WORKSHOP: MOTIVIC ALGEBRAIC TOPOLOGY

JANUARY 6-10, 2025

Titles and abstracts

Mini-courses

Amit Hogadi *Strictly $\mathbb{A}^1$-invariant sheaves*

Abstract: A celebrated theorem of Fabien Morel states that if the base field k is perfect, then any strongly $\mathbb{A}^1$-invariant sheaf of abelian groups is strictly $\mathbb{A}^1$-invariant. The aim of these lectures to outline a proof of this theorem, following T. Bachmann (arXiv:2406.11526).

Chetan Balwe *Introduction to Milnor-Witt motives*

Abstract: An essential feature of the development of motivic homotopy theory has been the role played by quadratic forms. The introduction of "quadratic refinements" of various pieces of machinery such as Milnor K-theory and Chow groups, etc. has led to the creation of machinery that successfully incorporates topological intuition from the complex as well as real geometry of varieties. Proceeding along this theme, Milnor-Witt motivic cohomology is a quadratic refinement of motivic cohomology, obtained by replacing finite correspondences by finite Milnor-Witt correspondences. These lectures will provide an introduction to the category of Milnor-Witt motives and review some of the fundamental results.

Research talks

Charanya Ravi *Virtual Kawasaki-Grothendieck-Riemann-Roch theorem*

Abstract: Verdier's version of the Grothendieck-Riemann-Roch theorem studies the interaction between Gysin pullbacks along local complete intersection morphisms in G-theory and rational Chow groups, under the Baum-Fulton-MacPherson Riemann-Roch transformation from G-theory to rational Chow groups. This result was extended to quasi-smooth morphisms of derived schemes by Khan. In this talk, we present an extension of this result for quasi-smooth morphisms of derived Deligne-Mumford stacks. As a corollary, we extend the virtual Hirzebruch–Riemann–Roch theorem of Fantechi and Göttsche to DM stacks. This extension provides a virtual analogue of Kawasaki's orbifold Riemann-Roch formula, an important tool for computing the holomorphic Euler characteristic of vector bundles on DM stacks. The talk is based on ongoing joint work with Adeel Khan.

Rakesh Pawar *Comparison of coniveau and Postnikov spectral sequences in motivic homotopy category*

Abstract: Classically, Federer (1956) and Oda-Shitanda (1986) studied homotopy groups of the space of pointed maps $map_*(X, Y)$ from a connected CW-complex X to a connected space Y. This was achieved by comparing on one hand, (skeletal) spectral sequence coming from cellular filtration on X and on the other hand the (Postnikov) spectral sequence coming from Postnikov tower of Y. Both spectral sequences conditionally converge to homotopy groups of $map_*(X,Y)$.

In the joint work with Frederic Deglise (ENS de Lyon), currently in preparation, we consider the above-mentioned results in the motivic homotopy theory context. We consider appropriate simplicial set $map_*(X_+, Y)$ for a smooth scheme X over a field k and Y a pointed simplicial sheaf. Inspired by the classical results above, we consider 'coniveau' spectral sequence on the one hand and the 'Postnikov' spectral sequence on the other hand and compare the two under appropriate assumptions on X, Y, and the base field k.

We present some consequences of studying these two spectral sequences in computing homotopy sets of maps.

Parnashree Ghosh *Higher dimensional $\mathbb{A}^1$-contractible affine varieties and the Cancellation Problem*

Abstract: Throughout k denotes a field of characteristic zero. One of the fundamental problems in Affine Algebraic geometry is the Cancellation Problem, which asks the following:

Question 1. Let X,Y be affine k-varieties of dimension n such that $X \times \mathbb{A}^1_k \cong Y \times \mathbb{A}^1_k$. Is then $X \cong Y$?

An affine variety X is said to be cancellative if for any affine variety Y, $X \times \mathbb{A}^1_k \cong Y \times \mathbb{A}^1_k$ implies $X \cong Y$. Question 1 has an affirmative answer in dimension 1, whereas for every dimension $n \geqslant 2$, there are several families of counterexamples to this problem. A special case of the Question 1, famously known as the Zariski Cancellation Problem, asks whether affine spaces are cancellative. More precisely it asks the following:

Question 2. Let X be an n dimensional affine k-variety with $X \times \mathbb{A}^1_k \cong \mathbb{A}^{n+1}_k$. Is $X \cong \mathbb{A}^n_k$?

It is worth noting that, a variety X satisfying the property in Question 2, must be isomorphic to an affine space in the unstable motivic homotopy category. In view of this observation it is worth investigating the following version of the Cancellation Problem:

Question 3. Let X,Y be $\mathbb{A}^1$- contractible smooth affine varieties of dimension n such that $X \times \mathbb{A}^1_k \cong Y \times \mathbb{A}^1_k$. Is then $X \cong Y$?

Another important problem in Affine Algebraic geometry is the Characterization Problem for affine spaces in every dimension. Over fields of characteristic zero, $\mathbb{A}^1$-contractibility of affine varieties played a major role to characterize the affine spaces in dimension $\leqslant 2$. In dimension 3, there exists an infinite family of pairwise non-isomorphic affine varieties, which are $\mathbb{A}^1$-contractible. This family contains the Koras-Russell threefolds which are potential counter-example to the Zariski Cancellation Problem in characteristic zero.

