xg$.

For the moment we drop the assumption that vector fields are smooth, i.e. “vector field” means “possibly discontinuous section of $\pi : TM \rightarrow M$ ”. A vector field v on G is **left-invariant** if

$$L_g^*v = v \quad \text{for all } g \in G$$

(equivalently, $L_{g*}(v) = v$ for all $g \in G$). In other notation,

$$v(gx) = (dL_g)_x(v(x)) \quad \text{for all } g, x \in G.$$

Let \mathfrak{g} or $Lie(G)$ denote the space of left-invariant vector fields on a Lie group G . Since G acts transitively on itself a left-invariant vector field is determined by $v(e) \in T_eG$; since the action is simply transitive the map $v \mapsto v(e)$ gives an isomorphism of vector spaces $\mathfrak{g} \cong T_eG$.

²³*Exercise.* If the multiplication $\mu : G \times G \rightarrow G$ is smooth then the inversion $\iota : G \rightarrow G$ is smooth. The idea is the following:

Lemma 5.2.1. (1) *Left-invariant vector fields are smooth.*

(2) *The bracket of two left-invariant vector fields is left-invariant.*

(3) *The space of left-invariant vector fields is a Lie algebra under bracket.*

Proof. (1) It is enough to show that if $f \in C^\infty(M)$ and v is left-invariant then $v(f) \in C^\infty(M)$. (Here $v(f)$ means: apply $v(g)$ to the germ of f at g .) We have

$$v(f)(g) = v(g) \cdot f = (dL_g)_e(v(e)) \cdot f = v(e) \cdot L_g^*f = v(e) \cdot f \circ L_g.$$

We must prove this is a smooth function of $g \in G$, which we will do using an auxiliary construction.

Let $i_g : G \rightarrow G \times G$ by $i_g(x) = (x, g)$ and let $j_g : G \rightarrow G \times G$ by $j_g(x) = (g, x)$. Then for tangent vectors $w_1, w_2 \in T_{(g_1, g_2)}(G \times G)$ and $h \in C^\infty(G \times G)$, we have

$$(w_1, w_2)(h) = (0, w_2)(h) + (w_1, 0)(h) = w_1(h \circ i_{g_2}) + w_2(h \circ j_{g_1})$$

(equality of functions on $G \times G$).

Choose a smooth vector field w on G such that $w(e) = v(e)$. Then $(0, w)$ is a smooth vector field on $G \times G$ and we have

$$\begin{aligned} ((0, w)(f \circ \mu))(i_e(g)) &= ((0, w)(g, e))(f \circ \mu) \\ &= (0(g))(f \circ \mu \circ i_e) + (w(e))(f \circ \mu \circ j_g) \\ &= w(e)(f \circ \mu \circ j_g) \\ &= w(e) \cdot f \circ L_g \\ &= v(e) \cdot f \circ L_g. \end{aligned} \tag{5.2.1}$$

where in the second equality we have used the observation above. Since the left-hand side is evidently smooth as a function of g we conclude that the right-hand side is smooth in g . By the previous identity $v(f)(g)$ is smooth in g .

(2) This follows from $L_g^*[v, w] = [L_g^*v, L_g^*w] = [v, w]$.

(3) follows from (1) and (2). \square

Thus with any Lie group G is canonically associated a finite-dimensional Lie algebra, namely the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ of left-invariant vector fields. Next we will show that this is functorial.

The following remark is often useful: The tangent bundle of a Lie group is trivial. ⁽²⁴⁾ More precisely, there is an isomorphism $TG \cong G \times T_eG$ given by $(g, v) \mapsto (g, (dL_{g^{-1}})_g(v))$. (Exercise: Check that this is an isomorphism.)

5.3. Homomorphisms of Lie groups. A homomorphism $\varphi : G \rightarrow H$ gives a map $d\varphi_e : T_eG \rightarrow T_eH$ on tangent spaces at the identity, and hence defines a linear map $d\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$. By definition, for $X \in \mathfrak{g}$, $d\varphi(X)$ is the unique left-invariant vector field on H with value $d\varphi_e(X(e))$ at $e \in H$. Thus we have

$$d\varphi(X)(h) = (dL_h)_e d\varphi_e(X(e)) \quad (h \in H).$$

Recall that if $f : M \rightarrow N$ is a smooth map and X, X' are vector fields on M and N respectively, we say X and X' are **f -related** if $df_p(X(p)) = X'(f(p))$ for all $p \in M$. The following was used earlier:

Exercise: If X is f -related to X' and Y is f -related to Y' , then $[X, Y]$ is f -related to $[X', Y']$.

Lemma 5.3.1. *The map $d\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a homomorphism of Lie algebras.*

²⁴A manifold M for which the tangent bundle is isomorphic to the trivial bundle $M \times \mathbb{R}^{\dim M}$ is called **parallelizable**. It is easy to see that an n -manifold M is parallelizable if and only if it admits n vector fields which are linearly independent at every point (in particular, nowhere-vanishing). Since even-dimensional spheres do not admit nowhere-vanishing vector fields they can never be Lie groups. (It is a famous theorem of Adams from the 1960s that the only parallelizable spheres are S^1, S^3, S^7 .)

Proof. Let us first see that any $X \in \mathfrak{g}$ is φ -related to $d\varphi(X) \in \mathfrak{h}$. Note that because φ is a group homomorphism, $\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$. This gives that

$$\begin{aligned} d\varphi_g(X(g)) &= d\varphi_g((dL_g)_e X(e)) \\ &= d(\varphi \circ L_g)_e X(e) \\ &= d(L_{\varphi(g)} \circ \varphi)_e X(e) \\ &= (dL_{\varphi(g)})_e (d\varphi)_e X(e) \\ &= (d\varphi(X))(\varphi(g)). \end{aligned}$$

Thus X and $d\varphi(X)$ are φ -related.

Let $X, Y \in \mathfrak{g}$. We must show that $d\varphi([X, Y]) = [d\varphi(X), d\varphi(Y)]$. By the previous observation and the exercise above we know that $[X, Y]$ and $d\varphi([X, Y])$ are φ -related. Then we have:

$$d\varphi_g([X, Y](g)) = (d\varphi[X, Y])(\varphi(g)). \quad (5.3.1)$$

Thus $d\varphi[X, Y]$ is the unique left-invariant vector field on H with value $d\varphi_e([X, Y](e))$ at $e = \varphi(e) \in H$.

On the other hand, $[d\varphi(X), d\varphi(Y)]$ is also φ -related to $[X, Y]$ on G (because X is φ -related to $d\varphi(X)$ and same for Y), hence it is the left-invariant vector field on H with value at the identity given by $d\varphi_e([X, Y](e))$. By uniqueness we have $d\varphi[X, Y] = [d\varphi(X), d\varphi(Y)]$. \square

We thus have a functor *Lie* from the category of Lie groups (and Lie group homomorphisms) to the category of Lie algebras (and Lie algebra homomorphisms). This functor is not an equivalence, as the following simple example shows: The Lie groups \mathbb{R} and S^1 have the same Lie algebra, namely $\mathfrak{g} = \mathbb{R}$.

5.4. Examples. Let us look at some simple examples. There are two one-dimensional connected Lie groups, namely \mathbb{R} (with addition) and S^1 (with multiplication). (This can be deduced e.g. from the fact that there are two connected one-manifolds up to diffeomorphism.) Their Lie algebras are both the unique one-dimensional Lie algebra \mathbb{R} . In the case S^1 the left-invariant vector field $\frac{\partial}{\partial \theta}$ spans the Lie algebra. In the case \mathbb{R} the Lie algebra is spanned by $\frac{\partial}{\partial t}$. The covering map $\mathbb{R} \rightarrow S^1$ by $t \mapsto \exp(2\pi it)$ induces an isomorphism of Lie algebras.

Another example is \mathbb{R}^* with multiplication. In this case the Lie algebra is again \mathbb{R} , spanned by the invariant vector field $t \frac{\partial}{\partial t}$. Note that the identity component \mathbb{R}_+ is diffeomorphic to \mathbb{R} under the map $\mathbb{R} \rightarrow \mathbb{R}_+$ by $t \mapsto e^t$ which also induces an isomorphism of Lie algebras.

Now consider the following examples of Lie subgroups in the torus $T = \mathbb{R}^2/\mathbb{Z}^2$. For $\alpha \in \mathbb{R}$ define a homomorphism of Lie groups $\mathbb{R} \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ by

$$\varphi_\alpha(t) = t(1, \alpha) \pmod{\mathbb{Z}^2}.$$

If $\alpha = p/q$ is rational then φ_α factors through the covering $\mathbb{R} \rightarrow S^1$ and the image is a compact circle in the torus. If α is irrational then $\varphi_\alpha : \mathbb{R} \rightarrow T$ is a non-embedded submanifold and injective group homomorphism. In either case the Lie algebra map is $\mathbb{R} \rightarrow \mathbb{R}^2$ by $1 \mapsto (1, \alpha)$.

A nonabelian example is given by the group of transformations of the real line of the form $x \mapsto ax + b$ for $a, b \in \mathbb{R}$. This can be given a smooth manifold structure using the obvious bijection with \mathbb{R}^2 , and this makes it into a Lie group.

Another nonabelian example is $GL(n, \mathbb{R})$, which is a manifold using its identification with an open subset of \mathbb{R}^{n^2} . The matrix multiplication formula and Cramer's rule for inversion show that it is also a Lie group.

5.5. Lie subgroups and Lie subalgebras, Lie's first theorem. We have the following notions of subobjects:

A **Lie subgroup** of G is a Lie group H which is a submanifold of G by an injective immersion $i : H \hookrightarrow G$ which is also a group homomorphism.

A **Lie subalgebra** of a Lie algebra \mathfrak{g} is a subspace $\mathfrak{h} \subset \mathfrak{g}$ which is closed under the bracket, i.e. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

If $i : H \rightarrow G$ is a Lie subgroup then $di(Lie(H))$ is a subspace of $Lie(G)$ closed under the bracket, i.e. a Lie subalgebra. The following theorem says that this gives a bijection between Lie subgroups and Lie subalgebras:

Theorem 5.5.1 (Lie's first theorem). *Let G be a Lie group. There is a one-to-one correspondence between connected Lie subgroups of G and Lie subalgebras of the Lie algebra of G , given by $H \mapsto Lie(H)$.*

Proof. The main point is to show that a Lie subalgebra of $Lie(G)$ comes from a (unique) Lie subgroup. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g} = Lie(G)$. Define a subbundle of the tangent bundle on G by

$$F_g = (dL_g)_e(\mathfrak{h}) \quad (g \in G).$$

This is a smooth subbundle because it is (globally) spanned by $\dim \mathfrak{h}$ smooth vector fields: Choose a basis $X_1, \dots, X_{\dim \mathfrak{h}}$ of \mathfrak{h} ; the left-invariant vector fields $v_i(g) = dL_g(X_i)$ are smooth and span F at every point. Note that $[v_i, v_j](g) = (dL_g)_e([X_i, X_j]) \in F_g$ for all g .

Let us check that F is integrable. Every vector field v such that $v(p) \in F_p$ for all p can be written as $v = \sum_i a_i v_i$ for $a_i \in C^\infty(G)$. Now

$$[a_i v_i, b_j v_j] = a_i b_j [v_i, v_j] + a_i v_i(b_j) v_j - b_j v_j(a_i) v_i$$

which lies (pointwise) in F since $[v_i, v_j](g) \in F_g$ for all g . Thus F is integrable.

Let $i : H \rightarrow G$ be the maximal connected integral manifold of F through e . Then

$$di_h(T_h H) = F_{i(h)} = (dL_{i(h)})_e(\mathfrak{h}) \quad (\text{for all } h \in H).$$

We must show that H is a Lie subgroup. Let us first see that it is closed under the group operation of G . For this note that for $h \in H$, $L_{i(h)^{-1}} \circ i : H \rightarrow G$ is again an integral submanifold of F through e . Indeed, for any $x \in H$,

$$d(L_{i(h)^{-1}} \circ i)_x(T_x H) = (dL_{i(h)^{-1}})_{i(x)} di_x(T_x H) = (dL_{i(h)^{-1}})_{i(x)} (dL_{i(x)})_e(\mathfrak{h}) = (dL_{i(h)^{-1}i(x)})_e(\mathfrak{h}).$$

By maximality of $i : H \rightarrow G$ we have that $L_{i(h)^{-1}i(H)} \subset i(H)$. This proves that $i(h)^{-1} \in i(H)$ and hence that $i(H)$ is a subgroup of G .

To prove that $i : H \rightarrow G$ is a Lie subgroup it remains only to show that the map defined by $\alpha(h_1, h_2) := i^{-1}(i(h_1)i(h_2)^{-1})$ is smooth. Consider the diagram

$$\begin{array}{ccc} H \times H & \xrightarrow{\alpha} & H \\ \parallel & & \downarrow i \\ H \times H & \xrightarrow{i \circ \alpha} & G \end{array} \quad (5.5.1)$$

By the factorization property of integral manifolds of integrable subbundles the map α is smooth.

Finally we must prove uniqueness. Suppose $i' : H' \rightarrow G$ is another Lie subgroup with $di'(Lie(H')) = \mathfrak{h}$. Then $i' : H' \rightarrow G$ is an integral manifold of the subbundle F . By maximality of H we have $i'(H') \subset i(H)$. Consider the diagram

$$\begin{array}{ccc} H' & \xrightarrow{\exists f} & H \\ \parallel & & \downarrow i \\ H' & \xrightarrow{i'} & G \end{array} \quad (5.5.2)$$

A smooth $f : H' \rightarrow H$ exists by the factorization property. It is a local diffeomorphism at e , hence it contains an open neighbourhood of $e \in H$ in its image. Therefore it is surjective by

the following elementary fact: A neighbourhood of the identity in a topological group generates the connected component of identity.

Hence f is a bijective homomorphism and is a local diffeomorphism everywhere (because it is a group homomorphism). Therefore it is an isomorphism of Lie groups. \square

Remark 5.5.2. As the example of the Lie subgroup $\varphi_\alpha : \mathbb{R} \rightarrow T$ with image the dense winding line on the torus for $\alpha \notin \mathbb{Q}$ shows, the Lie subgroup will not necessarily be embedded.

5.6. Example: $GL(n, \mathbb{R})$. Let us compute the Lie algebra of $G = GL(n, \mathbb{R})$. Let $\beta : \mathfrak{g} \rightarrow M(n, \mathbb{R})$ be the isomorphism defined by evaluating a left-invariant vector field at e :

$$\beta(X) = X(e) \in T_e M(n, \mathbb{R}) = M(n, \mathbb{R}).$$

We will show that β is a Lie algebra isomorphism if $M(n, \mathbb{R})$ is given the Lie algebra structures $[A, B] = AB - BA$. Let x_{ij} be the coordinate functions on G , so that $x_{ij}(g) = g_{ij}$. The essential property of β is that

$$\beta(X)_{ij} = X(e)(x_{ij}).$$

We will show that $\beta([X, Y]) = [\beta(X), \beta(Y)]$ (this will be enough since β is an isomorphism of vector spaces). For $Y \in \mathfrak{g}$ we have:

$$\begin{aligned} Y(x_{ij})(h) &= Y(h)(x_{ij}) \\ &= (dL_h)_e(Y(e))(x_{ij}) = Y(e)(x_{ij} \circ L_h) \\ &= Y(e) \left(\sum_k x_{ik}(h)x_{kj} \right) = \sum_k x_{ik}(h)Y(e)(x_{kj}) \\ &= \sum_k x_{ik}(h)\beta(Y)_{kj} \end{aligned} \tag{5.6.1}$$

where we have used the formula for matrix multiplication:

$$(x_{ij} \circ L_g)(h) = x_{ij}(gh) = \sum_{k=1}^n x_{ik}(g)x_{kj}(h).$$

Using this we have:

$$\begin{aligned} \beta([X, Y])_{ij} &= [X, Y](e)(x_{ij}) = X(e)Y(x_{ij}) - Y(e)X(x_{ij}) \\ &= \sum_k X(e)(x_{ik})\beta(Y)_{kj} - Y(e)(x_{ik})\beta(X)_{kj} \\ &= [\beta(X), \beta(Y)]_{ij}. \end{aligned} \tag{5.6.2}$$

In more basis-invariant terms this shows that the Lie algebra of $GL(V)$ is identified with $End(V)$ with the bracket $[A, B] = AB - BA$.

It follows that for the Lie algebra of any Lie subgroup of $GL(n, \mathbb{R})$ the Lie bracket on the corresponding subalgebra of $M(n, \mathbb{R})$ is given by $[X, Y] = XY - YX$.

5.7. Linear algebraic groups. ⁽²⁵⁾ A large class of groups (in fact, by far the most important ones), for example groups like $SO(n)$, $Sp(2n)$ etc. which are defined by polynomial conditions on matrices can be shown to be Lie groups by a direct method. We will prove this theorem first and then use it to discuss many examples.

Let $U \subset \mathbb{R}^n$ be open. Suppose that I is an ideal in $\mathbb{R}[x_1, \dots, x_n]$, which we know to be finitely generated by the Hilbert basis theorem. The set

$$V(I) = \{x \in U : f(x) = 0 \text{ for all } f \in I\}$$

²⁵This section is a bit of a diversion in that we use a real-analytic argument, following Chapter I of Varadarajan's *Lie Groups, Lie Algebras, and their Representations*. It is the quickest way I know of to prove that the various standard examples of matrix groups are in fact Lie groups all at once, i.e. without working out cases separately.

is called an **algebraic subset** of U . Conversely, given a set $S \subset U$ let $I(S)$ be the ideal of polynomials in $\mathbb{R}[x_1, \dots, x_n]$ which vanish on S . Then a set is algebraic if and only if $V(I(S)) = S$. ⁽²⁶⁾

While an algebraic subset need not be a manifold we will see that a large open subset of it is a manifold. Let X be an algebraic subset of $U \subset \mathbb{R}^n$ with ideal I . For $x \in X$ consider the differentials $df_x : T_x U = \mathbb{R}^n \rightarrow T_{f(x)} \mathbb{R} = \mathbb{R}$. Thinking of these as linear functionals on $T_x U$, let

$$r(x) = \text{dimension of span of } \{df_x \in (T_x U)^* : f \in I\}.$$

This is finite because if $I = (f_1, \dots, f_N)$ then the $\{df_{i,x}\}_{i=1, \dots, N}$ span the same space as $\{df_x\}_{f \in I}$ because of the identity

$$d(fg)_x = f(x)dg_x + g(x)df_x.$$

Now let

$$r := \max_{x \in X} r(x).$$

(The maximum exists because I is finitely generated.) Let

$$X^{reg} = \{x \in X : r(x) = r\}$$

be the **regular locus** of X . The complement $X^{sing} := X - X^{reg}$ is called the **singular locus** of X . The regular locus is open and nonempty. ⁽²⁷⁾

Theorem 5.7.1 (Whitney). *The open subset X^{reg} is a smooth manifold of dimension $n - r$ and an embedded real-analytic submanifold of U .*

Example 5.7.2. Consider the nodal cubic C given by $y^2 = x^3 - x^2$. Then $I = (f)$ for $f = y^2 - x^3 + x^2$, and $df_{(x,y)} = (-3x^2 + 2x, 2y)$. So $r((x, y)) = 1$ at all points except $(0, 0)$, where $r(0, 0) = 0$. Thus $r = 1$, the regular locus is $C - \{(0, 0)\}$ and the singular locus is $\{(0, 0)\}$.

Proof. Let $X \subset U \subset \mathbb{R}^n$ be as in the proposition and let I be the ideal of polynomials in x_1, \dots, x_n vanishing on X , so that $X = V(I)$.

Let $p \in X^{reg}$. Let x_1, \dots, x_n be coordinates on U centred at p and assume f_1, \dots, f_r are in I such that $df_{1,p}, \dots, df_{r,p}$ are linearly independent in $(T_p U)^*$. (Here r is the maximal possible value of $r(x)$.) Thus the $r \times n$ matrix of partials

$$\left(\frac{\partial f_i}{\partial x_j} \right)_{ij}$$

has rank r at 0; by renumbering coordinates we may assume that the first $r \times r$ columns are independent.

Define a map $\varphi : U \rightarrow \mathbb{R}^n$ by

$$\varphi(x_1, \dots, x_n) = (f_1, \dots, f_r, x_{r+1}, \dots, x_n)$$

This is a real-analytic diffeomorphism at 0, hence there is a neighbourhood V of $0 \in U$ such that $\varphi : V \rightarrow \mathbb{R}^n$ is a real-analytic diffeomorphism onto an open subset of \mathbb{R}^n . (Here we use the real-analytic version of the inverse function theorem.) Let y_1, \dots, y_n be the coordinates on V given by φ and let

$$V_0 = \{x \in V : y_1(\varphi(x)) = \dots = y_r(\varphi(x)) = 0\}.$$

Clearly this is a connected manifold of dimension $n - r$, as it is a slice of a coordinate neighbourhood. If we show that $V_0 \subset X$, in other words, the vanishing of f_1, \dots, f_r on V_0 guarantees

²⁶Note that for an ideal J , $I(V(J)) \supset J$, but the inclusion may be strict. For example, if $J = (x^2) \subset \mathbb{R}[x]$ then $I(V(J)) = (x)$. In general, for ideals in $\mathbb{C}[x_1, \dots, x_n]$ we have $I(V(J)) = \sqrt{J}$, where $\sqrt{J} = \{f : f^n \in J \text{ for some } n\}$ (Hilbert's Nullstellensatz). In the real case, an ideal $J \subset \mathbb{R}[x_1, \dots, x_n]$ is called *real* if $\sum_{i=1}^r f_i^2 \in J \Rightarrow f_i \in J$ for $i = 1, \dots, r$. Then $I(V(J)) = J$ iff J is real (this is called the real Nullstellensatz). We will not use these facts.

²⁷The singular locus is a closed algebraic subset of X : Indeed, if we fix generators f_1, \dots, f_N for $I = I(X)$ then the singular locus is given by the condition that the rank of the Jacobian $\left(\frac{\partial f_i}{\partial x_j} \right)$ with respect to the coordinates x_j is $< r$ (where r is as in the theorem). This is an algebraic set as it is the vanishing locus of the ideal generated by all $r \times r$ minors of the Jacobian matrix.

the vanishing of all $f \in I$ on V_0 , then we will have shown that $V_0 = V \cap X^{reg}$. (Note that $V \cap X^{reg} \subset V_0$.) Since V_0 is a manifold of dimension $n - r$ and $V_0 = V \cap X^{reg}$ is an open subset of X^{reg} , this will prove that X^{reg} is an embedded submanifold at p .

Let $\mathcal{A} = \mathcal{A}(V)$ be the algebra of all real-analytic functions on V . Let $\mathcal{I} = I \cdot \mathcal{A}$ be the ideal in A generated by $I = I(X)$.

Claim: For $k > r$ the operator $\frac{\partial}{\partial y_k}$ leaves \mathcal{I} invariant.

Proof of claim. Indeed, it is enough to prove that

$$\frac{\partial}{\partial y_k} I \subset \mathcal{I} \quad (5.7.1)$$

for $k > r$. Let $F \in I$. Define an n -tuple (f_1, \dots, f_n) by $f_i = y_i = x_i$ if $i > r$ and $i \neq k$ and let $f_k = F$. (Recall that f_1, \dots, f_r were already fixed as $y_i = f_i$ for $i \leq r$.) Then the Jacobians of partials are related by the chain rule

$$\det \left(\frac{\partial f_i}{\partial y_j} \right)_{1 \leq i, j \leq n} = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i, j \leq n} \cdot \det \left(\frac{\partial x_i}{\partial y_j} \right)_{1 \leq i, j \leq n}. \quad (5.7.2)$$

Now the left-hand side is simply $\frac{\partial F}{\partial y_k}$ (recall that $y_i = f_i$ for $i \leq r$) while on the right-hand side the second factor $\det \left(\frac{\partial x_i}{\partial y_j} \right)$ belongs to \mathcal{A} . The first factor on the right-hand side can be simplified as an $(r+1) \times (r+1)$ determinant:

$$\det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i, j \leq n} = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i \leq r, i \neq k, 1 \leq j \leq r, j \neq k}.$$

Now since $f_i \in I$ for $1 \leq i \leq r$ and $F \in I$, this vanishes on X . (If not there would be a point $q \in X$ at which $df_{1,q}, \dots, df_{r,q}, dF_q$ are linearly independent (since they are the columns of the matrix), violating the maximality of r .) Thus (5.7.2) shows that the left-hand side, namely $\frac{\partial}{\partial y_k}(F)$, belongs to $I \cdot \mathcal{A} = \mathcal{I}$. This proves (5.7.1) and hence the claim that $\frac{\partial}{\partial y_k} \mathcal{I} \subset \mathcal{I}$ for $k > r$. \square

Let us finish the proof of the theorem. It follows from the claim that for $F \in \mathcal{I}$ and any $(a_{r+1}, \dots, a_n) \in \mathbb{N}^{n-r}$, we have

$$\left(\frac{\partial}{\partial y_{r+1}} \right)^{a_{r+1}} \cdots \left(\frac{\partial}{\partial y_n} \right)^{a_n} (F) \in \mathcal{I}.$$

Since every element of I , and hence of \mathcal{I} , vanishes at 0 (because $p \in X$), any $F \in \mathcal{I}$ has all such derivatives vanishing at 0. Since F is an analytic function and V_0 is connected, we see that F vanishes on all of the slice V_0 . In particular, elements of I vanish on V_0 , proving that $V_0 \subset V(I) = X$. \square

Remark 5.7.3 (Complex case). The definition of algebraic set makes sense in the complex case: If $U \subset \mathbb{C}^n$ is open then a subset $S \subset U$ is called a **complex algebraic subset** if the ideal I of functions in $\mathbb{C}[x_1, \dots, x_n]$ vanishing on S has $V(I) = S$, where $V(I) = \{z \in \mathbb{C}^n : f(z) = 0 \text{ for all } f \in I\} \subset U$ is defined as before. The definitions above (of regular locus and the integer r) can be given for algebraic subsets of $U \subset \mathbb{C}^n$, except that we replace $T_x U$ by the complexified tangent space (which is spanned over \mathbb{C} by the $\frac{\partial}{\partial z_i}$ for local holomorphic coordinates z_1, \dots, z_n) and consider the elements $df_x \in T_x^{1,0} U \rightarrow T_{f(x)}^{1,0} \mathbb{C} = \mathbb{C}$ as elements of $Hom_{\mathbb{C}}(T_x^{1,0} U, \mathbb{C})$, which has as basis $\{dz_i\}$. The proof then also applies to the complex case, showing that if X is a complex algebraic subset of $U \subset \mathbb{C}^n$ then X^{reg} is a complex manifold of (complex) dimension $n - r$ and hence also a real manifold of dimension $2(n - r)$.

Remark 5.7.4 (Stratifications of algebraic sets). The theorem gives a canonical way to write an algebraic set in \mathbb{C}^n (or \mathbb{R}^n) as a union of locally closed subsets which are themselves complex manifolds (or real manifolds in the case \mathbb{R}). Indeed, consider the complement $X^{sing} = X - X^{reg}$. This is algebraic, so we can consider its singular locus $(X^{sing})^{sing}$ and continue. This produces

a decomposition $X = \bigsqcup_{i=1}^d S_i$ such that each S_i is a complex (or real) manifold and such that the closure of S_i is $\bigsqcup_{j \geq i} S_j$. (Such a decomposition is called a stratification.)

In practice this canonical stratification of varieties is not fine enough. The next example shows this.

Example 5.7.5 (Whitney's cusp). This is the zero set $X \subset \mathbb{C}^3$ of $x^3 + y^2 = x^2 z^2$. The singular locus is the z -axis. Thus the stratification defined above is $X = X - (z\text{-axis}) \sqcup (z\text{-axis})$. However, the singularity at origin is not the same as at other points. (How?)

To remedy this Whitney introduced his conditions (A) and (B).

Now define a (real) **linear algebraic group** to be a subgroup $G \subset GL(n, \mathbb{R})$ which is an algebraic subset. (A complex linear algebraic group is a subgroup of $GL(n, \mathbb{C})$ which is an algebraic subset.) Then we have:

Corollary 5.7.6. *A linear algebraic subgroup of $GL(n, \mathbb{R})$ (resp. $GL(n, \mathbb{C})$) is a closed Lie subgroup of $GL(n, \mathbb{R})$ (resp. of $GL(n, \mathbb{C})$).*

Proof. We will apply Whitney's theorem with $U = GL(n, \mathbb{R})$, an open subset of $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ with polynomial functions in x_{ij} . If $I = I(G)$ then for $f \in I$ we also have $L_g^* f \in I$ for $g \in G$. It is easy to see from this that the number $r(x)$ defined above is the same at all points of the group because of the left translation. So $G^{reg} = G$ and we are done. \square

Example 5.7.7. The group of 3×3 upper triangular matrices $\left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ is defined by the ideal $(x_{21}, x_{31}, x_{32}, x_{11} - 1, x_{22} - 1, x_{33} - 1)$ and so is a linear algebraic group. (That this is a Lie group can be proved directly, of course.)

The corollary gives us a number of examples of Lie groups which are linear algebraic.

5.8. Classical groups. The following more interesting examples of linear algebraic groups are usually referred to as the **classical groups**. Corollary 5.7.6 tells us that they are all Lie groups.

- (i) (special linear groups) $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det(A) = 1\}$ and $SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : \det(A) = 1\}$. (The topology is the one induced from $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.)
- (ii) (compact orthogonal groups) $O(n) = \{A \in GL(n, \mathbb{R}) : AA^T = A^T A = I\}$ and $SO(n) = O(n) \cap SL(n, \mathbb{R})$.
- (iii) (compact unitary groups) $U(n) = \{A \in GL(n, \mathbb{C}) : AA^* = A^* A = I\}$ and $SU(n) = U(n) \cap SL(n, \mathbb{C})$.
- (iv) (compact symplectic groups) Let \mathbb{H} denote the quaternions, i.e. the unique four-dimensional skew-field over \mathbb{R} . Then $Sp(g)$ is the group of matrices in $GL(g, \mathbb{H})$ which preserve the standard Hermitian form on $p, q \in \mathbb{H}^g$ given by $H(p, q) = \sum_{i=1}^g p_i \bar{q}_i$, i.e.

$$Sp(g) = \{g \in GL(g, \mathbb{H}) : g\bar{g}^T = I\}.$$

(Here \bar{q} denotes the quaternion conjugation: If $q = a + bi + cj + dk$ then $\bar{q} = a - bi - cj - dk$.) Another way to define the compact symplectic group is as

$$Sp(2g) = Sp(2g, \mathbb{C}) \cap U(2g, \mathbb{C}),$$

i.e. the group of $2g \times 2g$ unitary matrices which preserve the standard symplectic form on \mathbb{C}^{2g} . (See (vii) below for the definition of $Sp(2g, \mathbb{C})$.)

(*Exercise:* Check equivalence of the two definitions as follows: As \mathbb{C} -vector spaces we can write $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ (write $a + bi + cj + dk = a + bi + (c + di)j$). Check that the standard Hermitian form can be written as $H = h + Bj$ where h is the standard Hermitian form on \mathbb{C}^{2g} and B is an alternating form on \mathbb{C}^{2g} . Then a matrix in $GL(g, \mathbb{H})$ preserving H becomes a $GL(2g, \mathbb{C})$ matrix preserving both h and B .)

- (v) (orthogonal groups) Let $p + q = n$ for $p, q > 0$. Let B be the symmetric bilinear form on \mathbb{R}^n given by $B(x, y) = \sum_{i=1}^p x_i y_i - \sum_{i=p+1}^q x_i y_i$. Let $O(p, q) \subset GL(n, \mathbb{R})$ be the subgroup $\{A \in GL(n, \mathbb{R}) : B(Ax, Ay) = B(x, y)\}$. Let $SO(p, q) = O(p, q) \cap SL(n, \mathbb{R})$.

(On a real vector space of dimension n , there is up to equivalence a unique nondegenerate quadratic form of signature (p, q) for $p + q = n$ (Sylvester's theorem). So if Q is any quadratic form, the group $O(Q)$ of matrices preserving it is conjugate in $GL(n, \mathbb{R})$ to $O(p, q)$ for some p, q .)

The groups $SO(p, q)$ and $O(p, q)$ are compact if and only if $pq = 0$.

- (vi) (unitary groups) Let $p + q = n$ for $p, q > 0$. Let B be the Hermitian form on \mathbb{C}^n given by $B(z, w) = \sum_{i=1}^p z_i \bar{w}_i - \sum_{i=p+1}^q z_i \bar{w}_i$. Let $U(p, q) \subset GL(n, \mathbb{C})$ be the subgroup $\{A \in GL(n, \mathbb{C}) : B(Az, Aw) = B(z, w)\}$. Let $SU(p, q) = U(p, q) \cap SL(n, \mathbb{C})$.

- (vii) (symplectic groups) Let $Sp(2g, \mathbb{R})$ be the subgroup of matrices in $GL(2g, \mathbb{R})$ preserving the anti-symmetric form $B : \mathbb{R}^{2g} \times \mathbb{R}^{2g} \rightarrow \mathbb{R}$ given by $B(x, y) = \sum_{i=1}^g x_i y_{i+g} - x_{i+g} y_i$ for $x = (x_1, \dots, x_{2g})$ and $y = (y_1, \dots, y_{2g})$.

Similarly, looking at elements of $GL(2g, \mathbb{C})$ preserving B (extended linearly to $\mathbb{C}^{2g} \times \mathbb{C}^{2g}$) gives the complex symplectic group $Sp(2g, \mathbb{C})$.

(On a vector space of even dimension over any field there is a unique nondegenerate alternating form. So for any such form on \mathbb{R}^{2g} the group of matrices preserving the form is conjugate in $GL(2g, \mathbb{R})$ to $Sp(2g, \mathbb{R})$. Also, any matrix preserving an alternating form must have determinant one, so $Sp(2g, \mathbb{R}) \subset SL(2g, \mathbb{R})$.)

For each of these groups, the Lie algebra is a subalgebra of $M(n, \mathbb{R})$ (or $M(n, \mathbb{C})$ in the complex cases) and so the bracket is given by $[A, B] = AB - BA$.

Remark 5.8.1. There are linear algebraic groups which do not fall into the above classes, but aside from finitely many examples, any linear algebraic group which is simple (as a linear algebraic group, i.e. does not contain any proper nontrivial linear algebraic subgroup) falls into one of the above classical examples.

5.9. Coverings of Lie groups. If M is a smooth manifold then it has a universal cover \tilde{M} , which again has a natural structure of smooth manifold for which $\tilde{M} \rightarrow M$ is smooth. (It is obvious that \tilde{M} is Hausdorff and locally Euclidean; the only non-obvious part is that it is second-countable. This follows from the fact that $\pi_1(M)$ is countable. ⁽²⁸⁾)

Proposition 5.9.1. *The universal cover \tilde{G} of a Lie group G has a natural structure of Lie group such that $\pi : \tilde{G} \rightarrow G$ is smooth.*

Proof. We have remarked above that \tilde{G} has a smooth manifold structure for which π is smooth, so we must define the group structure. Fix $\tilde{e} \in \pi^{-1}(e)$. Let $\alpha : \tilde{G} \times \tilde{G} \rightarrow G$ be the map $\alpha(\tilde{x}, \tilde{y}) = xy^{-1}$. Consider the diagram of maps of pointed spaces

$$\begin{array}{ccc} (\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e})) & \xrightarrow{\exists \tilde{\alpha}} & (\tilde{G}, \tilde{e}) \\ \parallel & & \downarrow \pi \\ (\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e})) & \xrightarrow{\alpha} & (G, e). \end{array}$$

²⁸Here is a proof for a second countable topological manifold: Take a countable cover $\{U_n\}_n$ of M by coordinate charts homeomorphic to \mathbb{R}^n . For each m, n the intersection $U_m \cap U_n$ has at most countably many components; choose a point in each component and let C be the set of all such points (over all m, n). For each n and $x, y \in C$ such that $x, y \in U_n$ choose a path $\gamma_{n,x,y}$ from x to y in U_n . For a fixed basepoint $p \in C$ there are countably many loops which are finite products of paths of the form $\gamma_{n,x,y}$. It suffices to show that every element of $\pi_1(M, p)$ is of this form.

For a loop $\gamma : [0, 1] \rightarrow M$ based at p there is an N such that when $[0, 1]$ is subdivided into N equal intervals, each subinterval $[(k-1)/N, k/N]$ has image in some U_n . Reparametrizing $\gamma|_{[(k-1)/N, k/N]}$ we get paths γ_k such that $\gamma = \gamma_1 \cdots \gamma_N$. For each k , the point $\gamma(k/N)$ lies in some component $U_n \cap U_m$, in which we have a chosen point, call it $x_k \in C$. Now choose a path δ_k in $U_m \cap U_n$ from x_k to $\gamma(k/N)$ and let $\tilde{\gamma}_k = \delta_{k-1} \gamma \delta_k$. Then $[\gamma] = [\tilde{\gamma}_1] \cdots [\tilde{\gamma}_N]$. But for each k , $\tilde{\gamma}_k$ is a path in U_m from x_{k-1} to x_k , hence homotopic to the reference path γ_{n,x_{k-1},x_k} . Thus γ is homotopic to a product of paths of the form $\gamma_{n,x,y}$.

By the simple connectedness of $\tilde{G} \times \tilde{G}$ there is a unique lift $\tilde{\alpha}$ of α such that $\tilde{\alpha}(\tilde{e}, \tilde{e}) = \tilde{e}$. It is easy to see that this lift is locally smooth, hence smooth. Define the inversion map $\iota : \tilde{G} \rightarrow \tilde{G}$ by

$$\iota(\tilde{g}) := \tilde{\alpha}(\tilde{e}, \tilde{g}).$$

Define the group operation $m : \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$ by

$$m(\tilde{g}, \tilde{h}) = \tilde{\alpha}(\tilde{g}, \iota(\tilde{h})).$$

Verifying the group axioms is left as an exercise. \square

Lemma 5.9.2. *If $p : G' \rightarrow G$ is a covering of topological groups and G' is connected then $\ker(p)$ is contained in the centre of G' .*

Proof. If $p : G' \rightarrow G$ is a covering then $\ker(p)$ is a discrete normal subgroup of G' . For $g \in \ker(p)$ the map $h \mapsto hgh^{-1}$ defines a continuous map $G' \rightarrow \ker(p)$, which must then have constant value e if G' is connected. \square

Theorem 5.9.3. *A homomorphism $\varphi : H \rightarrow G$ of connected Lie groups is a covering map if and only if $d\varphi : \text{Lie}(H) \rightarrow \text{Lie}(G)$ is an isomorphism (equivalently, $d\varphi_e : T_e H \rightarrow T_e G$ is an isomorphism).*

Proof. Suppose first that the homomorphism $\varphi : H \rightarrow G$ is a covering map. We will show first that $d\varphi_e$ is injective. Indeed, if not then consider the subbundle defined by $F_h := \ker(d\varphi_h)$ on H . It is smooth and integrable. Thus the integral manifolds of F_h are submanifolds of H such that $d\varphi$ is uniformly zero on each one. Thus each integral manifold is contracted to a point in G . This contradicts the local one-to-one property of φ . Next let us see that $d\varphi_e$ is surjective. If not then $\dim H < \dim G$ and so (by Sard's theorem) the map $\varphi : H \rightarrow G$ cannot be surjective.

Conversely suppose that $\varphi : H \rightarrow G$ is a homomorphism of Lie groups with $d\varphi_e$ an isomorphism. Then $\ker(\varphi)$ is a discrete normal subgroup of H . Let V be a neighbourhood of e in H such that

$$V^{-1}V \cap \ker(\varphi) = \{e\}.$$

We will show that $\varphi(V)$ is a regularly covered neighbourhood of $e \in G$. Firstly $\varphi|_V$ is one-to-one. (Indeed, if $\varphi(h_1) = \varphi(h_2)$ for $h_1, h_2 \in V$ then $h_1^{-1}h_2 \in V \cap \ker(\varphi)$ and hence $h_1 = h_2$.) Since $d\varphi_h$ is an isomorphism for all $h \in H$ we see that $\varphi|_V$ is a diffeomorphism of V with the open neighbourhood $\varphi(V)$ of e in G . Next we claim that

$$\varphi^{-1}(\varphi(V)) = \bigsqcup_{\theta \in \ker(\varphi)} V\theta.$$

(Indeed, the inclusion \supset is obvious, so consider \subset . Let $h \in H$ with $\varphi(h) \in \varphi(V)$. Suppose $\varphi(h) = \varphi(h')$ for $h' \in V$. Then $\theta := (h')^{-1}h \in \ker(\varphi)$ and $h = h'(h')^{-1}h \in V\theta$. Moreover the union above is disjoint: if $h \in V\theta_1 \cap V\theta_2 \neq \emptyset$ then $h = h_1\theta_1 = h_2\theta_2$ for $h_1, h_2 \in V$ and then $h_1^{-1}h_2 \in V^{-1}V \cap \ker(\varphi)$ so $h_1 = h_2$ and hence $\theta_1 = \theta_2$.) This proves that $\varphi(V)$ is a regularly covered neighbourhood since $\varphi|_{V\theta} : V\theta \rightarrow \varphi(V)$ is a diffeomorphism for each θ . For a general $g = \varphi(h) \in G$ the neighbourhood $g\varphi(V) = \varphi(hV)$ is evenly covered by the disjoint union $\bigsqcup_{\theta \in \ker(\varphi)} hV\theta$. \square

We have also shown the following: For any Lie group G , the universal cover $\pi : \tilde{G} \rightarrow G$ is a Lie group, and $G = \tilde{G}/\ker(\pi : \tilde{G} \rightarrow G)$. Moreover, $\pi_1(G) = \ker(\pi : \tilde{G} \rightarrow G)$.

5.10. Example: $SU(2), SO(3), Sp(1)$, quaternions. The group $SU(2)$ of complex matrices $A \in SL(2, \mathbb{C})$ satisfying $AA^* = I$ (for A^* = conjugate transpose of A) is isomorphic to S^3 . This follows because any such matrix can be written as $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ where $z, w \in \mathbb{C}$ with $|z|^2 + |w|^2 = 1$, so that $SU(2)$ is the unit ball in \mathbb{C}^2 . (Alternately, the natural action of $SU(2)$ on \mathbb{C}^2 preserves the standard Hermitian form, so it preserves the unit ball S^3 of vectors of norm one in \mathbb{C}^2 . It

can be checked that the action of $SU(2)$ is simply transitive, so that there is an identification of $SU(2)$ with S^3 .)

There is a double covering $SU(2) \rightarrow SO(3)$. To see this, consider the space $su(2)$ of traceless skew-Hermitian matrices, i.e. A satisfying $A + A^* = 0$ and $\text{tr}(A) = 0$. This is the Lie algebra of $SU(2)$. This is a real vector space of dimension 3 and it has a natural bilinear form $B(A, B) = -\frac{1}{2}\text{tr}(AB)$.

- The form $B(-, -)$ is nondegenerate and positive definite.

There is an action of $SU(2)$ on $su(2)$ by linear transformations by $(g, A) \mapsto gAg^{-1}$. In other words, there is a homomorphism $\rho: SU(2) \rightarrow GL(su(2))$. The action preserves the symmetric bilinear form $B(A, B) = -\frac{1}{2}\text{tr}(AB)$. In other words, the homomorphism ρ has image inside $SO(q_B)$ where q_B is the inner product $q_B(v) = B(v, v)$ on the three-dimensional space $su(2)$. So we have produced a Lie group homomorphism $SU(2) \rightarrow SO(q_B)$, with kernel exactly $\pm I$. To see the homomorphism ρ explicitly, you must choose an orthonormal (w.r.t. q_B) basis of $su(2)$, which gives an isomorphism $su(2) \cong \mathbb{R}^3$. A popular choice among physicists is the basis $i\sigma_1, i\sigma_2, i\sigma_3$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are called the Pauli matrices. To show that ρ is surjective one has to only check that $d\rho_e$ is an isomorphism, for then the image contains a neighbourhood of e ; since $SO(3)$ is connected this neighbourhood generates the whole group, so ρ is surjective. ⁽²⁹⁾ As $SU(2) = S^3$, this establishes that $SO(3)$ is $\mathbb{R}P^3$. ⁽³⁰⁾ In physics, the fact that the group of rotational symmetries of physical space ($= \mathbb{R}^3$) is $SO(3)$, which has fundamental group $\mathbb{Z}/2\mathbb{Z}$, is responsible for the existence of particles of different “spin” in nature.

A third way to see the natural double cover $SU(2) \rightarrow SO(3)$ is using quaternions. Let \mathbb{H} denote the skew-field of quaternions, i.e. $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ where i, j, k are symbols satisfying the relations $i^2 = j^2 = k^2 = -1$, $ij = k, jk = i, ki = j$. The norm of a quaternion $a + bi + cj + dk$ is $(a^2 + b^2 + c^2 + d^2)^{1/2}$. The norm one quaternions are identified with $SU(2)$ by $a + bi + cj + dk \mapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$. The group of norm one quaternions is topologically the 3-sphere, and acts by multiplication on quaternions in \mathbb{H} with zero real part, preserving the norm. This gives the map $SU(2) \rightarrow SO(3)$ again.

The group $SU(2)$ acts naturally on \mathbb{C}^2 , and hence on $\mathbb{C}P^1 = S^2$. The isotropy of any vector is a closed subgroup isomorphic to $U(1)$, namely a conjugate of the group of matrices of the form $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. The orbit map $SU(2) \rightarrow \mathbb{C}P^1$ is then a map $S^3 \rightarrow S^2$ with $U(1) = S^1$ fibres, which is called the Hopf fibration. (This map is interesting because it is a map $S^3 \rightarrow S^2$ which can be shown to be not homotopic to a constant map (cf. Hatcher’s book for a proof). This shows that $\pi_3(S^2) \neq 0$, and in fact $\pi_3(S^2) = \mathbb{Z}$, the Hopf map is a generator.)

Finally, note that there is still one more description of S^3 : It is homeomorphic to the compact symplectic group $Sp(1)$. Indeed, $Sp(1) = SU(2)$ since both are identified with the unit quaternions.

5.11. Exponential map. Let G be a Lie group and $X \in \text{Lie}(G)$ a left-invariant vector field. Let $\gamma_X(t)$ be the integral curve of X with $\gamma_X(0) = e$, defined for $|t| < \epsilon$. So for any $|t| < \epsilon$, the tangent vector to γ_X at $\gamma_X(t)$ is $X(\gamma_X(t))$. Then the tangent vector to $t \mapsto L_g\gamma_X(t)$ at $g\gamma_X(t)$

²⁹One can also show that ρ is surjective onto $SO(q_B) \cong SO(3)$ by using that $SO(3)$ is generated by rotations and showing that every rotation is in the image of $SU(2)$.

³⁰Here is a direct proof that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$. Define a map $D^3 \rightarrow SO(3)$ by sending a point x to the matrix of the rotation by $|x|\pi$ around the axis through x (and sending 0 to the identity). This map is continuous and injective on the interior of D^3 , and is surjective since any element of $SO(3)$ is a rotation around an axis (Exercise). On the boundary ∂D^3 we see that two points go to the same matrix in $SO(3)$ if they are antipodal. This gives a bijective continuous map from the quotient space of D^3 by the relation $x \sim -x$ on ∂D^3 to $SO(3)$. This quotient space is $\mathbb{R}P^3$, and since the domain is compact the map is a homeomorphism.

is $(dL_g)_{\gamma_X(t)}X(\gamma_X(t)) = X(g\gamma_X(t))$. Thus $L_g\gamma_X(t)$ is the integral curve of X which is at g at $t = 0$.

Applying this to $g = \gamma_X(s)$ (for s small enough) we conclude that $t \mapsto \gamma_X(s)\gamma_X(t)$ is the integral curve of X starting at $\gamma_X(s)$ (i.e. which is at $\gamma_X(s)$ at $t = 0$). On the other hand $\gamma_X(s+t)$ is also an integral curve of X , and at $t = 0$ is at $\gamma_X(s)$. We conclude that

$$\gamma_X(s)\gamma_X(t) = \gamma_X(s+t) \quad (5.11.1)$$

for s, t small enough that both sides make sense. But then γ_X can be extended to a mapping $\gamma_X : \mathbb{R} \rightarrow G$ which is in fact a group homomorphism. It is also clear that γ_X is an integral curve of X for all t . Thus we see that *the integral curves of a left-invariant vector field on a Lie group are defined for all $t \in \mathbb{R}$.*

Define the **exponential map** $\exp : \text{Lie}(G) \rightarrow G$ by

$$\exp(X) := \gamma_X(1).$$

This has the following properties:

Lemma 5.11.1. *The exponential map $\exp : \text{Lie}(G) \rightarrow G$ has the following properties:*

- (i) $\gamma_X(t) = \exp(tX)$ for all t
- (ii) $\exp(tX)\exp(sX) = \exp((s+t)X)$ and $\exp(-tX) = \exp(tX)^{-1}$
- (iii) \exp is a smooth mapping
- (iv) $d\exp_0 : \text{Lie}(G) \rightarrow \text{Lie}(G)$ is the identity and hence \exp is a local diffeomorphism at 0
- (v) The flow of $X \in \text{Lie}(G)$ is given by $\alpha(t, g) = g\exp(tX)$ (i.e. $\alpha_t = R_{\exp(tX)}$).

Proof. (i) The integral curve for tX is $s \mapsto \gamma_X(st)$.

(ii) follows from (i) and (5.11.1).

(iii) Consider the smooth vector field on $G \times \mathfrak{g}$ defined by $v(g, X) = (X(g), 0)$. The integral curve of v through (g, X) is then $t \mapsto (g\exp(tX), X)$ and the flow is $\beta_t(g, X) = (g\exp(tX), X)$. Both these are defined for all t , hence β_1 is defined on $G \times \mathfrak{g}$ and then $\exp(X) = \pi_1 \circ \beta_1(e, X)$. But then $\exp(X)$ is a composition of smooth maps, hence smooth.

(iv) tX is a curve in \mathfrak{g} with tangent vector X at 0; by (i) $\gamma_X(t) = \exp(tX)$ is a curve in G with tangent vector X at e . Thus $d\exp_0(X) = X$.

(v) restates the fact that $L_g \circ \gamma_X$ is the integral curve of X starting at g . \square

Lemma 5.11.2. *Every Lie group homomorphism $\delta : \mathbb{R} \rightarrow G$ (i.e. every one-parameter subgroup of G) is of the form $t \mapsto \exp(tX)$ for some $X \in \text{Lie}(G)$.*

Proof. Let v be the tangent vector to $\delta : \mathbb{R} \rightarrow G$ at e , i.e. $v = d\delta_0 \left(\frac{\partial}{\partial t} \right) \in T_e G$. Let X be the left-invariant vector field on G associated with v . Thus $X(g) = (dL_g)_e v$ and $X(e) = v$. It suffices to show that δ is an integral curve for X , i.e. that $X(\delta(s)) = d\delta_s \left(\frac{\partial}{\partial s} \right)$. We compute:

$$\begin{aligned} X(\delta(s)) &= dL_{\delta(s)}X(e) \\ &= dL_{\delta(s)}d\delta_0 \left(\frac{\partial}{\partial s} \right) \\ &= d(L_{\delta(s)} \circ \delta)_0 \left(\frac{\partial}{\partial s} \right) \\ &= d(\delta \circ L_s)_0 \left(\frac{\partial}{\partial s} \right) \\ &= d\delta_s \left(\frac{\partial}{\partial s} \right) \end{aligned}$$

where we have used that $\delta \circ L_a = L_{\delta(a)} \circ \delta$ (because δ is a group homomorphism) and that $(dL_s)_0 \left(\frac{\partial}{\partial s} \right) = \frac{\partial}{\partial s}$ because $\frac{\partial}{\partial s}$ is translation-invariant on \mathbb{R} . \square

Proposition 5.11.3. *Let $\varphi : H \rightarrow G$ be a homomorphism of Lie groups. The diagram*

$$\begin{array}{ccc} \text{Lie}(H) & \xrightarrow{d\varphi} & \text{Lie}(G) \\ \text{exp} \downarrow & & \downarrow \text{exp} \\ H & \xrightarrow{\varphi} & G \end{array} \quad (5.11.2)$$

is commutative.

Proof. Let $X \in \text{Lie}(H)$. Then $t \mapsto \text{exp}(td\varphi(X))$ is an integral curve of $d\varphi(X)$ with tangent vector $d\varphi_e(X(e))$ at e . On the other hand $t \mapsto \varphi(\text{exp}(tX))$ is a homomorphism $\mathbb{R} \rightarrow G$ and as a curve has tangent vector $d\varphi_e(X)$ at e . Therefore (by the previous lemma) it is an integral curve, in fact we must have

$$\varphi(\text{exp}(tX)) = \text{exp}(td\varphi(X)).$$

Putting $t = 1$ gives the proposition. \square

An immediate consequence is the following:

Corollary 5.11.4. *Let H be a connected Lie group. If $\varphi_1, \varphi_2 : H \rightarrow G$ are Lie group homomorphisms such that $d\varphi_1 = d\varphi_2$ then $\varphi_1 = \varphi_2$.*

Proof. The group H is generated by any neighbourhood of e , in particular by a neighbourhood contained in the image of exp . \square

Remark 5.11.5 (Matrix exponential). For $X \in M(n, \mathbb{R})$ we have the familiar exponential defined by the absolutely convergent sum

$$e^X = \sum_{n \geq 0} \frac{X^n}{n!}$$

which gives an element of $GL(n, \mathbb{R})$. Now $t \mapsto e^{tX}$ is a one-parameter subgroup of G with derivative X at $t = 0$. By the previous proposition we conclude that $\text{exp}(X) = e^X$. Thus the exponential map of a Lie group specializes to the matrix exponential for $GL(n, \mathbb{R})$. The same also holds for any Lie subgroup of $GL(n, \mathbb{R})$ (by the proposition).

Remark 5.11.6 (Adjoint representation). If $\text{Int}(g) : G \rightarrow G$ is conjugation by g , i.e. $\text{Int}(g)(x) = gxg^{-1}$, let $\text{Ad}(g) := d\text{Int}(g)_e : \text{Lie}(G) \rightarrow \text{Lie}(G)$ be the induced map on the Lie algebra. This defines a homomorphism of Lie groups

$$\text{Ad} : G \rightarrow GL(\mathfrak{g})$$

which is called the **adjoint representation**. The previous proposition implies that

$$\text{exp} \circ \text{Ad}(g) = \text{Int}(g) \circ \text{exp}$$

for any $g \in G$. (In the case of $GL(n, \mathbb{R})$ this is simply the obvious identity $Ae^XA^{-1} = e^{AXA^{-1}}$, which is clear from the formula $e^X = \sum_{n \geq 0} X^n/n!$.)

Remark 5.11.7 (Surjectivity of exp). One can ask when the exponential map is surjective for a connected group. Show that:

- (1) exp is surjective for connected Abelian Lie groups. (Hint: Show that exp is a group homomorphism in this case and so $\text{exp}(\mathfrak{g})$ is a connected subgroup which contains a neighbourhood of the identity, hence $\text{exp}(\mathfrak{g}) = G$.)
(More generally, exp is surjective for all connected groups with nilpotent Lie algebra. (A Lie algebra is \mathfrak{g} is called nilpotent if its central series (defined by $\mathfrak{g}^{(i)} := [\mathfrak{g}, \mathfrak{g}^{(i-1)}]$ and $\mathfrak{g}^{(0)} = \mathfrak{g}$) terminates.) We will not discuss this here.)
- (2) If exp is surjective then any element of G has an n th root. (Hint: Use the argument in the proof of Proposition 5.11.8 below.) This is easily seen to fail for $G = SL(2, \mathbb{R})$.
- (3) $\text{exp} : M(n, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$ is surjective. (Hint: Jordan normal form.)

- (4) If $g \in SL(2, \mathbb{R})$ and $g = \exp(A)$ then $\text{trace}(g) \geq -2$. (Hint: The eigenvalues of an element of $A \in \mathfrak{sl}(2, \mathbb{R}) = \{A \in M(2, \mathbb{R}) : \text{tr}(A) = 0\}$ can be (1) both zero, or (2) $\pm x$ for $x \in \mathbb{R}$, or (3) $\pm\sqrt{-1}x$ for $x \in \mathbb{R}$. Consider each case.)
- (5) For $A \in \mathfrak{sl}(2, \mathbb{R}) = \{A \in M(2, \mathbb{R}) : \text{tr}(A) = 0\}$ the exponential is given by

$$\exp(A) = \cos(\sqrt{\det(A)}) I + \frac{\sin(\sqrt{\det(A)})}{\sqrt{\det(A)}} A.$$

Use this to show that $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ is not in the image of the exponential map for $SL(2, \mathbb{R})$ if $a < 0$ and $a \neq -1$.

- (6) \exp is surjective for $SU(n)$ and $SO(n)$. (Hint: Diagonalize.)

In fact \exp is surjective for any compact Lie group G , but we will not prove this here.

Proposition 5.11.8 (No-small-subgroups property). *Let G be a Lie group. There is a neighbourhood of the identity which contains no nontrivial subgroup.*

Proof. Put a Euclidean norm $\|\cdot\|$ on the finite-dimensional vector space \mathfrak{g} and for $\epsilon > 0$ let $B_\epsilon := \{X \in \mathfrak{g} : \|X\| < \epsilon\}$ be the ϵ -ball around 0. Since $d\exp_0 : \mathfrak{g} \rightarrow \mathfrak{g}$ is an isomorphism, we know that there exists δ such that \exp is a diffeomorphism on B_δ . Now consider the neighbourhood $U = \exp(B_{\delta/2})$. Let $e \neq g \in U$ and let $X \in B_{\delta/2}$ such that $\exp(X) = g$. There exists n such that $nX \in B_\delta - B_{\delta/2}$. But then $g^n = \exp(nX) \notin U$ since \exp is bijective on B_δ . Thus there is no subgroup contained in U except $\{e\}$. \square

Remark 5.11.9. The no-small-subgroups property implies that any continuous homomorphism from a profinite group to a real Lie group factors through a finite quotient. For example, any continuous representation of a Galois group (with the Krull topology) on a finite-dimensional real vector space factors through a finite quotient. So a group like $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ (where \mathbb{Q} is an algebraic closure of \mathbb{Q}), which is a central object of study in number theory, does not have a faithful finite-dimensional real or complex representation. To get faithful representations we must consider homomorphisms into $GL(n, \mathbb{Q}_p)$, which is a p -adic version of a Lie group. ⁽³¹⁾

Remark 5.11.10 (Hilbert's fifth problem). It is a remarkable fact that the following are equivalent for a locally compact Hausdorff topological group:

- (1) G is locally Euclidean (i.e. G is a topological manifold; no assumption is made about the group operations except their continuity)
- (2) G has the no-small-subgroups property
- (3) G is a Lie group.

Proving the implication (1) \Rightarrow (2) was Hilbert's fifth problem, and it was proved via the implications (1) \Rightarrow (2) and (2) \Rightarrow (3) by Gleason and Montgomery-Zippin (1952). ⁽³²⁾

5.12. Simply connected groups, Lie's third theorem, Cartan's theorem.

Theorem 5.12.1. *Let H be a simply connected Lie group and let G be a Lie group. Then for any Lie algebra homomorphism $\psi : \text{Lie}(H) \rightarrow \text{Lie}(G)$ there is a unique Lie group homomorphism $\varphi : H \rightarrow G$ such that $d\varphi = \psi$.*

Proof. Notice that giving a Lie group homomorphism $\varphi : H \rightarrow G$ is the same as giving its graph $\hat{H} := \{(h, \varphi(h)) : h \in H\} \subset H \times G$. The homomorphism $\varphi : H \rightarrow G$ is the composition of the second projection restricted to \hat{H} with the isomorphism $H \cong \hat{H}$ inverting the first projection restricted to \hat{H} . We will use this idea.

Let $\psi : \mathfrak{h} \rightarrow \mathfrak{g}$ be a Lie algebra homomorphism. Let $\hat{\mathfrak{h}} \subset \mathfrak{h} \times \mathfrak{g}$ be its graph, i.e. $\hat{\mathfrak{h}} = \{(X, \psi(X)) : X \in \mathfrak{h}\}$. This is a Lie subalgebra of $\text{Lie}(H \times G) = \mathfrak{h} \times \mathfrak{g}$, so there is a unique Lie subgroup $\hat{H} \subset H \times G$ with $\text{Lie}(\hat{H}) = \hat{\mathfrak{h}}$. The first projection $H \times G \rightarrow H$ restricts to a

³¹A good reference where the theories of p -adic Lie groups and real Lie groups are developed together is Serre's *Lie Groups and Lie Algebras*.

³²An excellent modern reference is T. Tao, *Hilbert's fifth problem and related topics*, AMS (2014).

homomorphism $\hat{H} \rightarrow H$ which is an isomorphism on Lie algebras. By a previous theorem it must be a covering of Lie groups. Since H is simply connected we have that $\hat{H} \cong H$. Let φ be the restriction of $H \times G \rightarrow G$ to \hat{H} . This is the required homomorphism, which is unique by the previous corollary. \square

At this stage we have proved:

Theorem 5.12.2 (Lie's third theorem). *The functor Lie from the category of (connected and) simply connected Lie groups to finite-dimensional real Lie algebras is fully faithful.*

The following theorem of Ado is not easy (for a proof see Jacobson's *Lie Algebras*):

Theorem 5.12.3 (Ado). *Every finite-dimensional Lie algebra (over \mathbb{R}) is linear, i.e. has an injective Lie algebra homomorphism to $M(n, \mathbb{R})$ for some n .*

Applying Lie's first theorem we see that every finite-dimensional Lie algebra is the Lie algebra of a Lie group. This gives the strong form of Lie's third theorem:

Theorem 5.12.4 (Cartan). *The functor Lie is an equivalence of categories from connected and simply-connected real Lie groups to finite-dimensional real Lie algebras.*

Remarks 5.12.5. (1) It is usually the weak form of Lie's third theorem that is used, because it is usually Lie groups which arise naturally.

(2) The essential surjectivity of Lie can be proved directly, i.e. without using Ado's theorem, which is what Cartan did. For a proof see Appendix B of Knapp's *Lie Groups Beyond an Introduction*.

(3) The analogue of Ado's theorem holds over arbitrary field, i.e. in characteristic $p > 0$ also (Iwasawa).

Let us summarize the picture: Any Lie group G is the quotient \tilde{G}/C of its simply connected cover by a subgroup C of the centre of \tilde{G} (which is necessarily $C \cong \pi_1(G)$). Different groups with the same Lie algebra come by choosing different subgroups of the centre. So in practice many questions about Lie groups reduce to considering simply connected groups (which can often be reduced to a purely Lie algebra question) and then considering the effect of the centre (which is a purely "discrete" question).

We can also use what we have proved so far to give an example of a Lie group which does not admit any embedding in a general linear group.

Remark 5.12.6 (Universal cover of $SL(2, \mathbb{R})$). Let $G = SL(2, \mathbb{R})$ and let $\pi : \tilde{G} \rightarrow G$ be its universal cover. Note that $\pi_1(G) = \pi_1(SO(2)) = \pi_1(S^1) \cong \mathbb{Z}$, so the kernel of $\tilde{G} \rightarrow G$ is infinite cyclic. We will see that any Lie group homomorphism $\rho : \tilde{G} \rightarrow GL(n, \mathbb{R})$ must be trivial on $\ker(\pi)$, i.e. must factor through π . In particular it cannot be injective. Thus not every Lie group can be thought of as a subgroup of a general linear group.

Suppose that $\rho : \tilde{G} \rightarrow GL(n, \mathbb{C})$ is a Lie group homomorphism. Let $d\rho : Lie(\tilde{G}) \rightarrow M(n, \mathbb{C})$ be the induced Lie algebra homomorphism. Composing with the inverse of the isomorphism $d\pi : Lie(\tilde{G}) \rightarrow sl(2, \mathbb{R})$ and complexifying gives a Lie algebra homomorphism

$$\psi := (d\rho \circ (d\pi)^{-1}) \otimes id_{\mathbb{C}} : sl(2, \mathbb{C}) \rightarrow M(n, \mathbb{C}).$$

Since $SL(2, \mathbb{C})$ is simply connected this can be integrated to give a Lie group homomorphism

$$\varphi : SL(2, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$$

with $d\varphi = \psi$. Then $\varphi|_{SL(2, \mathbb{R})} \circ \pi$ and ρ are two Lie group homomorphisms $\tilde{G} \rightarrow GL(n, \mathbb{C})$ and

$$d(\varphi|_{SL(2, \mathbb{R})} \circ \pi) = d\varphi|_{SL(2, \mathbb{R})} \circ d\pi = \psi|_{sl(2, \mathbb{R})} \circ d\pi = d\rho.$$

Thus $\rho = \varphi|_{SL(2, \mathbb{R})} \circ \pi$ and thus ρ factors through π .

Exercise: Show using a similar argument that the universal cover of $SL(n, \mathbb{R})$ has no injective Lie group homomorphism to a general linear group. You may use that $SU(n)$ is simply connected. (In contrast, the universal cover of the subgroup $SO(n) \subset SL(n, \mathbb{R})$ is necessarily linear since it is a compact group, and compact groups are always linear (as a consequence of the Peter-Weyl theorem).)

5.13. Closed subgroups (*). The following theorem will not be used, but it is important and often useful:

Theorem 5.13.1 (Cartan). *A closed subgroup of a Lie group is a Lie subgroup. (More precisely, if $A \subset G$ is a closed subgroup then there exists a smooth submanifold structure on A such that $A \subset G$ is a Lie subgroup. This structure is unique.)*

The correspondence between Lie subgroups and Lie subalgebras means that a closed subgroup must have a Lie algebra. It is easy to define it: If $A \subset G$ is a closed subgroup then $\mathfrak{a} = \{X \in \text{Lie}(G) : \exp(tX) \in A \text{ for all } t\}$. The proof requires showing that $\exp : \mathfrak{a} \rightarrow A$ is a homeomorphism near the identity, so that it can be used to define charts on A . For the proof see e.g. Warner p. 110ff.

Note that this theorem can also be used to see that the examples of linear algebraic groups given above are indeed Lie groups, since they are closed in $GL(n, \mathbb{R})$ (or $GL(n, \mathbb{C})$).

(In the sequel, the phrase “closed Lie subgroup” will be used; in fact by this theorem we can just say “closed subgroup”).

However, we will need one property of closed Lie subgroups. Here by a **closed Lie subgroup** we will mean a Lie subgroup $\iota : H \rightarrow G$ such that $\iota(H)$ is closed in G . As usual, we will drop ι from the notation and write $H \subset G$ etc.

Lemma 5.13.2. *A closed Lie subgroup of a Lie group is a embedded submanifold (and in fact a closed submanifold).*

Proof. Let H be a Lie subgroup of G . It is an integral manifold of the subbundle defined by $\text{Lie}(H) \subset \text{Lie}(G)$. Recall that in the proof of the factorization property for integral manifolds we showed that there is a neighbourhood U of $e \in G$ with coordinates x_1, \dots, x_n in which the integral manifolds are contained in slices, in which $U \cap H$ is given by a slice $\{x_{k+1} = \dots = x_n = 0\}$, and under the mapping $U \rightarrow \mathbb{R}^{n-k}$ given by $\pi(x_1, \dots, x_n) = (x_{k+1}, \dots, x_n)$ the image $\pi(U \cap H)$ is a countable subset. Now since H is closed, the image $\pi(U \cap H)$ is locally closed. A countable locally closed set in \mathbb{R}^{n-k} has at least one isolated point. Thus there is some component of $U \cap H$ on which the induced topology agrees with the topology of H . Translating using H implies this holds everywhere on H . This shows that H is embedded. \square

(The converse of this lemma also holds.)

5.14. Group actions and homogeneous manifolds. We will consider only left actions: A left action of a Lie group G on a smooth manifold M is a smooth map $\alpha : G \times M \rightarrow M$ such that (1) $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$ and $x \in M$ and (2) $\alpha(e, x) = x$ for all $x \in M$.

Theorem 5.14.1. *If H is a closed Lie subgroup of G then there is a manifold structure on the space of cosets G/H such that $\pi : G \rightarrow G/H$ is a smooth submersion and a fibre bundle (with fibres isomorphic to H). If H is a normal closed Lie subgroup then with this manifold structure G/H is a Lie group. The translation action $G \times G/H \rightarrow G/H$ is a transitive smooth left action.*

Conversely, if $G \times M \rightarrow M$ is a transitive smooth left action of G on a manifold M and $x \in M$ then the orbit mapping $G/G_x \rightarrow M$ is a diffeomorphism.

Let $\varphi : H \rightarrow G$ be a homomorphism of Lie groups. Applying this theorem to the action $H \times G \rightarrow G$ by $(h, g) \mapsto \varphi(h)g$ gives:

Corollary 5.14.2. *The kernel of a Lie group homomorphism $H \rightarrow G$ is a closed normal Lie subgroup of H . The image of a Lie group homomorphism $H \rightarrow G$ is a Lie subgroup of G .*

The image need not be closed, e.g. $\mathbb{R} \rightarrow S^1 \times S^1$ the dense winding line on a torus.

Examples 5.14.3 (Homogeneous spaces). The previous theorem gives a large number of examples of manifold structures on homogeneous spaces.

- (1) Consider the action of $SO(n)$ on S^{n-1} via its action on \mathbb{R}^n . The isotropy subgroup of a point is isomorphic to $SO(n-1)$ and hence we have $SO(n)/SO(n-1) \cong S^{n-1}$.
- (2) Consider the action of $SU(n)$ on S^{2n-1} via its action on \mathbb{C}^n . The isotropy subgroup of a point is isomorphic to $SU(n-1)$ and hence we have $SU(n)/SU(n-1) \cong S^{2n-1}$.
- (3) Consider the action of $Sp(n)$ on S^{4n-1} via its action on \mathbb{H}^n . The isotropy subgroup of a point is isomorphic to $Sp(n-1)$ and hence we have $Sp(n)/Sp(n-1) \cong S^{4n-1}$.
- (4) (real Grassmannians) Let $G = SL(n, \mathbb{R})$ and let H be the subgroup of matrices stabilizing the subspace $\mathbb{R}^k \subset \mathbb{R}^n$ given by the first k coordinates. The space G/H is identified with the space $Gr^{\mathbb{R}}(k, n)$ of k -dimensional subspaces of \mathbb{R}^n and is called the **real Grassmannian** of k -planes in n -space. Note that there is a map $Gr^{\mathbb{R}}(k, n) \rightarrow Gr^{\mathbb{R}}(n-k, n)$ given by $V \mapsto V^\perp$ (\perp taken w.r.t. the standard Euclidean inner product on \mathbb{R}^n). This map is smooth and hence diffeomorphism (exercise).

Another description of the Grassmannian is

$$Gr^{\mathbb{R}}(k, n) = O(n)/O(k) \times O(n-k) = SO(n)/S(O(k) \times O(n-k))$$

which shows that it is compact. In this description the isomorphism $Gr^{\mathbb{R}}(k, n) \cong Gr^{\mathbb{R}}(n-k, n)$ becomes obvious. When $k=1$ we have $\mathbb{R}P^{n-1} = SO(n)/O(n-1)$.

- (5) (complex Grassmannians) Let $G = SL(n, \mathbb{C})$ and let H be the subgroup of matrices stabilizing the subspace $\mathbb{C}^k \subset \mathbb{C}^n$ given by the first k coordinates. The space G/H is identified with the space of k -dimensional complex subspaces of \mathbb{C}^n and is called the **complex Grassmannian** of k -planes in \mathbb{C}^n and denoted $Gr(k, n)$. Once again, taking the \perp w.r.t. a Hermitian form on \mathbb{C}^n gives a diffeomorphism $Gr(k, n) \rightarrow Gr(n-k, n)$.

Another description of the complex Grassmannian is

$$Gr(k, n) = U(n)/U(k) \times U(n-k) = SU(n)/S(U(k) \times U(n-k))$$

which shows that it is compact. In this description the isomorphism $Gr(k, n) \cong Gr(n-k, n)$ becomes obvious. When $k=1$ we have $\mathbb{C}P^{n-1} = SU(n)/U(n-1)$.

- (6) (symplectic or Lagrangian Grassmannians) Let V be a $2n$ -dimensional vector space over $F = \mathbb{R}$ or $F = \mathbb{C}$ with a nondegenerate F -bilinear alternating form $V \times V \rightarrow F$. A subspace $I \subset V$ is called **isotropic** if $L^\perp \supset L$, in which case $\dim I \leq n$ and **Lagrangian** if $L^\perp = L$. A Lagrangian subspace is necessarily of dimension n . Let $\Lambda_F(V)$ be the set of Lagrangian subspaces of v , which is naturally a closed subspace of the Grassmannian $Gr_F(n, V)$. Then $\Lambda_F(V)$ is a homogeneous manifold for $Sp(2n, F) = \{g \in GL(2n, F) : (gv, gw) = (v, w) \text{ for all } v, w \in V\}$. (Equivalently, $Sp(2n, F)$ acts transitively on the Lagrangian subspaces of V .)

Alternate descriptions of $\Lambda_{\mathbb{R}}(V), \Lambda_{\mathbb{C}}(V)$ are:

$$\Lambda_{\mathbb{R}}(n) = U(n)/O(n), \quad \Lambda_{\mathbb{C}}(n) = Sp(n)/U(n)$$

which show that they are compact. (Alternately, they are closed subsets of $Gr_{\mathbb{R}}(n, V)$ and $Gr_{\mathbb{C}}(n, V)$, hence compact.)

- (7) (Stiefel manifolds) The space St_k of k -tuples of orthonormal vectors in \mathbb{R}^n is a homogeneous manifold for $O(n)$. The isotropy subgroup in $O(n)$ of the standard k -tuple is identified with $O(n-k)$. Thus $St_k = O(n)/O(n-k) = SO(n)/SO(n-k)$.

The corresponding complex Stiefel manifold classifying k -tuples of Hermitian-orthonormal vectors in \mathbb{C}^n is $U(n)/U(n-k) = SU(n)/SU(n-k)$.

- (8) (flag manifolds) Let $F = \mathbb{R}$ or $F = \mathbb{C}$. Let V be a vector space of dimension n over F . A (full) flag in V is a chain of F -subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

such that $\dim_F V_i = i$. Show that the space X of full flags (naturally a closed subset of the product of Grassmannians $Gr_F(1, n) \times \cdots \times Gr_F(n-1, n)$) is a homogeneous manifold for $SL(n, F)$ (or $GL(n, F)$).

- (9) (symplectic flag manifolds) Let V be an F -vector space of dimension $2n$. Let X be the set of chains of isotropic subspaces

$$\{0\} = I_0 \subset I_1 \subset \cdots \subset I_n$$

where $\dim_F I_k = k$. (So I_n is Lagrangian.) This is a closed subset of a product of n Grassmannians. This is a homogeneous manifold for $Sp(2n, F)$. Note that there is an $Sp(2n, F)$ -equivariant map $X \rightarrow \Lambda(n)$ to the Lagrangian Grassmannian.

5.15. π_0 and π_1 of some classical groups, spin groups (*). The following lemma is easy:

Lemma 5.15.1. *If G is a topological group and H is a closed connected subgroup such that G/H is connected (in the quotient topology) then G is connected.*

Corollary 5.15.2. *$SO(n), SU(n), Sp(n)$ are connected.*

Using $\det : O(n) \rightarrow \{\pm 1\}$ we see that $O(n)$ has two components. Using $\det : U(n) \rightarrow U(1)$ we see that $U(n)$ is connected.

Proposition 5.15.3. *$SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$ are connected.*

Proof. The Gram-Schmidt orthogonalization procedure, with parameters, shows that $O(n)$ is a deformation retract of $GL(n, \mathbb{R})$, which retracts the identity component $GL(n, \mathbb{R})^0$ onto $SO(n)$. The analogue in the Hermitian setting shows that $GL(n, \mathbb{C})$ has a deformation retraction onto $U(n)$ and $SL(n, \mathbb{C})$ has a deformation retraction onto $SU(n)$.

Instead of using this one can show that there is a diffeomorphism $GL(n, \mathbb{R})^+ \cong SO(n) \times Pos(n, \mathbb{R})$ where $GL(n, \mathbb{R})^+$ is the invertible matrices with determinant > 0 and $Pos(n)$ is the space of positive definite symmetric $n \times n$ real matrices. See Warner p. 131 for a proof of this. The analogue in the complex case is that $GL(n, \mathbb{C})$ is diffeomorphic to $U(n)$ times the space of positive definite Hermitian matrices. \square

Lemma 5.15.4. *If $H \subset G$ is a subgroup and H^0 is the identity component of H then $p : G/H \rightarrow G/H^0$ is a covering. (In particular, if G/H is simply-connected then H must be connected.)*

Proof. Since H^0 is open in H there is a neighbourhood V of e in G such that $V^{-1}V \cap H \subset H^0$. Refer to the following diagram to fix notation:

$$\begin{array}{ccc} G & \xrightarrow{\pi^0} & G/H^0 \\ \parallel & & \downarrow p \\ G & \xrightarrow{\pi} & G/H. \end{array}$$

Now I claim that for $g \in G$ the nhbd $\pi(gV)$ of gH in G/H is evenly covered by p .

Consider the open sets $\pi^0(gV\theta)$ as θ runs over a set of representatives $\Theta \subset H$ for the cosets of H^0 . Each of them maps homeomorphically onto $\pi(gV)$ under p , so it will be enough to show that

$$p^{-1}(\pi(gV)) = \bigsqcup_{\theta \in \Theta} \pi^0(gV\theta).$$

The inclusion \supset is clear, so let us prove \subset . If $p(xH^0) = \pi(gv) = gvH$ for $v \in V$ then $xH = gvH$ hence $v^{-1}g^{-1}x \in H$. Then $gvH^0v^{-1}g^{-1}x = xH^0$ and hence $xH^0 \in \pi^0(gV\theta)$. We leave the disjointness of the union as an exercise. \square

Lemma 5.15.5. *Let G be a connected Lie group and H a connected closed Lie subgroup.*

- (i) *If H is simply-connected and G/H is simply-connected then G is simply-connected.*
- (ii) *More generally, if G/H is simply-connected then $\pi_1(H) \rightarrow \pi_1(G)$ is surjective.*

Proof. (i) follows from (ii), so we prove (ii). Let $\pi : \tilde{G} \rightarrow G$ be the universal cover of G . Then π induces a map $\tilde{G}/\pi^{-1}(H) \rightarrow G/H$ which is easily seen to be a diffeomorphism (it is a bijective map which is a local diffeomorphism). Since $\tilde{G}/\pi^{-1}(H)$ is simply connected, by the previous lemma $\pi^{-1}(H)$ is connected. Thus $\pi : \pi^{-1}(H) \rightarrow H$ is a connected covering and hence the universal cover $\pi_H : \tilde{H} \rightarrow H$ of H factors as

$$\tilde{H} \xrightarrow{p} \pi^{-1}(H) \xrightarrow{\pi} H.$$

Now p is surjective (it is a covering) and hence induces a surjection of $\ker(\pi_H) = \pi_1(H)$ onto $\ker(\pi : \pi^{-1}(H) \xrightarrow{\pi} H) = \ker(\pi : \tilde{G} \rightarrow G) = \pi_1(G)$. (We leave as an exercise that this is the natural map $\pi_1(H) \rightarrow \pi_1(G)$, although it is not needed.) \square

Corollary 5.15.6. *The groups $SU(n), Sp(n), SL(n, \mathbb{C})$ are simply-connected. There is a surjection $\mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(SO(n)) = \pi_1(SL(n, \mathbb{R}))$ for $n \geq 3$.*

In fact we can do more if we use the long exact sequence of homotopy groups of a fibration. (We are assuming here that the quotient map $G \rightarrow G/H$ is a fibration. In fact it is even better; it is a fibre bundle.) We have seen that $SO(3)$ admits a double cover, namely $SU(2) \cong S^3$, so that $\pi_1(SO(3)) = \mathbb{Z}/2$. The homotopy long-exact sequence for the fibration $SO(n+1) \rightarrow S^n$, with fibres isomorphic to $SO(n)$ is:

$$\begin{aligned} \cdots \rightarrow \pi_3(SO(n)) \rightarrow \pi_3(SO(n+1)) \rightarrow \pi_3(S^n) \rightarrow \pi_2(SO(n)) \rightarrow \\ \rightarrow \pi_2(SO(n+1)) \rightarrow \pi_2(S^n) \rightarrow \pi_1(SO(n)) \rightarrow \pi_1(SO(n+1)) \rightarrow 0 \end{aligned}$$

One sees that if $n \geq 3$ then $\pi_1(SO(n+1)) = \pi_1(SO(3))$. By induction, we conclude that

$$\pi_1(SO(n)) = \mathbb{Z}/2 \quad (n \geq 3)$$

and more generally that $\pi_k(SO(n)) = \pi_k(SO(n+1))$ if $k+1 < n$. Since $SO(3) = \mathbb{R}P^3$, we get $\pi_2(SO(3)) = 0$ and hence $\pi_2(SO(4)) = 0$. Using the previous assertions we see that

$$\pi_2(SO(n)) = 0 \quad (\text{all } n).$$

One also sees from the sequence with $n = 3$ that the map $\pi_2(S^2) \rightarrow \pi_1(SO(2))$ must be $\mathbb{Z} \rightarrow \mathbb{Z}$ by multiplication by 2. and hence that $\pi_2(SO(3)) = 0$. Since $\pi_2(S^1) = \pi_3(S^1) = 0$, we get $\pi_3(SO(3)) = \pi_3(S^2) = \mathbb{Z}$. (The last group is generated by the Hopf map.)

- Use the fibrations $SU(n) \rightarrow S^{2n-1}$ and $Sp(n) \rightarrow S^{4n-1}$ to show that $\pi_2(SU(n)) = \pi_2(Sp(n)) = 0$.

In fact for any compact Lie group with zero-dimensional centre, $\pi_1(G)$ is finite, $\pi_2(G)$ is trivial, and $\pi_3(G) = \mathbb{Z}$.

The fact that $\pi_1(SO(n)) = \mathbb{Z}/2$ tells us that there is another sequence of simply-connected compact groups, namely the universal (=double) cover of $SO(n)$. These are called the **spin groups** and denoted

$$Spin(n).$$

In the case of $n \leq 6$ they are in fact already on our list above:

Example 5.15.7 (Spin(4)). There is an action of $S^3 \times S^3$ on \mathbb{H} given by thinking of S^3 as quaternions of norm one:

$$(\alpha, \beta) \cdot \theta = \alpha \theta \beta \quad (\theta \in \mathbb{H}).$$

This action (evidently) preserves the norm quadratic form on \mathbb{H} , which is positive definite. So we have a map

$$S^3 \times S^3 \rightarrow SO(4)$$

which is easily seen to be a group homomorphism. Check that the kernel is precisely $\{\pm(1, 1)\} = \mathbb{Z}/2$. By computing the Lie algebra map we can check that this is nonsingular at the identity, hence a double covering. It follows that

$$Spin(4) = S^3 \times S^3 = SU(2) \times SU(2).$$

There are two more isomorphisms

$$Spin(5) = Sp(4)$$

and

$$Spin(6) = SU(4)$$

both of which follow from the corresponding Lie algebra isomorphisms. (Exercise: What are these?) But for $n \geq 7$ the spin groups $Spin(n)$ are new simply connected compact groups, not in the list of examples above. These groups are linear, i.e. they can be embedded in $GL(n, \mathbb{R})$ but giving explicit embeddings requires the theory of Clifford algebras. They do not have a “canonical” embedding in $GL(n)$ like the series $SU(n)$ and $Sp(n)$, however.

Here is another application of the simple connectedness of $SU(n)$: (There are other ways to prove this result, of course.)

Proposition 5.15.8. *If G is a finite group then there is a compact manifold M with $\pi_1(M) = G$.*

Proof. Suppose that there is an injective homomorphism $G \hookrightarrow SU(n)$ for some $n \neq 2$. Then $M = SU(n)/G$ is a compact manifold with $\pi_1(M) \cong G$.

To show that $G \hookrightarrow SU(n)$ it is enough to treat the case of the symmetric group \mathfrak{S}_k on k letters (since there is an embedding $G \hookrightarrow \mathfrak{S}_k$ by somebody’s theorem in elementary group theory). We can embed \mathfrak{S}_k in $U(k)$ as the matrices of the permutation of the coordinates, i.e. σ is sent to the matrix of the linear transformation $\mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by $z_i \mapsto z_{\sigma(i)}$. The image is not in $SU(k)$, however, so we compose with the homomorphism $U(k) \hookrightarrow SU(2k)$ by $g \mapsto \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix}$ to get an injective homomorphism $\mathfrak{S}_k \hookrightarrow SU(2k)$. \square

Remark 5.15.9. The question of when a group can be realized as the fundamental group of a smooth projective complex variety is very interesting. Every finite group can be, by a result of Serre. If we drop the smoothness requirement then any finitely presented group is the fundamental group of a complex projective variety (Simpson), but there are nontrivial restrictions in the smooth case. No general answer is known (or conjectured) and this is an area of active research.

Remark 5.15.10 (Hopf’s theorem and compact Lie groups). The cohomology algebra $H^*(G, k)$ of a Lie group G with coefficients in a field k of characteristic zero, is a graded-commutative Hopf algebra of finite dimension. By the Hopf structure theorem (proved in the problem set) we have that $H^*(G, k)$ is an exterior algebra generated by elements of odd degree (equivalently, the cohomology algebra of a product of odd-dimensional spheres). In fact it is possible, for the compact Lie groups, to compute the degrees directly from the structure of the Lie group. The integral cohomology of the compact Lie groups is still not fully calculated (as far as I know).

6. DIFFERENTIAL FORMS

The material in this section is in many books, e.g. Warner, Bott-Tu etc. We will use the language of sheaves and presheaves although that will only appear formally in a later section.

6.1. Vector bundles. A (smooth) **vector bundle** of rank r on a manifold M is a smooth manifold E with a smooth map $\pi : E \rightarrow M$ together with:

- (i) an open covering $M = \cup_{\alpha} U_{\alpha}$ of M , together with diffeomorphisms

$$\psi_{\alpha} : \pi^{-1}(U_{\alpha}) \rightarrow U_{\alpha} \times \mathbb{R}^r$$

such that $\pi|_{\pi^{-1}(U_{\alpha})}$ is given by the first projection

- (ii) for α and β such that $U_{\alpha} \cap U_{\beta} \neq \emptyset$, a smooth map

$$h_{\alpha\beta} : U_{\beta} \cap U_{\alpha} \rightarrow GL(r, \mathbb{R})$$

such that the composite

$$\mathbb{R}^r \times U_{\alpha} \cap U_{\beta} \xrightarrow{\psi_{\beta}^{-1}} \pi^{-1}(U_{\alpha} \cap U_{\beta}) \xrightarrow{\psi_{\alpha}} \mathbb{R}^r \times U_{\alpha} \cap U_{\beta}$$

is given by $(h_{\alpha\beta}, \text{id})$.

The manifolds E and M are called the **total space** and the **base space**, respectively. Each $U_{\alpha}, \psi_{\alpha}$ is called a **local trivialization**, the collection $\{(U_{\alpha}, \psi_{\alpha})\}_{\alpha}$ is an **atlas** for the vector bundle. The $h_{\beta\alpha}$ are called the **transition functions** of the vector bundle. The **fibres**

$$E_x := \pi^{-1}(x)$$

are naturally vector spaces of dimension r . There are obvious notions of vector bundle homomorphisms and vector bundle isomorphisms. Note that the transition functions are the key data. They satisfy the following **cocycle condition**: If $U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \neq \emptyset$ then on this set we have the identity

$$h_{\alpha\beta} h_{\beta\gamma} = h_{\alpha\gamma}. \quad (6.1.1)$$

The vector bundle is determined up to isomorphism by the covering of M and the corresponding transition functions, as the following lemma shows:

Lemma 6.1.1. *Let $\{E_x\}_{x \in M}$ be a collection of vector spaces of dimension r indexed by points of M . Suppose that we are given a covering $\{U_{\alpha}\}$ of M and vector space isomorphisms $\rho_{\alpha,x} : E_x \rightarrow \mathbb{R}^r$ when $x \in U_{\alpha}$. If the functions $h_{\alpha\beta} : U_{\alpha} \cap U_{\beta} \rightarrow GL(r, \mathbb{R})$ defined by*

$$h_{\alpha\beta}(x) := \rho_{\alpha,x} \circ \rho_{\beta,x}^{-1} \in GL(r, \mathbb{R})$$

are smooth then there is a unique structure of rank r vector bundle on $E := \coprod_x E_x \rightarrow M$ for which

$$\left(E_{U_{\alpha}} = \coprod_{p \in U_{\alpha}} E_p, \quad \psi_{\alpha} : \coprod_{p \in U_{\alpha}} E_p \rightarrow U_{\alpha} \times \mathbb{R}^r \quad \text{by } \psi_{\alpha}(v) = (p, \rho_{\alpha,p}(v)) \text{ if } v \in E_p \right)$$

make an atlas and the $h_{\alpha\beta}$ are the transition functions.

Proof. First we must put a smooth manifold structure on the total space $E = \coprod_x E_x$. The proof is similar to the proof that the tangent bundle has a structure of a smooth manifold. Namely, for each member of the covering U_{α} we have a map

$$\psi_{\alpha} : E_{U_{\alpha}} := \coprod_{x \in U_{\alpha}} E_x \rightarrow U_{\alpha} \times \mathbb{R}^r \quad \text{by } \psi_{\alpha}|_{E_x} = \rho_{\alpha,x}.$$

We declare this to be a diffeomorphism, i.e. we give $E_{U_{\alpha}}$ the topology and smooth structure given by ψ_{α} . We leave to be checked that the topology on $E = \cup_{\alpha} E_{U_{\alpha}}$ defined in this way is second countable and paracompact. To check that we have a smooth atlas we must check that transition functions are smooth, i.e. if $U_{\alpha} \cap U_{\beta} \neq \emptyset$ then the map

$$U_{\beta} \cap U_{\alpha} \times \mathbb{R}^r \xrightarrow{\psi_{\beta}^{-1}} E_{U_{\alpha} \cap U_{\beta}} \xrightarrow{\psi_{\alpha}} U_{\beta} \cap U_{\alpha} \times \mathbb{R}^r$$

is smooth. By construction this is given by $(id, h_{\alpha\beta})$ which is smooth by hypothesis. This puts a structure of smooth manifold on E , and the map $\pi : E \rightarrow M$ is defined in the obvious way on each chart.

We leave the remaining verifications in the lemma to be checked. \square

The lemma allows standard constructions of linear algebra to be extended to vector bundles. If $E \rightarrow M$ and $E' \rightarrow M$ are vector bundles of rank r and r' then there are vector bundles

$$E^*, \quad E^{\otimes k}, \quad \Lambda^k E, \quad E \otimes E', \quad \text{Hom}(E, E'), \quad E \oplus E'$$

over M which, fibrewise, are

$$E_x^*, \quad E_x^{\otimes k}, \quad \Lambda^k E_x, \quad E_x \otimes E'_x, \quad \text{Hom}(E_x, E'_x), \quad E_x \oplus E'_x.$$

By the lemma, it is enough to give smooth transition functions. If $h_{\alpha\beta}$ (resp. $h'_{\alpha\beta}$) are transition functions for E (resp. E') then the transition functions for these bundles (with respect to an atlas which trivializes both E and E') are given by

$${}^T h_{\alpha\beta}^{-1} \in GL(r, \mathbb{R}), \quad h_{\alpha\beta} \otimes \cdots \otimes h_{\alpha\beta} \in GL(rr', \mathbb{R}), \quad \wedge^k h_{\alpha\beta} \in GL(\Lambda^k \mathbb{R}^r), \quad \dots$$

In many books (e.g. Bott-Tu) you will find an alternate way to construct a vector bundle out of transition functions satisfying the cocycle condition. Let $\{U_\alpha\}$ be a covering of M and let $\{h_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(r, \mathbb{R})\}_{\alpha, \beta}$ be a collection of smooth maps satisfying the cocycle condition (6.1.1). Define the space

$$E := \coprod_{\alpha} U_\alpha \times \mathbb{R}^r / \sim$$

where the relation \sim is as follows: For $(x, v) \in U_\alpha \times \mathbb{R}^r$ and $(y, w) \in U_\beta \times \mathbb{R}^r$ we have

$$(x, v) \sim (y, w) \iff x = y \text{ and } v = h_{\alpha\beta}(w).$$

This is an equivalence relation and the quotient defines a topological space with a map $\pi : E \rightarrow M$.

• Show that E defined in this way is a smooth manifold. (This amounts to showing that E is diffeomorphic to the construction in Lemma 6.1.1.)

Now suppose that E is a vector bundle, trivialized over $\{U_\alpha\}$ by ψ_α and suppose that ψ'_α is another family of trivializations. It is easy to check that the transition functions $\{h_{\alpha\beta}\}$ and $\{h'_{\alpha\beta}\}$ are related as follows: There exist maps $\lambda_\alpha : U_\alpha \rightarrow GL(r, \mathbb{R})$ such that

$$h_{\alpha\beta} = \lambda_\alpha h'_{\alpha\beta} \lambda_\beta^{-1} \quad \text{on } U_\alpha \cap U_\beta. \quad (6.1.2)$$

• Show that isomorphism classes of vector bundles on M which admit a trivialization over the cover $\{U_\alpha\}$ are in bijection with collections of maps $\{h_{\alpha\beta} : U_{\alpha\beta} \rightarrow GL(r, \mathbb{R})\}_{\alpha, \beta}$ satisfying the cocycle condition (6.1.1) modulo the relation (6.1.2). This implies the following:

Lemma 6.1.2. *Two vector bundles on M are isomorphic if and only if their transition functions with respect to some open cover of M are equivalent (in the sense of (6.1.2)).*

A **homomorphism** between the vector bundles $\pi : E \rightarrow M$ and $\pi' : E' \rightarrow M$ is a smooth map $f : E \rightarrow E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xlongequal{\quad} & M \end{array}$$

commutes and which is fibrewise linear, i.e. $E_x \rightarrow E'_x$ is linear. Note that the rank of $f|_{E_x}$ may not be constant in x , so that the kernel and cokernel are not necessarily vector bundles. An isomorphism of vector bundles is an invertible homomorphism. (Exs: A vector bundle homomorphism is an isomorphism if and only if it is an isomorphism fibrewise.)

A **subbundle** of a vector bundle $E \rightarrow M$ is a vector bundle $E' \rightarrow M$ with an injective homomorphism $E' \rightarrow E$. Put another way, a subbundle is a submanifold $E' \subset E$ such that

there exists an atlas (U_α, ψ_α) for E such that for each α , $\pi^{-1}(U_\alpha) \cap E' = \psi_\alpha^{-1}(\mathbb{R}^{r'} \times \{0\})$. (It is clear that such an E' is a vector bundle and there is an injective vector bundle homomorphism $E' \rightarrow E$.) The **quotient bundle** E/E' can be defined as follows: Take the union $\coprod_x E_x/E'_x$ with the isomorphisms $\bar{\rho}_{\alpha,x} : E_x/E'_x \rightarrow \mathbb{R}^{r-r'}$ given by

$$E_x/E'_x \rightarrow \mathbb{R}^r/(\mathbb{R}^{r'} \times \{0\}) \cong \mathbb{R}^{r-r'}.$$

and apply Lemma 6.1.1. There is an obvious exact sequence of vector bundles

$$0 \rightarrow E' \rightarrow E \rightarrow E/E' \rightarrow 0.$$

A **section** of a vector bundle $\pi : E \rightarrow M$ over $U \subset M$ is a smooth map $s : U \rightarrow \pi^{-1}(U) \subset E$ such that $\pi \circ s = \text{id}$. A section over $U = M$ will be called a **global section**.