In this talk, in every dimension $n \geqslant 4$ we will present an infinite family of pairwise non-isomorphic $\mathbb{A}^1$-contractible affine varieties which are not affine spaces and thus providing negative answer to Question 3 and also provides examples of new families of $\mathbb{A}^1$-contractible affine varieties in every dimension $\geqslant 4$.

This talk is based on an ongoing work with Adrien Dubouloz.

Biman Roy *Morel's conjecture on non-$\mathbb{A}^1$-uniruled affine surfaces*

Abstract: A presheaf of sets $\mathcal{F}$ on Sm/k, the category of smooth varieties over a field k, is called $\mathbb{A}^1$-invariant if the morphism $\mathcal{F}(U) \to \mathcal{F}(U \times_k \mathbb{A}^1_k)$, induced by the projection $U \times_k \mathbb{A}^1_k \to U$, is a bijection. Inspired by the discreteness of the topological connected components, Morel conjectured that $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant, for a space $\mathcal{X}$ on Sm/k. Ayoub constructed a space $\mathcal{X}$ such that $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is not $\mathbb{A}^1$-invariant. Thus Morel's conjecture is false in general. However Ayoub's counterexample is not a representable sheaf. Morel's conjecture is true in several cases: $\mathcal{X}$ is a motivic H-group or a homogeneous space for a motivic H-group over an infinite perfect field, $\mathcal{X}$ is a smooth projective surface over an algebraically closed field of characteristic zero, $\mathcal{X}$ is a smooth toric variety; due to the works of Choudhury, Balwe, Hogadi, Sawant and Wendt. It is a natural question to ask whether $\pi_0^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-invariant, for a smooth quasi-projective variety X over k? An affine k-variety X is said to be $\mathbb{A}^1$-uniruled or log-uniruled if there is a dominant generically finite morphism $H : \mathbb{A}^1_k \times_k Y \to X$ for some k-variety Y. In this talk I will present a proof to show that if X is a smooth affine surface over an algebraically closed field k and X is not an $\mathbb{A}^1$-uniruled, then $\pi_0^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-invariant.

Vivek Sadhu *Nil K-groups via binary complexes*

Abstract: Algebraic K-theory is not homotopy invariant in general. The Nil K-groups are obstruction groups for homotopy invariance of K-groups. In this talk, we will describe the generators of Nil K-groups using Grayson's machinery of binary complexes. In the first half, we will discuss Grayson's algebraic description of higher K-groups via binary complexes. This is a joint work with Sourayan Banerjee.

Devarshi Mukherjee *K-theory for analytic spaces*

Abstract: We introduce a version of algebraic K-theory and related localising invariants for bornological algebras, using Efimov's recently introduced continuous K-theory. In the commutative setting, our invariant satisfies descent for various topologies that arise in analytic geometry. If time permits, I will also discuss a version of the Grothendieck-Riemann-Roch Theorem for analytic spaces.

Nidhi Gupta *Field-valued sections of the sheaf of $\mathbb{A}^1$-connected components for varieties*

Abstract: In unstable $\mathbb{A}^1$-homotopy theory, there are two notions of connectedness for a variety X: the sheaf of $\mathbb{A}^1$-connected components $\pi_0^{\mathbb{A}^1}(X)$ and the sheaf of naive $\mathbb{A}^1$-connected components $\mathcal{S}(X)$. While these sheaves do not coincide in general, taking infinite iterations of $\mathcal{S}$ and then the direct limit yields the universal $\mathbb{A}^1$-invariant quotient:

$$\mathcal{L}(X) := \varinjlim_n \mathcal{S}^n(X).$$

By a result of Balwe, Rani, and Sawant, it is known that $\pi_0^{\mathbb{A}^1}(X)(K) = \mathcal{L}(X)(K)$ for any finitely generated separable field extension K of the base field. Thus, $\mathcal{L}$ provides a concrete formula for determining field-valued sections of the sheaf $\pi_0^{\mathbb{A}^1}(X)$.

In the first part of the talk, we will show that the infinite iterations of $\mathcal{S}$ involved in $\mathcal{L}$ are indeed necessary for varieties. For each n, we construct a variety X_n of dimension $n+1$ over $\mathbb{C}$ such that $\mathcal{S}^n(X_n)(\mathbb{C}) \neq \mathcal{S}^{n+1}(X_n)(\mathbb{C})$. In the second part of the talk, we determine the field-valued sections of the sheaf of $\mathbb{A}^1$-connected components of quadratic hypersurfaces X in $\mathbb{A}^n_k$ and show that $\pi_0^{\mathbb{A}^1}(X)(F) = \varinjlim_n \mathcal{S}^n(X)(F)$ stabilizes at $n = 2$ for all F/k. This is joint work with Dr. Chetan Balwe.

K Arun Kumar *Endomorphisms of equivariant algebraic K-theory*

Abstract: Equivariant motivic homotopy theory deals with extending tools and results from motivic homotopy theory to the category of schemes equipped with a group scheme action. There is a long history of research in generalised cohomology theories in this setting, and one of the important papers in the development of this subject is Thomason's work on algebraic K-theory of schemes with group actions. In this talk I will describe my work with Girja Tripathi on the endomorphisms of the equivariant algebraic K-theory space in the equivariant motivic homotopy category. Our main result is that in the case of actions by a finite group G whose order is invertible in the scheme S the endomorphisms are completely determined by the endomorphisms of the 0th K-group presheaf. This is a generalization of a result of Riou which uses the fact that algebraic K-theory is representable by the infinite Grassmannian in the motivic homotopy category.