Examples 6.1.3. (i) The trivial bundle of rank r , where $\pi : \mathbb{R}^r \times M \rightarrow M$ by projection. The transition functions of this bundle are $h_{\alpha\beta} = \text{Id}$ for any chart. (It follows from (6.1.2) that a bundle is trivial (i.e. isomorphic to the trivial bundle) if there exists a covering $\{U_\alpha\}_\alpha$ and smooth maps $\lambda_\alpha : U_\alpha \rightarrow GL(r, \mathbb{R})$ such that the transition functions are given by $h_{\alpha\beta} = \lambda_\alpha \lambda_\beta^{-1}$.)

(ii) The tangent bundle $TM \rightarrow M$ and the cotangent bundle $T^*M \rightarrow M$ are vector bundles of rank n . This gives a vector bundle $TM^{\otimes r} \otimes T^*M^{\otimes s}$ for any r, s . A section of this bundle is called a tensor field of type (r, s) on M .

The tangent bundle is trivialized by any atlas on the manifold. In an atlas $\{(U_\alpha, \varphi_\alpha)\}$ the transition functions of the tangent bundle are given by $h_{\alpha\beta} = \text{Jac}(\varphi_\alpha \circ \varphi_\beta^{-1})$. Since the cotangent bundle is its dual it has transition functions $g_{\alpha\beta} = {}^T \left(\text{Jac}(\varphi_\alpha \circ \varphi_\beta^{-1}) \right)^{-1}$ in this atlas.

(iii) The top exterior power $\Lambda^n T^*M$ of the cotangent bundle on a smooth n -manifold M is a line bundle (= rank one vector bundle). The transition functions are given by $h_{\alpha\beta} = \det(\text{Jac}(\varphi_\alpha \circ \varphi_\beta^{-1}))^{-1}$ in an atlas.

(iv) On any manifold M (not necessarily oriented) the **orientation line bundle** is defined, with respect to any atlas, by the transition functions

$$h_{\alpha\beta} = \frac{\det(\text{Jac}(\varphi_\alpha \circ \varphi_\beta^{-1}))}{|\det(\text{Jac}(\varphi_\alpha \circ \varphi_\beta^{-1}))|} \quad (= \text{sign of } \det(\text{Jac}(\varphi_\alpha \circ \varphi_\beta^{-1})))$$

i.e. it is the sign of the transition functions for $\Lambda^n T^*M$. The resulting line bundle is denoted or_M .

(v) The **tautological bundle** on $M = \mathbb{R}P^n$ is a line bundle whose fibre at a point in $\mathbb{R}P^n$ is the corresponding one-dimensional subspace of \mathbb{R}^{n+1} . Explicitly, it is the subbundle of the trivial $n+1$ -dimensional bundle defined by:

$$E := \{(v, x) \in \mathbb{R}^{n+1} \times \mathbb{R}P^n : v \in x\}.$$

with $\pi : E \rightarrow \mathbb{R}P^n$ by $\pi(v, x) = x$.

• Show that this is a vector bundle, i.e. find local trivializations and transition functions for it.

(iv) The **Moebius bundle** on $\mathbb{R}P^1$ is defined as the quotient of $S^1 \times \mathbb{R}$ by the action of $\mathbb{Z}/2$ by $(x, v) \mapsto (-x, -v)$.

• The tautological bundle over $\mathbb{R}P^1 = S^1$ is isomorphic to the Moebius bundle.

(vi) The tautological complex line bundle over $\mathbb{C}P^n$ is

$$E := \{(v, x) \in \mathbb{C}^{n+1} \times \mathbb{C}P^n : v \in x\}.$$

This is a vector bundle in which the fibres are of complex dimension one, so as a real vector bundle it has rank two. The associated circle bundle (in each fibre one takes the circle $S^1 \subset \mathbb{C} = \mathbb{R}^2$) has total space $S^{2n+1} \subset \mathbb{C}^{n+1}$.

(vii) There is a tautological bundle of rank k over the real Grassmannian $Gr(k, n)$. As in the case $Gr(1, n+1) = \mathbb{R}P^n$ it can be realized as the subbundle $\{(v, E) \in F^n \times Gr(k, n) : v \in E\}$ of the trivial bundle $\mathbb{R}^n \times Gr(k, n)$. Similarly for the complex Grassmannian.

Remark 6.1.4. The following obvious remark will be used repeatedly: A vector bundle of rank r is isomorphic to the trivial bundle if and only if it has r sections which are linearly independent at every point. If $E \rightarrow M$ is such a bundle and s_1, \dots, s_r are r such sections then $s_1(x), \dots, s_r(x)$ are a basis for E_x , so the map sending $v = \sum_i c_i s_i(x) \in E_x$ to (x, c_1, \dots, c_r) defines a map $E \rightarrow M \times \mathbb{R}^r$ which is an isomorphism.

6.2. Pullback. A useful construction with vector bundles is the pullback. Given a smooth map $f : M \rightarrow N$ and a vector bundle $\pi : E \rightarrow N$ the **pullback** of E by f is a vector bundle $f^*E \rightarrow M$ which has $(f^*E)_p = E_{f(p)}$. The total space is

$$f^*E := \{(v, p) \in E \times M \mid \pi(v) = f(p)\}$$

and projection to the second factor gives $f^*E \rightarrow M$. To define a vector bundle we must give an open covering and transition functions. If $\{U_\alpha\}_\alpha, \{h_{\alpha\beta}\}_{\alpha,\beta}$ are the data for E then the covering $\{f^{-1}(U_\alpha)\}_\alpha$ and functions $\{f^*h_{\alpha\beta} = h_{\alpha\beta} \circ f\}_{\alpha,\beta}$ define a vector bundle structure on $f^*E \rightarrow M$. (Check this.) There is a pullback map on sections: if s is a section of $E \rightarrow N$ then f^*s is the section of f^*E defined by $(f^*s)(p) = s(f(p)) \in E_{f(p)} = (f^*E)_p$. (Check that this defines a smooth section of $f^*E \rightarrow M$.)

Example 6.2.1. (i) If $i : W \rightarrow M$ is a submanifold and E is a vector bundle on M then i^*E is a vector bundle on W , simply the restriction of E to W .

(ii) If $i : W \rightarrow M$ is a submanifold then there is a vector bundle monomorphism $TW \rightarrow i^*TM$ and the quotient

$$N_{M/W} = i^*TM/TW$$

is called the **normal bundle** of W in M . If M is given a Riemannian metric then $(N_{M/W})_x$ can be identified with $(T_x W)^\perp$ in $T_x M$ and this identifies $N_{M/W}$ with a subbundle of i^*TM . However, this is not canonical and so you should not think of $N_{M/W}$ as a subbundle of i^*TM but only as a quotient.

(iii) There is a natural map $Gr(k, n) \rightarrow \mathbb{R}P^{\binom{n}{k}-1}$ by sending $E \subset \mathbb{R}^n$ to $\wedge^k E \subset \wedge^k \mathbb{R}^n$. What is the pullback of the tautological bundle?

6.3. Presheaves and sheaves. It will be convenient to use the language of sheaves in what follows.

Let X be a topological space. Let $Op(X)$ be the category with objects the open sets in X , and morphisms inclusions between open sets. A **presheaf of sets** on X is a contravariant functor $F : Op(X) \rightarrow Sets$ such that $F(\emptyset)$ is a one-element set and $F(id : U \rightarrow U)$ is the identity map of $F(U)$. A **presheaf of abelian groups** on X is a contravariant functor from the category $Op(X)$ to the category of abelian groups which takes the empty set \emptyset to $\{e\}$ and the morphism $U \subset U$ to the identity. Concretely, this means we have, for each open set $U \subset X$, an abelian group $F(U)$ and for each pair of open sets $V \subset U$ we have a homomorphism $\rho_V^U : F(U) \rightarrow F(V)$, such that for $W \subset V \subset U$ we have

$$\rho_W^V \circ \rho_V^U = \rho_W^U$$

and moreover $F(\emptyset) = \{e\}$ and $\rho_U^U = id$. The homomorphism ρ_V^U will be called “restriction from U to V ”. An element $s \in F(U)$ is referred to as a **section** of F over U .

Some examples of presheaves:

- (1) For a topological space X , the functor $U \mapsto C(U, \mathbb{R})$ (continuous real-valued functions on U) with ρ_V^U restriction of functions from V to $U \subset V$.
- (2) For a smooth manifold M the functor $U \mapsto C^\infty(U)$ with restriction of functions defines the sheaf of smooth functions.
- (3) For a complex manifold the functor $U \mapsto \{\text{holomorphic functions } U \rightarrow \mathbb{C}\}$ with restriction of functions.
- (4) For a smooth manifold M the functor $U \mapsto \mathcal{X}(U) = \text{smooth vector fields on } U$ with obvious restriction maps.

- (5) Let X be a topological space and $k \geq 0$ an integer. The functor $U \mapsto H^k(U, \mathbb{Z})$ (singular cohomology groups of U) defines a presheaf, where $\rho_U^V : H^k(V, \mathbb{Z}) \rightarrow H^k(U, \mathbb{Z})$ is the restriction in cohomology.
- (6) For $X = \mathbb{C}$ the functor $U \mapsto \{\text{bounded holomorphic functions on } U\}$ defines a presheaf.

A morphism between presheaves F and G has the obvious meaning, namely a natural transformation of functors from F to G . Concretely this means we are given for each $U \subset X$ a homomorphism $F(U) \rightarrow G(U)$ and for every pair $V \subset U$, the diagram

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ \rho_V^U \downarrow & & \downarrow \rho_V^U \\ F(V) & \longrightarrow & G(V) \end{array}$$

commutes.

A presheaf F on X is a **sheaf** if it satisfies the following two conditions:

- (S1) If $s_1, s_2 \in F(U)$ and $\{U_\alpha\}_\alpha$ is an open covering of U such that $\rho_{U_\alpha}^U(s_1) = \rho_{U_\alpha}^U(s_2)$ for all α , then $s_1 = s_2$
- (S2) If U is open in X , $\{U_\alpha\}_\alpha$ is an open covering of U and $\{s_\alpha \in F(U_\alpha)\}_\alpha$ a collection of sections satisfying $\rho_{U_\alpha \cap U_\beta}^{U_\alpha}(s_\alpha) = \rho_{U_\alpha \cap U_\beta}^{U_\beta}(s_\beta)$ for all α, β , then there is a section $s \in F(U)$ such that $s_\alpha = \rho_{U_\alpha}^U(s)$ for all α .

Informally, (S2) says that local sections of sheaves glue together to give sections, (S1) says that sections of sheaves are determined by what they are locally.

Among the examples of presheaves given above, (1)–(4) are sheaves. The example (5) for $k \geq 1$ fails (S1) in general (think of a manifold with nontrivial H^1). The example (6) also fails (S2). ⁽³³⁾

A morphism of sheaves is a morphism between them considered as presheaves, i.e. a natural transformation between the functors.

Given a vector bundle $\pi : E \rightarrow M$ on a manifold M , the functor on $Op(M)$

$$U \mapsto \{ \text{sections of } E \text{ over } U \}$$

defines a presheaf of abelian groups on M which is easily seen to be a sheaf. This is called the **sheaf of sections** of $E \rightarrow M$. It is a sheaf of modules over the sheaf of smooth functions on M .

- A homomorphism $E \rightarrow E'$ is the same as a section of the vector bundle $Hom(E, E')$.

A vector bundle $E \rightarrow M$ is trivial if and only if the sheaf of sections is isomorphic, as a module over the sheaf of smooth functions \mathcal{E}^0 , to the sheaf $\mathcal{E}^0 \oplus \cdots \oplus \mathcal{E}^0$ ($r = rk(E)$ copies).

Let F be a sheaf on X . The **stalk** of F at $x \in X$ is the direct limit

$$F_x := \varinjlim_{x \in U} F(U).$$

(In the case of smooth functions this is the space of germs of smooth functions at x .)

Lemma 6.3.1. *A homomorphism $F \rightarrow G$ of sheaves is an isomorphism if and only if the induced map on stalks $F_x \rightarrow G_x$ is an isomorphism for all $x \in X$.*

We will say more about sheaves later. For the moment we take as a definition that a sequence of sheaves $E \rightarrow F \rightarrow G$ is exact if the sequences of stalks $E_x \rightarrow F_x \rightarrow G_x$ is exact for all $x \in X$.

³³An example of a presheaf which fails (S2) and (S1) is the following: Let $X = \{x, y\}$ with the discrete topology. For an open set $U \subset X$ let $F(U) =$ functions from U to \mathbb{R} . For $V \subsetneq U$ let $\rho_V^U := 0$.

6.4. Differentials or differential 1-forms.

A differential or **differential 1-form** on a manifold M is a section of the cotangent bundle. (This is the dual notion to the notion of vector field.)

Suppose that we are in \mathbb{R}^n with coordinates x_1, \dots, x_n . The vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ form a basis for the tangent bundle over smooth functions. Let dx_1, \dots, dx_n be the dual basis of 1-forms defined by

$$dx_j \left(\frac{\partial}{\partial x_i} \right) = \delta_{ij}.$$

Then every 1-form on \mathbb{R}^n can be written as

$$\sum_i f_i dx_i$$

for smooth functions f_i . Thus on a smooth manifold a 1-form can be written locally in this form.

Let $f : M \rightarrow \mathbb{R}$ be a smooth function. Then $df : TM \rightarrow \mathbb{R}$ is a linear form on the tangent bundle, i.e. a section of the cotangent bundle. Let us introduce the notation

$$\begin{aligned} \mathcal{E}^0 &:= \text{sheaf of smooth functions} \\ \mathcal{E}^1 &:= \text{sheaf of smooth differential 1-forms.} \end{aligned}$$

Then $f \mapsto df$ defines a sheaf homomorphism

$$d : \mathcal{E}^0 \rightarrow \mathcal{E}^1$$

The kernel is the sheaf of constants, i.e. $\ker(d)(U)$ consists of functions that are constant on each connected component of U . The forms in the image are called **exact**.

Exercise. Show that in \mathbb{R}^n the 1-form $\omega = \sum_i a_i dx_i$ is exact if and only if $\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}$ for all i and j .

Exercise. Show that the differential $d\theta$ on S^1 is not exact. (Here $d\theta$ means the differential which is $d(\theta)$ in a coordinate patch - the function θ does not make sense globally on S^1 .)

Exercise. Show that the differential

$$\omega = \frac{-y}{(x^2 + y^2)^{\frac{1}{2}}} dx + \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} dy$$

on $\mathbb{R}^2 - \{0\}$ is not exact. (Hint: Integrate ω over the circle.)

6.5. Differential forms. A **differential k -form** on M is a section of the k th exterior power $\Lambda^k T^*M \rightarrow M$ of the cotangent bundle. Concretely, this means we have, for each point $p \in M$, an alternating multilinear form $(T_p^*M)^{\otimes k} \rightarrow \mathbb{R}$ which varies smoothly in p . Informally, a k -form is a machine which takes k vector fields and gives back a number, and it is alternating for the action of the symmetric group on k letters on $(T_p^*M)^{\otimes k}$. The assignment

$$\mathcal{E}^k(U) = \{ \text{differential } k\text{-forms on } U \} \quad (U \subset M \text{ open}).$$

defines a sheaf on M . This is a sheaf of modules over the sheaf of smooth functions \mathcal{E}^0 . Moreover, it is a sheaf of algebras under the exterior product

$$\mathcal{E}^k(U) \times \mathcal{E}^l(U) \rightarrow \mathcal{E}^{k+l}(U)$$

by $(\omega, \tau) \mapsto \omega \wedge \tau = (-1)^{k+l} \tau \wedge \omega$.

Suppose we are in \mathbb{R}^n with coordinates x_1, \dots, x_n . The choice of coordinates fixes a trivialization of the tangent bundle as $T\mathbb{R}^n \cong \mathbb{R}^n \times T_0\mathbb{R}^n$ which amounts to taking the n global vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. The dual basis of the cotangent bundle is the set of 1-forms dx_1, \dots, dx_n . Thus a basis (over the smooth functions) of the differential k -forms on an open subset of \mathbb{R}^n is given by the $\binom{n}{k}$ differential k -forms

$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

where $I = \{i_1, \dots, i_k\}$ runs over all subsets of $\{1, \dots, n\}$ of size k . Note that dx_I stands for the alternating multilinear form of k variables defined by:

$$dx_I(X_1, \dots, X_k) = \det(dx_{i_r}(X_s))$$

where X_1, \dots, X_k are sections of TM , i.e. vector fields. So for example $(dx_1 \wedge dx_2)(X_1, X_2) = \begin{vmatrix} dx_1(X_1) & dx_1(X_2) \\ dx_2(X_1) & dx_2(X_2) \end{vmatrix}$ and more generally $dx_I((X_i)_{i \in I}) = |(dx_i(X_j))_{i \in I, j \in I}|$.

If ω is a k -form on a manifold M and $\varphi : U \rightarrow V \subset \mathbb{R}^n$ is a chart, then there is a (unique) local expression

$$(\varphi^{-1})^*\omega = \sum_{|I|=k} \omega_I dx_I$$

for smooth functions ω_I in the local coordinates x_1, \dots, x_n on $V \subset \mathbb{R}^n$. (In words, we are using the basis dx_I to trivialize the k th exterior power of the cotangent bundle over V and writing a general section of this bundle in terms of the basis, with coefficients in \mathcal{E}^0 , as in the previous paragraph.) In the sequel we will suppress the chart φ from the notation and simply say that on U with local coordinates x_1, \dots, x_n , the form ω has an expression

$$\omega = \sum_{|I|=k} \omega_I dx_I$$

where $\omega_I(x_1, \dots, x_n)$ are smooth functions.

If $f : M \rightarrow N$ is a smooth manifold then there is a pullback map:

$$f^* : \mathcal{E}^k(N) \rightarrow \mathcal{E}^k(M)$$

Pointwise this is given by the multilinear form $(f^*\omega)_p$ defined by:

$$T_p M \times \dots \times T_p M \xrightarrow{df_p \times \dots \times df_p} T_{f(p)} N \times \dots \times T_{f(p)} N \xrightarrow{\omega_{f(p)}} \mathbb{R}.$$

In terms of sections of vector bundles the pullback comes from the pullback of sections of \mathcal{E}_N^k to sections of $f^*\mathcal{E}_N^k$ followed by the sheaf homomorphism $f^*\mathcal{E}_N^k \rightarrow \mathcal{E}_M^k$.

6.6. Exterior derivative. The homomorphism of sheaves

$$d : \mathcal{E}^0 \rightarrow \mathcal{E}^1$$

defined above extends to forms of higher order:

Theorem 6.6.1. *There is a unique \mathbb{R} -linear sheaf homomorphism $d : \mathcal{E}^k \rightarrow \mathcal{E}^{k+1}$ which agrees with $d : \mathcal{E}^0 \rightarrow \mathcal{E}^1$ for $k = 0$ and which satisfies*

- (i) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$ (Leibniz rule)
- (ii) $d^2 = 0$ (flatness).

The map d commutes with pullback: If $f : M \rightarrow N$ is a smooth map of manifolds then $d(f^(\alpha)) = f^*(d\alpha)$.*

Proof. We will first prove the uniqueness of such a map (over any manifold). Then we will define d in coordinate patches by an explicit formula. By uniqueness the definitions must agree on overlaps and hence we have a globally defined $d : \mathcal{E}^k \rightarrow \mathcal{E}^{k+1}$. (What we are really using here is the fact that the functor $Hom(\mathcal{E}^k, \mathcal{E}^{k+1})$ defined by $U \mapsto Hom(\mathcal{E}^k|_U, \mathcal{E}^{k+1}|_U)$ is itself a sheaf.) Then we will check (i) and (ii) by calculation in coordinate patches.

Let $U \subset M$ be open and $\omega \in \mathcal{E}^k(U)$. To verify that $d\omega$ is uniquely determined by (i) and (ii) it is enough to do so on a coordinate patch, i.e. we may assume U is contained in a chart with coordinates x_1, \dots, x_n . Write $\omega = \sum_I \omega_I dx_I$. Then $d dx_I = 0$ (reduce to the case $d dx_i = 0$ (which holds by (ii)) by repeated use of (i) and (ii)). Hence $d\omega = \sum_I d\omega_I \wedge dx_I$. This proves that $d\omega$ is uniquely determined, since $d\omega_I$ is determined.

To define d we cover M by charts and just use the formula $d\omega = \sum_I d\omega_I \wedge dx_I$ in each coordinate chart. If we verify the Leibniz rule and flatness conditions then by the uniqueness

just proved there is agreement on the overlaps of charts, so $d : \mathcal{E}^k \rightarrow \mathcal{E}^{k+1}$ is well-defined. For both it is enough to check on forms of the type fdx_I . We check flatness:

$$d(d(fd x_I)) = d(df \wedge dx_I) = d\left(\sum_i \frac{\partial f}{\partial x_i} dx_i \wedge dx_I\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I.$$

Now the equality of mixed partials and the relation $dx_i \wedge dx_j = -dx_j \wedge dx_i$ show this is zero. It only remains to check the Leibniz rule for forms of the form $\alpha = fdx_I$ and $\beta = gdx_J$, which is left as an exercise.

Compatibility with pullback is immediate for exact one-forms (i.e. df for f a smooth function) and for functions. The general case follows using that (1) locally on N every differential form is a linear combination of wedge products of exact one-forms with function coefficients, (2) f^* is a ring homomorphism, and (3) the Leibniz rule holds for d . \square

A k -form ω is called **closed** if $d\omega = 0$ and **exact** if $\omega = d\eta$ for a $k-1$ -form η .

Exercise. A 1-form $\omega \in \mathcal{E}^1(\mathbb{R}^n)$ is exact if and only if it is closed.

The following lemma gives an expression for the exterior derivative which does not depend on local coordinates:

Lemma 6.6.2. *Let α be a k -form on M . For vector fields X_0, \dots, X_k ,*

$$\begin{aligned} d\alpha(X_0, \dots, X_k) &= \sum_i (-1)^i X_i \alpha(X_0, \dots, \widehat{X}_i, \dots, X_k) \\ &\quad + \sum_{i < j} (-1)^{i+j-1} \alpha([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k). \end{aligned}$$

Proof. Check that the expression satisfies the Leibniz rule and flatness and conclude by uniqueness that this is the same. \square

Example 6.6.3. For a 1-form ω , we get $(d\omega)(X, Y) = X\omega(Y) - Y\omega(X) + \omega([X, Y])$.

6.7. Frobenius theorem revisited. Suppose F is a rank k integrable subbundle of TM . Let

$$\mathcal{I}^1(F) := \{\omega \in \mathcal{E}^1(M) : \omega(p) \text{ vanishes on } F_p \subset T_p M \text{ for all } p \in M\}.$$

Let $\mathcal{I}^*(F)$ be the ideal in $\mathcal{E}^*(M)$ generated by $\mathcal{I}^1(F)$. Then we have the following slightly more general version of Frobenius theorem (which we will not prove or use here):

Theorem 6.7.1 (Frobenius theorem). *A subbundle $F \subset TM$ has integral manifolds through every point if and only if $d\mathcal{I}^*(F) \subset \mathcal{I}^*(F)$.*

Proof. Exercise. (Use Lemma 6.6.2 and the Frobenius theorem.) \square

Remark 6.7.2. The reformulation suggests we should consider other graded ideals $\mathcal{I}^* \subset \mathcal{E}^*(M)$ with $d(\mathcal{I}^*) \subset \mathcal{I}^*$. Such a thing is called an exterior differential system. There is a notion of integral manifold – namely, a submanifold $i : I \hookrightarrow M$ such that $i^*(\mathcal{I}^*) = \{0\}$ – and the existence of integral manifolds is an interesting problem about which there is a classical theory. The case of systems generated in degree one (called Pfaffian systems) is already quite subtle. ⁽³⁴⁾

³⁴See e.g. the book *Exterior Differential Systems* by Bryant et. al.

7. DE RHAM COHOMOLOGY

7.1. Orientations. Recall that an orientation for a real vector space is an equivalence class of bases, under the relation that two bases are equivalent if the matrix relating them has positive determinant. There are thus two possible orientations for a real vector space.

The top exterior power $\Lambda^{n=\dim V} V$ is a one-dimensional \mathbb{R} -vector space. A choice of generator for $\Lambda^n V$ specifies an orientation uniquely: any generator of $\Lambda^n V$ is of the form $v_1 \wedge \cdots \wedge v_n$ and it gives the basis $\{v_1, \dots, v_n\}$. Conversely an orientation gives a generator of $\Lambda^n V$ (nonuniquely): If $\{v_1, \dots, v_n\}$ is any basis in the equivalence class then $v_1 \wedge \cdots \wedge v_n$ spans $\Lambda^n V$. The fact that for $g \in GL(V)$ the operator $\Lambda^n g : \Lambda^n V \rightarrow \Lambda^n V$ is multiplication by $\det(g)$ makes this well-defined.

An **oriented atlas** for a manifold M is a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ in which the transition functions satisfy the condition

$$\det(\text{Jac}(\varphi_\beta \circ \varphi_\alpha^{-1})) > 0 \quad \text{on } \varphi(U_\alpha \cap U_\beta) \subset \mathbb{R}^n.$$

If M admits an oriented atlas it is called **orientable**. Two oriented atlases which are compatible (in the sense that their union is a smooth atlas) are compatibly oriented if their union is oriented. This is an equivalence relation on oriented atlases; an equivalence class under this relation is an **orientation** on M . If we fix an orientation we call M **oriented**.

Here is an equivalent way to define an orientation: An orientation is the choice of an orientation for each $(T_x M)^*$ (or $T_x M$) which is locally constant in x . Here by locally constant we mean the following: given x there is a chart around U with coordinates x_1, \dots, x_n such that at any point in U the orientation of $(T_x M)^*$ is the same as $dx_1 \wedge \cdots \wedge dx_n$. It is an easy exercise to see that this is equivalent to the notion defined above.

Just as it is convenient to think of orientations of a vector space in terms of the top exterior power it is convenient to think of orientations in terms of sections of the bundle $\Lambda^n T^* M$. We have:

Lemma 7.1.1. *An n -manifold M is orientable if and only if $\Lambda^n T^* M$ has a nowhere-vanishing section.*

Proof. Suppose M is orientable. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a locally finite oriented atlas. Choose a partition of unity $\{f_\alpha\}$ subordinate to the covering. Define

$$\omega := \sum_{\alpha} f_{\alpha} \varphi_{\alpha}^*(dx_1 \wedge \cdots \wedge dx_n).$$

Let us check that ω is nowhere zero. Let $x \in M$ and let $U_{\alpha_1}, \dots, U_{\alpha_r}$ be the charts containing x . Then

$$\omega(x) = \sum_{i=1}^r f_{\alpha_i}(x) \varphi_{\alpha_i}^*(dx_1^i \wedge \cdots \wedge dx_n^i)$$

where x_1^i, \dots, x_n^i are the coordinates on $\varphi_{\alpha_i}(U_{\alpha_i}) \subset \mathbb{R}^n$ and the i th summand is understood to be extended by zero outside U_{α_i} . The coefficients $f_{\alpha_i}(x)$ are all positive, so it is enough to check that all the $\varphi_{\alpha_i}^*(dx_1^i \wedge \cdots \wedge dx_n^i)$ lie in the same component of $\Lambda^n T_x^* M - \{0\}$. But the change of coordinates from x_1^i, \dots, x_n^i to x_1^j, \dots, x_n^j is $\varphi_{\alpha_j} \circ \varphi_{\alpha_i}^{-1}$ and hence on $U_{\alpha_i} \cap U_{\alpha_j}$ we have

$$\varphi_{\alpha_i}^*(dx_1^i \wedge \cdots \wedge dx_n^i) = \det(\text{Jac}(\varphi_{\alpha_i} \circ \varphi_{\alpha_j}^{-1})) \varphi_{\alpha_j}^*(dx_1^j \wedge \cdots \wedge dx_n^j).$$

Since the atlas is oriented we know $\det(\text{Jac}(\varphi_{\alpha_i} \circ \varphi_{\alpha_j}^{-1})) > 0$ and we are done.

Conversely suppose ω is a nowhere-vanishing section of $\Lambda^n T^* M$. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas. For each α write ω on U_α as

$$\omega = g_\alpha \varphi_\alpha^*(dx_1 \wedge \cdots \wedge dx_n).$$

Then $g_\alpha \in C^\infty(U_\alpha)$ is either always > 0 or always < 0 . If, at $x \in M$, the $g_\alpha(x)$ for $U_\alpha \ni x$ all have the same sign then the transition functions $\varphi_{\alpha_i} \circ \varphi_{\alpha_j}^{-1}$ are all orientation preserving at x . Therefore it is enough to modify the atlas so that this holds. Indeed, this is easy to do: If $g_\alpha < 0$ on U_α then replace the chart $(U_\alpha, \varphi_\alpha)$ by the chart $(U_\alpha, s \circ \varphi_\alpha)$ where $s(x_1, x_2, \dots, x_n) = (x_2, x_1, \dots, x_n)$. Now all the functions g_α are > 0 and hence the new atlas is oriented. \square

A nowhere-vanishing top form on a manifold is called a **volume form**.

- Show that S^n is orientable and $\mathbb{R}P^n$ is orientable if and only if n is odd.

Remark 7.1.2. Suppose M is a complex manifold. Then its underlying real manifold is always orientable and in fact comes with a preferred choice of orientation. Indeed, consider first the case of \mathbb{C}^n or any open subset of it with coordinates $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$. The nowhere-vanishing $2n$ -form

$$(-1)^{\frac{n(n-1)}{2}} \frac{i^n}{2^n} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

gives an orientation on \mathbb{C}^n . Now any holomorphic map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ or between open subsets of \mathbb{C}^n automatically satisfies $\det(\text{Jac}(f)) \geq 0$. (For $n = 1$ the Jacobian of a holomorphic function $f(z) = u(x, y) + iv(x, y)$ is $\text{Jac}(f) = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$ which by the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ is equal to $\begin{pmatrix} u_x & v_x \\ -v_x & u_x \end{pmatrix}$ so $\det(\text{Jac}(f)) = u_x^2 + v_x^2 \geq 0$. The determinant is strictly positive everywhere since one cannot have $u_x = v_x = 0$ unless f is constant. For $n > 1$ one has to work with the generalization of the CR conditions.) So a holomorphic atlas of a complex manifold is automatically oriented, and the choice of the orientation above locally fixes a choice of orientation.

7.2. Integration on oriented manifolds. On \mathbb{R}^n there is a canonical measure, namely the Lebesgue measure, which is the unique (up to scalar multiple) translation-invariant measure, and with respect to which you can integrate functions. On a general manifold there is no canonical measure (obviously you cannot patch together the Lebesgue measures on charts), so we don't know how to integrate functions. The correct thing to integrate are top forms (on an oriented manifold).

First let us consider the case of an open subset $U \subset \mathbb{R}^n$. Then $\omega \in \mathcal{E}_c^n(U)$ can be written $\omega = f dx_1 \wedge \cdots \wedge dx_n$ and we make the definition

$$\int_U \omega := \int_{x_n=-\infty}^{\infty} \cdots \int_{x_2=-\infty}^{\infty} \int_{x_1=-\infty}^{\infty} f dx_1 dx_2 \cdots dx_n \quad (7.2.1)$$

where on the right-hand side one has the usual Lebesgue (or, in this case, Riemann) integral of a function. (f is extended by zero outside U .)

Now if $\varphi : V \rightarrow U$ is an orientation-preserving diffeomorphism, then, by assumption, the Jacobian determinant $\det \left(\frac{\partial \varphi_i}{\partial y_j} \right)$ is positive on V . Then by the usual change-of-variables formula for integration in \mathbb{R}^n we have

$$\int_V \varphi^*(\omega) = \int_U \omega. \quad (7.2.2)$$

If φ is orientation-reversing then this is true with a minus sign.

We will use the following simple lemma. (We could have used the earlier lemmas about partitions of unity, but this is even easier.)

Lemma 7.2.1. *Let $C \subset M$ be a compact subset of a manifold and let $\{U_\alpha\}_\alpha$ be a family of open subsets such that $C \subset \cup_\alpha U_\alpha$. Then there exist finitely many smooth functions $f_i : M \rightarrow [0, 1]$ such that each f_i has compact support contained in some U_{α_i} and $\sum_i f_i = 1$ on C .*

Proof. For each $x \in C$ choose $\alpha(x)$ with $x \in U_{\alpha(x)}$. Let $g_x : M \rightarrow [0, 1]$ be a smooth function with compact support $\text{supp}(g_x) \subset U_{\alpha(x)}$ and with g_x identically 1 on some open neighbourhood V_x of x . (If $\varphi : W \rightarrow \mathbb{R}^n$ is a chart neighbourhood with $W \subset U_{\alpha(x)}$ then take a bump function b which is constant 1 in a neighbourhood of $\varphi(x)$ and with $\text{supp}(b) \subset \varphi(W)$. Then take $g_x = \varphi^*b$ and extend it by zero outside W .)

Now $\{V_x\}_{x \in C}$ is an open cover of C hence has a finite subcover V_{x_1}, \dots, V_{x_r} . The product $\prod_i (1 - g_{x_i})$ is zero on $V_{x_1} \cup \cdots \cup V_{x_r}$ and hence $h = \sum_i g_{x_i} + \prod_i (1 - g_{x_i})$ is > 0 everywhere. Then the collection $f_i := g_{x_i}/h$ is as required. \square

Now let M be an oriented n -manifold, i.e. a manifold with a chosen orientation, and let $\omega \in \mathcal{E}_c^n(M)$ be a compactly supported n -form. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas of charts. The lemma gives a finite collection of functions f_i partitioning unity on C with $\text{supp}(f_i)$ compact, and for each i there is a α_i with $\text{supp}(f_i) \subset U_{\alpha_i}$. Define the **integral of ω over M** to be the finite sum:

$$\int_M \omega := \sum_i \int_{\varphi_{\alpha_i}(U_{\alpha_i})} (\varphi_{\alpha_i}^{-1})^*(f_i \omega). \quad (7.2.3)$$

The integrals on the right are defined by (7.2.1).

Lemma 7.2.2. (i) *The integral $\int_M \omega$ is well-defined, i.e. does not depend on the choice of oriented chart giving the fixed orientation or on the choice of partition of unity.*

(ii) *It depends on the choice of orientation only up to a sign, i.e. if the orientation on M is reversed then the integral changes sign (because each term in the sum (7.2.3) changes sign).*

(iii) *If the orientation of M is defined by the nowhere-vanishing form $\omega_0 \in \Lambda^n T^*M$ and $\omega = f\omega_0$ for $f \geq 0$, compactly supported, and not identically zero, then $\int_M \omega \neq 0$.*

Proof. Suppose $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are two oriented atlases giving the same orientation. Let $\omega \in \mathcal{E}_c^n(M)$ be a top form with compact support. Using the lemma we can find functions $\{f_i\}_{i=1}^r$ and $\{g_j\}_{j=1}^s$. Then we have

$$\sum_i \int_{U_{\alpha_i}} (\varphi_{\alpha_i}^{-1})^*(f_i \omega) = \sum_i \int_{U_{\alpha_i}} (\varphi_{\alpha_i}^{-1})^* \left(\sum_j g_j f_i \omega \right) = \sum_{i,j} \int_{\varphi_{\alpha_i}(U_{\alpha_i} \cap V_{\beta_j})} (\varphi_{\alpha_i}^{-1})^*(g_j f_i \omega)$$

while

$$\sum_j \int_{V_{\beta_j}} (\psi_{\beta_j}^{-1})^*(f_i \omega) = \sum_i \int_{V_{\beta_j}} (\psi_{\beta_j}^{-1})^* \left(\sum_i f_i g_j \omega \right) = \sum_{i,j} \int_{\psi_{\beta_j}(U_{\alpha_i} \cap V_{\beta_j})} (\psi_{\beta_j}^{-1})^*(g_j f_i \omega).$$

Now the diffeomorphism

$$\varphi_{\alpha_i} \circ \psi_{\beta_j}^{-1} : \psi_{\beta_j}(U_{\alpha_i} \cap V_{\beta_j}) \rightarrow \varphi_{\alpha_i}(U_{\alpha_i} \cap V_{\beta_j})$$

is orientation-preserving (by the assumption that the two atlases define the same orientation) and hence by (7.2.2) the i, j terms in each expression above are the equal. This proves the first statement.

The other statements of the lemma are easily checked. \square

7.3. Stokes's theorem.

This theorem is usually stated in the context of manifolds with boundary. To avoid the trouble of introducing this concept we will deal with a slightly simpler situation. A **domain** in an n -manifold M is an open subset $\Omega \subset M$ such that

- (i) the (topological) boundary $\partial\Omega = \overline{\Omega} - \Omega$ is an embedded submanifold of M of dimension $n - 1$ and
- (ii) there is an atlas of M such that for each (U, x_1, \dots, x_n) in the atlas for which $U \cap \partial\Omega \neq \emptyset$, the intersection $\Omega \cap U$ is given by $x_n \leq 0$ (and $U \cap \partial\Omega$ is given by $x_n = 0$).

(This terminology is not entirely standard.) So for example the unit ball B^n in \mathbb{R}^n is a domain; its boundary is S^{n-1} .

Note that the integral defined above can easily be extended to the case of a domain in an oriented manifold. For $\omega \in \mathcal{E}_c^n(M)$ and a domain $\Omega \subset M$, choose an oriented atlas with property (ii) above. Next choose functions f_i and charts U_{α_i} as before. If $U_{\alpha_i} \cap \partial\Omega \neq \emptyset$ the contribution to the sum (7.2.3) is simply defined to be the Riemann integral

$$\int_{\varphi_{\alpha_i}(U_{\alpha_i} \cap \Omega)} (\varphi_{\alpha_i}^{-1})^*(f_i \omega) = \int_{x_n=-\infty}^{x_n=0} \cdots \int_{x_1=-\infty}^{\infty} g \, dx_1 \cdots dx_n.$$

where g has been extended by zero outside $\varphi_{\alpha_i}(U_{\alpha_i} \cap \Omega)$.

The following lemma will allow us to consistently orient the boundary of a domain in an oriented manifold. The proof is left as an exercise.

Lemma 7.3.1. *If $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism which carries $\mathbb{H}^n := \{x_n \leq 0\}$ to itself then $\det(\text{Jac}(\varphi)) > 0$ implies $\det(\text{Jac}(\varphi|_{\partial\mathbb{H}^n})) > 0$.*

If M is an oriented manifold and $\Omega \subset M$ is a domain then there is an **induced orientation** on $\partial\Omega$. To define this, first consider the special case $(M, \Omega) = (\mathbb{R}^n, \mathbb{H}^n)$. In this case if the orientation of \mathbb{R}^n is defined by the n -form $dx_1 \wedge \cdots \wedge dx_n$ then the induced orientation of $\partial\mathbb{H}^n = \mathbb{R}^{n-1}$ is defined by $(-1)^{n-1} dx_1 \wedge \cdots \wedge dx_{n-1}$. In the general case suppose that M is oriented, i.e. we have fixed an oriented atlas $\{(U_\alpha, \varphi_\alpha)\}$. We may further assume that for a chart U_α such that $U_\alpha \cap \partial\Omega \neq \emptyset$ in the coordinates given by φ_α the domain $U_\alpha \cap \Omega$ is given by $x_n \leq 0$. Then by the previous lemma $\{(U_\alpha \cap \partial\Omega, \varphi_\alpha)\}$ is an oriented atlas for $\partial\Omega$. Locally in a coordinate chart U the orientation on $\partial\Omega \cap U$ is either the induced one defined in the previous paragraph (special case $(M, \Omega) = (\mathbb{R}^n, \mathbb{H}^n)$ or its reverse. The induced orientation is the one which locally looks like the one defined above for the special case. ⁽³⁵⁾

Theorem 7.3.2. (Stokes) *Let M be an oriented smooth manifold of dimension n and let Ω be a domain in M . Let $\partial\Omega$ be the boundary of Ω , with the induced orientation. Let ω be a compactly supported $n-1$ -form on M . Then*

$$\int_{\Omega} d\omega = \int_{\partial\Omega} i^* \omega.$$

In \mathbb{R} this is the fundamental theorem of calculus; in \mathbb{R}^2 it is Green's theorem; in \mathbb{R}^3 it contains the divergence theorem (Gauss's theorem) and the classical Stokes theorem. Taking $\Omega = M$ gives:

Corollary 7.3.3. *If $\partial M = 0$ then $\int_M d\omega = 0$.*

Proof. First consider the essential special case: $M = \mathbb{R}^n$ and $\Omega = \mathbb{H}^n = \{x_n \leq 0\}$. We assume \mathbb{R}^n is given the orientation $dx_1 \wedge \cdots \wedge dx_n$. Write

$$\omega = \sum_i g_i dx_1 \wedge \cdots \wedge \widehat{dx}_i \wedge \cdots \wedge dx_n.$$

Then

$$d\omega = \sum_i (-1)^{i-1} \frac{\partial g_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$$

and

$$\int_{\Omega} d\omega = \sum_i (-1)^{i-1} \int_{x_n=-\infty}^{x_n=0} \int_{x_{n-1}=-\infty}^{x_{n-1}=\infty} \cdots \int_{x_1=-\infty}^{x_1=\infty} \frac{\partial g_i}{\partial x_i} dx_1 \cdots dx_n.$$

Because each g_i vanishes outside a compact set all terms except the $i = n$ term are zero, giving:

$$\begin{aligned} \int_{\Omega} d\omega &= (-1)^{n-1} \int_{-\infty}^{x_{n-1}=\infty} \cdots \int_{x_1=-\infty}^{x_1=\infty} g_n(x_1, \dots, x_{n-1}, 0) dx_1 \cdots dx_{n-1} \\ &= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} i^* g_n dx_1 \cdots dx_{n-1}. \end{aligned}$$

On the other hand, $i^* \omega = i^* g_n dx_1 \wedge \cdots \wedge dx_{n-1}$ since $i^* dx_n = 0$ and $i^* dx_j = dx_j$ for $j < n$. So the equality $\int_{\Omega} d\omega = \int_{\partial\Omega} i^* \omega$ holds if we take into account the definition of induced orientation (i.e. the fact that $dx_1 \wedge \cdots \wedge dx_{n-1}$ is $(-1)^{n-1}$ -oriented for the induced orientation on \mathbb{R}^{n-1}). The same argument shows that if $\Omega = M = \mathbb{R}^n$ then $\int_{\mathbb{R}^n} d\omega = 0$ for compactly supported ω .

Now consider a general oriented manifold M and a domain $\Omega \subset M$. Choose an oriented atlas $\{U_\alpha\}$ in which each U_α is diffeomorphic to \mathbb{R}^n and such that if $U_\alpha \cap \partial\Omega \neq \emptyset$ then there are coordinates on U_α such that $U_\alpha \cap \partial\Omega$ is given by $x_n \leq 0$. (This can be done since Ω is a

³⁵The purpose of the sign in the induced orientation is to avoid a sign in the statement of Stokes's theorem. You will see a sign in books which have a different convention about the induced orientation.

domain.) Let $\omega \in \mathcal{E}_c^{n-1}(M)$. Choose a finite collection of functions $\{f_i\}$ which partition unity on $\text{supp}(\omega)$ and with each $\text{supp}(f_i) \subset U_{\alpha_i}$ compact. Write $\omega = \sum_i f_i \omega$ so that

$$d\omega = \sum_i d(f_i \omega).$$

Then

$$\begin{aligned} \int_M d\omega &= \sum_i \int_M d(f_i \omega) = \sum_i \int_{U_{\alpha_i}} d(f_i \omega) \\ &= \sum_i \int_{\varphi_{\alpha_i}(U_{\alpha_i} \cap \Omega)} (\varphi_{\alpha_i}^{-1})^*(d(f_i \omega)) \\ &= \sum_i \int_{\varphi_{\alpha_i}(U_{\alpha_i} \cap \Omega)} d((\varphi_{\alpha_i}^{-1})^*(f_i \omega)). \end{aligned}$$

The charts with $U_{\alpha_i} \subset \Omega$ contribute nothing by the special case above for $\Omega = \mathbb{R}^n$. The remaining charts (which intersect $\partial\Omega$) have coordinates in which Ω is given by $x_n \leq 0$. By the special case proved above we have, for such i ,

$$\begin{aligned} \int_{\varphi_{\alpha_i}(U_{\alpha_i} \cap \Omega)} d((\varphi_{\alpha_i}^{-1})^*(f_i \omega)) &= \int_{\varphi_{\alpha_i}(U_{\alpha_i} \cap \partial\Omega)} j^*(\varphi_{\alpha_i}^{-1})^*(f_i \omega) \\ &= \int_{\varphi_{\alpha_i}(U_{\alpha_i} \cap \partial\Omega)} (\varphi_{\alpha_i}^{-1})^* i^*(f_i \omega) \\ &= \int_{U_{\alpha_i} \cap \partial\Omega} i^*(f_i) i^* \omega. \end{aligned}$$

(Here j is the inclusion of $\varphi_{\alpha_i}(U_{\alpha_i} \cap \partial\Omega)$ in $\varphi_{\alpha_i}(U_{\alpha_i} \cap \Omega)$, i.e. $j = \varphi_{\alpha_i} \circ i \circ \varphi_{\alpha_i}^{-1}$.) Since $\{i^* f_i\}$ is a partition of unity for the oriented atlas $\{U_{\alpha_i} \cap \partial\Omega\}$, summing over i gives $\int_{\partial\Omega} i^* \omega$. \square

7.4. De Rham cohomology. The smooth differential forms on M with the exterior derivative d form a complex of sheaves of \mathbb{R} -vector spaces, the **de Rham complex of sheaves**:

$$0 \rightarrow \mathcal{E}^0 \rightarrow \mathcal{E}^1 \rightarrow \dots \rightarrow \mathcal{E}^n \rightarrow 0.$$

Taking global sections (i.e. sections over M) gives the **de Rham complex of M**

$$0 \rightarrow \mathcal{E}^0(M) \rightarrow \mathcal{E}^1(M) \rightarrow \dots \rightarrow \mathcal{E}^n(M) \rightarrow 0.$$

The homology groups of this complex $(\mathcal{E}^*(M), d)$ are the **de Rham cohomology groups of M** :

$$H_{dR}^i(M) := \frac{\ker(d : \mathcal{E}^i(M) \rightarrow \mathcal{E}^{i+1}(M))}{\text{im}(d : \mathcal{E}^{i-1}(M) \rightarrow \mathcal{E}^i(M))},$$

i.e. closed i -forms modulo exact i -forms. These are real vector spaces which may, a priori, be infinite-dimensional. Note that in degree zero we get just the locally constant functions, so that if M is connected

$$H_{dR}^0(M) = \mathbb{R}.$$

The formal properties of de Rham cohomology are summarized in the following:

Proposition 7.4.1. *The de Rham cohomology groups have the following properties:*

- (i) *The sum $H_{dR}^*(M) = \bigoplus_{k \geq 0} H_{dR}^k(M)$ is a graded-commutative algebra with the product defined by exterior product of differential forms: $[\omega] \wedge [\tau] = [\omega \wedge \tau]$.*
- (ii) *They are contravariantly functorial for smooth maps: For $f : M \rightarrow N$ there is a ring homomorphism*

$$f^* : H_{dR}^k(N) \rightarrow H_{dR}^k(M) \quad \text{by } [\omega] \mapsto [f^* \omega].$$

- (iii) *They are diffeomorphism invariant, i.e. if $f : M \rightarrow N$ is a diffeomorphism then f^* is an isomorphism.*

Proof. (i) We check that the product is well-defined: Suppose ω and τ are closed forms. If ω is replaced by $\omega + d\eta$ then

$$[(\omega + d\eta) \wedge \tau] = [\omega \wedge \tau] + [d\eta \wedge \tau] = [\omega \wedge \tau] + [d(\eta \wedge \tau) + (-1)^{\deg(\eta)+1} \eta \wedge d\tau] = [\omega \wedge \tau].$$

Graded-commutativity then follows from graded-commutativity of the wedge product of forms.

(ii) follows immediately from $f^*(d\omega) = d(f^*\omega)$.

(iii) $f^* : \mathcal{E}^*(N) \rightarrow \mathcal{E}^*(M)$ is an isomorphism of complexes. \square

One can also consider the complex of sheaves of k -forms with compact support:

$$0 \rightarrow \mathcal{E}_c^0 \rightarrow \mathcal{E}_c^1 \rightarrow \dots \rightarrow \mathcal{E}_c^n \rightarrow 0$$

where $\mathcal{E}_c^k(U)$ is the space of k -forms with support in a compact subset of U . The homology groups of the global sections are the **de Rham cohomology groups with compact support**:

$$H_{dR,c}^i(M) = \frac{\ker(d : \mathcal{E}_c^i(M) \rightarrow \mathcal{E}_c^{i+1}(M))}{\operatorname{im}(d : \mathcal{E}_c^{i-1}(M) \rightarrow \mathcal{E}_c^i(M))}.$$

Of course $H_{dR,c}^i(M) = H_{dR}^i(M)$ if M is compact. If M is noncompact then $H_{dR,c}^0(M) = 0$.

Proposition 7.4.2. *The groups $H_{dR,c}^*(M)$ have the following properties:*

- (i) *The sum $H_{dR,c}^*(M) = \bigoplus_{k \geq 0} H_{dR,c}^k(M)$ is a ring under wedge of differential forms and also a module over $H_{dR}^*(M)$.*
- (ii) *The groups $H_{dR,c}^*(M)$ are contravariantly functorial for proper maps of manifolds.*
- (iii) *The groups $H_{dR,c}^*(M)$ are covariantly functorial for inclusions of open subsets.*
- (iv) *For oriented M , integration defines a nonzero map $\int_M : H_{dR,c}^n(M) \rightarrow \mathbb{R}$.*
- (v) *If an orientation on M is fixed then*

$$(\omega, \tau) \mapsto \int_M \omega \wedge \tau$$

defines a nonzero pairing

$$H_{dR}^k(M) \times H_{dR,c}^{n-k}(M) \rightarrow \mathbb{R}.$$

Proof. (i) is straightforward from the Leibniz rule for d .

(ii) comes from the fact that if $\omega \in \mathcal{E}_c^k(N)$ and $f : M \rightarrow N$ is proper then $f^*(\omega) \in \mathcal{E}_c^k(M)$.

(iii) comes as follows: if $U \subset M$ is open and $\omega \in \mathcal{E}_c^k(U)$ then extending ω by zero outside U defines a form $i_*(\omega) \in \mathcal{E}_c^k(M)$. Clearly $d(i_*(\omega)) = i_*(d\omega)$ so that this gives a map in cohomology $H_{dR,c}^k(U) \rightarrow H_{dR,c}^k(M)$.

(iv) follows from the corollary to Stokes's theorem.

(v) check that the pairing descends to cohomology (Stokes's theorem again). This is the same as the product

$$H_{dR}^k(M) \otimes H_{dR,c}^{n-k}(M) \rightarrow H_{dR,c}^n(M) \xrightarrow{\int_M} \mathbb{R}$$

which is nonzero by earlier observations. \square

Remark 7.4.3. For a connected oriented manifold the integration map in (iv) is an isomorphism $H_{dR,c}^n(M) \cong \mathbb{R}$. It is a version of Poincaré duality that for a connected oriented manifold the pairing in (v) is nondegenerate. (We will not prove these facts directly here but deduce them from the de Rham theorem later; see Bott-Tu for a direct proofs in the setting of de Rham cohomology.)

Exercise. Show that $H_{dR,c}^1(\mathbb{R}) = \mathbb{R}$ and $H_{dR,c}^0(\mathbb{R}) = 0$. More generally, show that $H_{dR,c}^k(\mathbb{R}^n) =$

$$\begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n. \end{cases}$$

7.5. Poincaré lemma.

Lemma 7.5.1. *For any smooth manifold M the projection $\pi : M \times \mathbb{R} \rightarrow M$ induces an isomorphism $\pi^* : H_{dR}^k(M) \rightarrow H_{dR}^k(M \times \mathbb{R})$.*

Proof. Let $i_t : M \rightarrow M \times \mathbb{R}$ be $i_t(x) = (x, t)$. Since $\pi \circ i_t = \text{id}$ we know that π^* is injective. It remains to prove it is surjective.

A k -form on $M \times \mathbb{R}$ can be written as $\omega = \omega_1 + dt \wedge \omega_2$ where ω_1 and ω_2 involve no dt . (Both ω_1 and ω_2 may depend on t .) Consider the form

$$\sigma := \omega - d \left(\int_0^t \omega_2(s) ds \right).$$

This means the following: In local coordinates on any coordinate patch of $M \times \mathbb{R}$, which we assume to be of the form (x_1, \dots, x_n, t) where x_i are local coordinates on a patch in M , we can write ω_2 as a sum of terms like $\sum_I f_I(x_1, \dots, x_n, t) dx_I$ running over subsets $I \subset \{1, \dots, n\}$ with $|I| = k$. Then the above integral means: In each such coordinate patch take $\sum_I \left(\int_0^t f(x_1, \dots, x_n, s) ds \right) dx_I$ where the integral here is an ordinary Riemann integral. These expressions obviously agree on overlaps (as one varies the patch in M) and hence define a differential k -form, which is what is meant by $\int_0^t \omega_2(s) ds$. Note that the form dt does not appear in σ . (Indeed, the second term in σ gives only one term involving dt , namely $(-1)^{\deg(\omega_2)} \frac{\partial}{\partial t} \left(\int_0^t \omega_2(s) ds \right) \wedge dt = dt \wedge \omega_2$. This is cancelled by the summand $dt \wedge \omega_2$ coming from the first term ω , so that σ has no dt .)

Now $d\sigma = d\omega = 0$ and clearly σ and ω represent the same element of $H_{dR}^k(M \times \mathbb{R})$ as they differ by a coboundary. The form dt does not appear in σ , so if, in local coordinates, $\sigma = \sum_{|I|=k} \sigma_I dx_I$, then $d\sigma = 0$ implies that $\frac{\partial}{\partial t}(\sigma_I) \equiv 0$ for all I . Hence σ is actually constant in t , i.e. $\sigma = \pi^*(\tau)$ for τ a closed form on M . This proves the lemma. \square

Since any point on a manifold has neighbourhoods diffeomorphic to \mathbb{R}^n , we get:

Corollary 7.5.2 (Poincaré lemma). $H_{dR}^k(\mathbb{R}^n) = 0$ for $k > 0$ and $H_{dR}^0(\mathbb{R}^n) = \mathbb{R}$.

Corollary 7.5.3 (Poincaré lemma). *The following is an exact sequence of sheaves on M :*

$$0 \rightarrow \mathbb{R}_M \rightarrow \mathcal{E}^0 \rightarrow \mathcal{E}^1 \rightarrow \dots \rightarrow \mathcal{E}^n \rightarrow 0$$

where \mathbb{R}_M is the constant sheaf on M (i.e. the sheaf of locally constant functions). In other words, the **de Rham complex** (\mathcal{E}^\bullet, d) is a resolution of the constant sheaf.

Proof. The exactness of a sequence of sheaves can be checked on stalks. Fix $p \in M$. The stalk of \mathcal{E}^k at p is $\text{colim}_{p \ni U} \mathcal{E}^k(U)$. An element of this stalk killed by d means giving a k -form ω on some neighbourhood U of p such that $d\omega = 0$. Since U contains a neighbourhood V diffeomorphic to \mathbb{R}^n , we may write $\omega = d\eta$ for $\eta \in \mathcal{E}^{k-1}(V)$. But then $(V, \eta) \in \text{colim}_{p \ni U} \mathcal{E}^{k-1}$ goes to (U, ω) , proving exactness. \square

7.6. Homotopy invariance. Two smooth maps $f, g : M \rightarrow N$ are **smoothly homotopic** if there exists a smooth map $F : \mathbb{R} \times M \rightarrow N$ such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$. (Here \mathbb{R} can be replaced by any open interval containing $0, 1$.)

Lemma 7.6.1. *If f and g are smoothly homotopic then $f^* = g^*$.*

Proof. Let $i_t : M \rightarrow M \times \mathbb{R}$ be the inclusion $i_t(x) = (x, t)$ and let $\pi : M \times \mathbb{R} \rightarrow M$ be the projection. Then $\pi \circ i_t = \text{id}$ and hence $i_t^* \pi^* = \text{id}$. We saw earlier that π^* is an isomorphism, hence $i_t^* = (\pi^*)^{-1}$ is independent of t . In particular $i_0^* = i_1^*$ and hence $f^* = i_0^* F^* = i_1^* F^* = g^*$. \square

It is a fact that in any (continuous) homotopy class of maps between two smooth manifolds there is a smooth map, and two smooth maps in the class are smoothly homotopic. Thus we see that a (continuous) homotopy class of maps between manifolds induces a well-defined map in de Rham cohomology.

7.7. Mayer-Vietoris sequence. Let $U, V \subset M$ be open subsets (hence manifolds themselves) such that $U \cup V = M$. Let $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ be the inclusions. For each k there is a short-exact sequence

$$0 \rightarrow \mathcal{E}^k(M) \rightarrow \mathcal{E}^k(U) \oplus \mathcal{E}^k(V) \xrightarrow{i^* - j^*} \mathcal{E}^k(U \cap V) \rightarrow 0.$$

Exactness at the first two places is obvious, so we must only check surjectivity at the end. Let $\omega \in \mathcal{E}^k(U \cap V)$. Let $\rho_U + \rho_V = 1$ be a partition of unity subordinate to $\{U, V\}$. Then $\rho_U \omega$ defines an element of $\mathcal{E}^k(U)$ by extending it by zero on $V - U \cap V$. Similarly $\rho_V \omega$ defines an element of $\mathcal{E}^k(V)$. Then

$$\sigma := (-\rho_V \omega, \rho_U \omega) \in \mathcal{E}^k(U \sqcup V)$$

goes to ω under $i^* - j^*$.

Since the maps in the short exact sequence are compatible with differentials we have a short-exact sequence of complexes

$$0 \rightarrow \mathcal{E}^*(M) \rightarrow \mathcal{E}^*(U) \oplus \mathcal{E}^*(V) \xrightarrow{i^* - j^*} \mathcal{E}^*(U \cap V) \rightarrow 0.$$

and hence a long exact sequence in de Rham cohomology

$$\cdots \rightarrow H_{dR}^k(M) \rightarrow H_{dR}^k(U \sqcup V) \rightarrow H_{dR}^k(U \cap V) \rightarrow H_{dR}^{k+1}(M) \rightarrow \cdots .$$

A **good cover** of a manifold is a covering $\{U_\alpha\}$ such that all finite intersections $\cap_{i=1}^k U_{\alpha_i}$ are diffeomorphic to \mathbb{R}^n . The Mayer-Vietoris sequence, the Poincaré lemma, and induction on the cardinality of the good cover give:

Theorem 7.7.1. *The de Rham cohomology groups of a manifold with a finite good cover are finite-dimensional. Moreover, they agree with the singular cohomology $H^*(M, \mathbb{Z}) \otimes \mathbb{R} = H^*(M, \mathbb{R})$.*

There is a standard argument to show that a compact manifold always has a (finite) good cover, which uses a Riemannian metric and geodesically convex sets. However, we will not give it here.

There is a Mayer-Vietoris sequence for compactly supported de Rham cohomology: Let $\{U, V\}$ be an open covering of M . The Mayer-Vietoris sequence

$$\cdots \rightarrow H_{dR,c}^i(U \cap V) \rightarrow H_{dR,c}^i(U \sqcup V) \rightarrow H_{dR,c}^i(M) \rightarrow \cdots$$

comes from the following short exact sequence of complexes

$$0 \rightarrow \mathcal{E}_c^*(U \cap V) \rightarrow \mathcal{E}_c^*(U \sqcup V) \xrightarrow{i_* - j_*} \mathcal{E}_c^*(M) \rightarrow 0$$

where $\{i : U \hookrightarrow M, j : V \hookrightarrow M\}$ is an open cover of M . (The maps i_*, j_* here come from the covariant functoriality of $\mathcal{E}_c^k(-)$ under open inclusion; verify the exactness of the sequence.)

Example 7.7.2. It follows easily from the M-V sequence and an induction that

$$H_{dR}^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

We can write down an explicit generator for $H_{dR}^n(S^n)$ as follows: Consider the form $\omega \in \mathcal{E}^n(\mathbb{R}^{n+1})$ defined by

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_{\{1, \dots, n+1\} - \{i\}}.$$

Then $d\omega = (n+1)dx_{\{1, \dots, n+1\}}$ is a volume form on \mathbb{R}^{n+1} . Let $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion. Then

$$\sigma := i^* \omega \in \mathcal{E}^n(S^n)$$

is closed (being a top form) and $[\sigma]$ is a generator of $H_{dR}^n(S^n)$ because

$$\int_{S^n} \sigma = \int_{B^{n+1}} d\omega \neq 0$$

where B^{n+1} is the ball of radius one.

Example 7.7.3. Let us write down a generator for $H_{dR}^*(\mathbb{C}P^n)$. (At this stage we do not yet know that $H_{dR}^*(\mathbb{C}P^n) = \mathbb{R}[x]/(x^{n+1})$, so assume this.) Consider the map $p : S^{2n+1} \rightarrow \mathbb{C}P^n$ which is the restriction of $\mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{C}P^n$. For $i = 0$ to n let $dz_i = dx_i + \sqrt{-1}dy_i$ and $d\bar{z}_i = dx_i - \sqrt{-1}dy_i$ so that $dz_i \wedge d\bar{z}_i = -2\sqrt{-1}dx_i \wedge dy_i$. Restricting $\frac{\sqrt{-1}}{2} \sum_{i=0}^n dz_i \wedge d\bar{z}_i = \sum_i dx_i \wedge dy_i$ to S^{2n+1} gives a closed real 2-form σ .

- Show that there is a closed 2-form ω on $\mathbb{C}P^n$ such that $p^*(\omega) = \sigma$.
- Show that ω is invariant under the natural action of $U(n+1)$ on $\mathbb{C}P^n$.
- Show that ω^k is nonzero for any $k \leq n$.

7.8. De Rham theorem. The de Rham cohomology groups of a smooth manifold depend, *a priori*, on the smooth structure of the manifold. In fact, they depend only on the underlying topological space and they compute the singular cohomology groups, tensored with \mathbb{R} :

Theorem 7.8.1 (De Rham). *There are natural isomorphisms $H_{dR}^*(M) \cong H^*(M, \mathbb{R})$ and $H_{dR,c}^*(M) \cong H_c^*(M, \mathbb{R})$.*

Proof. For the general case this will be proved in the next section using sheaf-theoretic methods. Here let's prove it for compact manifolds using the existence of a finite good cover. Indeed, if $\{U_i\}_{i=1}^n$ is a finite good cover, then a straightforward induction using the Mayer-Vietoris sequence and the Poincaré lemma proves the theorem. \square

The isomorphism of the theorem is functorial with respect to smooth maps, but we will not prove this now.

Note that for an oriented manifold M , the isomorphism $H_{dR,c}^n(M) \cong H_c^n(M, \mathbb{R})$ relates the integration map $\int_M : H_{dR,c}^n(M) \rightarrow \mathbb{R}$ to the map $PD_M : H_c^n(M, \mathbb{R}) \rightarrow H_0(M, \mathbb{R}) = \mathbb{R}$ used in the Poincaré duality theorem. (In the case M is compact PD_M is cap product with the fundamental class $[M]$ given by the orientation). Thus we can conclude that if M is connected then $H_{dR,c}^n(M) \cong \mathbb{R}$. Also, we can apply the Poincaré duality theorem to conclude that the pairing

$$H_{dR}^k(M) \times H_{dR,c}^{n-k}(M) \rightarrow \mathbb{R}.$$

by $(\omega, \tau) \mapsto \int_M \omega \wedge \tau$ is a nondegenerate pairing. (For a purely de Rham cohomology proof of this duality see Bott&Tu.)

7.9. Relative de Rham cohomology (*). Let $f : M \rightarrow N$ be a smooth map. We will define de Rham cohomology groups $H^*(f : M \rightarrow N)$. When f is an embedding this will give back the relative cohomology $H^*(M, N, \mathbb{R})$ under a relative version of the de Rham theorem.

THIS IS STILL TO BE WRITTEN

8. MORE DIFFERENTIAL FORMS

8.1. Lie derivative. Let X be a vector field on M . The **contraction** operator $i_X : \mathcal{E}^k \rightarrow \mathcal{E}^{k-1}$ is defined as follows: For a k -form ω we have

$$i_X \omega(X_1, \dots, X_{k-1}) := \omega(X, X_1, \dots, X_{k-1}).$$

(When $k = 0$ the contraction of a function is defined to be zero.) Evidently $i_X^2 = 0$. Notice that contraction is a derivation of degree -1 of the graded algebra $\mathcal{E}^* = \bigoplus_{k \geq 0} \mathcal{E}^k$ differential forms in the sense that

$$i_X(\alpha \wedge \beta) = i_X(\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge i_X(\beta)$$

for homogeneous α, β .

The **Lie derivative** operator $L_X : \mathcal{E}^k \rightarrow \mathcal{E}^k$ is defined by:

$$L_X \omega := di_X \omega + i_X d\omega = (d + i_X)^2 \omega$$

This is **Cartan's formula**. If $k = 0$, i.e. we have a function, then we have

$$L_X f = i_X df = Xf$$

(the vector field applied to the function). The formal properties of the Lie derivative are easy to verify:

Lemma 8.1.1. *The Lie derivative satisfies:*

- (i) $dL_X = L_X d$
- (ii) $L_X(\omega_1 \wedge \omega_2) = L_X \omega_1 \wedge \omega_2 + (-1)^{\deg(\omega_1)} \omega_1 \wedge L_X \omega_2$ (*Leibniz rule*)
- (iii) If $\varphi : M \rightarrow M$ is a diffeomorphism then $\varphi^*(L_X \omega) = L_{\varphi^*(X)}(\varphi^* \omega)$.
- (iv) For vector fields X, Y we have $i_{[X, Y]} = L_X \circ i_Y - i_Y \circ L_X$.

Proof. (i) follows from $L_X = di_X + i_X d$ and $d^2 = 0$. (ii) follows from the Leibniz rule for i_X and d . (iii) and (iv) are left as exercises. \square

Lemma 8.1.2. *Suppose X is a vector field with associated local flow α . Then*

$$\alpha_t^*(L_X \omega) = \frac{d}{dt} \alpha_t^*(\omega). \quad (8.1.1)$$

At $t = 0$ we get

$$L_X \omega = \left. \frac{d}{dt} \right|_{t=0} \alpha_t^*(\omega)$$

(which is sometimes given as the definition of L_X). ⁽³⁶⁾

Proof. Both sides of (8.1.1) are derivations of the algebra of differential forms, and both sides commute with d , so it suffices to check it for functions. Writing out the derivative, we have

$$\frac{d}{dt} \phi_t^*(f) = \lim_{h \rightarrow 0} \frac{\phi_{t+h}^*(f) - \phi_t^*(f)}{h} = \phi_t^* \left(\lim_{h \rightarrow 0} \frac{\phi_h^*(f) - f}{h} \right) = \phi_t^*(Xf) = \phi_t^*(L_X f)$$

which is what we want. \square

Example 8.1.3. The Lie derivative generalizes familiar objects: Let α_t be a local flow on an open subset of \mathbb{R}^n with associated vector field X . Then one has the identity

$$L_X(dx_1 \wedge \dots \wedge dx_n) = \operatorname{div}(X) dx_1 \wedge \dots \wedge dx_n$$

where $\operatorname{div}(X)$ is the classical divergence of a vector field, defined by $\operatorname{div}(\sum_i f_i \frac{\partial}{\partial x_i}) = \sum_i \frac{\partial f_i}{\partial x_i}$. Thus if a vector field is divergence-free then its flow preserves the volume form (by (8.1.1)).

³⁶This identity/definition is reminiscent of the definition of Lie bracket in Lemma 3.4.1. Indeed, obviously we can make sense of $L_X \tau$ for any tensor field, i.e. τ a section of $\wedge^r TM \otimes \wedge^s T^*M$ for any $r, s \geq 0$.

8.2. Explicit homotopy operators. Let M be a smooth manifold and X a vector field with α_t its associated local flow. Assume that α_t is defined for $t \in [0, 1]$ (e.g. if X has relatively compact support). By the previous lemma,

$$\alpha_1^*(\omega) - \alpha_0^*(\omega) = \int_0^1 \frac{d}{dt} \alpha_t^*(\omega) = \int_0^1 \alpha_t^*(L_X \omega) dt.$$

Define an operator $H : \mathcal{E}^k(M) \rightarrow \mathcal{E}^{k-1}(M)$ by

$$H(\omega) = \int_0^1 i_X \alpha_t^*(\omega) dt.$$

Then

$$(dH + Hd)(\omega) = \int_0^1 L_X(\alpha_t^* \omega) = \int_0^1 \alpha_t^*(L_X \omega) = \alpha_1^*(\omega) - \alpha_0^*(\omega).$$

Thus H gives an explicit homotopy operator between α_1^* and $\alpha_0^* = id$.

This gives another way of showing that homotopic maps induce the same map in de Rham cohomology. Suppose $f_0, f_1 : M \rightarrow N$ are smooth maps and suppose they are smoothly homotopic, i.e. there is a smooth map $F : M \times \mathbb{R} \rightarrow N$ such that $F|_{M \times \{0\}} = f_0$ and $F|_{M \times \{1\}} = f_1$. Then for $i_t : M \rightarrow M \times \mathbb{R}$ by $i_t(p) = (p, t)$ we have

$$f_0 = F \circ i_0, \quad f_1 = F \circ i_1 = F \circ \alpha_1 \circ i_0$$

where α_t is the flow of the vector field $\frac{\partial}{\partial t}$ on $M \times \mathbb{R}$. Then we define the homotopy operator on $M \times \mathbb{R}$ as above $H : \mathcal{E}^k(M \times \mathbb{R}) \rightarrow \mathcal{E}^{k-1}(M \times \mathbb{R})$, and consider the operator $\tilde{H} := i_0^* H F^* : \mathcal{E}^k(N) \rightarrow \mathcal{E}^{k-1}(M)$, i.e. the composition

$$\tilde{H} := i_0^* H F^* := \left(\mathcal{E}^k(N) \xrightarrow{F^*} \mathcal{E}^k(M \times \mathbb{R}) \xrightarrow{H} \mathcal{E}^{k-1}(M \times \mathbb{R}) \xrightarrow{i_0^*} \mathcal{E}^{k-1}(M) \right).$$

This satisfies

$$d\tilde{H} + \tilde{H}d = i_0^*(\alpha_1 - \alpha_0)F = f_1^* - f_0^*$$

i.e. it is an explicit homotopy between f_0^* and f_1^* .

8.3. Isotopies and time-dependent vector fields. Recall that we observed that flows and vector fields were closely related. A flow can be thought of as a one-parameter group of diffeomorphisms, i.e. a group homomorphism $\mathbb{R} \rightarrow \text{Diff}(M)$ which is smooth for a suitable smooth structure on $\text{Diff}(M)$. The notion of **isotopy** is the weaker version of this without the group structure, i.e. a one-parameter family of diffeomorphisms $\phi_t : M \rightarrow M$ for $t \in \mathbb{R}$. (So we need not have $\phi_s \circ \phi_t = \phi_{s+t}$.) The analogous notion on the vector field side is that of time-dependent vector field.

By definition, a **time-dependent vector field** is a smooth mapping from an open subset of $\mathbb{R} \times M$ to TM , which we will write as

$$(t, p) \mapsto v_t(p) \in T_p M.$$

An integral curve of a time-dependent vector field v_t is a mapping $\gamma : (a, b) \rightarrow M$ satisfying the equation

$$d\gamma_t \left(\frac{\partial}{\partial t} \right) = v_t(\gamma(t))$$

for $t \in (a, b)$. Integral curves starting at a given point of M exist for sufficiently small times, by the same existence-uniqueness theorem for ODE as in the earlier case. (Notice that there are many integral curves passing through a given point, depending on the time at which they are passing through it, in contrast with the time-independent case.)

The associated operation “follow the integral curve for small times” is then a family of local diffeomorphisms $\phi_t : U \rightarrow M$, defined for $|t| < \epsilon$ for some ϵ on relatively compact U . This is no longer necessarily a local flow, i.e. the property $\phi_t \circ \phi_s = \phi_{s+t}$ need not hold. We will still refer to ϕ_t as a flow, though one should keep in mind that this property is not available. If ϵ can be chosen uniformly at all points of M then we have an isotopy $\phi_t : M \rightarrow M$. This is, in

particular, the case if v_t has relatively compact support (independent of t). The time-dependent vector field and the isotopy are related by

$$v_t(f) = \left. \frac{d}{ds} \right|_{s=t} \phi_s^* \phi_{-t}^*(f).$$

Conversely, given an isotopy $\phi_t : M \rightarrow M$ we can go in the other direction, i.e. we get a time-dependent vector field using this formula.

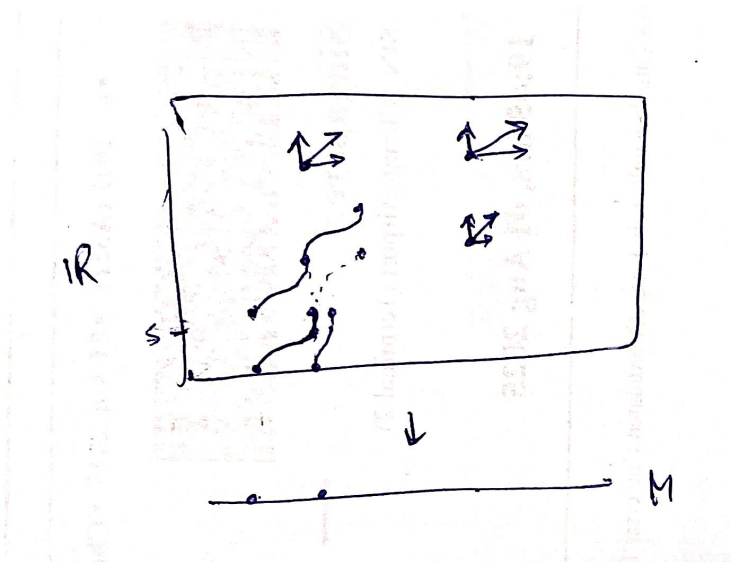
A formally convenient way to think of a time-dependent vector field v on M is using the ordinary (i.e. time-independent) vector field \tilde{v} on $\mathbb{R} \times M$ given by

$$\tilde{v}(t, p) = \frac{\partial}{\partial t} + v_t(p) \in T_{(t,p)}(\mathbb{R} \times M) = T_t\mathbb{R} \times T_pM.$$

An integral curve of \tilde{v} is of the form $\gamma(s) = (s, \gamma_2(s))$ where $\gamma_2(s)$ satisfies the differential equation $d\gamma_{2,s} \left(\frac{\partial}{\partial s} \right) = v_s(\gamma_2(s))$, so that

$$d\gamma_s \left(\frac{\partial}{\partial s} \right) = \left(\frac{\partial}{\partial s}, d\gamma_{2,s} \left(\frac{\partial}{\partial s} \right) \right) = \left(\frac{\partial}{\partial s}, v_s(\gamma_2(s)) \right) = \tilde{v}(\gamma(s)).$$

The picture is as follows:



(Note that the integral curve starting at (t, p) is not a translate of the one starting at $(0, p)$, i.e. the picture is not invariant under translations in the vertical direction.) So projection $\pi : \mathbb{R} \times M \rightarrow M$ relates the integral curves of \tilde{v} and the integral curves of v_t : The projection of the integral curve of \tilde{v} starting at (t, p) gives the integral curve of v_t starting at p . The relation between flows is as follows:

$$\tilde{\phi}_s(0, p) = (s, \phi_s(p))$$

or $\phi_s(p) = \pi(\tilde{\phi}_s(0, p))$.

Lemma 8.3.1. *Let X_t be a time-dependent vector field on M and ϕ_t its flow. Then*

$$\frac{d}{dt} \phi_t^*(\omega) = \phi_t^*(L_{X_t}\omega)$$

for any differential form ω . ⁽³⁷⁾

Proof. This is like (8.1.1) except that things are time-dependent. For the form $\pi^*(\omega)$ on $\mathbb{R} \times M$ and the vector field \tilde{X} associated with X_t as above, we have

$$\frac{d}{dt} \tilde{\phi}_t^*(\pi^*(\omega)) = \tilde{\phi}_t^*(L_{\tilde{X}}(\pi^*(\omega))).$$

³⁷If ω_t is a k -form depending smoothly on a parameter t , then $\frac{d\omega_t}{dt}$ is the k -form given by differentiating the coefficient functions in a chart, i.e. we write locally $\omega_t = \sum_I \omega_I(x, t) dx_I$ and then $\frac{d\omega_t}{dt} = \sum_I \frac{d\omega_I(x, t)}{dt} dx_I$.

This gives the identity on projection to the second factor. \square

Remark 8.3.2 (Homotopy operator from an isotopy). The homotopy operator H defined earlier can be extended to this setting: If ϕ_t is the flow of a time-dependent vector field X_t (e.g. ϕ_t is an isotopy) then let

$$H(\omega) = \int_0^1 i_{X_t} \phi_t^*(\omega) dt.$$

Then we have

$$(dH + Hd)(\omega) = \int_0^1 L_{X_t} \phi_t^*(\omega) = \int_0^1 \frac{d}{dt} \phi_t^*(\omega) dt = \phi_1^*(\omega) - \phi_0^*(\omega)$$

giving a homotopy between ϕ_1^* and ϕ_0^* .

We can also allow for a time-varying differential form in the setup, which will be useful later:

Lemma 8.3.3. *Let X_t be a smooth time-dependent vector field and ϕ_t be the associated flow. Then for a family of smooth differential forms ω_t , we have*

$$\frac{d}{dt}(\phi_t^* \omega_t) = \phi_t^*(L_{X_t} \omega_t) + \phi_t^* \left(\frac{d}{dt} \omega_t \right).$$

Proof. Consider $\phi_r^* \omega_s$ as a function of $(r, s) \in \mathbb{R}^2$. We wish to restrict it to $r = s = t$ and compute the derivative. By the chain rule we have that

$$\begin{aligned} \frac{d}{dt}(\phi_t^* \omega_t) &= \frac{d}{dr} \Big|_{r=t} (\phi_r^* \omega_t) + \frac{d}{ds} \Big|_{s=t} (\phi_t^* \omega_s) \\ &= \phi_t^*(L_{X_t} \omega_t) + \phi_t^* \left(\frac{d}{dt} \omega_t \right) \end{aligned} \tag{8.3.1}$$

using the previous lemma for the first term. \square

8.4. Moser's trick and Moser's theorem.

Lemma 8.4.1 (Moser's trick). *Let ω_t be a (smooth) family of smooth differential forms on M . If there exists a time-dependent vector field X_t on M for $t \in I$ such that*

$$L_{X_t} \omega_t + \frac{d}{dt} \omega_t = 0 \tag{8.4.1}$$

then there exists an isotopy ϕ_t with $\phi_0 = id$ and $\phi_t^(\omega_t) = \omega_0$ for all t .*

Proof. Let ϕ_t be the flow of the time-dependent vector field X_t . Then by the previous lemma $\frac{d}{dt}(\phi_t^* \omega_t) = 0$, so that $\phi_t^* \omega_t = \omega_0$. \square

The point of the trick is that the condition (8.4.1), if satisfied, allows us to construct an isotopy. By Cartan's formula we can rewrite the condition (8.4.1) as

$$d(i_{X_t} \omega_t) + i_{X_t} d\omega_t + \frac{d}{dt} \omega_t = 0. \tag{8.4.2}$$

Now if ω_t is closed for all t then this becomes the condition

$$d(i_{X_t} \omega_t) + \frac{d}{dt} \omega_t = 0.$$

In other words, if the time derivative of ω_t is exact then we can find an isotopy with $\phi_t^*(\omega_t) = \omega_0$. Here is an example:

Theorem 8.4.2 (Moser⁽³⁸⁾). *Let M be a compact oriented n -manifold and let $\omega_0, \omega_1 \in \mathcal{E}^n(M)$ be two volume forms (i.e. nowhere-vanishing n -forms). Then*

$$\int_M \omega_0 = \int_M \omega_1 \quad (8.4.3)$$

if and only if there is an orientation-preserving diffeomorphism $\varphi : M \rightarrow M$ such that $\varphi^(\omega_1) = \omega_0$.*

Proof. We assume M is connected, which is no loss of generality. If $\varphi^*(\omega_1) = \omega_0$ then

$$\int_M \omega_0 = \int_M \varphi^*(\omega_1) = \int_M \omega_1$$

by the change of variables formula, so we need to consider the other direction. In fact we will show that if $\int_M \omega_0 = \int_M \omega_1$ then there is an isotopy $\varphi_t : M \rightarrow M$ such that $\varphi_0 = id$ and $\varphi := \varphi_1$ satisfies $\varphi^*(\omega_1) = \omega_0$.

Since $\int_M \omega_0 = \int_M \omega_1$, the class of $\omega_1 - \omega_0$ in $H_{dR}^n(M) \cong \mathbb{R}$ is zero. Thus there exists $\beta \in \mathcal{E}^{n-1}(M)$ such that

$$d\beta = \omega_1 - \omega_0.$$

Let

$$\omega_t := t\omega_0 + (1-t)\omega_1.$$

Since $\omega_t = f\omega_0$ for a smooth function which is everywhere positive, ω_t is a nowhere-vanishing top form for $0 < t < 1$. The previous lemma will construct an isotopy for us with the desired property as long as we can satisfy the condition (8.4.2), which in the case of interest is

$$0 = d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t = d(i_{X_t}\omega_t + \beta)$$

since $\frac{d}{dt}\omega_t = \omega_1 - \omega_0$. It is enough to solve the equation

$$0 = i_{X_t}\omega_t + \beta$$

for X_t , which we can always do since the ω_t are volume forms. \square

Let M be a smooth compact $2n$ manifold. Let ω be a smooth 2-form on M which is closed (i.e. $d\omega = 0$) and nondegenerate (by definition, this means ω^n is nowhere vanishing, i.e. a volume form on M). Such an ω is called a **symplectic form** on M . Notice that $[\omega] \in H_{dR}^{2n}(M)$ is nonzero since $\int_M \omega^n \neq 0$. The symplectic form gives an isomorphism of vector bundles $TM \cong T^*M$, and hence of vector fields (= sections of TM) with one-forms (= sections of T^*M).

Theorem 8.4.3 (Moser). *Let M be compact and let $\omega_t = \omega_0 + d\beta_t$ be a smooth family of symplectic forms. Then there exists a family of diffeomorphisms $\varphi_t : M \rightarrow M$ such that $\varphi_t^*\omega_t = \omega_0$.*

Proof. We use the Moser trick. The equation to be satisfied is

$$0 = d(i_{X_t}\omega_t) + \frac{d}{dt}(d\beta_t) = d\left(i_{X_t}\omega_t + \frac{d}{dt}\beta_t\right).$$

By nondegeneracy of the symplectic form ω_t we can solve the equation

$$i_{X_t}\omega_t + \frac{d}{dt}\beta_t = 0$$

for X_t and hence the previous equation. Then Moser's trick gives the theorem. \square

Thus one cannot nontrivially deform the symplectic form in the same de Rham cohomology class. (This does not hold in the noncompact case.)

³⁸J. Moser, On the volume elements on a manifold, *Transactions of the AMS* 120 (1965).

Remark 8.4.4. In both theorems above, it was key that the Moser condition reduced to $n - 1$ equations (because the β or β_t are $n - 1$ forms or 1-forms, hence have $n - 1$ coefficient functions) in $n - 1$ unknowns (the coefficients of X_t). This made solving for X_t an exercise in linear algebra.

9. BASICS OF SYMPLECTIC MANIFOLDS

There are many references for this material, e.g. D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Chapters 2 and 3, or A. Cannas da Silva, *Lectures on Symplectic Geometry*, Parts I – III (both of which I have borrowed from here). A beautiful account which may be a little harder to learn from is R. Bryant’s 1991 Park City lectures *Introduction to Lie Groups and Symplectic Geometry*.

9.1. Symplectic linear algebra. A **symplectic form** on a (real) vector space V is a nondegenerate alternating bilinear form $\omega : V \times V \rightarrow \mathbb{R}$, or, equivalently, a nonzero element of $\Lambda^2 V^*$. We shall see presently that having such a form forces $\dim_{\mathbb{R}} V$ to be even, and there is a unique form in each even dimension up to suitable equivalence. The nondegeneracy of the form means that it gives isomorphisms $V \rightarrow V^*$ and $H : V^* \rightarrow V$ dual to each other. This H has the property that

$$\omega(v, H(\ell)) = \ell(v) \quad (v \in V, \ell \in V^*).$$

Example 9.1.1. Given a vector space W the sum $W \oplus W^*$ has a symplectic form given by

$$\omega((w, \ell), (w', \ell')) = \ell'(w) - \ell(w').$$

for $w, w' \in W$ and $\ell, \ell' \in W^*$. From another point of view, $\Lambda^2(W \oplus W^*)^* = \Lambda^2 W^* \otimes W + W^* \otimes W + W \otimes \Lambda^2 W$ contains the canonical element $id \in Hom(W, W) \cong W^* \otimes W$ and this is simply ω . (Note that $-\omega$ is also a symplectic form.)

Let (V, ω) be a symplectic space. For a subspace $W \subset V$ define

$$W^\perp := \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}.$$

Lemma 9.1.2. $\dim W + \dim W^\perp = \dim V$ and $(W^\perp)^\perp = W$.

Proof. Exercise. □

A subspace $W \subset V$ is called:

- (1) **symplectic** if $W \cap W^\perp = \{0\}$, in which case $(W, \omega|_W)$ is again a symplectic space.
- (2) **isotropic** if $W^\perp \supset W$, in which case ω descends to W^\perp/W and $(W^\perp/W, \omega)$ is a symplectic space
- (3) **coisotropic** if $W^\perp \subset W$, in which case $(W/W^\perp, \omega)$ is a symplectic space
- (4) **Lagrangian** if it is both isotropic and coisotropic, i.e. if $W^\perp = W$.

Example 9.1.3. The subspaces W, W^* in $W \oplus W^*$ with the symplectic form as in Example 9.1.1 are both Lagrangian.

Lemma 9.1.4. *If V carries a symplectic form ω then it has even dimension $2n$ and there is a basis $\{e_i, f_i\}_{1 \leq i \leq n}$ in which the form is given by $\omega(e_i, e_j) = 0$, $\omega(f_i, f_j) = 0$, and $\omega(e_i, f_j) = \delta_{ij}$.*

Proof. Choose vectors e_1, f_1 with $\omega(e_1, f_1) \neq 0$. Then $W = \mathbb{R}e_1 \oplus \mathbb{R}f_1$ is a symplectic subspace and W^\perp is a symplectic subspace of smaller dimension. The lemma follows by induction. □

A basis of this type is called a **symplectic basis** for V . In this basis the form is given by $\omega(v, w) = v^T J w$ where $J = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$. The 2-form is given by

$$\omega = \sum_j f_j^* \wedge e_j^* \in \Lambda^2 V^*$$

where $\{e_i^*, f_i^*\}$ is the dual basis. Notice that ω is nondegenerate (as a bilinear form) if and only if $\omega^n \neq 0$. (Indeed, if $\omega = \sum_i f_i^* \wedge e_i^*$ then $\omega^n = \pm e_1^* \wedge \cdots \wedge e_n^* \wedge f_1^* \wedge \cdots \wedge f_n^*$.)

Example 9.1.5. Let \mathbb{R}^{2n} be given the standard symplectic form. The graph of $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, considered as a linear subspace of \mathbb{R}^{2n} , is Lagrangian if and only if A is a symmetric matrix. ⁽³⁹⁾

A linear isomorphism $\varphi : V \rightarrow V'$ of symplectic spaces is called a **symplectomorphism** if $\varphi^*(\omega') = \omega$. So the previous lemma says that (V, ω) is symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$ where ω_0 is the standard form.

Lemma 9.1.6. *Let (V, ω) be symplectic. Then every isotropic subspace is contained in a Lagrangian subspace. If L, L' are Lagrangian then any linear map $L \rightarrow L'$ extends to a symplectomorphism of V .*

Proof. Exercise. □

Remark 9.1.7. If L is a Lagrangian subspace of (V, ω) then $(V, \omega) \cong (L \oplus L^*, \pm\omega_{can})$ where ω_{can} is given as in the example above.

Example 9.1.8. Let (V, ω) be a symplectic space. A linear map $\phi : V \rightarrow V$ is a symplectomorphism if and only if the graph $\Gamma = \{(v, \phi(v)) : v \in V\}$ is a Lagrangian subspace of $V \times V$ with the symplectic form $-\omega \times \omega$. ⁽⁴⁰⁾

The group of symplectomorphisms of (V, ω) with itself is called the **symplectic group** of (V, ω) and denoted $Sp(V, \omega)$. The choice of symplectic basis as in the lemma shows that

$$Sp(V, \omega) = Sp(2n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) : A^T J A = J\}. \quad (9.1.1)$$

This is the real points of a linear algebraic group, hence a Lie group. To calculate its dimension, we can either try to show that its Lie algebra is

$$\mathfrak{sp}(2n, \mathbb{R}) = \{A \in M(2n, \mathbb{R}) : A^T J + J A = 0\}$$

which is straightforward, or we can argue that $Sp(2n, \mathbb{R})$ is the stabilizer in $GL(2n, \mathbb{R})$ of the vector $\omega \in \wedge^2 \mathbb{R}^{2n}$. Since $GL(2n, \mathbb{R})$ acts with an open orbit on $\wedge^2 \mathbb{R}^{2n}$ the dimension of the stabilizer is $\dim GL(2n, \mathbb{R}) - \dim \wedge^2 \mathbb{R}^{2n} = (2n)^2 - \binom{2n}{2} = 4n^2 - 2n(2n - 1)/2 = n(2n + 1)$.

Lemma 9.1.9. $Sp(V, \omega) \subset SL(V)$.

Proof. From $A^T J A = J$ one gets $\det(A)^2 = 1$, but we must show $\det(A) = 1$. Recall from algebra that associated with an alternating matrix S (by definition, this means $S^T = -S$ and $S_{ii} = 0$) over any ring is a quantity called the Pfaffian, which is a polynomial expression in the entries of S , satisfying (1) $Pf(S)^2 = \det(S)$ and (2) $Pf(B^T S B) = \det(B) Pf(S)$ for any matrix B . ⁽⁴¹⁾ But now using $S = J$, for $A \in Sp(2n, \mathbb{R})$ we get $Pf(J) = \det(A) Pf(J)$, and hence $\det(A) = 1$. □

³⁹Indeed, for $v, w \in \mathbb{R}^n$, we have

$$\omega_0((v, Av), (w, Aw)) = (v^T, (Av)^T) \begin{pmatrix} & -I \\ I & \end{pmatrix} \begin{pmatrix} w \\ Aw \end{pmatrix} = (v^T A^T, -v^T) \begin{pmatrix} w \\ Aw \end{pmatrix} = v^T (A^T - A)w.$$

⁴⁰Indeed, for $v, w \in V$, $(-\omega, \omega)((v, \phi(v)), (w, \phi(w))) = -\omega(v, w) + \omega(\phi(v), \phi(w)) = 0$.

⁴¹Here is a proof of the existence of the Pfaffian over any ring. Consider the general alternating matrix $X = (x_{ij})$ with $x_{ij} = -x_{ji}$ for $i < j$ and $x_{ii} = 0$. We consider this as an $n \times n$ matrix with entries from the polynomial ring $\mathbb{Z}[x_{ij}]_{i < j}$ in $n(n - 1)/2$ variables over \mathbb{Z} . Now $\det(X)$ is a square in the quotient field $\mathbb{Q}(x_{ij})$. (Indeed, over any field X may be put in the standard form $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ by the proof of Lemma 9.1.4, in other words there is an invertible matrix C with entries in $\mathbb{Q}(x_{ij})$ such that $C^T X C = J$. Then $\det(C)^2 \det(X) = \det(J) = 1$ or 0 , hence $\det(X)$ is a square.) Since $\mathbb{Z}[x_{ij}]$ is a unique factorization domain, it must be a square in $\mathbb{Z}[x_{ij}]$ (Gauss lemma). Thus there is a polynomial $Pf(x_{ij})$ such that $\det(X) = Pf(x_{ij})^2$. Evidently Pf is unique only up to multiplication by -1 , so we will fix $Pf(x_{ij})$ uniquely by requiring $Pf(J) = -1$, where $J = \begin{pmatrix} & -I \\ I & \end{pmatrix}$. Now for a general ring R and an antisymmetric matrix S over R , we have an obvious homomorphism $\mathbb{Z}[x_{ij}] \rightarrow R$ by $x_{ij} \mapsto S_{ij}$ and we define $Pf(S)$ to be the image of $Pf(X)$ under $\mathbb{Z}[x_{ij}] \rightarrow R$. For the identity $Pf(B^T S B) = \det(B) Pf(S)$ we note that it holds up to a (universal) sign since it holds after squaring both sides. Thus it is enough to check that it is $+1$ for some choice of B and S . For this take $S = J$ and $B = I$. □

Example 9.1.10. If $\dim V = 2$ then we have $Sp(V, \omega) = SL(V)$ for any ω . Thus symplectomorphisms are simply area-preserving transformations of V .

Lemma 9.1.11. *A matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n, \mathbb{R})$ is in $Sp(2n, \mathbb{R})$ if and only if its inverse is of the form $\begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}$.*

Proof. Write out the equations. □

Example 9.1.12. $O(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R})$ consists of matrices of the form $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ where $A^T B = B^T A$ and $A^T A + B^T B = I_n$. In other words, $A + \sqrt{-1}B$ is a unitary matrix, i.e. $O(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R}) \cong U(n)$.

Proposition 9.1.13. *The subgroup $U(n) \subset Sp(2n, \mathbb{R})$ is a maximal compact subgroup. The quotient $Sp(2n, \mathbb{R})/U(n)$ is identified with the space $S\text{Pos}_{2n}$ of symmetric positive definite symplectic matrices. The space $Sp(2n, \mathbb{R})/U(n) \cong S\text{Pos}_{2n}$ is contractible. (In fact, it is diffeomorphic to $\mathbb{R}^{n(n+1)}$.)*

Proof. This is a version for the symplectic group of the analogous facts for $GL(N, \mathbb{R})$ and its maximal compact subgroup $O(N, \mathbb{R})$, so as a warmup we recall how those facts are proved using polar decomposition.

Given $A \in GL(N, \mathbb{R})$ the matrix $A^T A$ is symmetric and positive definite. It is therefore diagonalizable by orthogonal matrices, i.e. there is a matrix $C \in O(N, \mathbb{R})$ such that $D = C(A^T A)C^T$ is diagonal with positive diagonal entries. (When you think of $A^T A$ as giving a positive definite bilinear form, i.e. inner product, then this is the same as saying there is an orthogonal basis, which can be constructed e.g. using the Gram-Schmidt process.) Then we can define D^r for any real r and hence we can define $(A^T A)^r := C^T D^r C$ for any r . Let

$$P := (A^T A)^{1/2}, \quad U := (A^T A)^{-1/2} A.$$

Then P is symmetric positive definite, U is orthogonal, and $A = PU$ is the unique way to write A as a product of a symmetric positive definite matrix and an orthogonal matrix. This is the polar decomposition of A . It follows that the action of $GL(N, \mathbb{R})$ on Pos_N by $(g, P) \mapsto g^T P g$ is transitive and the orbit map of I_N induces a bijection $GL(N, \mathbb{R})/O(N, \mathbb{R}) \cong \text{Pos}_N$.⁽⁴²⁾ To see that Pos_N is contractible note that $P \mapsto P^s$ defines (for $s \in [0, 1]$) a deformation retraction of Pos_N to the point $\{I_N\}$. The statement that Pos_N is diffeomorphic to $\mathbb{R}^{\frac{N(N+1)}{2}}$ requires more work – one proves that the exponential map is a diffeomorphism of the space of all symmetric matrices onto Pos_N . We do not need it so we will not do it here.

Now let us adapt the argument to $Sp(2n, \mathbb{R})$. The main point is to observe that in the polar decomposition $A = PU$ of $A \in Sp(2n, \mathbb{R})$, both P and U belong to $Sp(2n, \mathbb{R})$. If $P \in Sp(2n, \mathbb{R})$ then $U \in O(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R}) = U(n)$, so let us see that $P \in Sp(2n, \mathbb{R})$. For this we note some facts about a symplectic matrix. The characteristic polynomial of a symplectic matrix of size $2n$ is real and palindromic, i.e. the coefficients of t^k and t^{2n-k} are equal. (Indeed, the coefficient of t^k is, up to the sign $(-1)^k$, the trace of the action of the matrix in $\wedge^k \mathbb{R}^{2n}$, and if B preserves a symplectic form then $\wedge^k \mathbb{R}^{2n} \cong (\wedge^{2n-k} \mathbb{R}^{2n})^*$.) Thus the eigenvalues of B come in pairs, in the sense that λ, λ^{-1} appear with equal multiplicity. (Alternately, note that $B^T J B = J$ implies $B^T = J B^{-1} J^{-1}$, so that B and B^{-1} have the same eigenvalues.) The multiplicity of -1 is even since $\det(B) = 1$, and so the multiplicity of $+1$ is also even since the total dimension is even. Decomposing \mathbb{R}^{2n} into eigenspaces of B , we see that the λ and λ' eigenspaces are ω -orthogonal unless $\lambda\lambda' = 1$, since $\omega(v, v') = \omega(Bv, Bv') = \omega(\lambda v, \lambda' v') = \lambda\lambda' \omega(v, v')$.

Now consider the polar decomposition $P = (A^T A)^{1/2} (A^T A)^{-1/2} A$ of A . If A is symplectic, so are A^T and $A^T A$. Then $(A^T A)^s$ as defined above is symplectic for all s : Indeed, this follows from what was just established about the $B = A^T A$ -eigenspace decomposition of \mathbb{R}^{2n} .

⁴²In fact you can use the polar decomposition to show that there is a diffeomorphism $GL(N, \mathbb{R}) \cong O(N, \mathbb{R}) \times \text{Pos}_N$. Since Pos_N is contractible, this shows that $O(N, \mathbb{R}) \subset GL(N, \mathbb{R})$ is a homotopy equivalence, a fact that was earlier proved using the deformation retraction given by parametrized Gram-Schmidt orthogonalization. An equivalent way to see this is using the deformation retraction $A \mapsto P^r U$ where $A = PU$ is the polar decomposition.

Now let us finish the proof of the proposition. For the first assertion suppose $K \subset Sp(2n, \mathbb{R})$ is compact. Let dk be a Haar measure on K . Then

$$P := \text{vol}(K)^{-1} \int_K k^T k \, dk$$

is a symmetric positive definite matrix in $Sp(2n, \mathbb{R})$. (Think of $k \mapsto k^T k$ as a function on K valued in the vector space of symmetric matrices to define the integral. It is easily seen to be symmetric positive definite and symplectic.) It satisfies $k^T P k = P$ for $k \in K$. Then for $k \in K$, $P^{1/2} k P^{-1/2} \in O(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R}) = U(n)$, hence K is conjugated into $U(n)$ by $P^{1/2}$. It follows that $U(n)$ is maximal compact, and all maximal compact subgroups are conjugate to $U(n)$. For the contractibility of $S\text{Pos}_{2n}$ note that $P \mapsto P^t$ gives a deformation retraction to the identity. (For the diffeomorphism with $\mathbb{R}^{n(n+1)}$ see the remarks below.) \square

Remark 9.1.14 (Riemannian symmetric spaces). $Sp(2n, \mathbb{R})/U(n)$ and $SL(2n, \mathbb{R})/SO(n)$ are examples of (irreducible) Riemannian symmetric spaces. A Riemannian manifold (M, g) is globally symmetric if for any point $p \in M$, the endomorphism $-I \in T_p M$ can be realized by a global isometry, i.e. there is an isometry ϕ of (M, g) fixing p and such that $d\phi_p = -I$ on $T_p M$. (E.g. S^n is globally symmetric for any n .) It is irreducible if it cannot be written as a product (with the product metric) in a nontrivial way. A trivial example is \mathbb{R}^n with the Euclidean metric. Aside from this, any irreducible globally symmetric space M has as its group of isometries a (simple) Lie group, and for M noncompact we can identify M as the quotient of the group of Riemannian isometries by a maximal compact subgroup or, equivalently (because all maximal compact subgroups are conjugate), as the space of maximal compact subgroups. The irreducible Riemannian globally symmetric spaces were classified by Cartan in 1926/27 (see e.g. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*).

9.2. Hermitian, symplectic, orthogonal. Let V be a finite-dimensional complex vector space.

An **Hermitian form** on V is a map $h : V \times V \rightarrow \mathbb{C}$ which is symmetric (i.e. $h(v, w) = \overline{h(w, v)}$) and sesquilinear (i.e. $h(\lambda v, \mu w) = \lambda \bar{\mu} h(v, w)$). We say h is positive definite if $h(v, v) > 0$. We may write h in terms of its real and imaginary parts:

$$h = g + \sqrt{-1}\omega, \quad (\text{for } g = \text{Re}(h), \omega = \text{Im}(h)).$$

Then g is a symmetric bilinear form on $V_{\mathbb{R}}$ and ω is an alternating form on $V_{\mathbb{R}}$. (Here $V_{\mathbb{R}}$ denotes V considered as a real vector space of dimension $2 \dim_{\mathbb{C}} V$.) Note that g and ω determine each other by $g(v, \sqrt{-1}w) = \text{Re } h(v, \sqrt{-1}w) = \text{Im } h(v, w) = \omega(v, w)$. If h is positive definite, as we will always assume, then g is positive definite (i.e. it is an inner product) and ω is nondegenerate.

Given (V, h) as above we have three subgroups of $GL(V_{\mathbb{R}}) = GL(2n, \mathbb{R})$:

- (1) $GL(V) \quad (\cong GL(n, \mathbb{C}))$
- (2) $Sp(V_{\mathbb{R}}, \omega) = \{A \in GL(V_{\mathbb{R}}) : \omega(Av, Aw) = \omega(v, w) \text{ for } v, w \in V_{\mathbb{R}}\} \quad (\cong Sp(2n, \mathbb{R}))$
- (3) $O(V_{\mathbb{R}}, g) = \{A \in GL(V_{\mathbb{R}}) : g(Av, Aw) = g(v, w) \text{ for } v, w \in V_{\mathbb{R}}\} \quad (\cong O(2n, \mathbb{R}))$

In fact any linear transformation of $V_{\mathbb{R}}$ preserving two of these three structures (inner product, complex structure, symplectic structure) also preserves the third:

Lemma 9.2.1. *The pairwise intersections coincide:*

$$O(V_{\mathbb{R}}, g) \cap GL(V) = O(V_{\mathbb{R}}, g) \cap Sp(V_{\mathbb{R}}, \omega) = Sp(V_{\mathbb{R}}, \omega) \cap GL(V)$$

and are isomorphic to $U(V, h) = \{g \in GL(V) : h(gv, gw) = h(v, w)\} \cong U(n)$.

Proof. We may choose an h -orthonormal \mathbb{C} -basis v_1, \dots, v_n for V so that h is given by

$$h\left(\sum_i z_i v_i, \sum_i \zeta_i v_i\right) = \sum_i z_i \bar{\zeta}_i.$$

Let $z_i = x_i + \sqrt{-1}y_i$, so that x_i, y_i are a basis for $V_{\mathbb{R}}$. In this basis of $V_{\mathbb{R}}$ the multiplication by $\sqrt{-1}$ is given by $J = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$. In this basis the subgroups in the lemma are:

- (1) $GL(V) = \{A \in GL(2n, \mathbb{R}) : AJ = JA\}$
- (2) $Sp(V_{\mathbb{R}}, \omega) = \{A \in GL(2n, \mathbb{R}) : A^T JA = J\}$
- (3) $O(V_{\mathbb{R}}, g) = \{A \in GL(2n, \mathbb{R}) : A^T A = I\}$.

The equalities can be checked by hand, e.g. if $A \in (1) \cap (2)$ then $A^T AJ = A^T(JA) = J$ and hence $A^T A = I$.

From the description $U(V, h) = O(V_{\mathbb{R}}, g) \cap Sp(V_{\mathbb{R}}, \omega)$ we can deduce that its elements are of the form $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ such that $A^T B = B^T A$ and $A^T A + B^T B = I_n$. In other words, $A + \sqrt{-1} B$ is a unitary matrix. Thus $U(V, h) \cong U(n)$. \square

Remark 9.2.2. Given (V, h) and the associated ω as above, an n -dimensional subspace $L \subset V$ is Lagrangian for ω if and only if it is totally real for h (i.e. $h(v, w) \in \mathbb{R}$ for all $v, w \in L$).

9.3. Compatible complex structures. Let V be a finite-dimensional real vector space of even dimension $2n$.

A complex structure on V is the same as a linear operator $J : V \rightarrow V$ such that $J^2 = -I$. Let $\mathbf{J}(V) = \{J \in GL(V) | J^2 = -I\}$ denote the set of complex structures on V . A choice of basis $V \cong \mathbb{R}^{2n}$ identifies $\mathbf{J}(V)$ with $\mathbf{J}(\mathbb{R}^{2n})$ and choosing $J \in \mathbf{J}(\mathbb{R}^{2n})$ identifies it with $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$. Thus there are two components

$$\mathbf{J}(\mathbb{R}^{2n}) = \mathbf{J}^+(\mathbb{R}^{2n}) \sqcup \mathbf{J}^-(\mathbb{R}^{2n}),$$

where the $+$ component contains $J_0 = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$ and the $-$ component contains $-J_0$. Thus $\mathbf{J}^+(\mathbb{R}^{2n}) \cong GL(2n, \mathbb{R})^+/GL(n, \mathbb{C})$ where $GL(2n, \mathbb{R})^+$ is the identity component of matrices (i.e. with determinant > 0).

Example 9.3.1 ($n = 1$). In this case the set of complex structures is naturally identified with the union of upper and lower half-planes $\mathbb{H} \sqcup (-\mathbb{H})$ in \mathbb{C} .

Let (V, ω) be a symplectic space. We have the following notions of compatible complex and metric structures:

- (1) A complex structure $J : V \rightarrow V$ is **compatible** with ω if

- $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in V$
- $\omega(v, Jv) > 0$ for $0 \neq v \in V$.

Then $h(v, w) := \omega(v, Jw) + \sqrt{-1}\omega(v, w)$ defines a Hermitian form with imaginary part ω and the real part $g_J(v, w) := \omega(v, Jw)$ is an inner product on V . The various automorphism groups are related as in the previous lemma.

Let $\mathbf{J}(V, \omega)$ be the space of complex structures on V compatible with ω . It is easy to see that $\mathbf{J}(V, \omega) \subset \mathbf{J}^+(V)$.

- (2) An inner product $g : V \times V \rightarrow \mathbb{R}$ on V is **compatible** with ω if the linear transformation J of V defined by $g(v, w) = \omega(Jv, w)$ (for all v, w) satisfies $J^2 = -I$. Notice that if g is compatible with ω then $g(Jv, w) = \omega(v, w) = -\omega(w, v) = -g(Jw, v) = -g(v, Jw)$ so that J is skew-adjoint with respect to (V, g) .

Let $\mathbf{M}(V)$ denote the space of all metrics on V and let $\mathbf{M}(V, \omega)$ denote the metrics compatible with ω . Note that $\mathbf{M}(\mathbb{R}^{2n}) \cong GL(2n, \mathbb{R})^+/SO(2n)$. The map $J \mapsto g_J$ defined in (1) defines a map $\mathbf{J}(V, \omega) \rightarrow \mathbf{M}(V)$ with image $\mathbf{M}(V, \omega)$.

Example 9.3.2. If $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$ then $J_0 = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$ belongs to $\mathbf{J}(\mathbb{R}^{2n}, \omega_0)$. Fixing J_0 as a basepoint, we have

$$\mathbf{J}(\mathbb{R}^{2n}, \omega_0) = \{J \in GL(2n, \mathbb{R}) : J^2 = -I, v^T(-J_0 J)v > 0 \text{ for } v \neq 0\} \subset \mathbf{J}^+(\mathbb{R}^{2n}).$$

Let $S\text{Pos}_{2n} = \{A \in Sp(2n, \mathbb{R}) : A \text{ is symmetric and positive definite}\}$ be the space of positive definite symmetric symplectic matrices of size $2n$. Then the map

$$\mathbf{J}(\mathbb{R}^{2n}, \omega_0) \longrightarrow S\text{Pos}_{2n} \quad \text{by } J \mapsto -J_0 J$$

is a bijection, with inverse given by $P \mapsto -J_0^{-1}P = J_0P$. This is equivariant for the actions of $Sp(2n, \mathbb{R})$ by $J \mapsto g^{-1}Jg$ on $\mathbf{J}(\mathbb{R}^{2n}, \omega_0)$ and $P \mapsto g^T P g$ on $S\text{Pos}_{2n}$.

Proposition 9.3.3. *The space of complex structures on \mathbb{R}^{2n} compatible with the standard symplectic structure is $\mathbf{J}(\mathbb{R}^{2n}, \omega_0) = S\text{Pos}_{2n} = Sp(2n, \mathbb{R})/U(n)$. This space is contractible (and in fact diffeomorphic to Euclidean space).*

Proof. These follow from Proposition 9.1.13. □

The following diagram summarizes the discussion, except for the retraction r :

$$\begin{array}{ccc} \mathbf{J}^+(V) & \longleftarrow & \mathbf{J}(V, \omega) & \simeq & \frac{GL(2n, \mathbb{R})^+}{GL(n, \mathbb{C})} & \longleftarrow & Sp(2n, \mathbb{R})/U(n) = S\text{Pos}_{2n} \\ & & \downarrow \left. \begin{array}{l} J \mapsto g_J \\ \end{array} \right\} r & & & & \downarrow \left. \begin{array}{l} \\ \end{array} \right\} r \\ & & \mathbf{M}(V) & & & & GL(2n, \mathbb{R})^+/SO(2n) = \text{Pos}_{2n} \end{array}$$

Let us define the retraction $r : \mathbf{M}(V) \rightarrow \mathbf{J}(V, \omega)$. First define an action of $Sp(2n, \mathbb{R})$ on $\mathbf{M}(V)$ by pullback, i.e. $(b^*g)(v, w) = g(bv, bw)$.

Lemma 9.3.4. *There is a smooth map $r : \mathbf{M}(V) \rightarrow \mathbf{J}(V, \omega)$ such that (1) $r(g_J) = J$ and $r(b^*g) = b^{-1}r(g)b$ for $b \in Sp(2n, \mathbb{R})$.*

Proof. Let $g \in \mathbf{M}(V)$. For $A \in GL(V)$ denote by A^* the g -adjoint of A , i.e. it is defined by $g(Av, w) = g(v, A^*w)$ for $v, w \in V$.

Now if we define $A \in GL(V)$ by $\omega(v, w) = g(Av, w)$, then A is skew-adjoint with respect to g , since $g(Av, w) = \omega(v, w) = -\omega(w, v) = -g(Aw, v) = -g(v, Aw)$. Thus $P = A^*A = -A^2$ is g -positive definite. It follows that there is a unique g -self-adjoint and g -positive definite $Q : V \rightarrow V$ such that $Q^2 = P$. (Choosing a g -orthonormal basis the assertions reduce to the previous situation of positive definite symmetric matrices where we showed how to take a square root.) Then

$$r(g) = Q^{-1}A$$

is a complex structure since $Q^{-1}AQ^{-1}A = Q^{-2}A^2 = -I$. (Here we use that A commutes with $A^*A = -A^2$ and hence with Q .)

For compatibility with ω note first that

$$\begin{aligned} \omega(r(g)v, r(g)w) &= \omega(Q^{-1}Av, Q^{-1}Aw) = g(AQ^{-1}Av, Q^{-1}Aw) \\ &= g(-Qv, Q^{-1}Aw) = g(-v, Aw) = \omega(v, w) \end{aligned} \tag{9.3.1}$$

since Q is g -self-adjoint. For the second condition note that

$$\omega(v, r(g)v) = g(Av, Q^{-1}Av) = g(Q^{-1}Av, Av) > 0$$

since Q is g -positive definite.

Thus $r : \mathbf{M}(V) \rightarrow \mathbf{J}(V, \omega)$ has been defined (it is left to the reader to justify its continuity). Let us check (1) and (2). For (1) note that if $g = g_J$ for a complex structure then $A = J$ and $Q = I$, and hence $r(g_J) = J$. For (2) note that if we replace g by $(b^*g)(v, w) := g(bv, bw)$ for $b \in Sp(2n, \mathbb{R})$, then A becomes $b^{-1}Ab$ and hence $r(b^*g) = b^{-1}r(g)b$. □

Remark 9.3.5. In dimension two (i.e. $n = 1$) the horizontal map is a bijection and the vertical map is the inclusion of $\mathbb{H} = SL(2, \mathbb{R})/SO(2)$ in $GL(2, \mathbb{R})^+/SO(2) \cong \mathbb{H} \times \mathbb{R}_+$ as $\mathbb{H} \times \{1\}$. The retraction is the projection.

Remark 9.3.6. A different construction of a retraction r as in the diagram is as follows: First one shows that the exponential map $\exp : M(N, \mathbb{R}) \rightarrow GL(N, \mathbb{R})$ maps the subspace of skew-symmetric matrices $\mathfrak{p} = \{A \in M(N, \mathbb{R}) : A^T + A = 0\}$ diffeomorphically onto Pos_N . Now \exp restricts to the exponential map $\exp : \mathfrak{sp}(2n, \mathbb{R}) \rightarrow Sp(2n, \mathbb{R})$ and we consider the subspace

$$\mathbf{Sp} := \mathfrak{p} \cap \mathfrak{sp}(2n, \mathbb{R}) = \{A \in M(2n, \mathbb{R}) : A^T J + JA = 0, A + A^T = 0\}.$$

The exponential map restricts to a diffeomorphism $S\mathfrak{p} \rightarrow S\text{Pos}_{2n}$. Then r can be defined using a linear projection to the subspace $\mathfrak{p} \rightarrow S\mathfrak{p}$. (I think this is the same retraction as above, but I haven't checked.)

9.4. Lagrangian Grassmannian. Let (V, ω) be a symplectic space. Let $\Lambda(V, \omega)$ be the set of Lagrangian subspaces of (V, ω) . By what we said earlier, there is a transitive action of $Sp(V, \omega)$ on $\Lambda(V, \omega)$, identifying it as a homogeneous space $Sp(V, \omega)/Stab(L)$, where $L \subset V$ is any Lagrangian. When $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$ and L is chosen to be $\mathbb{R}^n \times \{0\}$ we have a short exact sequence

$$1 \longrightarrow \left\{ \begin{pmatrix} I & S \\ & I \end{pmatrix} : S^T = S \right\} \longrightarrow Stab_{Sp(2n, \mathbb{R})}(L) \longrightarrow GL(L) \longrightarrow 1$$

by calculation.

Given a compatible complex structure J on V and Hermitian form h on V with respect to J , we get an alternate description of $\Lambda(V, \omega)$ using compact groups:

Lemma 9.4.1. $\Lambda(V, \omega) = U(V, h)/U(V, h) \cap Stab(L) \cong U(n)/O(n)$.

Proof. It is enough to show that $U(n)$ acts transitively on Lagrangian subspaces. This is left as an exercise. \square

9.5. Symplectic vector bundles. The results of the previous subsections work in parameters, i.e. for vector bundles.

TO BE FINISHED

9.6. Symplectic manifolds. We start with the definition:

A **symplectic manifold** is a pair (M, ω) consisting of a $2n$ -dimensional smooth manifold M and a 2-form ω which is closed ($d\omega = 0$) and nondegenerate, namely $\omega^n \neq 0$ everywhere, i.e. ω^n is a volume form. If (M, ω) and (M', ω') are two symplectic manifolds then a smooth map $\varphi : M \rightarrow M'$ is called a **symplectomorphism** if it is a diffeomorphism and $\varphi^*(\omega') = \omega$.

Example 9.6.1. $(\mathbb{R}^{2n}, \omega_0 = \sum_i dx_i \wedge dy_i)$ where x_i, y_i are coordinates. Equivalently, any symplectic space (V, ω) is a symplectic manifold, symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Example 9.6.2. Open subsets of symplectic manifolds are symplectic manifolds and products of symplectic manifolds are symplectic manifolds.

Remark 9.6.3. A compact symplectic manifold has $H^2(M) \neq 0$ since $\int_M \omega^n \neq 0$.

Example 9.6.4. In dimension $n = 2$ a symplectic structure is a volume/area form, so any oriented surface carries one. By Moser's theorem, ω is unique up to symplectomorphism once $\int_M \omega \in \mathbb{R}$ is fixed, at least if M is compact.

Example 9.6.5. Consider $\mathbb{R}^2 - \{0\}$ with form $dx \wedge dy$ and $\mathbb{R} \times S^1$ with form $dr \wedge d\theta$. They are diffeomorphic but not symplectomorphic. Indeed, a homologically nontrivial smoothly embedded S^1 in $\mathbb{R}^2 - \{0\}$ divides it into two regions, one of which has finite area, whereas for the diffeomorphic image in the cylinder both regions have infinite area.

A symplectic form on M defines a vector bundle map $TM \cong T^*M$ which is fibrewise an isomorphism, hence an isomorphism. It follows that there is a canonical bijection between vector fields (= sections of TM) and one-forms (= sections of T^*M), given explicitly by $X \mapsto i_X\omega$. Thus there are two special types of vector fields on M :

- If $0 = L_X\omega = d(i_X\omega)$, i.e. $i_X\omega$ is closed, then X is called a **symplectic vector field**. It is easy to see that the flow of a symplectic vector field is by symplectomorphisms ⁽⁴³⁾.
- If $i_X\omega$ is further exact, then X is called a **Hamiltonian vector field**. If $i_X\omega = dH$ then the flow of X respects the level sets of H . ⁽⁴⁴⁾

⁴³Indeed, $\frac{d}{dt}\phi_t^*(\omega) = \phi_t^*(L_X\omega) = \phi_t^*(d(i_X\omega)) = 0$ so $\phi_t^*(\omega) = \phi_0^*(\omega) = \omega$.

⁴⁴Warning: Some references use the definition $i_X\omega = -dH$ instead.

Note that if $H_{dR}^1(M) = 0$ then a symplectic vector field is Hamiltonian.

Example 9.6.6 (Cotangent bundles). The cotangent bundle T^*L of a smooth manifold carries a canonical symplectic structure. An invariant way to see this structure is the following. Let $\pi : T^*L \rightarrow L$ be the projection map. The tautological one-form λ is defined as follows: For $(p, \xi) \in T^*L$ the map

$$d\pi_{(p,\xi)} : T_{(p,\xi)}(T^*L) \longrightarrow T_pL$$

is surjective with kernel $T_\xi(T_p^*L)$, which is canonically identified with T_p^*L . Thus a linear form on T_pL pulls back to a linear form π^* on $T_{(p,\xi)}(T^*L)$ vanishing on T_p^*L . Define λ to be the one-form on T^*L whose value at (p, ξ) is $d\pi^*\xi$. Concretely, it is defined by

$$\lambda((p, \xi)) = \xi \circ d\pi_{(p,\xi)} : T_{(p,\xi)}(T^*L) \rightarrow T_pL \xrightarrow{\xi} \mathbb{R}.$$

The symplectic form is $\omega := -d\lambda$. (The nondegeneracy of ω will follow from the local expression below.)

Let us write this out in local coordinates. Let x_1, \dots, x_n be local coordinates on $U \subset L$ and let y_1, \dots, y_n be the dual coordinates in the cotangent bundle, i.e. the coefficients of a cotangent vector in the basis dx_1, \dots, dx_n dual to $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. Then $d\pi_{(p,\xi)}(\frac{\partial}{\partial y_i}) = 0$ and $d\pi_{(p,\xi)}(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}$. Hence

$$\lambda = \sum_i y_i dx_i, \quad \omega = -d\lambda = \sum_i dx_i \wedge dy_i.$$

This shows that $\omega = -d\lambda = \sum_i dx_i \wedge dy_i$ locally, hence it is nondegenerate.

Here is another way to see this structure pointwise: There is a canonical identification $T_{(x,\xi)}(T^*L) \cong T_xL \times T_x^*L$ and this carries a canonical symplectic structure by Example 9.1.1.

Example 9.6.7 (Smooth projective varieties). A smooth projective variety $i : V \hookrightarrow \mathbb{C}P^n$ is a symplectic manifold with the form $\omega = i^*\omega_0$ where ω_0 is the form on $\mathbb{C}P^n$ descended from $\frac{\sqrt{-1}}{2} \sum_{i=0}^n dz_i \wedge d\bar{z}_i = \sum_i dx_i \wedge dy_i$ as in Example 7.7.3. This is special case of the next example.

Example 9.6.8 (Kähler manifolds). These are complex manifolds which have several equivalent definitions.

Let X be a complex manifold and $T^{1,0}X$ its holomorphic tangent bundle. A **Hermitian metric** on X is a smooth mapping $h : T^{1,0}X \times T^{1,0}X \rightarrow \mathbb{C}$ which restricts to a positive definite Hermitian form $T_x^{1,0}X \times T_x^{1,0}X \rightarrow \mathbb{C}$ for each $x \in X$. As in the earlier discussion, this determines both an inner product $g(v, w) := \operatorname{Re} h(v, w)$ and a symplectic form $\omega(v, w) = -\operatorname{Im} h(Jv, w) = -\operatorname{Im} h(v, w)$, both varying smoothly.

These satisfy $g(Jv, Jw) = g(v, w)$ and $\omega(Jv, Jw) = \omega(v, w)$ for the complex structure on $T^{1,0}X$.

TO BE FINISHED

The inner product $g(v, w) = \operatorname{Re} h(v, w)$ is defined on $T^{1,0}X \times T^{1,0}X$ but extending it \mathbb{C} -linearly to $T^{1,0}X \otimes \mathbb{C} = TX \otimes \mathbb{C}$, one sees that it is the complexification of its restriction to the real tangent bundle TX . In other words, the Hermitian metric gives a Riemannian metric on the underlying smooth manifold.

We say that X is a **Kähler manifold** if at every point x there exist local holomorphic coordinates such that the Hermitian metric looks like the standard holomorphic form on \mathbb{C}^n up to order 2, i.e. $h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = \delta_{ij} + O(|z|^2)$ -terms on a neighbourhood of x . Under this condition $\omega = -\operatorname{Im}(h)$ is closed (this requires a calculation), so that it gives a symplectic structure. (An equivalent version of the Kähler condition is that TX carries Hermitian form h which is positive definite everywhere and for which form $\omega = -\operatorname{Im}(h)$ is closed.)

(Aside: Complex projective varieties are Kähler by restricting the canonical form from $\mathbb{C}P^n$. Conversely, according to Kodaira's embedding theorem, a compact Kähler manifold such that the associated 2-form defines an integral cohomology class, i.e. $[\omega] \in H^2(M, \mathbb{Z})$, is projective, i.e. there is an embedding $i : X \hookrightarrow \mathbb{C}P^N$ such that $i^*(\omega_0) = \omega$, where ω_0 is as in Example 7.7.3.)

Example 9.6.9. An alternate definition of a Kähler manifold is the following: A Kähler manifold is a symplectic manifold with a compatible almost-complex structure J which is integrable (i.e. comes from an actual complex structure).

Example 9.6.10 (Coadjoint orbits). Let G be a connected Lie group and \mathfrak{g} its Lie algebra. The adjoint representation $Ad : G \rightarrow GL(\mathfrak{g})$ by $Ad(g) = dInt(g)$ where $Int(g)(x) = gxg^{-1}$. The dual representation is the coadjoint representation $Ad^* : G \rightarrow GL(\mathfrak{g}^*)$, in which g acts by $Ad^*(g) = Ad(g^{-1})^T$. We have $d(Ad)(X) = ad(X)$ and

Let $O \subset \mathfrak{g}^*$ be an orbit of the coadjoint representation. For $\alpha \in O$ we can consider $(X, Y) \mapsto \alpha([X, Y])$.

TO BE FINISHED

Example 9.6.11. Let M and N be manifolds with $\dim M < \dim N$ or with $\dim M = \dim N$ and M noncompact. According to the Hirsch-Smale theorem, regular homotopy classes of immersions $M \rightarrow N$ (i.e. homotopy through immersions) are in bijection with homotopy classes of bundle monomorphisms $TM \rightarrow TN$. As a consequence, if $f : M \rightarrow N$ is a smooth map such that $TM \rightarrow f^{-1}TN$ is a monomorphism, then there is an immersion of M in N homotopic to f . This can be used to construct some symplectic structures: If (N, ω) is symplectic and $f : M \rightarrow N$ is such that $TM \cong f^{-1}TN$, then there is an immersion ι homotopic to f and $\iota^*(\omega)$ gives a symplectic structure on M . In particular, if M is a noncompact manifold of dimension $2n$ with trivial tangent bundle then each homotopy class of trivializations of TM gives a symplectic structure on M by pulling back ω_0 using a homotopic immersion $M \rightarrow \mathbb{R}^{2n}$.

Remark 9.6.12. There is no obvious symplectic category. For a smooth map $f : (M, \omega) \rightarrow (N, \tau)$ between symplectic manifolds, requiring that $f^*(\tau) = \omega$ would (by nondegeneracy) force $\dim M \leq \dim N$, and in fact force that f is an immersion. Nevertheless there is a proposed symplectic category, which we will discuss below.

9.7. Darboux's theorem. The following theorem says that locally any symplectic manifold looks like the standard example:

Theorem 9.7.1 (Darboux). *Let (M, ω) be a symplectic manifold and $p \in M$. Then there are local coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ near p such that $\omega = \sum_i dp_i \wedge dq_i$.*

Thus there are no local invariants of symplectic manifolds – in contrast to the Riemannian case, where there are curvature etc. – only their global topology is constrained.

We give two proofs of the theorem, one by induction on the dimension and another using Moser's trick.

First proof. This proof is by induction on $n = \dim M/2$. Let $p \in M$. Choose a function y_1 on a neighbourhood U of p such that $dy_1 \neq 0$ on U . Let X be the vector field on U defined by $i_X\omega = dy_1$. Let x_1 be a function on U such that $X(x_1) \equiv 1$. If Y is a vector field on U such that $i_Y\omega = -dx_1$, then

$$L_X\omega = di_X\omega + i_Xd\omega = d(dy_1) = 0.$$

On the other hand,

$$i_{[X, Y]}\omega = L_X(i_Y\omega) - i_Y(L_X\omega) = L_X(-dx_1) = -d(X(x_1)) \equiv 0.$$

(Here we use the identity $L_X \circ i_Y - i_Y \circ L_X = i_{[X, Y]}$ as derivations on differential forms.) Since ω is nondegenerate, this implies that $[X, Y] = 0$ on U . It follows (since we have a pair of commuting vector fields) that we can find coordinates $x_1, y_1, z_1, \dots, z_{2n-2}$ near p such that $X = \frac{\partial}{\partial x_1}$ and $Y = \frac{\partial}{\partial y_1}$. Consider the form

$$\omega' := \omega - dx_1 \wedge dy_1.$$

Then $d\omega' = 0$ and $0 = i_X\omega' = L_X\omega' = i_Y\omega' = L_Y\omega'$. It follows that ω' only involves the variables z_1, \dots, z_{2n-2} . Moreover, since $\omega^n = n dx_1 \wedge dy_1 \wedge (\omega')^{n-1}$ we have that $(\omega')^{n-1}$ is nowhere-vanishing. We are done by induction applied to ω' . \square

Second proof (using Moser's trick). Let $p \in M$ and choose a chart $\varphi : U \rightarrow \mathbb{R}^{2n}$ centred at p such that $(\varphi^*\omega_0)(p) = \omega(p)$, i.e. the pullback of the standard symplectic structure agrees with ω at p (as symplectic forms on T_pM). Let $\omega' = \varphi^*(\omega_0)$. Thus ω, ω' are two symplectic structures on U which agree at p . Let

$$\omega_t = t\omega + (1-t)\omega'.$$

This is closed and remains nondegenerate in some neighbourhood of p , call it U' , which we may assume contractible. Since U' is contractible, we may find $\alpha \in \mathcal{E}^1(U')$ such that

$$d\alpha = \omega - \omega' = \frac{d}{dt}\omega_t.$$

Moreover, we may arrange $\alpha(p) = 0$ by subtracting off $\alpha(p)$, thought of as a translation-invariant (hence closed) one-form on \mathbb{R}^n . Now Moser's trick will give us an isotopy ϕ_t with $\phi_t^*\omega_t = \omega$, in particular $\phi_1^*\omega = \omega'$, provided the vector field X_t satisfies

$$d(i_{X_t}\omega_t + \alpha) = 0.$$

But we can solve $i_{X_t}\omega_t + \alpha = 0$ for the vector field X_t using the nondegeneracy of ω_t . Since $\alpha(p) = 0$ we see that $X_t(p) = 0$, so that the flow ϕ_t fixes p . Then there is a neighbourhood $U'' \subset U'$ of p such that $\phi_t(U'') \subset U'$ for $0 \leq t \leq 1$, and on this neighbourhood we have $\phi_1^*\omega = \omega'$. \square

Here is a consequence:

Proposition 9.7.2. *The symplectomorphism group of a connected symplectic manifold (M, ω) acts transitively on M . In fact the subgroup of Hamiltonian symplectomorphisms acts transitively on M .*

Proof. By the Darboux theorem we are reduced to showing that in $(\mathbb{R}^{2n}, \omega_{std})$ any two points can be related by a Hamiltonian symplectomorphism which is the identity outside some large ball. Suppose $x, y \in \mathbb{R}^{2n}$ and let $v = y - x$. Then $x \mapsto x + tv$ is the flow of an obvious vector field X , namely the constant vector field $x \mapsto v$, and the time one map takes x to y . This vector field is Hamiltonian, i.e. X is given by a function $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$. (This can easily be written down explicitly, but we can also note that the flow of X on \mathbb{R}^{2n} preserves ω , so that $0 = L_X\omega = i_X\omega = dH$ for some H since $H_{dR}^1(\mathbb{R}^{2n}) = 0$. *Exercise.* Give an explicit H .) Now the time one flow of X takes x to y but it is not the identity outside a large set so we must modify it. For this simply take open neighbourhoods $U \subset V$ of the segment $[x, y]$ with $\bar{U} \subset V$ and a function $\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ with $\text{supp}(\phi) \subset V$ and $\phi|_U \equiv 1$. Then $H' := \phi \cdot H$ has associated vector field X' defined by $i_{X'}\omega = dH'$ which takes x to y and is zero outside V . The time one flow of X' does the job. \square

9.8. Weinstein's theorem. The idea of proof using Moser's trick gives a stronger version of Darboux's theorem due to Weinstein. First we prove the following lemma using homotopy operators.

Lemma 9.8.1. *Let $i : N \hookrightarrow M$ be a closed submanifold of a smooth manifold and let $U \supset N$ be a tubular neighbourhood of N . Let $\omega \in \mathcal{E}^k(U)$ be a closed differential form such that $i^*\omega = 0$. Then there exists $\sigma \in \mathcal{E}^{k-1}(U)$ with $d\sigma = \omega$ and such that σ vanishes on N (as a section of $\wedge^{k-1}T^*U|_N$). If ω vanishes on N (as a section of $\wedge^kT^*U|_N$) then σ can be chosen to vanish to second order on N .*

Proof. Notice that $i^*[\omega] = [i^*\omega] = 0$ in $H^k(N) \cong H^k(U)$, so we can certainly find $\sigma \in \mathcal{E}^{k-1}(U)$ such that $d\sigma = \omega$. The content of the lemma is that we can choose σ to vanish along N .

We may identify the tubular neighbourhood U of N with the unit disc bundle (for some metric) in the normal bundle \mathcal{N} of N , with coordinates (x, v) for $x \in N$ and $v \in \mathcal{N}_x$. There is an obvious retraction $r_t : U \rightarrow U$ defined by $r_t(x, v) = (x, tv)$ so that r_1 is the identity and $r_0 = i \circ (\text{projection from } U \text{ to } N)$.

Let X be the vector field generating the retraction r_t , i.e. $X(x, tv) = v$ for $(x, v) \in U$. We use a homotopy operator as in previous discussions to define σ :

$$\sigma(x, v) := \int_0^1 i_X(r_t^*(\omega)) dt = \int_0^1 r_t^*(i_{(0,v)}\omega) dt$$

(using $r_t^*(X) = X$ in the second equality). If $v = 0$ the integrand is zero, so this vanishes along the zero section N , and if ω vanishes to first order along N then σ vanishes to second order. Finally,

$$\begin{aligned} d\sigma &= \int_0^1 di_X r_t^*(\omega) dt = \int_0^1 L_X(r_t^*(\omega)) dt = \int_0^1 r_t^*(L_X\omega) dt \\ &= \int_0^1 \frac{d}{dt}(r_t^*(\omega)) dt = r_1^*(\omega) - r_0^*(\omega) = \omega - 0 = \omega \end{aligned}$$

where we use that $r_t^*(X) = X$ in the third equality and $r_t^*(L_X\omega) = \frac{d}{dt}(r_t^*(\omega))$ (cf. (8.1.1)) in the fourth. This proves the lemma. \square

Theorem 9.8.2 (Weinstein's symplectic neighbourhood theorem). *Let M be a symplectic manifold and let ω_0, ω_1 be two symplectic forms on M . Let $\iota : N \hookrightarrow M$ be a compact submanifold such that $\omega_0|_N = \omega_1|_N$ (equality as sections of $\wedge^2 T^*M|_N$). Then there exist neighbourhoods U_0, U_1 of N in M and a diffeomorphism $\varphi : U_0 \rightarrow U_1$ with $\varphi|_N = id$, such that $\varphi^*(\omega_1) = \omega_0$.*

Proof. The idea is to use Moser's trick. Let $\omega_t = t\omega_0 + (1-t)\omega_1$ for $0 < t < 1$. This is a closed 2-form which restricts to $\omega_0|_N = \omega_1|_N$ on N , hence there is a neighbourhood U of N in M , which we may assume is tubular, in which ω_t continues to be nondegenerate, i.e. gives a symplectic structure on U for all $t \in [0, 1]$. By the lemma above, there exists a one-form $\alpha \in \mathcal{E}^1(U)$ which vanishes on N such that

$$\frac{d}{dt}\omega_t = \omega_0 - \omega_1 = d\alpha.$$

Let X_t be a time-dependent vector field on U such that

$$i_{X_t}\omega_t = \alpha.$$

This can be solved because ω_t is nondegenerate. Then using that ω_t is closed, we have

$$L_{X_t}\omega_t = d(i_{X_t}\omega_t) = d\alpha = \omega_0 - \omega_1 = \frac{d}{dt}\omega_t.$$

Since α vanishes on N , we see that $X_t = 0$ on N , so that its flow preserves N pointwise. So shrinking U to U_0 , also a neighbourhood of N , we may assume that the flow of X_t is defined on U_0 for $0 \leq t \leq 1$. Let ϕ_t be the associated flow. Then let $U_1 = \phi_1(U_0)$ and by Moser's trick $\phi_1^*(\omega_1) = \omega_0$. \square

Remark 9.8.3. The condition $\omega_0|_N = \omega_1|_N$ can be weakened in a special case: If N is Lagrangian for both ω_0 and ω_1 (i.e. $i^*\omega_0 = i^*\omega_1 = 0$), then no further condition is required, as we will see below.

Proof of Darboux's theorem from Weinstein's theorem. For $p \in M$ choose a symplectic basis of T_pM , which then gives coordinates p_i, q_i in a neighbourhood of p . There are two symplectic forms on this neighbourhood, namely the standard one $\sum dp_i \wedge dq_i$, and the one coming from ω . Applying Weinstein's theorem with $N = \{p\}$ we immediately get Darboux's theorem. \square

9.9. Submanifolds of symplectic manifolds. The submanifolds of interest in a symplectic manifold are of the following types:

- (1) **Symplectic submanifolds** are submanifolds for which the 2-form restricts to a nondegenerate 2-form. These can have any even dimension $< \dim M$.
- (2) **Isotropic submanifolds** are submanifolds for which the 2-form restricts to zero, i.e. $W \subset M$ is isotropic if $T_pW \subset T_pM$ is an isotropic subspace for all p . Necessarily these have dimension $\leq \dim M/2$.

- (3) **Coisotropic submanifolds** are submanifolds for which $T_p W \subset T_p M$ is a coisotropic subspace for all p . Necessary these have dimension $\geq \dim M/2$.
- (4) **Lagrangian submanifolds** $L \subset M$ are those which are isotropic and coisotropic, i.e. $T_x L \subset T_x M$ is a Lagrangian subspace for all $x \in M$. These have dimension $\dim M/2$.

Lagrangian submanifolds are the most important, for reasons which will become clear. Note that a generic submanifold will be symplectic, while Lagrangians are special.

Example 9.9.1. Let M be an oriented surface with a fixed volume form. Any embedded smooth curve is Lagrangian.

Example 9.9.2. The zero section in the cotangent bundle $M \subset T^*M$ is Lagrangian. A fibre T_p^*M of $T^*M \rightarrow M$ is Lagrangian.

Example 9.9.3 (Conormal bundles). Let $i : W \hookrightarrow M$ be a smooth submanifold. The **conormal bundle** to W is $(i^*TM/TW)^*$, which is naturally identified with

$$N_W^* = \{(p, v) \in T^*M : p \in W, v \in T_p^*M \text{ such that } v(w) = 0 \text{ for all } w \in T_p W\}$$

and is a Lagrangian in T^*M . The extreme cases $W = M$ and $W = \{p\}$ are the zero section and the fibre of $T^*M \rightarrow M$, respectively.

Example 9.9.4 (Conormal varieties). The previous construction can be generalized to the case of cotangent bundles of algebraic varieties. Let $X \subset \mathbb{C}^n$ be a nonsingular algebraic variety. For any closed subvariety $Z \subset X$ we have a stratification (see Remark 5.7.4) $Z = \sqcup_i Z_i$ in which the Z_i are nonsingular varieties. Thus there is a natural conormal variety $N_Z^* := \sqcup_i N_{Z_i}^*$ to Z in X . (Actually, one has to make sure the stratification is fine enough for this to be a good definition, i.e. it should satisfy the conditions (A) and (B) mentioned after Remark 5.7.4.)

Example 9.9.5. Let M be a smooth n -manifold. A one-form α is a section of T^*M , hence its image is an n -manifold in T^*M . It is Lagrangian if and only if α is closed (exercise).

Example 9.9.6. Given symplectic manifolds (M, ω) and (M', ω') , a diffeomorphism $f : M \rightarrow M'$ is a symplectomorphism if and only if its graph $\Gamma \subset \overline{M} \times M'$ is a Lagrangian submanifold, where \overline{M} means M with symplectic form $-\omega$. (This boils down to the same statement for vector spaces which was in Example 9.1.8.)

Exercise. Show that $(\mathbb{R}^{2n}, \omega_0)$ contains no compact symplectic submanifold.

Much deeper is the following theorem of Gromov: $T^*\mathbb{R}^n$ contains no compact exact Lagrangian submanifolds. (Here $L \subset T^*M$ is **exact** if the canonical one-form λ (i.e. $\lambda = \sum_i p_i dq_i$ in local symplectic coordinates) restricts to an exact one-form on L .)

Lemma 9.9.7. *Suppose (M, ω) is symplectic and $L \subset M$ is Lagrangian. Then the normal bundle $N_L = TM|_L/TL$ is isomorphic to T^*L . Thus there is a short exact sequence of vector bundles $0 \rightarrow TL \rightarrow T^*M|_L \rightarrow T^*L \rightarrow 0$ on M :*

Proof. Indeed, $TM|_L \cong T^*M|_L \rightarrow T^*L$ is surjective with kernel TL . (The kernel contains TL as L is Lagrangian, hence is equal to TL by dimension.) Thus $N_L \cong T^*L$. \square

Recall that if V is a symplectic vector space and $L \subset V$ is Lagrangian then $V \cong L \oplus L^*$. The isomorphism is not canonical; it can be given by choosing either a compatible complex structure or a compatible inner product on V . This generalizes to the global situation. The following strengthens the symplectic neighbourhood theorem in the Lagrangian case:

Theorem 9.9.8 (Weinstein). *Let M be a smooth manifold and ω_0, ω_1 two symplectic structures on it. Let $L \subset M$ be compact Lagrangian for both ω_0 and ω_1 . Then there exist open neighbourhoods U_0 and U_1 of L and a diffeomorphism $\varphi : U_0 \rightarrow U_1$ such that $\varphi|_L = \text{id}$ and such that $\varphi^*(\omega_1) = \omega_0$.*

Corollary 9.9.9 (Weinstein's Lagrangian tubular neighbourhood theorem). *Let (M, ω) be a symplectic manifold and $L \subset M$ a compact Lagrangian submanifold. Then there exist:*

- a neighbourhood U_0 of L in M
- a neighbourhood U_1 of L in T^*L , and
- a diffeomorphism $\varphi : U_0 \rightarrow U_1$ with $\varphi|_L = id$

such that $\varphi^*(\omega) = \omega_0$, where ω_0 is the canonical symplectic form on T^*L .

Proof. Combine the previous theorem and the tubular neighbourhood theorem. \square

Before coming to the proof of Weinstein's theorem let us first understand the situation in a symplectic vector space. The following preparatory lemma will be useful:

Lemma 9.9.10. *Let (V, ω) be a symplectic vector space and $L \subset V$ a Lagrangian subspace. Given $W \subset V$ such that $V = L \oplus W$ there is a functorial construction of a Lagrangian complement to L . ⁽⁴⁵⁾*

Proof. The pairing ω restricts to a nondegenerate pairing on $L \times W$ and hence fixes an isomorphism $\phi : L \rightarrow W^*$ induced by $v \mapsto \omega(v, -)$. Now suppose we write the sought-for complement in the form $W' = (I + A)W = \{w + Aw : w \in W\}$ for a linear operator $A : W \rightarrow L$. Then W' is Lagrangian iff for all $w, w' \in W$, we have

$$\begin{aligned} 0 &= \omega(w + Aw, w' + Aw') \\ &= \omega(w, w') + \omega(w, Aw') + \omega(Aw, w') + \omega(Aw, Aw') \\ &= \omega(w, w') + \omega(w, Aw') + \omega(Aw, w'). \end{aligned}$$

This can be written in terms of the composition $\phi \circ A : W \rightarrow W^*$ as the equation

$$\begin{aligned} \omega(w, w') &= \omega(Aw', w) - \omega(Aw, w') \\ &= (\phi \circ A)(w')(w) - (\phi \circ A)(w)(w'). \end{aligned}$$

This is solved by setting $(\phi \circ A)(v) := -\frac{1}{2}\omega(v, -) \in W^*$. Since ϕ is an isomorphism this gives a canonical $A : W \rightarrow L$ and Lagrangian complement $W' = (I + A)W$. \square

Lemma 9.9.11. *Let V be a vector space with two symplectic forms ω_0, ω_1 . Let $L \subset V$ be a Lagrangian subspace for both ω_0 and ω_1 . Let $W \subset V$ be a subspace such that $V = L \oplus W$. Then there exists an isomorphism $\lambda : V \rightarrow V$ such that $\lambda|_L = id$ and $\lambda^*(\omega_1) = \omega_0$.*

Proof. By the previous lemma there are functorially constructed complements W_0 and W_1 to L which are Lagrangian for ω_0 and ω_1 respectively. There are also isomorphisms $L \cong W_0^*$ and $L \cong W_1^*$ using ω . This gives an isomorphism $W_1^* \rightarrow W_0^*$ and its dual $\phi : W_0 \rightarrow W_1$. (Explicitly, ϕ satisfies $\omega_1(\phi(w_0), \ell) = \omega_0(w_0, \ell)$ for $w_0 \in W_0, \ell \in L$.) Define $\lambda := id_L \oplus \phi$ on $V = L \oplus W_0$. Then $\lambda^*(\omega_1) = \omega_0$. \square

Proof of Theorem 9.9.8. Choose a Riemannian metric on M . Then along L we have a orthogonal splitting $TM|_L = TL \oplus W$. The construction of the previous lemma gives us an isomorphism $\lambda : TM|_L \rightarrow TM|_L$ such that $\lambda^*(\omega_1) = \omega_0$. (The pointwise construction of λ varies smoothly, as is easily seen from the explicit formula.) Now we use the following, which will be proved afterwards:

Proposition 9.9.12. *Let M be a smooth manifold and $i : N \hookrightarrow M$ a compact submanifold. Let λ be a section of $End(TM|_N)$ such that $\lambda|_{TN} \equiv id$. Then there exists a neighbourhood U of N and a mapping $\varphi : U \rightarrow M$ which is a diffeomorphism onto its image such that $\varphi|_N = id$ and $d\varphi_p = \lambda(p)$ for all $p \in N$.*

⁴⁵The lemma defines a map from $\{W \in Gr(n, V) : W \cap L = \{0\}\}$ to $\{\Lambda \in \Lambda(V) : \Lambda \cap L = \{0\}\}$.

The proposition gives an embedding $\varphi : U \rightarrow M$ of a neighbourhood U of L in M with the property that $d\varphi_p = \lambda(p)$ for $p \in L$. Thus for $p \in L$,

$$\varphi^*(\omega_1)(p) = d\varphi_p^*(\omega_1(p)) = \lambda(p)^*(\omega_1(p)) = \omega_0(p).$$

Thus $\varphi^*(\omega_1)$ and ω_0 are symplectic forms on U_0 such that they agree along L . By the symplectic neighbourhood theorem (i.e. Theorem 9.8.2), there are neighbourhoods U_0 and U_1 of L in U and a diffeomorphism $\phi : U_0 \rightarrow U_1$ with $\phi|_L = id$ such that $\phi^*(\omega_1) = \omega_0$. This proves the theorem, modulo the proposition. \square

Proof of Proposition 9.9.12. We will use the exponential map from Riemannian geometry here as it gives coordinates on a tubular neighbourhood of N . Choose a Riemannian metric on M . Then there is a smooth mapping $Exp : TM \rightarrow M$ which is the identity on the zero section $M \subset TM$ and such that $dExp_{(p,v)} : T_{(p,v)}(TM) \rightarrow T_pM$ restricts to the identity on $\{0\} \times T_pM \subset T_{(p,v)}(TM) \cong T_pM \oplus T_pM$. (The curve $t \mapsto Exp(p, tv)$ is a geodesic for the metric, i.e. a locally distance-minimizing curve, starting at p and with tangent vector $v \in T_pM$.) In particular, if $\mathcal{N} \subset TM|_N$ is the orthogonal complement of TN in $TM|_N$ with respect to the metric, then the map $\mathcal{N} \rightarrow M$ by $(p, v) \mapsto Exp_p(v)$ is a diffeomorphism on a neighbourhood U of $N \subset \mathcal{N}$. (Thus we get a coordinatized tubular neighbourhood of N in M .) Now define the map $\varphi : U \rightarrow M$ by $(p, v) \mapsto Exp_p(\lambda(p)(v))$. \square

This concludes the proof of Theorem 9.9.8 and Corollary 9.9.9. Here is a nice consequence:

Theorem 9.9.13. *Let (M, ω) be a symplectic manifold. Then a neighbourhood of the identity in the symplectomorphism group is identified with a neighbourhood of zero in the space of closed one-forms on M .*

Sketch of proof. Here the symplectomorphism group is given the induced topology from its inclusion in all diffeomorphisms. (The topology on the diffeomorphism group was discussed in a problem set, essentially it is defined by the family (for $k \in \mathbb{N}$) of seminorms which take into account all derivatives of order $\leq k$.)

The diagonal $\Delta \subset \overline{M} \times M$ is a Lagrangian, so by the previous theorem it has a neighbourhood which is symplectomorphic with a convex neighbourhood U_0 of the zero section of $M \cong \Delta$ in the cotangent bundle T^*M , with the canonical form $\omega = -d\lambda$. If $\varphi : M \rightarrow M$ is a symplectomorphism then its graph gives a Lagrangian submanifold of T^*M . If φ is C^1 -sufficiently close to the identity then its graph is in U_0 . Now any such submanifold of T^*M is the graph of a one-form, which is closed since the submanifold is Lagrangian. (If $\alpha : M \rightarrow T^*M$ is the one-form then $\alpha^*(\omega) = d\alpha$.) We leave to be checked that this identification is smooth for the natural smooth structures. \square

Remark 9.9.14 (Automorphism groups). Symplectic manifolds lie somewhere between complex manifolds and smooth manifolds in terms of their rigidity or flexibility. Here is an illustration:

- (1) The group of holomorphic automorphisms of a compact complex manifold is a finite-dimensional Lie group, possibly with infinitely many connected components. The Lie algebra (i.e. tangent space at the identity) is naturally identified with the space of (global) holomorphic vector fields. ⁽⁴⁶⁾
- (2) The group of symplectomorphisms of a compact symplectic manifold is an infinite-dimensional Frechet Lie group ⁽⁴⁷⁾ with tangent space at the identity identified with the

⁴⁶Another rigid setting: The group of Riemannian isometries of a compact Riemannian manifold (M, g) is a finite-dimensional Lie group (Myers-Steenrod) with Lie algebra given by the Killing vector fields (:= vector fields whose infinitesimal flow preserves the metric).

⁴⁷A Frechet (resp. Banach) manifold is locally modelled on a Frechet (resp. Banach) space. A Frechet (resp. Banach) Lie group is a Frechet (resp. Banach) manifold G such that the multiplication and inversion are smooth. The basic theory of Banach manifolds is very similar to the usual case, mainly because the inverse function theorem and the existence-uniqueness theorems for ODE hold. The theory of Frechet manifolds is more delicate. The symplectomorphism and diffeomorphism groups are Frechet Lie groups but they are *not* Banach Lie groups. For all this see Milnor, *Remarks on infinite-dimensional Lie groups* (1983).

space of symplectic vector fields, or, equivalently, with the space of closed one-forms (cf. Theorem 9.9.13).

- (3) The group of diffeomorphisms of a compact manifold is an infinite-dimensional Fréchet Lie group. The natural candidate for its tangent space at the identity is the space of all smooth vector fields, with the Lie bracket of vector fields (with a minus sign).

⁽⁴⁸⁾ ⁽⁴⁹⁾ In all three cases the natural exponential map is given by taking the flow, i.e. one-parameter group of diffeomorphisms associated to a vector field. In cases (1) and (2) this exponential map is an isomorphism in a neighbourhood of zero. In case (3) the map is neither locally injective (i.e. 1-1) nor locally surjective. On the other hand, the automorphism group in case (1) is typically small (e.g. the automorphism group of a generic compact complex curve of genus ≥ 2 is finite), and may not give much information, while in case (2) it is always at least the space of closed forms, hence infinite-dimensional. (3) is obviously infinite-dimensional. Symplectomorphism groups are an active area of current research. ⁽⁵⁰⁾

Remark 9.9.15 (Hamiltonian symplectomorphisms). The subgroup of symplectomorphisms of a compact manifold given by the time one flow (i.e. ϕ_1 for ϕ_t) of a Hamiltonian vector field is the group of Hamiltonian symplectomorphisms. It is a normal subgroup contained in the connected component of identity. In the identification of the previous theorem it corresponds to the space of exact one-forms in M .

9.10. Miscellaneous remarks. Arnol'd made some important conjectures (1970s/80s), of which the following is a special case which remains very much open:

Conjecture 9.10.1. *If M, N are compact orientable manifolds and there is a symplectic isomorphism of cotangent bundles $T^*M \cong T^*N$ then M is diffeomorphic to N .*

Recall that given a smooth function $H : M \rightarrow \mathbb{R}$ we must get a Hamiltonian vector field X_H corresponding to dH defined by $i_{X_H}\omega = dH$ and the flow of X_H gives a one-parameter group of symplectomorphisms. More generally, we may allow a time-dependent function $H_t : M \rightarrow \mathbb{R}$ and the corresponding time-dependent Hamiltonian vector field X_{H_t} defined by

$$i_{X_{H_t}}\omega = dH_t.$$

The corresponding isotopy is called a **Hamiltonian isotopy**. Another case of Arnol'd's conjectures is the following (recall that a Lagrangian $L \subset T^*M$ is **exact** if the one-form $\lambda|_L$ is exact), usually called the “nearby Lagrangian conjecture”:

Conjecture 9.10.2. *A closed exact Lagrangian in a cotangent bundle of a closed manifold is Hamiltonian isotopic to the zero section.*

(Here closed means compact without boundary. A fibre of $T^*M \rightarrow M$ is exact Lagrangian, but the conjecture obviously does not apply to it.)

The conjecture remains largely open: The only cases where it is known are $M = S^1$ (which is elementary) and $M = S^2$ and $S^1 \times S^1$. (!) It has been proved (2012) that any closed exact Lagrangian is homotopy equivalent to the zero section, so that, e.g. they have the same cohomology.

Example 9.10.3. Consider the simplest example, namely $T^*S^1 \cong S^1 \times \mathbb{R}$. The symplectic form is the area form $d\theta \wedge dy$ and $\lambda = y d\theta$. Now any submanifold of dimension one is Lagrangian, and a closed Lagrangian is a finite union of smoothly embedded closed curves. Check that every exact closed Lagrangian is the graph of a 1-form $f(\theta) d\theta$ where $\int f(\theta) d\theta = 0$. Check that such a Lagrangian is Hamiltonian isotopic to the zero section.

⁴⁸So for example if we take a compact Kähler manifold, we can consider its automorphisms in any of the contexts (1)–(3), and because the symplectic form and metric determine each other we have inclusions (1) \subset (2) \subset (3) for the automorphism groups in each context.

⁴⁹In fact, symplectomorphisms are C^0 -closed in diffeomorphisms, by a theorem of Gromov-Eliashberg.

⁵⁰See e.g. McDuff, *Lectures on symplectomorphism groups*, arXiv:math/0201032v2 for an older survey.

Remark 9.10.4 (Symplectic category). The idea is to take as objects symplectic manifolds and for morphisms between two symplectic manifolds M_1 and M_2 we take the Lagrangian submanifolds of $\overline{M}_1 \times M_2$. How would one compose morphisms? Well, given Lagrangians $L_{12} \subset \overline{M}_1 \times M_2$ and $L_{23} \subset \overline{M}_2 \times M_3$, consider the fibre product

$$L_{12} \times_{M_1} L_{23} := (L_{12} \times L_{23}) \cap (\overline{M}_1 \times \Delta_2 \times M_3)$$

(where $\Delta_2 \subset \overline{M}_1 \times M_3$ is the diagonal) and define the composition

$$L_{12} \circ L_{23} := \pi_{13}(L_{12} \times_{M_1} L_{23})$$

where $\pi_{13} : \overline{M}_1 \times M_2 \times \overline{M}_2 \times M_3 \rightarrow \overline{M}_1 \times M_3$ is the projection. ⁽⁵¹⁾ If the intersection above is transverse then the fibre product exists as a manifold and this will define a Lagrangian in $\overline{M}_1 \times M_3$. However, most of the time this is not true, and so constructing the symplectic category presents a lot of technical problems.

Remark 9.10.5 (Embedding theorems). A compact symplectic manifold (M, ω) with integral symplectic form (i.e. the class $[\omega] \in H_{dR}^2(M) = H^2(M, \mathbb{R})$ should come from $H^2(M, \mathbb{Z})$) can be embedded in $\mathbb{C}P^N$, i.e. there is an embedding $i : M \hookrightarrow \mathbb{C}P^N$ such that $\omega = i^*\omega_0$, where ω_0 is the canonical form (Gromov, Tischler). A deep theorem of Gromov says that every noncompact symplectic manifold can be embedded if the topological obstruction vanishes, i.e. if $[\omega] = 0$ in $H^2(M, \mathbb{R})$ then there is an embedding in \mathbb{R}^N .

Remark 9.10.6 (Complex symplectic manifolds). One can ask about the analogue of symplectic manifolds in the complex setting. A complex symplectic manifold is a complex manifold X of even complex dimension $2n$ with a nowhere-vanishing holomorphic 2-form ω with $\omega^n \neq 0$ everywhere. Compact examples include K3 surfaces, and noncompact examples include holomorphic cotangent bundles. The Darboux theorem holds in the holomorphic sense, but some things are different in this world, e.g. the holomorphic analogue of the Lagrangian neighbourhood theorem fails, which is not surprising as the tubular neighbourhood theorem fails. ⁽⁵²⁾

⁵¹This kind of definition is familiar in algebraic geometry where we consider categories of correspondences in which the morphisms from X to Y are linear combinations of subvarieties of $X \times Y$ of dimension $\dim X$, modulo an adequate equivalence relation (usually rational equivalence). The key point is that there are pullback and pushforward operations on these groups of morphisms and a product structure so that the composition formula makes sense.

⁵²See e.g. the discussions at <https://mathoverflow.net/questions/322447/holomorphic-version-of-darboux-theorem> and <https://mathoverflow.net/questions/347256/holomorphic-weinstein-lagrangian-neighborhood-theorem>

10. SHEAVES AND COHOMOLOGY I

In this section we will discuss sheaves on topological spaces and their cohomology and apply them to manifolds. Good references for the material in this section are Voisin's *Hodge Theory and Complex Geometry*, vol. I or Ramanan's *Global Calculus*.

(Note that the definitions we use, which are the ones in these books, are sometimes different from those of Warner's book and some other books.)

10.1. Presheaves and sheaves. Let X be a topological space. Let $Op(X)$ be the category with objects the open sets in X , and morphisms inclusions between open sets. A **presheaf of sets** on X is a contravariant functor $F : Op(X) \rightarrow Sets$ such that $F(\emptyset)$ is a one-element set and $F(id : U \rightarrow U)$ is the identity map of $F(U)$. A **presheaf of abelian groups** on X is a contravariant functor from the category $Op(X)$ to the category of abelian groups which takes the empty set \emptyset to $\{e\}$ and the morphism $U \subset U$ to the identity. Concretely, this means we have, for each open set $U \subset X$, an abelian group $F(U)$ and for each pair of open sets $V \subset U$ we have a homomorphism $\rho_V^U : F(U) \rightarrow F(V)$, such that for $W \subset V \subset U$ we have

$$\rho_W^V \circ \rho_V^U = \rho_W^U$$

and moreover $F(\emptyset) = \{e\}$ and $\rho_U^U = id$. The homomorphism ρ_V^U will be called "restriction from U to V ". An element $s \in F(U)$ is referred to as a **section** of F over U .

A morphism between presheaves F and G has the obvious meaning, namely a natural transformation of functors from F to G . Concretely this means we are given for each $U \subset X$ a homomorphism $F(U) \rightarrow G(U)$ and the diagram below commutes for every pair $V \subset U$:

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ \rho_V^U \downarrow & & \downarrow \rho_V^U \\ F(V) & \longrightarrow & G(V). \end{array}$$

A presheaf of rings (or algebras) has the obvious meaning: a covariant functor from $Op(X)$ to rings (or algebras) such that $F(\emptyset) = \{0\}$ and $F(U \subset U) = id$.

Examples 10.1.1. Here are some examples of presheaves:

(1) The assignment $U \mapsto C^0(U, \mathbb{R})$ (continuous functions from U to \mathbb{R} with restriction of functions) defines a presheaf of abelian groups. Here \mathbb{R} may be replaced by any topological space Y to get a presheaf of sets.

(2) Fix an abelian group A . The assignment $U \mapsto A$ defines the constant presheaf, where all $\rho_V^U = id$.

(3) Let $X = \mathbb{C}$, or more generally any complex manifold. For an open subset $U \subset X$ define $\mathcal{O}(U)$ to be the ring of holomorphic functions on U , and let $\rho_V^U : \mathcal{O}(U) \rightarrow \mathcal{O}(V)$ be the restriction of functions.

(4) Let M be a smooth manifold and let $\mathcal{E}(U)$ be the smooth functions from U to \mathbb{R} , and ρ_V^U is restriction of functions. This defines a presheaf \mathcal{E} on M .

(5) For $X = \mathbb{C}$ let \mathcal{B} be the presheaf defined by $\mathcal{B}(U) =$ bounded holomorphic functions on U , $\rho_V^U =$ restriction of functions.

(6) Fix a point $x \in X$ and an abelian group A . Define F by $F(U) = A$ if $x \in U$ and $F(U) = \{0\}$ if $x \notin U$. The restriction maps are the obvious ones (identity or zero). This presheaf is called the **skyscraper presheaf** at x with value A .

(7) Let X be a smooth manifold. The functor $U \mapsto \mathcal{X}(U) =$ smooth vector fields on U defines a presheaf, as does the functor of sections of any smooth vector bundle on X .

(8) Let X be a topological space and $k \geq 0$ an integer. The functor $U \mapsto H^k(U, \mathbb{Z})$ (singular cohomology groups of U) defines a presheaf. Restriction maps are restriction maps in cohomology.

A **sheaf** on X is a presheaf F on X which further satisfies the following two conditions:

- (S1) If $s_1, s_2 \in F(U)$ and $\{U_\alpha\}_\alpha$ is an open covering of U such that $\rho_{U_\alpha}^U(s_1) = \rho_{U_\alpha}^U(s_2)$ for all α , then $s_1 = s_2$
- (S2) If U is open in X , $\{U_\alpha\}_\alpha$ is an open covering of U and $\{s_\alpha \in F(U_\alpha)\}_\alpha$ a collection of sections satisfying $\rho_{U_\alpha \cap U_\beta}^{U_\alpha}(s_\alpha) = \rho_{U_\alpha \cap U_\beta}^{U_\beta}(s_\beta)$ for all α, β , then there is a section $s \in F(U)$ such that $s_\alpha = \rho_{U_\alpha}^U(s)$ for all α .

Informally, (S2) says that local sections of sheaves glue together to give sections, (S1) says that sections of sheaves are determined by what they are locally. The sheaves on X form a category; the morphisms between two sheaves are the as the morphisms of presheaves between them, i.e. natural transformations of functors.

Examples 10.1.2. Among the examples of presheaves given above, (1), (3), (4), (6), (7) are sheaves and (8) is a sheaf in the case $k = 0$. (2) and (5) are not sheaves because they fail to satisfy (S2). (An example of a presheaf which fails (S2) and (S1) is the following: Let $X = \{x, y\}$ with the discrete topology. For an open set $U \subset X$ let $F(U) =$ functions from U to \mathbb{R} . For $V \subsetneq U$ let $\rho_V^U := 0$.) The presheaf (8) for $k \geq 1$ fails (S1) in general, e.g. if X is a manifold or locally contractible.

Example 10.1.3. For any topological space Y , the presheaf $U \mapsto C^0(U, Y)$ defines a sheaf of sets on X . If Y is a(n abelian) topological group then this defines a sheaf of (abelian) groups.

Example 10.1.4. For any smooth manifolds M and N the presheaf $U \mapsto C^\infty(U, N)$ defines a sheaf of sets on M . If N is a(n abelian) Lie group then this defines a sheaf of (abelian) groups.

10.2. Stalks, sheafification. Sheaves are (as we shall see) much more convenient to work with than presheaves, although it is usually presheaves which appear “in nature”. So it is useful to have a canonical (i.e. functorial) way to associate a sheaf to a presheaf.

Let F be a presheaf on X . The **stalk** of F at $x \in X$ is the direct limit:

$$F_x := \varinjlim_{U \ni x} F(U).$$

For $U \ni x$ denote the natural map to the stalk by

$$\rho_x^U : F(U) \rightarrow F_x.$$

Now if F satisfies (S1) then the collection $(\rho_x^U(s))_{x \in U}$ determines the section $s \in F(U)$, in the sense that the map

$$F(U) \rightarrow \prod_{x \in U} F_x$$

is injective. In fact this is equivalent to (S1). This suggests how, given a presheaf, we should make a sheaf out of it.

A **continuous section** of F over U is an element $(s_x)_{x \in U} \in \prod_{x \in U} F_x$ such that for each $x \in U$ there exists a V and $\sigma \in F(V)$ such that for all $y \in V$ we have $s_y = \rho_y^U(\sigma)$.⁽⁵³⁾ Define the presheaf \tilde{F} by

$$\tilde{F}(U) = \{ \text{continuous sections of } F \text{ over } U \}.$$

The restriction maps $\tilde{F}(U) \rightarrow \tilde{F}(V)$ are given by $(s_x)_{x \in U} \mapsto (s_x)_{x \in V}$. There is an obvious presheaf homomorphism

$$\theta : F \rightarrow \tilde{F}$$

(since an actual section of F over U gives a continuous section).

Let us see that \tilde{F} defined like this is a sheaf:

⁵³Here “continuous” does not refer to a topology; it is simply a terminology for us. However there is a natural topology on the coproduct $\prod_{x \in X} F_x$ such that these are exactly the continuous sections over U of the natural map $\prod_{x \in X} F_x \rightarrow X$ which takes the stalk F_x to x . (Exercise) The space $\prod_{x \in X} F_x$ is called the *espace étalé* of F , cf. e.g. Hartshorne, Chapter II, Exs.

Lemma 10.2.1. *The sheafification $F \mapsto \tilde{F}$ has the following properties:*

- (1) *The construction $F \mapsto \tilde{F}$ is functorial.*
- (2) *The map θ induces an isomorphism $F_x \cong \tilde{F}_x$ of stalks.*
- (3) *If F is a sheaf then $\theta : F \rightarrow \tilde{F}$ is an isomorphism.*
- (4) *The construction has a universal property: If G is a sheaf and $\phi : F \rightarrow G$ is a presheaf homomorphism then there is a unique sheaf homomorphism $\tilde{\phi} : \tilde{F} \rightarrow G$ such that $\phi = \tilde{\phi} \circ \theta$. More precisely, sheafification is left adjoint to the forgetful functor from sheaves to presheaves.*
- (5) *$\text{Hom}(F, G) = \text{Hom}(\tilde{F}, \tilde{G})$, i.e. sheafification is a faithful functor.*

Example 10.2.2. (i) Consider the examples (2) and (5) above which were not sheaves. It is easy to see that the sheafification of the constant presheaf A is the sheaf which assigns to U the group of locally constant functions $U \rightarrow A$, i.e. $A^{\pi_0(U)}$. This is called the constant sheaf with value A . The sheafification of (5) is the sheaf of all holomorphic functions.

(ii) Note that a presheaf can have nonzero global sections even if all its stalks are zero. (This cannot happen for a sheaf by (S1).) An example is the presheaf $U \mapsto H^k(U)$ of de Rham or singular cohomology groups on a manifold for $k > 0$. By the Poincaré lemma (for de Rham cohomology) or the contractibility of balls (for singular cohomology) this has zero stalk at all points, but it has global sections e.g. if $H^k(M) \neq 0$. This presheaf fails to satisfy (S1) because any two classes in $H^k(M)$ have the same (=zero) restriction to small enough coordinate neighbourhoods. In this case the associated sheaf is zero.

Example 10.2.3. Here is an important example of a sheaf arising from differential equations. Let $X = \mathbb{C}^*$ and fix $\alpha \in \mathbb{C}$. Let F be the sheaf of holomorphic solutions of the differential equation

$$z \frac{d}{dz} f = \alpha f \tag{10.2.1}$$

i.e. over an open set $U \subset \mathbb{C}^*$ the group $F(U)$ is the holomorphic functions on U satisfying the differential equation. The restriction maps are given by restricting functions. This defines a presheaf and it is easy to see that it is a sheaf, a subsheaf of the sheaf of all holomorphic functions on \mathbb{C} .

(1) Suppose first that $\alpha \in \mathbb{Z}$. In this case $f(z) = z^\alpha$ is a holomorphic solution to (10.2.1) which makes sense in all of \mathbb{C}^* . Moreover it is clear that every solution is a multiple of this one. Therefore the sheaf F is isomorphic to the constant sheaf \mathbb{C} .

(2) Suppose now that $\alpha \notin \mathbb{Z}$. Then $z^\alpha = \exp(\alpha \log(z))$ defines a solution of (10.2.1) in any domain U in which we can choose a branch of the logarithm function, i.e. any domain not containing a real half-line. Moreover in such a domain any solution is a multiple of z^α , so $F(U) \cong \mathbb{C}$ with a basis given by z^α . On the other hand $F(U) = \{0\}$ if U contains a punctured disk. Thus F is not the constant sheaf, but it is locally constant in the sense that any point has a neighbourhood in which F is isomorphic to the constant sheaf \mathbb{C} . It follows that the stalks are all isomorphic to \mathbb{C} .

This example can be generalized by looking at solutions to

$$\left(z \frac{d}{dz} - \alpha \right)^n f = 0$$

for $n \geq 1$. For example if $\alpha \in \mathbb{Z}$ then there are n linearly independent solutions defined on all of \mathbb{C}^* and the sheaf is isomorphic to the constant sheaf \mathbb{C}^n . For $\alpha \notin \mathbb{Z}$ the sheaf of solutions is locally constant and locally isomorphic to the constant sheaf \mathbb{C}^n .

Example 10.2.4. Consider the previous example with $\alpha \notin \mathbb{Z}$ but with domain $X = \mathbb{C}$, i.e. let G be the sheaf of local solutions of (10.2.1) on \mathbb{C} . Now $G(U) = \{0\}$ if $0 \in U$. Thus the stalk at 0 is zero. The stalks at other points are the same since $i^{-1}G = F$ where $i : \mathbb{C}^* \rightarrow \mathbb{C}$ is the inclusion and F is as earlier.

10.3. Kernels, cokernels, images. Let $F \rightarrow G$ be a homomorphism of sheaves. The presheaf kernel, presheaf cokernel, and presheaf image are the functors

$$\begin{aligned} U &\mapsto \ker(F(U) \rightarrow G(U)), \\ U &\mapsto \operatorname{im}(F(U) \rightarrow G(U)), \\ U &\mapsto \operatorname{coker}(F(U) \rightarrow G(U)). \end{aligned}$$

It is easy to see that the first is already a sheaf, called the kernel sheaf and denoted $\ker(F \rightarrow G)$. However, simple examples show that the presheaf image and presheaf cokernel are not necessarily sheaves. ⁽⁵⁴⁾ So we define the **image sheaf** and the **cokernel sheaf** to be the sheaves associated with the image presheaf and cokernel presheaf respectively. We will use the notation $\operatorname{im}(F \rightarrow G)$ and $\operatorname{coker}(F \rightarrow G)$ for these objects as we will never need to use the presheaf image and presheaf cokernel.

In general, if $F \subset G$ is a subsheaf then we will define the **quotient sheaf** to be the sheaf associated with the presheaf $U \mapsto G(U)/F(U)$. In this way the kernel and cokernel become the sheaves associated with the presheaf kernel and cokernel. With this definition, we have

Lemma 10.3.1. (1) *A homomorphism of sheaves $F \rightarrow G$ is an isomorphism if and only if it is an isomorphism on all stalks.*

(2) *Two subsheaves F' and F'' of F are equal if and only if $F'_x = F''_x$ in F_x .*

(3) *A sequence $F_1 \rightarrow F_2 \rightarrow F_3$ of sheaves is exact if and only if the corresponding sequence of stalks $F_{1,x} \rightarrow F_{2,x} \rightarrow F_{3,x}$ is exact for every $x \in X$.*

Example 10.3.2 (Exponential sequence). Here is a basic example of a short exact sequence of sheaves on any smooth manifold X . For any abelian Lie group G let \underline{G}_X be the sheaf on X defined by $U \mapsto C^\infty(U, G)$. Then there is a short exact sequence

$$0 \rightarrow \underline{\mathbb{Z}}_X \rightarrow \underline{\mathbb{C}}_X \rightarrow \underline{\mathbb{C}}^*_X \rightarrow 0$$

where the second map is defined by $f \mapsto \exp(2\pi i f)$. That this map is surjective on stalks is easy to check and comes down to the fact that for small enough open neighbourhoods of any point in \mathbb{C}^* the exponential map can be inverted. This also gives an example of a short exact sequence of sheaves such that taking sections over an open set does not necessarily give a short exact sequence of abelian groups.

10.4. Global sections functor. From now on we will mainly consider sheaves of abelian groups and let $Sh(X)$ denote the category of such. Most of what is said goes through for sheaves valued in any abelian category with enough injectives.

Recall the notions of exact functor and left-exact functor. A functor $T : Sh(X) \rightarrow AbGps$ or $T : Sh(X) \rightarrow Sh(Y)$ is called **exact** if it takes short-exact sequences to short-exact sequences. It is **left-exact** if $0 \rightarrow T(A) \rightarrow T(B) \rightarrow T(C)$ is exact whenever $0 \rightarrow A \rightarrow B \rightarrow C$ is exact. It is **right-exact** if $T(A) \rightarrow T(B) \rightarrow T(C) \rightarrow 0$ is exact whenever $A \rightarrow B \rightarrow C \rightarrow 0$ is exact.

The **global sections functor** is the functor

$$\Gamma(X, -) : Sh(X) \rightarrow AbGps$$

defined by taking sections over X . So in the earlier notation, $\Gamma(X, F) = F(X)$.

Lemma 10.4.1. *The global sections functor is left-exact.*

Proof. This is straightforward. □

⁵⁴Here is an example for cokernel: Let $F = G = \mathcal{O}$ be the sheaf of holomorphic functions on $X = \mathbb{C}^*$ and let $\mathcal{O} \rightarrow \mathcal{O}$ be the operator $\frac{d}{dz}$. Then every point $x \in \mathbb{C}^*$ has a neighbourhood U for which the presheaf cokernel has sections over U equal to $\{0\}$ (any small enough neighbourhood of x will do). But the presheaf cokernel has nonzero sections over $X = \mathbb{C}^*$ since e.g. $\frac{1}{z}$ is not in the image of $\frac{d}{dz}$ on \mathbb{C}^* . Thus the presheaf cokernel is not a sheaf.

Examples 10.4.2. Here are examples to show that $\Gamma(X, -)$ is not an exact functor:

(1) Let $X = \mathbb{C}^*$ and consider the exponential sequence on X . The map $\Gamma(X, \underline{\mathbb{C}}_X) \rightarrow \Gamma(X, \underline{\mathbb{C}}^*_X)$ is not since e.g. the function z has no logarithm in X .

(2) Let $X = \mathbb{C}P^1$ and \mathcal{O} the sheaf of holomorphic functions on X . Let $x, y \in X$ be two distinct points. Then there is a short exact sequence

$$0 \rightarrow \mathcal{I}_{\{x,y\}} \rightarrow \mathcal{O} \rightarrow i_{x*}\mathbb{C} \oplus i_{y*}\mathbb{C} \rightarrow 0.$$

(Here the second map is $f \mapsto (f(x), f(y))$ and $\mathcal{I}_{\{x,y\}}$ is the kernel, i.e. the ideal of functions vanishing at x and y .) The only global sections of \mathcal{O} are constant functions, so the map $\mathbb{C} = \Gamma(X, \mathcal{O}) \rightarrow \Gamma(X, i_{x*}\mathbb{C} \oplus i_{y*}\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$ is not surjective.

10.5. Pullback, pushforward, internal Hom, tensor product. Let $f : X \rightarrow Y$ be a continuous map of topological spaces. Given a sheaf F on X we can define a presheaf on Y by

$$U \mapsto F(f^{-1}(U)).$$

This is already a sheaf, called the **pushforward** or **direct image** of F by f , and denoted f_*F . (*Exercise:* Check that this defines a sheaf.) Note that a special case of direct image is $\Gamma(X, -)$, which is direct image by the map $X \rightarrow pt$.

Given a sheaf G on Y we can define a presheaf on X by

$$U \mapsto \varinjlim_{V \supset f(U)} G(V).$$

This is not a sheaf in general, so we define the **pullback** or **inverse image** sheaf $f^{-1}G$ to be the sheaf associated with this presheaf. The stalks are related by $(f^{-1}G)_x = G_{f(x)}$ for $x \in X$. For any abelian group A we have $f^{-1}A_Y = A_X$.

Remark 10.5.1. Note that if $i_x : \{x\} \rightarrow X$ is the inclusion of a point then for an abelian group A the skyscraper sheaf at x with value A is $i_{x*}A$. For a sheaf F we have $i_x^{-1}F = F_x$ (the stalk at x .)

Lemma 10.5.2. *The inverse image functor $f^{-1} : Sh(Y) \rightarrow Sh(X)$ is exact. The direct image functor $f_* : Sh(X) \rightarrow Sh(Y)$ is left-exact.*

Proof. The first assertion follows from the fact that $(f^{-1}F)_x = F_y$ and the fact that a short-exact sequence of sheaves is exact if and only if it is exact on every stalk. The second assertion is left as an exercise. (Note that if $f : X \rightarrow pt$ is the map to a point then $f_*F = \Gamma(X, F)$.) \square

There are natural transformations of functors $id \rightarrow f_*f^{-1}$ and $f^{-1}f_* \rightarrow id$ which define homomorphisms $Hom(f^{-1}F, G) \leftrightarrow Hom(F, f_*G)$. The pair (f^{-1}, f_*) is an **adjoint pair** of functors, which means (by definition) that there is a natural isomorphism

$$Hom(f^{-1}G, F) = Hom(G, f_*F)$$

for $F \in Sh(X), G \in Sh(Y)$. (*Exercise:* Verify this.) We say f^{-1} is **left adjoint** to f_* or f_* is **right adjoint** to f^{-1} .

Lemma 10.5.3. (1) *If L is an exact functor and R is right adjoint to L then R takes injective objects to injective objects.*

(2) *f_* takes injective sheaves to injective sheaves.*

(3) *f_* takes flasque sheaves to flasque sheaves.*

Proof. Exercise. \square

Given two sheaves F, G of abelian groups on a space X we have the **internal Hom** presheaf defined by

$$U \mapsto \underline{Hom}(F, G)(U) := Hom(F|_U, G|_U)$$

where $F|_U$ means the pullback by the inclusion $U \hookrightarrow X$. (55)

⁵⁵One might think of defining it by $\underline{Hom}(F, G)(U) = Hom(F(U), G(U))$ but this does not make any sense: There are no natural restriction maps for $V \subset U$.

Lemma 10.5.4. *If F and G are sheaves then $\underline{Hom}(F, G)$ is a sheaf.*

Proof. Exercise. □

Something to keep in mind: In general, the natural map $\underline{Hom}(F, G)_x \rightarrow Hom(F_x, G_x)$ is neither injective nor surjective. ⁽⁵⁶⁾

The adjointness relation between f_* and f^{-1} then has a sheaf version, which gives the previous one after taking global sections:

Lemma 10.5.5. *If $f : X \rightarrow Y$ is a continuous map of topological spaces and $F \in Sh(X)$ and $G \in Sh(Y)$ then $f_*\underline{Hom}(f^{-1}G, F) \cong \underline{Hom}(F, f_*G)$.*

Proof. Exercise. □

For $F, G \in Sh(X)$ the **tensor product** $F \otimes G$ is the sheaf associated with the presheaf

$$U \mapsto F(U) \otimes G(U).$$

(The presheaf is not in general a sheaf, so we must sheafify.) The stalks of the tensor product are given by $(F \otimes G)_x = F_x \otimes G_x$. (Easy exercise.)

A **complex of sheaves** on a topological space X is a sequence of sheaves $(F^i)_{i \in \mathbb{Z}}$ on X and morphisms $F^i \rightarrow F^{i+1}$ ($i \in \mathbb{Z}$) such that the composition of any two successive morphisms is zero. We write:

$$\cdots \rightarrow F^{-1} \rightarrow F^0 \rightarrow F^1 \rightarrow F^2 \rightarrow \cdots$$

We say “ F^i appears in degree i ”. Taking sections over any open set $U \subset X$ gives a complex of abelian groups $\cdots \rightarrow \Gamma(U, F^i) \rightarrow \Gamma(U, F^{i+1}) \rightarrow \cdots$.

Example 10.5.6. The de Rham complex of sheaves \mathcal{E}_M^* on a smooth manifold M .

Let F be a sheaf on X . A **resolution** of F is a complex of sheaves $0 \rightarrow K^0 \rightarrow K^1 \rightarrow \cdots$ such that the following sequence is exact:

$$0 \rightarrow F \rightarrow K^0 \rightarrow K^1 \rightarrow \cdots$$

Example 10.5.7. The de Rham complex of sheaves \mathcal{E}_M^* on a smooth manifold M is a resolution of the constant sheaf \mathbb{R}_M . (Here $\mathbb{R}_M \subset \mathcal{E}_M^0$ as the locally constant functions.)

10.6. Injective sheaves and injective resolutions. An **injective sheaf** is an injective object in the abelian category of sheaves, i.e. a sheaf $I \in Sh(X)$ for which the functor $Hom(-, I)$ is exact. It is easy to see that this is equivalent to the following property: In the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & \downarrow & \searrow & \uparrow \\ & & I & & \exists \end{array}$$

if $0 \rightarrow A \rightarrow B$ is exact, equivalently $A \rightarrow B$ is a sheaf monomorphism, then there exists a homomorphism $B \rightarrow I$ making the diagram commute. A short exact sequence $0 \rightarrow I \rightarrow G \rightarrow H \rightarrow 0$ with I injective has a splitting. (Look at the identity $I \rightarrow I$ to get a morphism $G \rightarrow I$ which splits the sequence.)

Example 10.6.1. (i) Let I be an injective abelian group. Then the skyscraper sheaf $i_{x*}I$ is injective. Indeed, this follows from the equality $Hom(F, i_{x*}I) = Hom(F_x, I)$ (which is a special case of the adjointness of i_x^{-1} and i_{x*}).

(ii) The product of injective sheaves is injective. (Indeed, this follows from $Hom(-, \prod_{\alpha} F_{\alpha}) = \prod_{\alpha} Hom(-, F_{\alpha})$.)

The following lemma is usually quoted as saying “ $Sh(X)$ has enough injectives”:

⁵⁶Here is an example: Let $X = \mathbb{C}$ and $i : \{0\} \hookrightarrow \mathbb{C}$ the inclusion. There is an injection $i_*\mathbb{Z} \hookrightarrow \mathbb{Z}_X$; let Q be the cokernel. Then $\mathbb{Z}_X \rightarrow Q$ gives a nonzero element of $\underline{Hom}(\mathbb{Z}_X, Q)_0$ while $Hom(\mathbb{Z}, Q_0) = 0$ since $Q_0 = 0$.

Lemma 10.6.2. *Any sheaf admits a monomorphism to an injective sheaf.*

Proof. Recall that any abelian group admits a monomorphism to an injective (=divisible) abelian group. Thus for each $x \in X$ the stalk F_x admits an injection $j_x : F_x \hookrightarrow I_x$ for some injective I_x . Now define $I = \prod_{x \in X} i_{x*} I_x$. This is an injective sheaf. Consider the morphism

$$F \hookrightarrow \prod_{x \in X} i_{x*} F_x \hookrightarrow \prod_{x \in X} i_{x*} I_x = I.$$

defined on U by composing $\prod_{x \in U} \rho_x^U$ with the product $\prod_{x \in U} j_x$. This is injective on stalks and hence is injective. \square

A resolution

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

where each I^k is injective is called an **injective resolution** of F .

Lemma 10.6.3. (i) *Every sheaf of abelian groups has an injective resolution.*

(ii) *Given two sheaves F and G and resolutions $F \rightarrow I^\bullet$ and $G \rightarrow J^\bullet$ such that J^\bullet is injective, any homomorphism $\alpha : F \rightarrow G$ lifts to a homomorphism $I^\bullet \rightarrow J^\bullet$ such that*

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots \\ & & \alpha \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \dots \end{array} \quad (10.6.1)$$

commutes. Any two such liftings are homotopy equivalent, i.e. if $\varphi = (\varphi^i : I^i \rightarrow J^i)_i$ and $\psi = (\psi^i : I^i \rightarrow J^i)_i$ are two liftings then there is a homomorphism $H = (H^i : I^i \rightarrow J^{i-1})_i$ from I^\bullet to $J^{\bullet-1}$ such that $d_J H + H d_I = \varphi - \psi$. (Here $d_I = (d_I : I^i \rightarrow I^{i+1})_i$ and $d_J = (d_J^i)$ are the morphisms in the resolutions.)

(iii) *An injective resolution of a sheaf is unique up to homotopy, i.e. given two injective resolutions $F \rightarrow I^\bullet$ and $F \rightarrow J^\bullet$ there are maps $\varphi : I^\bullet \rightarrow J^\bullet$ and $\psi : J^\bullet \rightarrow I^\bullet$ such that $\varphi\psi$ and $\psi\varphi$ are both homotopic to the identity.*

(iv) *Given an injective homomorphism $\alpha : F \hookrightarrow G$ we can find injective resolutions $F \rightarrow I^\bullet$ and $G \rightarrow J^\bullet$ and injective homomorphisms $I^k \hookrightarrow J^k$ such that the diagram (10.6.1) commutes.*

Proof. (i) Existence is proved in the usual way: First let

$$F \hookrightarrow I^0$$

be a monomorphism to an injective sheaf. The cokernel of $F \rightarrow I^0$ can be embedded in injective sheaf I^1 . The morphism

$$I^0 \rightarrow I^1$$

is the composition $I^0 \rightarrow \text{coker}(F \rightarrow I^0) \rightarrow I^1$. If $I^0 \rightarrow I^1$ is not surjective embed its cokernel in I^2 and continue.

(ii) is straightforward, (iii) follows from (ii).

(iv) We will first prove the following: Given $F \hookrightarrow G$ one can find $F \hookrightarrow I$ and $G \hookrightarrow J$ (injections into injectives) such that $I \hookrightarrow J$ and the natural map $I/F \rightarrow J/G$ is injective. Indeed, look at the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \xrightarrow{\phi} & G & \xrightarrow{\psi} & Q & \longrightarrow & 0 \\ & & \downarrow u & & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & I & \longrightarrow & J := I \oplus K & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

Here $u : F \hookrightarrow I$ and $w : Q \hookrightarrow K$ with I, K injective have been chosen. The map $u : F \rightarrow I$ extends to a map $\alpha : G \rightarrow I$ with $\alpha\phi = u$ (by injectivity of I). Define $J = I \oplus K$ and $v : G \hookrightarrow J$ by $v = (w\psi, \alpha)$. The maps in the bottom row are the obvious ones.

Now suppose $F \hookrightarrow G$ is injective. Applying the above we can find injectives I^0 and J^0 and maps such that all rows and columns in the diagram below are exact:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F & \longrightarrow & I^0 & \longrightarrow & \operatorname{coker}(F \rightarrow I^0) \longrightarrow 0 \\
 & & \downarrow \phi & & \downarrow v & & \downarrow w \\
 0 & \longrightarrow & G & \longrightarrow & J^0 & \longrightarrow & \operatorname{coker}(G \rightarrow J^0) \longrightarrow 0
 \end{array}$$

Now apply the above observation to the injection $\operatorname{coker}(F \rightarrow I^0) \hookrightarrow \operatorname{coker}(G \rightarrow J^0)$ we find $I^1 \hookrightarrow J^1$ and continue as before. \square

10.7. Derived functors, cohomology of sheaves. Let $T : Sh(X) \rightarrow AbGps$ be a left-exact functor to abelian groups (or any abelian category). The **derived functors** of T are defined as follows: Let F be a sheaf. Take an injective resolution $F \rightarrow I^\bullet$. Apply the functor T to get a complex of abelian groups

$$0 \rightarrow T(I^0) \rightarrow T(I^1) \rightarrow \dots$$

and take cohomology:

$$R^i T(F) := H^i(T(I^\bullet)) = \frac{\ker(T(I^i) \rightarrow T(I^{i+1}))}{\operatorname{im}(T(I^{i-1}) \rightarrow T(I^i))}.$$

(We set $T(I^{-1}) = 0$.) Note that since T is left-exact we have $R^0 T(F) = T(F)$.

The main example of a left-exact functor is the global sections functor $\Gamma(X, -) : Sh(X) \rightarrow AbGps$. The cohomology groups $H^i(X, F)$ are defined as its derived functors, i.e.

$$H^i(X, F) := R^i \Gamma(X, F).$$

In other words, given a sheaf F choose a resolution by injective sheaves

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

then apply $\Gamma(X, -)$ to get a complex of abelian groups

$$0 \rightarrow \Gamma(X, I^0) \rightarrow \Gamma(X, I^1) \rightarrow \dots$$

and then take cohomology to define

$$H^i(X, F) := \frac{\ker(\Gamma(X, I^k) \rightarrow \Gamma(X, I^{k+1}))}{\operatorname{im}(\Gamma(X, I^{k-1}) \rightarrow \Gamma(X, I^k))}$$

It follows from Lemma 10.6.3 above that this definition does not depend on the choice of injective resolution.

The following proposition summarizes the properties of derived functors:

Proposition 10.7.1. *The objects $R^i T(F)$ are determined up to canonical isomorphism. The assignment $F \rightarrow R^i T(F)$ has the following properties:*

- (1) $R^0 T(F) = T(F)$
- (2) If I is an injective sheaf then $R^i T(I) = 0$ for $i > 0$.
- (3) A short exact sequence

$$0 \rightarrow F \rightarrow G \rightarrow Q \rightarrow 0$$

of sheaves gives a long exact sequence

$$0 \rightarrow T(F) \rightarrow T(G) \rightarrow T(Q) \rightarrow R^1 T(F) \rightarrow R^1 T(G) \rightarrow R^1 T(Q) \rightarrow \dots$$

- (4) If $F \rightarrow G$ is a homomorphism of sheaves and injective resolutions $F \rightarrow I^\bullet$ and $G \rightarrow J^\bullet$ are chosen then there is a canonical induced homomorphism $R^i F \rightarrow R^i G$.

Proof. (1), (2), (4) follow from the definitions and Lemma 10.6.3.

(3) is proved using the last part of Lemma 10.6.3: Choose injective resolutions $F \rightarrow I^\bullet$ and $G \rightarrow J^\bullet$ such that $I^\bullet \hookrightarrow J^\bullet$ compatibly with $F \hookrightarrow G$. Then the quotients

$$K^i := J^i / I^i$$

is a resolution of Q by injective objects. (The quotient of an injective object by an injective subobject is injective: use the splitting property to write $J^i = I^i \oplus K^i$ and then show that $\text{Hom}(-, K^i)$ is exact). Moreover the short exact sequences

$$0 \rightarrow I^i \rightarrow J^i \rightarrow K^i \rightarrow 0$$

are split (because I^i is injective). Thus applying the functor T gives split short exact sequences

$$0 \rightarrow T(I^i) \rightarrow T(J^i) \rightarrow T(K^i) \rightarrow 0$$

Thus we have a short exact sequence of complexes $0 \rightarrow T(I^\bullet) \rightarrow T(J^\bullet) \rightarrow T(K^\bullet) \rightarrow 0$ which splits at each i and hence by the usual argument (used, for example, to produce the long exact sequence in cohomology for a pair (X, A) from the short exact sequence of cochain complexes) gives a long exact sequence in cohomology:

$$\cdots \rightarrow H^i(T(I^\bullet)) \rightarrow H^i(T(J^\bullet)) \rightarrow H^i(T(K^\bullet)) \rightarrow H^{i+1}(T(I^\bullet)) \rightarrow \cdots$$

This is the desired long exact sequence. \square

Remarks 10.7.2. (i) One can ask whether it is possible to left-derive right-exact (covariant) functors, for example the functor $\text{Hom}(-, F) : A \mapsto \text{Hom}(A, F)$ e.g. in the way one does for modules over a ring in order to define Ext^i . One problem is that the category $\text{Sh}(X)$ does not usually have enough projectives, i.e. not every object admits a surjection $P \rightarrow F$ from a projective object. (A projective object is one for which $\text{Hom}(P, -)$ is exact.) ⁽⁵⁷⁾ So we will usually derive such bifunctors by resolving the second variable by injectives. (Another option, when working with sheaves of modules over a fixed sheaf of rings R , is to use resolutions by sheaves of R -flat modules.)

(ii) Although we call the $R^i T$ derived functors, they are not really functors because the objects $R^i T(F)$ depend on the choice of injective resolution, thus they have only been defined up to canonical isomorphism. Nevertheless, the property (4) means they behave essentially like functors.

We note the following fact for later use:

Lemma 10.7.3. *If $i : Z \hookrightarrow X$ is the inclusion of a closed subspace then i_* is an exact functor and*

$$H^k(Z, F) = H^k(X, i_* F)$$

for $F \in \text{Sh}(X)$.

Proof. ⁽⁵⁸⁾ The exactness of i_* is easily checked on stalks. Now if $F \rightarrow I^\bullet$ is an injective resolution of F then $i_* I^\bullet$ is a resolution of $i_* F$ and each $i_* I^k$ is injective (because i_* is the right adjoint to the exact functor i^{-1} it takes injectives to injectives). Thus $i_* I^\bullet$ is an injective resolution of $i_* F$ and so one has

$$H^k(Z, F) = H^k(\Gamma(Z, I^\bullet)) = H^k(\Gamma(X, i_* I^\bullet)) = H^k(X, i_* F)$$

⁵⁷Here is an example: Suppose X is Hausdorff and F is a nonzero sheaf on X . Choose $x \in X$ such that $F_x \neq 0$. Then $i_{x*} F_x$ admits no epimorphism from a projective sheaf. Indeed, suppose $P \rightarrow i_{x*} F_x$ is such an epimorphism. Then there is an open set U for which $P(U) \rightarrow F_x$ is nonzero. Let $y \in U$ be a point distinct from x and let $j : U - \{y\} \hookrightarrow X$ be the inclusion. Then $j_! j^{-1} F \rightarrow i_{x*} F_x$ is surjective (it is surjective on stalks). Thus the map $P \rightarrow i_{x*} F_x$ lifts to $P \rightarrow j_! j^{-1} F$. Applying $\Gamma(U, -)$ gives that $P(U) \rightarrow F_x$ is factors as $P(U) \rightarrow \Gamma(U, j_! j^{-1} F) \rightarrow F_x$. But $\Gamma(U, j_! j^{-1} F) = 0$ so this is a contradiction.

This shows that on any reasonable space there are no projectives in the category of sheaves. Note however that there are often plenty of projectives in the category of presheaves.

⁵⁸The proof here shows the following: If $\mathfrak{A} \xrightarrow{T} \mathfrak{B} \xrightarrow{U} \mathfrak{C}$ are left-exact functors between abelian categories and T is exact and takes injectives to injectives then $R^k(U \circ T) = R^k U \circ T$. In general the derived functors of $U \circ T$ are computed in terms of those of U and of T by a spectral sequence.

because $\Gamma(Z, I^k) = \Gamma(X, i_* I^k)$ for each k . □

10.8. Acyclic resolutions. Let T be a left-exact functor from $Sh(X)$ to $AbGps$ or $Sh(Y)$ as before. A sheaf F is called T -**acyclic** if $R^i T(F) = 0$ for $i > 0$.

Proposition 10.8.1. *Let $0 \rightarrow F \rightarrow A^\bullet$ be a resolution of F by T -acyclic objects. Then*

$$R^i T(F) = H^i(T(A^\bullet)) = \frac{\ker(T(A^i) \rightarrow T(A^{i+1}))}{\operatorname{im}(T(A^{i-1}) \rightarrow T(A^i))}, \quad (10.8.1)$$

i.e. the derived functors of T can be computed using T -acyclic resolutions. More precisely, if $F \hookrightarrow A^\bullet$ is a resolution and $F \hookrightarrow I^\bullet$ is an injective resolution then the (canonical) induced map

$$H^i(T(A^\bullet)) \rightarrow H^i(T(I^\bullet)) = R^i T(F)$$

is an isomorphism when A^\bullet consists of T -acyclic objects.

Proof. We prove by induction on i that $R^i T(F)$ is computed by the formula (10.8.1) for any F . The base case $i = 0$ is an immediate consequence of the left-exactness of T . We make the induction hypothesis that $R^k T$ for $k \leq i - 1$ is computed by acyclic resolutions. We want to prove the same for $R^i T$.

Let $F \in Sh(X)$ and let $F \rightarrow A^\bullet$ be an acyclic resolution of F . Look at the short exact sequence:

$$0 \rightarrow F \xrightarrow{d_0} A^0 \rightarrow \operatorname{coker}(d_0) \rightarrow 0.$$

The long exact sequence of $R^i T$ is:

$$\begin{aligned} 0 \rightarrow T(F) \rightarrow T(A^0) \rightarrow T(\operatorname{coker}(d_0)) \rightarrow R^1 T(F) \rightarrow R^1 T(A^0) \rightarrow \dots \\ \dots R^{i-1} T(A^0) \rightarrow R^{i-1} T(\operatorname{coker}(d_0)) \rightarrow R^i T(F) \rightarrow R^i T(A^0) \rightarrow \dots \end{aligned}$$

From T -acyclicity of A^0 one gets:

$$\begin{aligned} R^1 T(F) &= \operatorname{coker}(T(A^0) \rightarrow T(\operatorname{coker}(d_0))) \\ R^i T(F) &= R^{i-1} T(\operatorname{coker}(d_0)) \quad (i \geq 2). \end{aligned} \quad (10.8.2)$$

On the other hand applying T to the resolution $F \rightarrow A^\bullet$ gives the complex

$$0 \rightarrow T(F) \rightarrow T(A^0) \rightarrow T(A^1) \rightarrow T(A^2) \rightarrow \dots$$

which is exact at the first two places. Thus $H^1(T(A^\bullet)) = \frac{\ker(T(A^1) \rightarrow T(A^2))}{\operatorname{im}(T(A^0) \rightarrow T(A^1))}$. But the sequence

$$0 \rightarrow \operatorname{coker}(d_0) \rightarrow A^1 \rightarrow A^2$$

is also exact and hence

$$0 \rightarrow T(\operatorname{coker}(d_0)) \rightarrow T(A^1) \rightarrow T(A^2)$$

is exact and hence $\ker(T(A^1) \rightarrow T(A^2)) = \operatorname{im}(T(\operatorname{coker}(d_0) \hookrightarrow T(A^1)))$ and so

$$\begin{aligned} H^1(T(A^\bullet)) &= \frac{\operatorname{im}(T(\operatorname{coker}(d_0) \hookrightarrow T(A^1)))}{\operatorname{im}(T(A^0) \rightarrow T(A^1))} \\ &= \frac{T(\operatorname{coker}(d_0))}{\operatorname{im}(T(A^0) \rightarrow T(\operatorname{coker}(d_0)))} = R^1 T(F). \end{aligned} \quad (10.8.3)$$

Now $\operatorname{coker}(d_0)$ has the resolution by T -acyclic objects

$$0 \rightarrow \operatorname{coker}(d_0) \rightarrow A^1 \rightarrow A^2 \rightarrow \dots$$

Then for $i \geq 2$ (10.8.2) and the induction hypothesis give:

$$R^i T(F) = R^{i-1} T(\operatorname{coker}(d_0)) = H^{i-1}(T(A^1) \rightarrow T(A^2) \rightarrow \dots) = H^i(T(A^\bullet))$$

and we are done. □

10.9. Flasque sheaves, canonical flasque resolution. A sheaf is called **flasque** (or sometimes **flabby**) if the restriction maps $F(X) \rightarrow F(U)$ are surjective for all U (equivalently, all restrictions $F(V) \rightarrow F(U)$ for $U \subset V$ are surjective).

Proposition 10.9.1. *A flasque sheaf is Γ -acyclic.*

Proof. Let F be flasque. We can find a short-exact sequence

$$0 \rightarrow F \xrightarrow{\alpha} I \xrightarrow{\beta} G \rightarrow 0$$

where I is both flasque and injective. (Indeed, the proof of Lemma 10.6.2 showed this: the sheaf $\prod_x i_{x*} I_x$ in that proof is automatically flasque.) Suppose we show that

$$I(U) \xrightarrow{\beta} G(U) \text{ is surjective for all } U \subset X. \quad (10.9.1)$$

It would then follow that G is also flasque and we could use the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(X, F) \rightarrow H^0(X, I) \rightarrow H^0(X, G) \rightarrow H^1(X, F) \rightarrow H^1(X, I) \rightarrow \\ \dots \rightarrow H^{i-1}(X, I) \rightarrow H^{i-1}(X, G) \rightarrow H^i(X, F) \rightarrow H^i(X, I) \rightarrow \dots \end{aligned}$$

The surjectivity of $I(X) \rightarrow G(X)$ would show that $H^1(X, F) = 0$ and then the dimension shift $H^i(X, F) \cong H^{i-1}(X, G)$ plus flasqueness of G would allow an induction to show that $H^i(X, F) = 0$ for all $i > 1$.

Let us prove (10.9.1). Let $\sigma \in G(U)$. Suppose that $V, W \subset U$ are open and $\tau_V \in I(V)$ and $\tau_W \in I(W)$ map (under β) to $\rho_V^U(\sigma)$ and $\rho_W^U(\sigma)$ respectively. On the overlap $V \cap W$ we have

$$\rho_{V \cap W}^V(\tau_V) - \rho_{V \cap W}^W(\tau_W) \xrightarrow{\beta} \rho_{V \cap W}^V \rho_V^U(\sigma) - \rho_{V \cap W}^W \rho_W^U(\sigma) = 0$$

and hence there is $\eta_{V \cap W} \in F(V \cap W)$ such that

$$\eta_{V \cap W} \xrightarrow{\alpha} \rho_{V \cap W}^V(\tau_V) - \rho_{V \cap W}^W(\tau_W) \in I(V \cap W).$$

Since F is flasque there is an $\eta \in F(V)$ restricting to $\eta_{V \cap W}$ (i.e. with $\rho_{V \cap W}^V(\eta) = \eta_{V \cap W}$). Define

$$\tau'_V = \tau_V - \eta.$$

Then $\tau'_V \in I(V)$ and $\tau_W \in I(W)$ agree on $V \cap W$ and $\tau'_V \mapsto \rho_V^U(\sigma)$ and $\tau_W \mapsto \rho_W^U(\sigma)$. Thus there is a section $\tau \in I(V \cup W)$ which goes to $\rho_{V \cup W}^U(\sigma)$, i.e we have increased the domain on which we can lift a section $\sigma \in G(U)$. Now consider the set of pairs $(W, \tau \in I(W))$ where $\tau \mapsto \rho_W^U(\sigma)$, partially ordered according to

$$(W, \tau) \leq (W', \tau') \Leftrightarrow W \subset W' \text{ and } \rho_{W'}^{W'}(\tau') = \tau.$$

Since any increasing chain in this poset has a maximal element (by the sheaf axiom (S2)), the poset has a maximal element (Zorn's lemma). By the previous argument its domain must be all of U . In other words, there is a $\tau \in I(U)$ such that $\tau \mapsto \sigma$. This proves (10.9.1) and the lemma. ⁽⁵⁹⁾ \square

As a corollary we see that flasque resolutions compute sheaf cohomology. Looking back at the proof of the existence of injective resolutions we see that the same idea gives the **canonical flasque resolution**, usually called the **Godement resolution**: For a sheaf F let

$$C^0(F) := \prod_{x \in X} i_{x*} F_x$$

This is flasque and functorial in F . The natural morphism of sheaves $F \rightarrow C^0(F)$ is injective (look at stalks) and we have the cokernel sheaf $\text{coker}(F \rightarrow C^0(F))$. Define

$$C^1(F) := C^0(\text{coker}(F \rightarrow C^0(F))).$$

⁵⁹Note that we have really proved that if $0 \rightarrow F \rightarrow G \rightarrow Q \rightarrow 0$ is exact and F and G are flasque then $Q = G/F$ is flasque.

The map $C^0(F) \rightarrow C^1(F)$ is given by the composition

$$C^0(F) \rightarrow \text{coker}(F \rightarrow C^0(F)) \hookrightarrow C^0(\text{coker}(F \rightarrow C^0(F))) = C^1(F).$$

Now set

$$C^2(F) := C^0(\text{coker}(C^0(F) \rightarrow C^1(F)))$$

and $C^1(F) \rightarrow C^2(F)$ is induced by

$$C^1(F) \rightarrow \text{coker}(C^0(F) \rightarrow C^1(F)) \rightarrow C^0(\text{coker}(C^0(F) \rightarrow C^1(F))) = C^2(F).$$

Proceeding in this way gives a complex $C^\bullet(F)$ which is a resolution of F . Because all the terms in the resolution come by the $C^0(-)$ construction they are all flasque. ⁽⁶⁰⁾

We thus have a canonical resolution computing the cohomology of any sheaf. Moreover, as is obvious from the construction, it is functorial in the sheaf (i.e. in F). From this (and Prop. 10.8.1) we see that $F \mapsto R^i\Gamma(X, F) = H^i(X, F)$ is a functor. ⁽⁶¹⁾

10.10. Singular cohomology and sheaf cohomology. Let X be a topological space. For an abelian group A we will use $C^*(-, A)$ to denote the complex of singular cochains with values in A , i.e. $C^*(-, A) = \text{Hom}(C_*(-), A)$ where $C_*(-)$ is the singular chain complex. For notational simplicity we restrict to $A = \mathbb{Z}$, but everything below works the same for general A .

For each $i \geq 0$, the assignment

$$U \mapsto C^i(U, \mathbb{Z})$$

defines a presheaf of abelian groups on X . This presheaf is flasque and satisfies (S2) (i.e. gluing). It does not satisfy (S1).

Let \mathcal{C}^i be the sheafification of $U \mapsto C^i(U, \mathbb{Z})$. By functoriality of sheafification the coboundary maps $\delta : C^i(U, \mathbb{Z}) \rightarrow C^{i+1}(U, \mathbb{Z})$ give sheaf homomorphisms $\mathcal{C}^i \rightarrow \mathcal{C}^{i+1}$. The relation $\delta \circ \delta = 0$ means we have a complex of sheaves

$$0 \rightarrow \mathcal{C}^0 \rightarrow \mathcal{C}^1 \rightarrow \dots$$

There is an inclusion $\mathbb{Z}_X \hookrightarrow \mathcal{C}^0$ coming from the presheaf inclusion given by $\mathbb{Z} \hookrightarrow C^0(U, \mathbb{Z})$ for any U as the constant cochain, i.e. n goes to the cochain which takes the constant value n on every chain in U .

Lemma 10.10.1. *For a locally contractible space X this gives a resolution of the constant sheaf \mathbb{Z}_X , i.e. the sequence*

$$0 \rightarrow \mathbb{Z}_X \rightarrow \mathcal{C}^0 \rightarrow \mathcal{C}^1 \rightarrow \dots$$

is an exact sequence of sheaves.

Proof. Recall that the stalks of a presheaf and of the associated sheaf are the same. So it suffices to prove that each $x \in X$ has a sequence of neighbourhoods U for which the sequence

$$0 \rightarrow \mathbb{Z} \rightarrow C^0(U, \mathbb{Z}) \rightarrow C^1(U, \mathbb{Z}) \rightarrow \dots$$

is exact. This follows immediately from the local contractibility of X since this complex computes the reduced singular cohomology of U . \square

The following point-set fact is Proposition 1.14 of Ramanan's *Global Calculus*:

⁶⁰In the case where we work with sheaves of vector spaces over a fixed field, a flasque sheaf is injective. So in this case the Godement resolution is a canonical (and functorial) injective resolution. Thus all derived functors are actually functors. In general, there is no canonical and functorial injective resolution in an abelian category, and this can be a source of subtle problems.

⁶¹The existence of a canonical and functorial flasque resolution means that we could have *defined* the cohomology groups using it, i.e. we could have defined $H^i(X, F)$ by $H^i(X, F) := H^i(\Gamma(X, C^\bullet(F)))$. This would have the advantage of obviously being a functor and one can directly prove its properties. (This is the approach taken in some books, e.g. Chapter 3 of Wells, *Differential Analysis on Complex Manifolds*.)

Lemma 10.10.2. *If X is a paracompact Hausdorff space in which every open set is paracompact⁽⁶²⁾ and F is a presheaf on X satisfying (S2), then $F(U) \rightarrow \tilde{F}(U)$ for all open $U \subset X$.*

Proof. Let $U \subset X$ be open. Recall that a section of $\tilde{F}(U)$ is a collection of germs $(s_x)_{x \in U} \in \prod_{x \in U} F_x$ satisfying a continuity condition. So giving an element $s \in \tilde{F}(U)$ amounts to giving a cover $\{U_i\}_{i \in I}$ of U and sections $s_i \in F(U_i)$ such that for each $x \in U_i$, $s_i \mapsto s_x$ in $F_x = \tilde{F}_x$. We would like to show that under the assumptions of the lemma s comes from an element of $F(U)$. Since U is paracompact (by our assumption on X) we may assume that the cover $\{U_i\}$ is locally finite.

Suppose we have a covering $\{W_x\}$ of U by open subsets, indexed by $x \in U$, such that

- for all x , W_x is an open neighbourhood of x with the property that $W_x \subset U_i$ for some i
- if $W_x \cap W_y \neq \emptyset$ then $W_x \cup W_y \subset U_i$ for some i .

Indeed, if so then $s_i|_{W_x}$ defines a collection of sections of F which can evidently be glued to give an element $\sigma \in F(U)$ which has stalks $\sigma_x = s_x$ for all $x \in U$, hence $\sigma \mapsto s \in \tilde{F}(U)$. (There is a small point here: Given x , the U_i containing W_x may not be unique, so that if $W_x \subset U_j$ then we do not know whether to take $s_i|_{W_x}$ or $s_j|_{W_x}$. However, since s_i and s_j go to the same element s_x in the stalk at x , we can ensure this by shrinking W_x , and this does not affect the other properties of $\{W_x\}_{x \in U}$.)

To construct the covering W_x we proceed as follows. First choose a refinement $\{V_i\}_{i \in I}$ of $\{U_i\}_{i \in I}$ with the same indexing set and such that $\bar{V}_i \subset U_i$.⁽⁶³⁾ Since $\{V_i\}$ is also locally finite, we can choose for each $x \in U$ a neighbourhood W'_x such that $\{i \in I : W'_x \cap V_i \neq \emptyset\}$ is finite. Now define the following, which is open since it is a finite intersection of opens:

$$W_x := W'_x \cap \left(\bigcap_{W'_x \cap V_i \neq \emptyset, x \notin \bar{V}_i} U - \bar{V}_i \right) \cap \left(\bigcap_{W'_x \cap V_i \neq \emptyset} U_i \right).$$

The first intersection makes sure that if $W_x \cap V_i \neq \emptyset$ then $x \in \bar{V}_i$, while the second makes sure that if $W_x \cap V_i \neq \emptyset$ then $W_x \subset U_i$. Now if $W_x \cap W_y \neq \emptyset$, say $z \in W_x \cap W_y$, then there exists i such that $z \in V_i$, whence $z \in W_x \cap V_i \neq \emptyset$, so that $W_x \subset U_i$. The same argument applies to W_y , so $W_x \cup W_y \subset U_i$.

We are now in the situation we wanted. The covering $\{W_x\}_{x \in U}$ allows us to glue the sections $s_i|_{W_x} \in F(W_x)$ (which by the remark made earlier we may assume is independent of i) into a section in $F(U)$, which maps to s under $F(U) \rightarrow \tilde{F}(U)$. \square

The lemma implies that on such a space X , if a presheaf F satisfying (S2) is flasque (i.e. $F(X) \rightarrow F(U)$ for all U), then the associated sheaf \tilde{F} is flasque. This follows immediately from the commutative diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & \tilde{F}(X) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & \tilde{F}(U) \end{array} \quad (10.10.1)$$

In particular, on such a space X , the sheaves \mathcal{C}^i are flasque. In particular, this holds for smooth manifolds (since any open set is again a smooth manifold).

⁶²It is a fact, which we do not use here, that if X is a topological space such that every open subset is paracompact, then in fact every subspace of X is paracompact. (Closed subsets of paracompact spaces are automatically paracompact.) Such a space is called *hereditarily paracompact*.

⁶³This is a general property of paracompact Hausdorff spaces, which can be proved e.g. using the fact that they are normal, and on normal spaces using Urysohn's lemma we can find such refinements. In the situation of manifolds we proved this directly in the course of constructing smooth partitions of unity.

Theorem 10.10.3. *If X is a locally contractible and paracompact Hausdorff space in which any open set is paracompact, then for any abelian group A there are canonical isomorphisms*

$$H_{sing}^*(X, A) \cong H^*(X, A_X)$$

between the singular cohomology groups of X with coefficients in A and the sheaf cohomology groups of the constant sheaf A_X .

Proof. We will assume $A = \mathbb{Z}$, the proof is the same in general. Since $\mathbb{Z}_X \rightarrow \mathcal{C}^\bullet$ is a flasque resolution we know that

$$H^*(X, \mathbb{Z}_X) \cong H^*(\Gamma(X, \mathcal{C}^\bullet)).$$

Thus it will suffice to show that

$$C^\bullet(X, \mathbb{Z}) \rightarrow \Gamma(X, \mathcal{C}^\bullet)$$

induces an isomorphism in cohomology. Let

$$C^i(X, \mathbb{Z})_0 := \{\sigma \in \Gamma(X, \mathcal{C}^i) : \sigma \mapsto 0 \in \mathcal{C}_x^i \text{ for all } x \in X\}.$$

We first show that the following sequence is short exact:

$$0 \rightarrow C^i(X, \mathbb{Z})_0 \rightarrow C^i(X, \mathbb{Z}) \rightarrow \Gamma(X, \mathcal{C}^i) \rightarrow 0$$

The surjectivity on the right comes from the previous lemma. That $C^i(X, \mathbb{Z})_0 \subset \ker(C^i(X, \mathbb{Z}) \rightarrow \Gamma(X, \mathcal{C}^i))$ is clear. Equality follows from the fact that $\Gamma(X, \mathcal{C}^i) \hookrightarrow \prod_x \mathcal{C}_x^i$ by the sheaf axiom (S1).

Thus it is enough to prove that the complex $C^\bullet(X, \mathbb{Z})_0$ is acyclic, i.e. has zero cohomology. This will be deduced from the theorem of small chains. Recall the statement: If X is a topological space, $\mathcal{U} = \{U_\alpha\}_\alpha$ is an open cover of X and $C_\bullet^\mathcal{U}(X)$ denotes the complex of singular chains generated by singular simplices with image contained in some element of \mathcal{U} , then

$$C_\bullet^\mathcal{U}(X) \hookrightarrow C_\bullet(X)$$

is a chain homotopy equivalence. From the universal coefficient theorem we get short-exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ext^1(H_{i-1}(X, \mathbb{Z}), \mathbb{Z}) & \longrightarrow & H^i(X, \mathbb{Z}) & \longrightarrow & Hom(H_i(X, \mathbb{Z}), \mathbb{Z}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \pi_\mathcal{U}^* & & \downarrow \\ 0 & \longrightarrow & Ext^1(H_{i-1}^\mathcal{U}(X, \mathbb{Z}), \mathbb{Z}) & \longrightarrow & H_\mathcal{U}^i(X, \mathbb{Z}) & \longrightarrow & Hom(H_i^\mathcal{U}(X, \mathbb{Z}), \mathbb{Z}) \longrightarrow 0 \end{array}$$

where the maps are induced by $C_\bullet^\mathcal{U}(X) \subset C_\bullet(X)$ or its dual. (The upper sequence is the usual one; the lower sequence comes from exactly the same homological algebra applied to the complexes $C_\bullet^\mathcal{U}(X)$ and $Hom(C_\bullet^\mathcal{U}(X), \mathbb{Z})$.) Hence we get the dual assertion that the morphism

$$\pi_\mathcal{U} : C^\bullet(X, \mathbb{Z}) \rightarrow C_\mathcal{U}^\bullet(X, \mathbb{Z})$$

induces an isomorphism in cohomology and hence $\ker(\pi_\mathcal{U})$ is acyclic. Now $\ker(\pi_\mathcal{U})$ consists precisely of cochains which map to zero in $C^\bullet(U_\alpha, \mathbb{Z})$ for every α , so

$$C^\bullet(X, \mathbb{Z})_0 = \varinjlim_\mathcal{U} \ker(\pi_\mathcal{U})$$

where the limit is over all coverings of X . Since taking cohomology commutes with direct limits we conclude that $C^\bullet(X, \mathbb{Z})_0$ is acyclic. \square

Corollary 10.10.4. *If M is a smooth manifold then $H^*(M, \mathbb{Z}_M) \cong H_{sing}^*(M, \mathbb{Z})$.*

Remark 10.10.5. In fact it is true for any locally contractible space that singular cohomology and sheaf cohomology of the constant sheaf agree, i.e. the assumption that every open in X is paracompact can be dropped in Theorem 10.10.3. A correct proof of this is surprisingly recent, see Y. Sella, arXiv:1602.06674v3. Sella also gives an example where Lemma 10.10.2 fails without some hypothesis, i.e. the sheaves \mathcal{C}^k are not flasque.

Remark 10.10.6. An important example to which the previous theorem applies is that of a possibly singular complex algebraic variety. This requires some justification, which we do not give here.

10.11. Fine sheaves and de Rham's theorem. We would like to relate de Rham cohomology to sheaf cohomology, just as was done above for singular cohomology. The de Rham sheaves \mathcal{E}_M^* are certainly not flasque. Nevertheless, they are acyclic for the global sections functor. The key property that ensures this is that $\mathcal{E}^0(M)$ contains partitions of unity. Abstracting this we say that a sheaf R of (commutative) rings with unity⁽⁶⁴⁾ on a space X **admits partitions of unity** if for every locally finite covering $\{U_\alpha\}_\alpha$ of X there exists a partition of unity $\{f_\alpha\}$ subordinate to it, i.e. elements $f_\alpha \in R(X)$ with $\text{supp}(f_\alpha) \subset U_\alpha$ such that

$$\sum_{\alpha} f_{\alpha} = 1.$$

(This equation is to be interpreted as making sense locally: In a neighbourhood of any point it is a finite sum and everything makes sense.) Our main example is the sheaf of smooth functions \mathcal{E}^0 on a smooth manifold, which admits partitions of unity. The sheaf of holomorphic functions on any complex manifold is an example of a sheaf of rings with unity which does not admit partitions of unity.

A **fine sheaf** is a sheaf of modules over a sheaf of rings R which admits partitions of unity.⁽⁶⁵⁾ ⁽⁶⁶⁾ ⁽⁶⁷⁾ The main example is \mathcal{E}^k for $k \geq 0$. More generally, the sheaf of sections of any smooth vector bundle is a sheaf of \mathcal{E}^0 -modules, hence it is fine.

Proposition 10.11.1. *A fine sheaf on a paracompact space is Γ -acyclic.*

Proof. Let F be the fine sheaf, which by definition is a sheaf of modules over a sheaf of rings R . We will use the Godement resolution $F \rightarrow C^\bullet = C^\bullet(F)$ (the canonical flasque resolution). Note that the differentials $D : C^i \rightarrow C^{i+1}$ are, by construction, R -module homomorphisms. By the Γ -acyclicity of flasque sheaves we have

$$H^i(X, F) = \frac{\ker(\Gamma(X, C^i) \rightarrow \Gamma(X, C^{i+1}))}{\text{im}(\Gamma(X, C^{i-1}) \rightarrow \Gamma(X, C^i))}.$$

Suppose $\sigma \in \Gamma(X, C^i)$ is a cocycle with $i > 0$. By exactness of the Godement resolution this means that there is a covering $\{U_\alpha\}$ and elements $\sigma_\alpha \in C^{i-1}$ such that

$$\sigma|_{U_\alpha} = D\sigma_\alpha.$$

By refining the covering we may assume (since X is paracompact) that it is locally finite. Let $\{f_\alpha\}$ be a partition of unity for R subordinate to the covering. Let

$$\tilde{\sigma} := \sum_{\alpha} f_{\alpha} \sigma_{\alpha}.$$

Here $f_\alpha \sigma_\alpha$ is extended by zero outside U_α , so this defines a section in $\Gamma(X, C^{i-1})$. Then

$$D\tilde{\sigma} = \sum_{\alpha} D(f_{\alpha} \sigma_{\alpha}) = \sum_{\alpha} f_{\alpha} \sigma|_{U_{\alpha}} = \sigma.$$

This proves that $H^i(X, F) = 0$ for $i > 0$. □

We have the following important corollary:

Corollary 10.11.2. *The de Rham cohomology groups of a smooth manifold M are naturally isomorphic to the sheaf cohomology of the constant sheaf \mathbb{R}_M .*

Combining this with the identification of singular cohomology with sheaf cohomology we get:

⁶⁴A sheaf of rings has the obvious meaning: a contravariant functor $R : \text{Op}(X) \rightarrow \text{Rings}$ satisfying the sheaf conditions. We assume that there is a section $1 \in R(X)$ which restricts to the unit in $R(U)$ for every U .

⁶⁵A sheaf of modules F over a sheaf of rings R means: Each $\Gamma(U, F)$ is a module over $\Gamma(U, R)$ and the module multiplication maps $\Gamma(U, R) \otimes \Gamma(U, F) \rightarrow \Gamma(U, F)$ are compatible with restrictions from U to $V \subset U$.

⁶⁶A more natural definition of fine would be the following: F is fine if the sheaf of rings $\underline{\text{Hom}}(F, F)$ admits partitions of unity. Everything here works with this definition also.

⁶⁷There are several different definitions of fine sheaf in the literature; the one we use here is the easiest for us.

Theorem 10.11.3 (de Rham). *There is a canonical isomorphism*

$$H_{dR}^*(M) \cong H_{sing}^*(M, \mathbb{R}) = H_{sing}^*(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

between the de Rham cohomology and the singular cohomology with \mathbb{R} -coefficients.

The isomorphism in the theorem is canonical.⁽⁶⁸⁾ The isomorphism is also functorial for smooth maps $f : M \rightarrow N$ and (it follows that) it is a ring isomorphism when $H_{sing}^*(M, \mathbb{Z})$ is given the cup product and $H_{dR}^*(M)$ is given the product given by exterior product of differential forms. These facts will be proved later.

10.12. De Rham's theorem and integration over smooth chains. We will discuss a refinement of the de Rham theorem without giving full details, for which we refer to e.g. Warner's book.

Let M be a smooth manifold. A **smooth k -simplex** is a smooth map from a neighbourhood of the standard k -simplex Δ^k in \mathbb{R}^n to the manifold M . A smooth k -chain is a (finite) formal linear sum with integer coefficients of smooth k -simplices. The boundary of a smooth k -chain is a smooth $k - 1$ -chain and we have the complex of smooth chains:

$$0 \rightarrow C_0^{sm}(M) \rightarrow C_1^{sm}(M) \rightarrow \dots$$

Taking $Hom(-, \mathbb{Z})$ we get the complex of smooth cochains:

$$0 \rightarrow C_{sm}^0(M, \mathbb{Z}) \rightarrow C_{sm}^1(M, \mathbb{Z}) \rightarrow C_{sm}^2(M, \mathbb{Z}) \rightarrow \dots$$

The definition can be sheafified: Let \mathcal{C}_{sm}^k be the sheaf associated with the presheaf $U \mapsto C_{sm}^k(U, \mathbb{Z})$. Then we have a complex of sheaves on M

$$0 \rightarrow \mathcal{C}_{sm}^0 \rightarrow \mathcal{C}_{sm}^1 \rightarrow \dots$$

and an inclusion $\mathbb{Z}_M \hookrightarrow \mathcal{C}_{sm}^0$ as before. To show that this is a resolution of the constant sheaf we must prove that it is exact on stalks in degrees ≥ 1 . This requires some argument which I will not give here, referring instead to pp. 194-195 of Warner's book.

It follows from this that \mathcal{C}_{sm}^\bullet is a resolution of the constant sheaf \mathbb{Z}_M . The sheaves \mathcal{C}_{sm}^k are flasque by exactly the same argument as before, and hence we have an isomorphism

$$H^k(M, \mathbb{Z}_M) \cong H^k(C_{sm}^\bullet(M, \mathbb{Z})).$$

As a corollary we also see that the surjection $C^\bullet(M, \mathbb{Z}) \rightarrow C_{sm}^\bullet(M, \mathbb{Z})$ induces an isomorphism in cohomology, i.e. smooth singular cochains compute cohomology.

Now suppose σ is a smooth k -simplex in $U \subset M$, i.e. a smooth map $\sigma : nbhd(\Delta^k) \rightarrow U$. If $\omega \in \mathcal{E}^k(U)$ then $\sigma^*(\omega)$ can be integrated over Δ^k . There is a minor point here: The k -simplex for $k > 1$ is not quite a domain in \mathbb{R}^n since its boundary is not a manifold at the vertex points. Nevertheless, we can integrate over it, as you can easily check by going over the arguments defining integration over domains. Moreover a slight generalization of Stokes's theorem to this situation says that for a smooth k -chain σ and smooth k -form ω we have:

$$\int_{\sigma} d\omega = \int_{\partial\sigma} \omega|_{\partial\sigma}.$$

Thus integration defines a homomorphism of complexes of sheaves

$$\mathcal{E}^\bullet \rightarrow \mathcal{C}_{sm}^\bullet.$$

It is easy to check that this induces the de Rham isomorphism (indeed, this follows by the footnote after Theorem 10.11.3), implying the following:

⁶⁸In general if T is a left-exact functor, given two different T -acyclic resolutions $F \rightarrow A^\bullet$ and $F \rightarrow B^\bullet$ there is a canonical isomorphism between $H^*(T(A^\bullet))$ and $H^*(T(B^\bullet))$: Choose an injective resolution $F \rightarrow I^\bullet$; by Lemma 10.6.3 there are morphisms of resolutions $A^\bullet \rightarrow I^\bullet \leftarrow B^\bullet$. By Prop. 10.8.1 these induce canonical isomorphisms $H^i(T(A^\bullet)) \rightarrow H^i(T(I^\bullet)) \leftarrow H^i(T(B^\bullet))$. The isomorphism $H^i(T(A^\bullet)) \cong H^i(T(B^\bullet))$ is independent of the choice of I^\bullet , again by using Lemma 10.6.3.

Moreover, if A^\bullet and B^\bullet are acyclic resolutions and $A^\bullet \rightarrow B^\bullet$ lifts the identity $F \rightarrow F$ then it induces the same isomorphism as the previous one.

Theorem 10.12.1 (Explicit de Rham theorem). *Integration over smooth chains defines a non-degenerate pairing*

$$H_{dR}^i(M) \otimes H_i^{sing}(M, \mathbb{R}) \rightarrow \mathbb{R}$$

which induces the de Rham isomorphism $H_{dR}^i(M) \cong H_{sing}^i(M, \mathbb{R})$.

The explicit version of de Rham's theorem shows that the de Rham isomorphism is functorial in M . Let $f : M \rightarrow N$ be a smooth map of smooth manifolds. If $\sigma : nbhd(\Delta^k) \rightarrow M$ is a smooth k -simplex in M and $\omega \in \mathcal{E}^k(N)$ then

$$\int_{\sigma} f^* \omega = \int_{f \circ \sigma} \omega.$$

This shows that the isomorphism $H_{dR}^k(M) \rightarrow H_{sing}^k(M, \mathbb{R})$ induced by the integration pairing is functorial. In particular, compatibility with cup products follows using the description of cup product via pullback along the diagonal $M \subset M \times M$.

11. SHEAVES AND COHOMOLOGY II

In this section we introduce hypercohomology, which generalizes cohomology to complexes of sheaves. The necessary formalism is a little cumbersome but we will use it to establish that the de Rham isomorphism is functorial and one of rings. We describe applications to the homology of fibrations and to the Thom isomorphism and discuss some important examples of these. The presentation mostly follows Voisin’s book.

In this section we fix an abelian category \mathfrak{A} , which is always either $Sh(X)$ or $AbGps$. The word “object” will usually mean object of \mathfrak{A} and “homomorphism” will mean homomorphism in \mathfrak{A} , unless explicitly stated otherwise. We will consider complexes of objects in \mathfrak{A} . All complexes K^\bullet considered below will be in nonnegative degrees, i.e. $K^i = 0$ for $i < 0$. (The proofs work with trivial changes for all complexes bounded below, i.e. if there is a r such that $K^i = 0$ for $i < r$.)

11.1. Cohomology objects, quasi-isomorphisms. Let F^\bullet be a complex of objects of \mathfrak{A} . The i th **cohomology object** of F^\bullet is:

$$H^i(F^\bullet) = \frac{\ker(F^i \rightarrow F^{i+1})}{\text{im}(F^{i-1} \rightarrow F^i)}.$$

(Here if $\mathfrak{A} = Sh(X)$ then the image and quotient are taken in the sheaf sense.) A homomorphism of complexes of sheaves $F^\bullet \rightarrow G^\bullet$ is a **quasi-isomorphism** if the induced homomorphism on cohomology objects $H^i(F^\bullet) \rightarrow H^i(G^\bullet)$ is an isomorphism for all i .

Example 11.1.1. On a manifold M the inclusion $\mathbb{R}_M \rightarrow \mathcal{E}_M^\bullet$ is a quasi-isomorphism.

Example 11.1.2. More generally, if $F \hookrightarrow A^\bullet$ is a resolution then $F \rightarrow A^\bullet$ is a quasi-isomorphism, where we think of F as the complex $0 \rightarrow F \rightarrow 0$ with F in degree 0.

A **double complex** $K^{\bullet,\bullet}$ (in \mathfrak{A}) is a collection of objects $(K^{p,q})_{(p,q) \in \mathbb{Z}^2}$ (in \mathfrak{A}) with homomorphisms $D_2 : K^{p,q} \rightarrow K^{p,q+1}$ (the vertical differential) and $D_1 : K^{p,q} \rightarrow K^{p+1,q}$ (the horizontal differential) such that

$$D_1^2 = 0 = D_2^2 \quad \text{and} \quad D_1 D_2 = D_2 D_1.$$

One should keep in mind the following picture:

$$\begin{array}{ccccc} & & \uparrow & & \uparrow \\ & & K^{p,q+1} & \xrightarrow{D_1} & K^{p+1,q+1} & \longrightarrow \\ & & \uparrow D_2 & & \uparrow D_2 \\ & & K^{p,q} & \xrightarrow{D_1} & K^{p+1,q} & \longrightarrow \\ & & \uparrow & & \uparrow \end{array}$$

The **total complex** of $K^{\bullet,\bullet}$ is the associated single complex $Tot(K^{\bullet,\bullet})^\bullet$ with

$$Tot(K^{\bullet,\bullet})^i = \bigoplus_{p+q=i} K^{p,q}$$

and differential

$$D|_{K^{p,q}} = D_1 + (-1)^p D_2$$

(The sign is put so that $D^2 = 0$.)

An example of this construction is the **tensor product** of complexes: If K^\bullet and L^\bullet are complexes then the tensor product complex is defined to be the total complex

$$Tot(K^\bullet \otimes L^\bullet)^i := \bigoplus_{p+q=i} K^p \otimes L^q$$

with the total differential $D_K \otimes id + (-1)^p id \otimes D_L$. Note that there is a map

$$H^p(K^\bullet) \otimes H^q(L^\bullet) \longrightarrow H^{p+q}(Tot(K^\bullet \otimes L^\bullet))$$

of cohomology objects. We will frequently drop the “Tot” from the notation when we are dealing with tensor products of complexes; this will not usually cause any confusion.

11.2. Complexes of injectives. In the case F is a single object taking an injective resolution $F \hookrightarrow I^\bullet$ means we can replace F by a quasi-isomorphic complex of injective objects. We will do this for any complex F^\bullet in this subsection (cf. Proposition 11.2.3).

Lemma 11.2.1. *Let F^\bullet be a complex bounded below (say $F^k = 0$ for $k < 0$). There is a double complex $(I^{\bullet,\bullet}, D_1, D_2)$ consisting of injective objects and a monomorphism of complexes $F^\bullet \hookrightarrow I^{\bullet,0}$*

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & I^{0,2} & \longrightarrow & I^{1,2} & \longrightarrow & I^{2,2} & \longrightarrow \dots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & I^{0,1} & \xrightarrow{D_1} & I^{1,1} & \longrightarrow & I^{2,1} & \longrightarrow \dots \\
 & \uparrow D_2 & & \uparrow D_2 & & \uparrow & \\
 & I^{0,0} & \xrightarrow{D_1} & I^{1,0} & \longrightarrow & I^{2,0} & \longrightarrow \dots \\
 i^0 \uparrow & & & i^1 \uparrow & & i^2 \uparrow & \\
 F^0 & \xrightarrow{d_F} & F^1 & \xrightarrow{d_F} & F^2 & \xrightarrow{d_F} & \dots
 \end{array}$$

such that:

- (1) each column is exact (i.e. $I^{k,\bullet}$ is an injective resolution of F^k)
- (2) the inclusion $F^\bullet \hookrightarrow I^{\bullet,0}$ is a homomorphism of complexes.

Proof. It is enough to prove the following: A complex F^\bullet admits a monomorphism to a complex of injectives J^\bullet . Indeed, if so then we can construct the first row $I^{\bullet,0}$ of the double complex with the inclusion $F^\bullet \hookrightarrow I^{\bullet,0}$. Then we can embed the cokernel $\text{coker}(F^\bullet \rightarrow I^{\bullet,0})$ in another complex of injectives which will be the row $I^{\bullet,1}$ and proceed from there.

So let us construct the first row. We start with $i^0 : F^0 \hookrightarrow I^{0,0}$ a monomorphism to an injective object. Next consider

$$(i^0, -d_F) : F^0 \hookrightarrow I^{0,0} \oplus F^1.$$

Choose a monomorphism

$$\text{coker}(i^0, -d_F) \hookrightarrow I^{1,0}$$

to an injective object. Then the homomorphisms D_1 and i^1 are given by:

$$\begin{array}{ccc}
 I^{0,0} & & \\
 \downarrow & & \\
 I^{0,0} \oplus F^1 & \longrightarrow & \text{coker}(i^0, -d_F) \hookrightarrow I^{1,0} \\
 \uparrow & & \\
 F^1 & &
 \end{array}$$

The identity $D_1 \circ i^0 = i^1 \circ d_F$ holds because $i^0(f)$ and $d_F(f)$ are the same in the module $\text{coker}(i^0, -d_F)$. The injectivity of $F^1 \rightarrow I^{1,0}$ is easy.

Now to continue we take

$$(\bar{i}^1, -d_F) : F^1 \rightarrow \text{coker}(D_1) \oplus F^2$$

where \bar{i}^1 is the composition $F^1 \rightarrow I^{1,0} \rightarrow \text{coker}(D_1)$ and choose a monomorphism

$$\text{coker}(\bar{i}^1, -d_F) \hookrightarrow I^{2,0}.$$

The maps $D_1 : I^{1,0} \rightarrow I^{2,0}$ and $i^2 : F^2 \rightarrow I^{2,0}$ are given by:

$$\begin{array}{ccc}
 I^{1,0} & & \\
 \downarrow & & \\
 I^{1,0} \oplus F^2 & \longrightarrow & \operatorname{coker}(\bar{i}^1, -d_F) \hookrightarrow I^{2,0} \\
 \uparrow & & \\
 F^2 & &
 \end{array}$$

As before we see that $D_1 \circ i^1 = i^2 \circ d_F$ by construction, and i^2 is injective. Also $D_1^2 = 0$ by construction.

Proceeding in this fashion constructs the first row. □

Lemma 11.2.2. *Suppose that $I^{\bullet,\bullet}$ is a double complex of objects in \mathfrak{A} and $F^\bullet \hookrightarrow I^{\bullet,0}$ is a monomorphism of complexes such that $F^k \hookrightarrow I^{k,\bullet}$ is a resolution of F^k for each k . Then $F^\bullet \hookrightarrow \operatorname{Tot}(I^{\bullet,\bullet})^\bullet$ is a quasi-isomorphism.*

Proof. We give the proof in $\mathfrak{A} = \operatorname{AbGps}$; the same proof works in general with some rephrasing (one cannot work with elements).

First let us see that $H^k(F^\bullet) \twoheadrightarrow H^k(\operatorname{Tot}(I)^\bullet)$ for all k . A class in $H^k(\operatorname{Tot}(I)^\bullet)$ is represented by $\alpha = \sum_{p+q=k} \alpha_{p,q} \in \bigoplus_{p+q=k} I^{p,q}$ with $D\alpha = 0$. Then $D_2(\alpha_{0,k}) = 0$ and by exactness of the first column $\alpha_{0,k} = D_2(\beta)$ for $\beta \in I_{0,k-1}$. Then $\alpha' = \alpha - D\beta$ represents the same class as α but $\alpha'_{0,k} = 0$. But then $D_2(\alpha'_{k-1,0}) = 0$ and we can run the same argument to produce α'' in the same class as α and with $\alpha''_{1,k-1} = 0$. Proceeding in this fashion we can assume $\alpha \in I^{k,0}$. Then $D\alpha = 0$ implies $D_2\alpha = D_1\alpha = 0$. Hence $\alpha = i^k(f)$ for $f \in F^k$, and $i^{k+1}(d_F(f)) = D_1 i^k(f) = D_1\alpha = 0$. Then the class of f maps to the class of α . This proves surjectivity.

Next let us see that $H^k(F^\bullet) \rightarrow H^k(\operatorname{Tot}(I)^\bullet)$ is injective for all k . If $k = 0$ then this follows from the injectivity of i^0 . Let $k > 0$ and let $f \in F^k$ with $d_F(f) = 0$ and suppose that $i^k(f) = D\beta$ for some $\beta \in \operatorname{Tot}(I)^{k-1}$. Then $\beta \in I^{k-1,0}$ and hence $D\beta = i^k(f)$ implies that $D_2(\beta) = 0$ and $D_1\beta = i^k(f)$. Since the columns are exact there is a $e \in F^{k-1}$ with $i^{k-1}(e) = \beta$. Then

$$i^k d_F(e) = D_1 i^{k-1}(e) = D_1\beta = i^k(f).$$

Since i^k is injective we have $d_F(e) = f$. This proves the injectivity. □

Note that the objects in the double complex here were not required to be injective in this lemma.

Combining the two lemmas we get the first part of the following proposition:

Proposition 11.2.3. (1) *If F^\bullet is a bounded below complex there is a monomorphism $F^\bullet \hookrightarrow I^\bullet$ to a complex of injective objects which is a quasi-isomorphism.*

(2) *If $F^\bullet \hookrightarrow I^\bullet$ is as in (1) and $F^\bullet \rightarrow J^\bullet$ is any other quasi-isomorphism with J^\bullet a complex of injectives, then there exists a morphism of complexes $\phi : I^\bullet \rightarrow J^\bullet$ making the diagram*

$$\begin{array}{ccc}
 F^\bullet & \xrightarrow{id} & F^\bullet \\
 \downarrow & & \downarrow \\
 I^\bullet & \xrightarrow{\phi} & J^\bullet
 \end{array} \text{ commute.}$$

Proof. (2) Exercise. □

11.3. Derived functors and hypercohomology. For a left-exact functor T define the *i th (hyper)derived functor* on F^\bullet as follows: Choose a monomorphic quasi-isomorphism $F^\bullet \hookrightarrow I^\bullet$ to a complex of injectives and set:

$$R^i T(F^\bullet) := H^i(T(I^\bullet)).$$

In the special case where T is the global sections functor these are called the **hypercohomology groups of F^\bullet** and are denoted:

$$\mathbb{H}^i(X, F^\bullet) = R^i \Gamma(X, F^\bullet) = H^i(\Gamma(X, I^\bullet)).$$

As in the case of derived functors these depend, a priori, on the choice of $F^\bullet \hookrightarrow I^\bullet$. The second part of the previous lemma can be used to show that they are well-defined up to canonical isomorphism and that they have the same functorial property as derived functors.

Thus if F^\bullet consists of a single sheaf F placed in degree zero, then

$$R^i T(F^\bullet) = R^i T(F) \quad \text{and} \quad \mathbb{H}^i(X, F^\bullet) = H^i(X, F).$$

If I^\bullet is a complex of injectives then

$$R^i T(I^\bullet) = H^i(T(I^\bullet)) \quad \text{and} \quad \mathbb{H}^i(X, I^\bullet) = H^i(\Gamma(X, I^\bullet)).$$

In particular, an exact complex of injectives has zero higher ($i > 0$) derived functors.

The proof of the following proposition will be given later:

Proposition 11.3.1. *If $F^\bullet \rightarrow J^\bullet$ is a quasi-isomorphism (but not necessarily a monomorphism) with a complex of injective objects and T is a left-exact functor then it induces canonical isomorphisms*

$$R^i T(F^\bullet) \rightarrow R^i T(J^\bullet) = H^i(T(J^\bullet))$$

for all i .

We have the following immediate corollary:

Corollary 11.3.2. *If $F^\bullet \rightarrow K^\bullet$ is a quasi-isomorphism and T is a left-exact functor then it induces canonical isomorphisms $R^i T(F^\bullet) \rightarrow R^i T(K^\bullet)$ for all i .*

Proof. Suppose $\phi : K^\bullet \rightarrow L^\bullet$ is a quasi-isomorphism of complexes. Choose a monomorphism $i : L^\bullet \hookrightarrow I^\bullet$ which is a quasi-isomorphism to a complex of injectives. Then $i \circ \phi : K^\bullet \rightarrow I^\bullet$ is a quasi-isomorphism to a complex of injectives. By Prop. 11.3.1 there are canonical isomorphisms

$$R^i T(K^\bullet) = H^i(T(I^\bullet)) = R^i T(L^\bullet).$$

This proves the corollary. □

The proof of the following proposition (which generalizes Prop. 10.8.1) will be given later:

Proposition 11.3.3. *If T is a left-exact functor and $F^\bullet \rightarrow A^\bullet$ is a quasi-isomorphism to a complex of T -acyclic objects then it induces canonical isomorphisms*

$$R^i T(F^\bullet) \rightarrow R^i T(A^\bullet) = H^i(T(A^\bullet))$$

for all i .

For example, applying the corollary to the quasi-isomorphism $\mathbb{R}_M \rightarrow \mathcal{E}_M^\bullet$ on a smooth manifold we get:

$$H^i(M, \mathbb{R}_M) \cong \mathbb{H}^i(M, \mathcal{E}_M^\bullet) = H^i(\Gamma(M, \mathcal{E}_M^\bullet))$$

where we have also used the second proposition. This gives the isomorphism between de Rham cohomology and sheaf cohomology of \mathbb{R}_M .

Suppose F^\bullet is a complex of sheaves. The Godement resolution is functorial hence we get a double complex $C^\bullet(F^\bullet)$ in which each column $C^\bullet(F^k)$ is a resolution of F^k . By Lemma 11.2.2 we conclude that $F^\bullet \hookrightarrow \text{Tot}(C^\bullet(F^\bullet))^\bullet$ is a quasi-isomorphism. Since the terms in the Godement resolutions are flasque we get a canonical isomorphism:

$$\mathbb{H}^i(X, F^\bullet) = \mathbb{H}^i(X, \text{Tot}(C^\bullet(F^\bullet))^\bullet).$$

As in the case of sheaf cohomology, we see that the functoriality of Godement resolutions shows that hypercohomology is actually a functor from complexes of sheaves to abelian groups.

11.4. Pullback maps in (hyper)cohomology. First let $f : Y \rightarrow X$ be a continuous map of topological spaces and F a sheaf on X . Then there is an obvious map ⁽⁶⁹⁾

$$\Gamma(X, F) \rightarrow \Gamma(Y, f^{-1}F).$$

For each $i \geq 0$ there is a canonical pullback map

$$H^i(X, F) \rightarrow H^i(Y, f^{-1}F) \quad (11.4.1)$$

which is the previous map when $i = 0$. This is defined as follows: Choose injective resolutions $F \hookrightarrow I^\bullet$ and $f^{-1}F \hookrightarrow f^{-1}I^\bullet$. Then $f^{-1}I^\bullet$ is a resolution of $f^{-1}F$ (because f^{-1} is exact) and hence there is a homomorphism of complexes $f^{-1}I^\bullet \rightarrow J^\bullet$, unique up to homotopy. Applying Γ we have homomorphisms of complexes

$$\Gamma(X, I^\bullet) \rightarrow \Gamma(Y, f^{-1}I^\bullet) \rightarrow \Gamma(Y, J^\bullet)$$

which induce the desired homomorphisms (11.4.1) in cohomology. It is straightforward to check that these are independent of the choices made.

Now let F^\bullet be a complex of sheaves on X . Then there is a canonical pullback map

$$\mathbb{H}^i(X, F^\bullet) \rightarrow \mathbb{H}^i(Y, f^{-1}F^\bullet) \quad (11.4.2)$$

defined as follows: Choose injective resolutions $F^\bullet \hookrightarrow I^\bullet$ and $f^{-1}F^\bullet \rightarrow J^\bullet$ as in (1) of Proposition 11.2.3. Since f^{-1} is exact $f^{-1}F^\bullet \hookrightarrow f^{-1}I^\bullet$ is also a quasi-isomorphism ⁽⁷⁰⁾, so by (2) of the same lemma there exists a homomorphism $f^{-1}I^\bullet \rightarrow J^\bullet$, unique up to homotopy. Thus the maps on global sections give maps

$$\Gamma(X, I^\bullet) \rightarrow \Gamma(Y, f^{-1}I^\bullet) \rightarrow \Gamma(Y, J^\bullet)$$

and hence a canonical map

$$\mathbb{H}^i(X, F^\bullet) = H^i(\Gamma(X, I^\bullet)) \rightarrow H^i(\Gamma(Y, J^\bullet)) = \mathbb{H}^i(Y, f^{-1}F^\bullet).$$

It is easy to check that this pullback map is independent of choices and is natural in homomorphisms of complexes of sheaves $F^\bullet \rightarrow G^\bullet$. It obviously reduces to (11.4.1) in case F^\bullet and G^\bullet are single sheaves in degree zero.

In some cases this map is a familiar one: For example, if $f : Y \rightarrow X$ is a continuous map of topological spaces then $f^{-1}\mathbb{Z}_X \cong \mathbb{Z}_Y$. Thus there is a pullback map

$$H^i(X, \mathbb{Z}_X) \rightarrow H^i(Y, \mathbb{Z}_Y).$$

If X and Y are locally contractible then under the isomorphisms $H^i(-, \mathbb{Z}_X) \cong H^i_{\text{sing}}(-, \mathbb{Z})$ this is the usual pullback map f^* in singular cohomology. (Exercise.)

11.5. Composition of functors. Let

$$\mathfrak{A} \xrightarrow{T} \mathfrak{B} \xrightarrow{U} \mathfrak{C}$$

be left-exact functors between abelian categories. Consider the three conditions:

- (1) T carries injective objects to injective objects
- (2) T carries injective objects to U -acyclic objects
- (3) T carries $U \circ T$ -acyclic objects to U -acyclic objects.

⁶⁹Indeed, a section of $f^{-1}F$ over Y consists of a covering of Y and for each U in the covering an element of $\lim_{\rightarrow V \text{ open, } V \supset f(U)} \Gamma(V, F)$. An element of $\Gamma(X, F)$ gives such a section trivially.

⁷⁰An exact functor takes quasi-isomorphisms to quasi-isomorphisms (see Lemma 11.6.2 below).

Let $F \rightarrow A^\bullet$ be a resolution by injectives. Then if either (1) or (2) hold, we know that

$$R^k(U \circ T)(F) = H^k(U \circ T(A^\bullet)) \quad (11.5.1)$$

(by definition), while $T(A^\bullet)$ is a complex of U -acyclic objects and hence the derived functors of U on the object $T(A^\bullet)$ can be computed using the “resolution” $T(A^\bullet) \xrightarrow{id} T(A^\bullet)$, by U -acyclics, giving:

$$R^k U(T(A^\bullet)) = H^k(U(T(A^\bullet))). \quad (11.5.2)$$

Thus under (1) or (2) there are natural isomorphisms for all k

$$R^k(U \circ T)(F) = R^k U(T(A^\bullet)). \quad (11.5.3)$$

If $F \rightarrow A^\bullet$ is a resolution by $U \circ T$ -acyclics and (3) holds then (11.5.1) holds (derived functors are computed by acyclic resolutions) and (11.5.2) holds (because $T(F) \rightarrow T(A^\bullet)$ is a quasi-isomorphism to a complex of U -acyclics) and hence (11.5.3) holds in this case also. ⁽⁷¹⁾

We will use this observation (rather, a similar argument) in the following situation: $f : Y \rightarrow X$ is a continuous map and we consider the composition $\Gamma(Y, -) = \Gamma(X, -) \circ f_*$:

$$Sh(Y) \xrightarrow{f_*} Sh(X) \xrightarrow{\Gamma(X, -)} AbGps.$$

Then we have:

Lemma 11.5.1. *For $F \in Sh(Y)$ and $F \rightarrow A^\bullet$ a flasque or injective resolution, we have:*

$$H^k(Y, F) = \mathbb{H}^k(X, f_* A^\bullet).$$

(This also holds if F^\bullet is a complex of sheaves and $F^\bullet \rightarrow A^\bullet$ is a quasi-isomorphism to a complex of flasque or injective sheaves.)

Proof. The injective case is (1) above. In the flasque case, since f_* takes flasque sheaves to flasque sheaves and flasques are acyclic for the global sections functor, this follows by the same argument as above, with the variant that we start with flasques (which we know compute $H^k(Y, F)$). \square

The important point here is that the cohomology of any sheaf on Y (or any complex of sheaves on Y) can be computed by a complex of sheaves on X , namely the direct image of a flasque or injective resolution. For the latter we may, for instance, take the Godement resolution.

Remark 11.5.2. In general, the relation between derived functors is most naturally expressed in terms of total derived functors and takes the simple form $R(U \circ T) \cong RU \circ RT$, but we have not discussed total derived functors. In any case, deducing a relation between R^k requires spectral sequences or spectral objects, which are avoided here for special situations using the arguments above.

11.6. Proofs. We now go back and prove various unproven statements. We will need some extra notation.

If A^\bullet is a complex then the shifted complex $A^\bullet[1]$ is defined by

$$A[1]^i := A^{i-1} \quad \text{and} \quad d_{A[1]}^i = (-1)^i d_A^{i-1}.$$

It has the property that $H^i(A[1]^\bullet) = H^{i-1}(A^\bullet)$. The functor $A \mapsto A[1]$ from complexes to complexes is called the **shift functor**.

If $\phi : A^\bullet \rightarrow B^\bullet$ is a morphism of complexes then the **cone of ϕ** is the complex

$$C^i(\phi) := A^i \oplus B^{i-1} \quad \text{with} \quad d_C := \begin{pmatrix} d_A^i & (-1)^i \phi^i \\ & d_B^{i-1} \end{pmatrix}.$$

⁷¹A special case of (11.5.3) was seen earlier: If T is actually exact and satisfies (1) then any object is acyclic for T and hence taking $A^\bullet = F$ (in degree zero) we get $R^k(U \circ T)(F) = R^k U(T(F))$. This was proved in Lemma 10.7.3 (see also the footnote there).

There is a split short-exact sequence of complexes

$$0 \rightarrow B^\bullet[1] \rightarrow C(\phi)^\bullet \rightarrow A^\bullet \rightarrow 0.$$

We will also need the following facts:

Lemma 11.6.1. (1) If I^\bullet is an exact complex of injective objects and T is a left-exact functor then $T(I^\bullet)$ is exact.

(2) If $\phi : K^\bullet \rightarrow L^\bullet$ is a quasi-isomorphism then $C(\phi)^\bullet$ is acyclic, i.e. $H^*(C(\phi)^\bullet) = 0$, and conversely.

(3) If A^\bullet is an exact complex of T -acyclic objects for a left-exact functor T then $T(A^\bullet)$ is an exact complex.

Proof. (1) If I^\bullet is an exact complex of injective objects then there is a homotopy between the identity map of I^\bullet and the zero map, constructed as follows: Look at the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & I^2 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \dots \end{array}$$

Since $I^0 \rightarrow I^1$ is a monomorphism and I^0 is injective there is a map $H^1 : I^1 \rightarrow I^0$ extending the identity of I^0 , i.e. with $H^1 d^0 = id_{I^0}$. Now $\ker(H^1)$ is a complement to $d^0(I^0)$, i.e. $I^1 \rightarrow \text{coker}(d^0)$ gives an isomorphism $\ker(H^1) \cong \text{coker}(d^0)$. By the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \text{coker}(d^0) = \ker(H^1) & \xrightarrow{d^1} & I^2 \\ & & \downarrow & & \\ & & I^1 & & \end{array}$$

and injectivity of $\text{coker}(d^0)$ we get $H^2 : I^2 \rightarrow I^1$ satisfying

$$H^2 \circ d^1 = Id_{I^1} - d^0 \circ H^1.$$

Continuing in this way constructs a homotopy $H^k : I^k \rightarrow I^{k-1}$ such that $d^{k-1}H^k + H^{k+1}d^k = id^k$.

Now applying T to the homotopy gives a homotopy of the identity of $T(I^\bullet)$ with the zero map. Thus $T(I^\bullet)$ is exact.

(2) The split short exact sequence $0 \rightarrow B[1]^\bullet \rightarrow C(\phi)^\bullet \rightarrow A^\bullet \rightarrow 0$ gives the long exact sequence

$$\dots \longrightarrow H^i(B[1]^\bullet) \longrightarrow H^i(C(\phi)^\bullet) \longrightarrow H^i(A^\bullet) \longrightarrow H^{i+1}(B[1]^\bullet) \longrightarrow \dots$$

The connecting homomorphism $H^i(A^\bullet) \rightarrow H^{i+1}(B[1]^\bullet) = H^i(B^\bullet)$ is easily checked to be the map induced by ϕ . Since this is an isomorphism for all i (because ϕ is a quasi-isomorphism) we have that $H^i(C(\phi)^\bullet) = 0$ for all i , i.e. the cone is acyclic. The converse also clearly holds.

(3) Suppose that

$$0 \longrightarrow A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \longrightarrow \dots$$

is an exact complex of T -acyclics. The sequences

$$0 \longrightarrow \text{im}(d^{i-1}) = \ker(d^i) \longrightarrow A^i \longrightarrow \ker(d^{i+1}) \longrightarrow 0$$

are then short-exact for all i . Induction and the long exact sequence for $R^i T(-)$ shows that $\ker(d^i)$ is T -acyclic for all i . Thus applying T gives short-exact sequences

$$0 \longrightarrow T(\ker(d^i)) \longrightarrow T(A^i) \longrightarrow T(\ker(d^{i+1})) \longrightarrow 0$$

for all i . It follows that $T(\ker(d^i)) = \ker(T(d^{i+1}))$ for all i and putting these short exact sequences gives exactness of the complex $T(A^\bullet)$. \square

The following statements about exact functors were used above:

Lemma 11.6.2. (1) If K^\bullet is an exact complex and T is an exact functor then $T(K^\bullet)$ is an exact complex.

(2) If $K^\bullet \rightarrow L^\bullet$ is a quasi-isomorphism and T is an exact functor then $T(K^\bullet) \rightarrow T(L^\bullet)$ is a quasi-isomorphism.

Proof. (1) Exercise. (Break the complex up into short exact sequences and apply T , similar to the proof of part (3) of the previous lemma.)

(2) By (2) of the previous lemma a morphism $\phi : K^\bullet \rightarrow L^\bullet$ is a quasi-isomorphism if and only if the cone $C(\phi)^\bullet$ is exact, i.e. $H^*(C(\phi)^\bullet) = 0$. By (1), $T(C(\phi)^\bullet) = C(T(\phi))^\bullet$ is exact, and hence (by (2) of the previous lemma) $T(\phi)$ is a quasi-isomorphism. \square

Proof of Proposition 11.3.1. Suppose that $j : F^\bullet \rightarrow J^\bullet$ is a quasi-isomorphism to a complex of injective objects and $i : F^\bullet \rightarrow I^\bullet$ is a monomorphism which is a quasi-isomorphism to a complex of injective objects. By (2) of Proposition 11.2.3 there is a map $\phi : I^\bullet \rightarrow J^\bullet$ such that $\phi \circ i = j$. Then ϕ is also a quasi-isomorphism. By (2) of the previous lemma the cone $C(\phi)^\bullet$ is acyclic. It also consists of injective objects. By (1) of that lemma $T(C(\phi)^\bullet)$ is again an exact complex. Since the short exact sequence

$$0 \longrightarrow J[1]^\bullet \longrightarrow C(\phi)^\bullet \longrightarrow I^\bullet \longrightarrow 0$$

is split exact, so is the sequence

$$0 \longrightarrow T(J[1]^\bullet) \longrightarrow T(C(\phi)^\bullet) \longrightarrow T(I^\bullet) \longrightarrow 0.$$

Using the long exact cohomology sequence of this sequence and the fact that $T(C(\phi)^\bullet)$ is exact we get isomorphisms

$$H^i(T(I^\bullet)) \cong H^{i+1}(T(J[1]^\bullet)) = H^i(T(J^\bullet)).$$

Since $R^iT(I^\bullet) = H^i(T(I^\bullet))$ (by definition), this proves the proposition. \square

Proof of Proposition 11.3.3. Let $F^\bullet \rightarrow A^\bullet$ be a quasi-isomorphism to a complex of acyclic objects for the left-exact functor T . By Corollary 11.3.2 we know that

$$R^iT(F^\bullet) \cong R^iT(A^\bullet)$$

for all i . Thus we must show that for a complex of acyclic objects we have $R^iT(A^\bullet) = H^i(T(A^\bullet))$. Let $A^\bullet \hookrightarrow I^\bullet$ be a monomorphic quasi-isomorphism to a complex of injective objects. Let $Q^\bullet = I^\bullet/A^\bullet$. The sequence

$$0 \longrightarrow A^\bullet \longrightarrow I^\bullet \longrightarrow Q^\bullet \longrightarrow 0$$

shows (via the long-exact sequence in cohomology) that Q^\bullet is an exact complex. Applying T gives (because each A^i is acyclic) a short-exact sequence of complexes

$$0 \longrightarrow T(A^\bullet) \longrightarrow T(I^\bullet) \longrightarrow T(Q^\bullet) \longrightarrow 0.$$

Each Q^i is T -acyclic (by the long-exact sequence for $R^iT(-)$ because A^i and I^i are T -acyclic). Taking cohomology and applying (4) of the previous lemma to $T(Q^\bullet)$ gives

$$H^i(T(A^\bullet)) = H^i(T(I^\bullet)) = R^iT(F^\bullet)$$

proving the proposition. \square

11.7. Application: Functoriality of the de Rham isomorphism. Let us see how this follows nicely using the formalism of hypercohomology. For a map of smooth manifolds $f : M \rightarrow N$ we shall see that the pullback map in either singular or de Rham cohomology is given by the map

$$H^i(N, \mathbb{R}_N) \rightarrow H^i(N, f^{-1}\mathbb{R}_N) \cong H^i(M, \mathbb{R}_M)$$

under the respective isomorphisms $H_{sing}^i(?, \mathbb{R}) \cong H^i(?, \mathbb{R}_?)$ and $H_{dR}^i(?) \cong H^i(?, \mathbb{R}_?)$. This will prove the functoriality of the de Rham isomorphism.

First consider de Rham cohomology. We have a diagram

$$\begin{array}{ccc} f^{-1}\mathbb{R}_N & \longrightarrow & \mathbb{R}_M \\ \downarrow & & \downarrow \\ f^{-1}\mathcal{E}_N^\bullet & \longrightarrow & \mathcal{E}_M^\bullet \end{array}$$

The lower map is given by pullback of differential forms from (open sets of) N to (open sets of) M . Both vertical maps are quasi-isomorphisms (the right by the Poincaré lemma on M , the left by the Poincaré lemma on N plus exactness of f^{-1}). The upper map is an isomorphism. The commutativity of the diagram is obvious. Taking hypercohomology gives the right square of:

$$\begin{array}{ccccc} \mathbb{H}^k(N, \mathbb{R}_N) & \longrightarrow & \mathbb{H}^k(M, f^{-1}\mathbb{R}_N) & \xlongequal{\quad} & \mathbb{H}^k(M, \mathbb{R}_M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^k(N, \mathcal{E}_N^\bullet) & \longrightarrow & \mathbb{H}^k(M, f^{-1}\mathcal{E}_N^\bullet) & \longrightarrow & \mathbb{H}^k(M, \mathcal{E}_M^\bullet) \end{array}$$

The outer vertical maps give the isomorphism between sheaf cohomology and de Rham cohomology on N and M respectively. The horizontal maps in the left square are pullback maps. Both squares commute, thus the outer isomorphisms are compatible. Checking that the composition across the bottom row is the same as the map induced by pullback of differential forms from N to M is left as an exercise.

Now consider singular cohomology. The same argument using the complex of sheaves of continuous cochains proves that the pullback

$$H^i(N, \mathcal{E}_N^\bullet) = \mathbb{H}^i(N, \mathcal{E}_N^\bullet) \rightarrow \mathbb{H}^i(M, f^{-1}\mathcal{E}_N^\bullet) \cong \mathbb{H}^i(M, \mathcal{E}_M^\bullet) = H^i(M, \mathcal{E}_M^\bullet)$$

is the sheaf pullback. Now this map is clearly compatible with the usual pullback

$$H_{sing}^i(N, \mathbb{Z}) \rightarrow H_{sing}^i(M, \mathbb{Z})$$

Thus we have:

Proposition 11.7.1. *The de Rham isomorphism $H_{dR}^*(M) \cong H_{sing}^*(M, \mathbb{R})$ is functorial for smooth maps of manifolds.*

Remark 11.7.2 (Cup products). Similar arguments with hypercohomology can be used to show that the de Rham isomorphism relates cup products in singular cohomology with the wedge product of differential forms in de Rham cohomology.

11.8. Application: Leray-Hirsch theorem. We look at the cohomology of locally trivial fibre bundles. These are usually studied using spectral sequences, but (following Voisin's book) we will avoid these for some types of fibre bundles using a sheaf-theoretic argument with hypercohomology. The main result is the Leray-Hirsch theorem, which we then use to establish the Thom isomorphism. The next subsection contains some computations using the Leray-Hirsch theorem.

We will need the following lemma, which describes the pushforward of a complex:

Lemma 11.8.1. *Let $f : Y \rightarrow X$ be a continuous map and K^\bullet a complex of sheaves on Y . Then $R^i f_* K^\bullet$ is the sheaf on X associated with the presheaf $U \mapsto \mathbb{H}^i(f^{-1}(U), K^\bullet)$.*

Proof. Let $K^\bullet \hookrightarrow I^\bullet$ be a quasi-isomorphism to a complex of injective sheaves. By definition,

$$R^i f_* K^\bullet = H^i(f_* I^\bullet).$$

But $H^i(f_* I^\bullet)$ is the sheaf associated with the presheaf

$$U \mapsto H^i(\Gamma(U, f_* I^\bullet)) = H^i(\Gamma(f^{-1}(U), I^\bullet)) = \mathbb{H}^i(f^{-1}(U), K^\bullet).$$

This proves the lemma. □

We will now use the formalism above to prove a useful theorem about the homology of certain manifolds.

A map $f : Y \rightarrow X$ of topological spaces is **locally trivial** if for each $x \in X$ there is a neighbourhood U_x of x in X and a homeomorphism $f^{-1}(U_x) \cong U_x \times f^{-1}(x)$ such that $f|_{f^{-1}(U_x)}$ is given by the first projection. An example of a locally trivial map is a fibre bundle: A map $\pi : E \rightarrow B$ is a **fibre bundle** if there exists a space F (the fibre) and a covering $\{U_\alpha\}$ of B and homeomorphisms $\pi^{-1}(U_\alpha) \cong U_\alpha \times F$ such that $\pi|_{\pi^{-1}(U_\alpha)}$ is given by the first projection. An example of a fibre bundle is a vector bundle.

Theorem 11.8.2 (Leray-Hirsch). *Suppose that X and Y are locally contractible ⁽⁷²⁾ and $f : Y \rightarrow X$ is a locally trivial map. Assume that*

- (1) *for each $x \in X$ the cohomology groups $H^*(f^{-1}(x), \mathbb{Z})$ are torsion-free*
- (2) *there exist classes $\alpha_1, \dots, \alpha_N \in H^*(Y, \mathbb{Z})$ such that for all $x \in X$ the restrictions $\alpha_i|_{f^{-1}(x)}$ form a basis of $H^*(f^{-1}(x), \mathbb{Z})$.*

Let A^ be the graded subgroup of $H^*(Y, \mathbb{Z})$ generated by $\alpha_1, \dots, \alpha_N$. (Thus $A^* \cong H^*(f^{-1}(x), \mathbb{Z})$ for any x). Then the map*

$$A^* \otimes H^*(X, \mathbb{Z}) \rightarrow H^*(Y, \mathbb{Z}) \quad \text{by } \sum_i c_i \alpha_i \otimes \omega \mapsto \sum_i c_i \alpha_i \cdot f^*(\omega)$$

is an isomorphism of graded groups. In particular, $H^(Y, \mathbb{Z})$ is freely generated as a graded $H^*(X, \mathbb{Z})$ -module by A^* .*

Proof. Let I^\bullet be a flasque resolution of the constant sheaf \mathbb{Z}_Y . We know that $H^*(Y, \mathbb{Z}) = H^*(Y, \mathbb{Z}_Y) = H^*(\Gamma(Y, I^\bullet))$. Let $\tilde{\alpha}_1, \dots, \tilde{\alpha}_N \in \Gamma(Y, I^\bullet)$ be cocycles representing the classes $\alpha_1, \dots, \alpha_N$. We may assume that the α_i are all homogeneous of a single degree, and that the same holds for the $\tilde{\alpha}_i$. Then they generate a subgroup of $\Gamma(Y, I^\bullet)$ isomorphic to A^* and they give an inclusion

$$A_X^* \hookrightarrow f_* I^\bullet$$

which is an inclusion of complexes of sheaves. (Here A_X^k is the constant sheaf on X with stalks A^k and the complex A_X^* has zero differentials.) The local triviality of f and assumption (2) in the theorem imply that this is a quasi-isomorphism (use Lemma 11.8.1). Thus we have isomorphisms in hypercohomology

$$\mathbb{H}^i(X, A_X^*) \cong \mathbb{H}^i(X, f_* I^\bullet)$$

for all i . The right-hand side is $H^*(Y, \mathbb{Z})$ by the lemma above, while the right-hand side is

$$\mathbb{H}^i(X, A_X^*) = \bigoplus_k \mathbb{H}^{i-k}(X, \mathbb{Z}_X) \otimes A^k = \bigoplus_k H^{i-k}(X, \mathbb{Z}) \otimes A^k$$

by the assumption (1) on A^* . Thus we have $H^*(X, \mathbb{Z}) \otimes A^* \cong H^*(Y, \mathbb{Z})$. □

Corollary 11.8.3 (Künneth formula). *Let X and Y be locally contractible spaces and assume that $H^*(X, \mathbb{Z})$ is torsion-free. Then $H^*(X \times Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z})$ (as graded \mathbb{Z} -modules).*

Remarks 11.8.4. (1) The proof given here works with any ring R as coefficients.

(2) Note in particular that the map $f^* : H^*(X, \mathbb{Z}) \rightarrow H^*(Y, \mathbb{Z})$ is an injective ring homomorphism under the conditions of the theorem. Moreover, the image is a direct summand if the fibres are connected: a complement is given by choosing a basis of $A^{>0}$.

(3) The Leray-Hirsch theorem does not claim that the ring structure on $H^*(Y, \mathbb{Z})$ is the tensor product, in fact that is not true in general. We will see an example below. In the product case, it is true that the ring structure is given by the tensor product, but the proof here does not prove this.

⁷²The proof will use the identification of singular cohomology with sheaf cohomology for X and Y . Since this was only proved in these notes for locally contractible spaces under the additional hypothesis that every open subset is paracompact, we should probably add this assumption here. In any case, in the applications all spaces will be manifolds.

(4) If $f : X \rightarrow Y$ is a smooth map of manifolds and the fibres of $Y \rightarrow X$ are manifolds then everything holds in de Rham cohomology. In fact, a proper surjective smooth submersion of smooth manifolds with connected fibres always satisfies the local triviality condition in the Leray-Hirsch theorem, by Ehresmann's theorem.

(5) An easy example to see that condition (2) in the theorem is not always satisfied is the Hopf fibration $S^3 \rightarrow S^2$, which is locally trivial with fibre S^1 .

11.9. Example: Homogeneous manifolds for compact Lie groups. Let G be a compact connected Lie group. The de Rham cohomology of a compact Lie group is computed by the bi-invariant differential forms, i.e.

$$H_{dR}^*(G) = (\Lambda^* \text{Lie}(G)^*)^{AdG},$$

where $Ad : G \rightarrow GL(\text{Lie}(G))$ is the adjoint representation. (This is an exercise on the problem set.) If H is a closed connected (Lie) subgroup then $\text{Lie}(G)^* \twoheadrightarrow \text{Lie}(H)^*$ but it need not be the case that

$$\Lambda^k(\text{Lie}(G)^*)^{Ad(G)} \twoheadrightarrow \Lambda^k(\text{Lie}(H)^*)^{Ad(H)} \quad \text{for all } k. \quad (11.9.1)$$

Now consider the fibration $\pi : G \rightarrow G/H$. It is locally trivial since the smooth manifold structure on G/H is such that π has local smooth sections. The fibre over eH is identified with H . If (11.9.1) holds then one can choose classes $\alpha_1, \dots, \alpha_N \in H_{dR}^*(G)$ which map isomorphically to a basis of $H_{dR}^*(H)$. Since the fibre $\pi^{-1}(gH) = gH = L_g(H)$ and the classes $\alpha_1, \dots, \alpha_N$ may be represented by left-invariant differential forms, we see that $\alpha_1, \dots, \alpha_N$ restrict to a basis of $H_{dR}^*(\pi^{-1}(gH))$ for any fibre. Thus the Leray-Hirsch theorem applies and tells us that

$$H^*(H, \mathbb{R}) \otimes H^*(G/H, \mathbb{R}) \cong H^*(G, \mathbb{R}). \quad (11.9.2)$$

So if (11.9.1) holds then we can conclude (11.9.2).

(An example where (11.9.1) fails is $G = SU(2) \supset H = U(1)$, with $k = 1$. The corresponding map $S^3 = SU(2) \rightarrow SU(2)/U(1) = S^2$ is the Hopf fibration.)

Example 11.9.1 (Unitary groups). Assume that (11.9.1) holds for the pair $U(n-1) \subset U(n)$ for all $n \geq 1$. Then applying the isomorphism (11.9.2) to the fibration

$$U(n) \rightarrow U(n)/U(n-1) = S^{2n-1}$$

and induction on n shows that

$$H^*(U(n), \mathbb{R}) = \Lambda_{\mathbb{R}}[c_1, c_3, \dots, c_{2n-1}]$$

is the free exterior algebra on generators in each odd degree $\leq 2n-1$. Note that the ring structure is also determined since all products are zero by graded-commutativity of the cup product. For $SU(n)$ the same argument gives

$$H^*(SU(n), \mathbb{R}) = \Lambda_{\mathbb{R}}[c_3, \dots, c_{2n-1}]$$

(since $SU(1) = \{e\}$ the induction begins with $SU(2) = S^3$ so the first generator is in degree 3). The same assertions hold with \mathbb{Q} coefficients.

We will verify (11.9.1) for the pair $U(n-1) \subset U(n)$ for $n \geq 3$ later.

Remarks 11.9.2. (1) These results illustrate Hopf's theorem, viz. the cohomology of a compact connected Lie group is an exterior algebra on odd-degree generators.

(2) The same method can be used for the sequence of fibrations involving the compact symplectic group: $Sp(1) \subset Sp(2) \subset \dots$.

(3) The method does not work for $SO(n)$. For example, the fibration $SO(3) \rightarrow S^2 = SO(3)/SO(2)$ is not homologically like a product, i.e. the Leray-Hirsch theorem does not apply. (This has to be the case because according to the Hopf theorem $H^*(SO(n), \mathbb{R})$ is also an exterior algebra on odd degree generators. Computing the actual degrees of the exponents is not as easy as in the $SU(n), Sp(n)$ cases; the answer is different for even n and odd n .)

11.10. Example: Flag manifolds. A k -flag in \mathbb{C}^n is a chain $V_1 \subset V_2 \subset \dots \subset V_k$ of subspaces of \mathbb{C}^n with $\dim(V_i) = i$. The set $Fl(k, n)$ of k -flags in \mathbb{C}^n is a homogenous manifold: The Lie group $GL(n, \mathbb{C})$ acts transitively on it and the isotropy is a certain closed Lie subgroup. This implies that it has a manifold structure.

It will be more convenient for us to think of a k -flag as an ordered k -tuple (ℓ_1, \dots, ℓ_k) of orthogonal lines in \mathbb{C}^n . (By orthogonal we mean with respect to the standard Hermitian inner product. The relation with the previous description is the obvious one: $V_i = \ell_1 + \dots + \ell_i$.)

For each $i = 1, \dots, k$ there is a map

$$p_i : Fl(k, n) \rightarrow \mathbb{C}P^{n-1} \quad \text{by } (\ell_1, \dots, \ell_k) \mapsto \ell_i.$$

Pulling back the generator of $H^2(\mathbb{C}P^{n-1})$ for each $i = 1, \dots, k$ gives elements $x_1, \dots, x_k \in H^2(Fl(k, n))$.

Lemma 11.10.1. *The cohomology of $Fl(k, n)$ is given by:*

$$H^*(Fl(k, n), \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_k] / (x_1^n, x_2^{n-1}, \dots, x_k^{n-k+1}) = H^*(\mathbb{C}P^{n-1} \times \dots \times \mathbb{C}P^{n-k}, \mathbb{Z}).$$

Proof. Forgetting the last line defines a map to $Fl(k-1, n)$; the fibre over a $k-1$ tuple $(\ell_1, \dots, \ell_{k-1})$ consists of lines in $(\ell_1 + \dots + \ell_{k-1})^\perp$, i.e. it is isomorphic to $\mathbb{C}P^{n-k}$. It is easy to see that

$$\begin{array}{ccc} \mathbb{C}P^{n-k} & \longrightarrow & Fl(k, n) \\ & & \downarrow \\ & & Fl(k-1, n) \end{array}$$

is a fibre bundle with fibre $\mathbb{C}P^{n-k}$. Now the element x_k restricts to any fibre as the generator of H^2 . Indeed, the fibre over $(\ell_1, \dots, \ell_{k-1})$ consists of lines in the space $(\ell_1 + \dots + \ell_{k-1})^\perp$, which is a copy of $\mathbb{C}P^{n-k}$ sitting in the $\mathbb{C}P^{n-1}$ consisting of choices of ℓ_k , from which the generator x_k was pulled back. Thus the restriction to the fibre is surjective in cohomology and the Leray-Hirsch theorem applies to $Fl(k, n) \rightarrow Fl(k-1, n)$.

Let us prove the lemma by induction on k . The induction hypothesis gives that $H^*(Fl(k-1, n), \mathbb{Z})$ is generated by x_1, \dots, x_{k-1} with relations as in the theorem. The Leray-Hirsch theorem then tells us that $H^*(Fl(k, n), \mathbb{Z})$ is freely generated over $H^*(Fl(k-1, n), \mathbb{Z})$ by x_k , with the relation $x_k^{n-k+1} = 0$. This proves the lemma. \square

For the space of k -flags in an infinite-dimensional space $\mathbb{C}^\infty = \varinjlim_n \mathbb{C}^n$, viz. the space $Fl(k, \infty) = \varinjlim_n Fl(k, n)$ the same proof shows that the cohomology is

$$H^*(Fl(k, \infty), \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \dots]$$

i.e. a polynomial ring in the generators defined by pulling back the generator of $H^2(\mathbb{C}P^\infty)$ under the k maps $Fl(k, \infty) \rightarrow \mathbb{C}P^\infty$. Notice that this is simpler than $H^*(Fl(k, n), \mathbb{Z})$. ⁽⁷³⁾

11.11. Example: Grassmannians. Consider the Grassmannian $Gr(k, n)$ of k -dimensional subspaces of \mathbb{C}^n . There is a fibre bundle

$$\begin{array}{ccc} Fl(k, k) & \longrightarrow & Fl(k, n) \\ & & \downarrow \pi \\ & & Gr(k, n) \end{array}$$

where the vertical map π sends (ℓ_1, \dots, ℓ_k) to $\ell_1 + \dots + \ell_k$. The fibre over a subspace $E \in Gr(k, n)$ consists of full flags in E , which is simply $Fl(k, k)$. We have seen above that the restriction map

$$H^*(Fl(k, n), \mathbb{Z}) \rightarrow H^*(Fl(k, k), \mathbb{Z})$$

⁷³We are being a little careless here: If $Fl(k, \infty)$ is defined as the direct limit of $Fl(k, n)$ as $n \rightarrow \infty$, then it is not clear that the cohomology satisfies $H^i(Fl(k, \infty), \mathbb{Z}) = H^i(Fl(k, n), \mathbb{Z})$ for n large. Rather we are talking about the value $H^i(Fl(k, n), \mathbb{Z})$ for n large, which is independent of n for n large. This same comment applies to $Gr(k, \infty)$ below. Nevertheless, this is okay, see e.g. Milnor-Stasheff.

is surjective and this applies to any fibre. Thus the Leray-Hirsch theorem applies and we deduce that the pullback

$$\pi^* : H^*(Gr(k, n), \mathbb{Z}) \rightarrow H^*(Fl(k, n), \mathbb{Z})$$

is injective and in fact the image is a direct summand.

Now we consider the case of $Gr(k, \infty)$, in which case we have $H^*(Gr(k, \infty), \mathbb{Z}) \hookrightarrow \mathbb{Z}[x_1, \dots, x_k]$.

Lemma 11.11.1. *The image of $\pi^* : H^*(Gr(k, \infty), \mathbb{Z}) \rightarrow H^*(Fl(k, \infty), \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_k]$ is the subalgebra of symmetric polynomials.*

Proof. First we show that the image consists of symmetric polynomials. For a permutation σ of $\{1, \dots, k\}$ there is a map $\sigma : Fl(k, n) \rightarrow Fl(k, n)$ by permuting the lines. This map induces the automorphism $x_i \mapsto x_{\sigma(i)}$ on $H^*(Fl(k, n), \mathbb{Z})$. Now $\pi \circ \sigma = \pi$ because $\ell_1 + \dots + \ell_k = \ell_{\sigma(1)} + \dots + \ell_{\sigma(k)}$, so that $\sigma^* \pi^* = \pi^*$ and thus the image consists of symmetric polynomials.

To show that the image is the subalgebra of symmetric polynomials we need a definition. For a free graded \mathbb{Z} -module $A^* = \bigoplus_{k \geq 0} A^k$ with $A^i = 0$ for odd i the Poincaré series is the formal power series

$$P_A(t) := \sum_{k \geq 0} \text{rank}(A^{2k}) t^k \in \mathbb{Z}[[t]].$$

For a graded tensor product $A^* \otimes B^*$ we have:

$$P_{A \otimes B} = P_A P_B.$$

Thus $P_{H^*(Fl(k, \infty))} = (1 - t)^{-k}$. By the computation of the cohomology of $Fl(k, k)$ we have

$$P_{H^*(Fl(k, k))} = (1 + t)(1 + t + t^2) \cdots (1 + t + \dots + t^{k-1}) = \prod_{k=1}^k \frac{1 - t^i}{1 - t} = (1 - t)^{-k} \prod_{i=1}^k (1 - t^i).$$

Thus by the isomorphism of graded groups

$$H^*(Fl(k, \infty), \mathbb{Z}) = H^*(Fl(k, k), \mathbb{Z}) \otimes H^*(Gr(k, \infty), \mathbb{Z})$$

coming from Leray-Hirsch, the Poincaré series $Q = P_{H^*(Gr(k, \infty))}$ satisfies:

$$Q \cdot (1 - t)^{-k} \prod_{i=1}^k (1 - t^i) = (1 - t)^{-k}$$

and hence

$$Q = \prod_{i=1}^k \frac{1}{1 - t^i}.$$

Now we use the classical fact (see e.g. Lang's *Algebra*) that the symmetric polynomials are freely generated, as a \mathbb{Z} -algebra, by the elementary symmetric polynomials c_1, \dots, c_k where:

$$\begin{aligned} c_1 &= x_1 + \dots + x_k \\ c_2 &= \sum_{1 \leq i < j \leq k} x_i x_j \\ &\vdots \\ c_k &= x_1 \cdots x_k. \end{aligned} \tag{11.11.1}$$

Thus the Poincaré series of the algebra of invariant polynomials is

$$(1 + t + t^2 + \dots)(1 + t^2 + t^4 + \dots) \cdots (1 + t^k + t^{2k} + \dots) = \prod_{i=1}^k \frac{1}{1 - t^i}.$$

This is exactly Q and hence we have proved the lemma. \square

We have proved:

Theorem 11.11.2. $H^*(Gr(k, \infty), \mathbb{Z})$ is a polynomial ring over \mathbb{Z} in k generators c_1, \dots, c_k where $\deg(c_i) = 2i$.

Note that the generators are canonical (under π^* they map to the symmetric functions in x_1, \dots, x_k , which are themselves canonically defined).

The same arguments as above, using the fact that $H^*(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[x]/(x^{n+1})$, give the $\mathbb{Z}/2\mathbb{Z}$ -cohomology of the infinite real Grassmannian:

Theorem 11.11.3. $H^*(Gr_{\mathbb{R}}(k, \infty), \mathbb{Z}/2\mathbb{Z})$ is the polynomial ring over $\mathbb{Z}/2\mathbb{Z}$ in k generators w_1, \dots, w_k where $\deg(w_i) = 2i$.

These two theorems are of fundamental importance in the theory of characteristic classes, because $Gr(k, \infty)$ and $Gr_{\mathbb{R}}(k, \infty)$ are the classifying spaces for vector bundles: On any reasonable space X (e.g. a manifold or a CW complex) any rank r complex (resp. real) vector bundle E is the pullback of the tautological rank r bundle on $Gr(k, \infty)$ (resp. $Gr_{\mathbb{R}}(k, \infty)$) by a mapping h_E of X to $Gr(k, \infty)$ (resp. to $Gr_{\mathbb{R}}(k, \infty)$) which is unique up to homotopy. The Chern classes $c_i(E) := h_E^*(c_i) \in H^{2i}(X, \mathbb{Z})$ in the complex case (resp. Stiefel-Whitney classes $w_i(E) := h_E^*(w_i) \in H^{2i}(X, \mathbb{Z}/2\mathbb{Z})$ in the real case) are then important invariants of the vector bundle. For example, a famous theorem of Thom says that a complex manifold X is the boundary of a manifold (i.e. cobordant to the empty set) if and only if all the Chern classes of its tangent bundle are trivial. For more on this see Milnor-Stasheff, *Characteristic Classes*.

11.12. Application: Thom isomorphism theorem (*). A (smooth) vector bundle $E \rightarrow M$ of rank r is called **oriented** if we are given a nowhere vanishing section of $\wedge^r E \rightarrow M$. Informally, we have an orientation of each fibre vector space $E_x = \pi^{-1}(x)$ varying smoothly in $x \in M$. This also gives an orientation in the topological sense, i.e. a class in $H^r(E_x, E_x - \{x\})$. Notice that together with an orientation of M , this determines an orientation of the manifold E .

Theorem 11.12.1 (Thom isomorphism). *Let $\pi : E \rightarrow M$ be an oriented smooth vector bundle of rank r on a (compact, oriented) manifold M . Then*

- (1) *There exists a unique class $u \in H^r(E, E - M)$ (called the **Thom class**) such that for each $x \in M$, the image of u under $H^r(E, E - M) \rightarrow H^r(E_x, E_x - \{x\})$ is the orientation class of the fibre E_x .*
- (2) *The map*

$$H^{*-r}(M) \longrightarrow H^*(E, E - M) \quad \alpha \mapsto \pi^*(\alpha) \cdot u$$

is an isomorphism of $H^(M)$ -modules.*

Proof. Put a metric on the vector bundle, i.e. a norm on the vector spaces $E_x = \pi^{-1}(x)$ which varies smoothly in $x \in M$. (*Exercise:* This can be constructed using a trivialization and a partition of unity.) Let D_E be the unit disk bundle, i.e. in each fibre of π we take the vectors of norm ≤ 1 , and let S_E be the sphere bundle (vectors of norm = 1 in each fibre). Then D_E is a smooth manifold of dimension $\dim M + r$ with boundary $\partial D_E = S_E$. By excision and a deformation retraction we have

$$H^*(E, E - M) \cong H^*(D_E, S_E).$$

Then we have isomorphisms

$$H^k(D_E, S_E) \xrightarrow{\cap [D_E]} H_{\dim M + r - k}(D_E) \cong H_{\dim M + r - k}(M) \xleftarrow{\cap [M]} H^{k-r}(M)$$

by Lefschetz duality for D_E , the fact that $M \hookrightarrow D_E$ is a homotopy equivalence, and Poincaré duality for M .

(1) Now for $k = r$ we see that $H^r(E, E - M) \cong H^0(M) = \mathbb{Z}$ (assuming M is connected, which case is enough to consider). Let $u \mapsto 1$ under this isomorphism. The sequence of isomorphisms used above is compatible with restriction to a single fibre of $D_E \rightarrow M$ (use the local triviality to reduce to the case of products), and this shows that u restricts to the orientation class in each fibre.

(2) Form the quotient space Q of D_E in which each fibre of S_E is collapsed to a point. (To show this exists, construct it locally using a trivialization of E and then glue.) This is a fibre bundle over M with fibre the r -sphere $D_{E,x}/S_{E,x} \cong S^r$. The Leray-Hirsch theorem applies – check that the existence of the class u gives the condition needed to apply that theorem. This gives that $H^*(S^r) \otimes H^*(M) \rightarrow H^*(Q)$ is an isomorphism, and this restricts to the isomorphism in the theorem on the summand $H^r(S^r) \otimes H^*(M) = H^{*+r}(M)$ since $H^r(S^r) = H^r(D_E, S_E)$. \square

Applying this to the case of the normal bundle of a submanifold, we get:

Corollary 11.12.2. *Let $i : M \hookrightarrow N$ be an embedding of oriented smooth manifolds with M compact and let c be the codimension of M in N . Then $H^*(N, N - M) \cong H^{*-c}(M)$.*

Proof. Use the tubular neighbourhood theorem and excision to replace the pair $(N, N - M)$ by the disk bundle of the normal bundle to M in N , and apply the previous theorem. (An orientation of the normal bundle is given by the orientations of M and N .) \square

Remark 11.12.3. The Thom isomorphism also holds in the case M is not oriented (but E is an oriented vector bundle). See Milnor-Stasheff. It also holds without any assumption if we use $\mathbb{Z}/2\mathbb{Z}$ coefficients (and the proofs above work).