AN EQUIVARIANT VERSION OF THE GABBER PRESENTATION LEMMA

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Abstract. We extend the C_2-equivariant version of the Gabber presentation lemma proved in [1] to the case of a general finite abelian group under suitable hypotheses.

1. Introduction

In [4], Gabber proved a geometric presentation lemma in order to prove the effacement theorem for étale cohomology and as a result, the exactness of the Gersten complex for certain classes of sheaves. This lemma allows one to reduce a problem to the situation of an affine line over a base and a closed subset finite over the base and is of fundamental importance to motivic homotopy theory as developed in [7]. It may be seen as an analogue of the tubular neighbourhood theroem in differential geometry.

The Gabber presentation lemma was proved over an infinite base field in [4], but it was extended to the finite field case in [6] and to the case where the base ring is a of a Noetherian domain with infinite residue fields in [3]. In [1], Bachmann proved a C_2-equivariant version of the lemma, where C_2 is the cyclic group with two elements and used it to prove a modified form of Gersten injectivity and thereby, extend some standard results in motivic homotopy theory about stable $\mathbb{A}^1$-connectivity and the homotopy t-structure to the C_2-equivariant setting.

Conventions 1.1. Throughout this paper, X will denote a smooth scheme of finite type over an infinite field k and G will denote a finite abelian group with of order ℓ such that ℓ is invertible in k and that k has primitive lth roots of unity. Let X be endowed with a G-action.

Now suppose $G = \mathbb{Z}/\ell$ for some prime power ℓ and let $\sigma \in G$ be a generator and let α be a primitive lth root of unity. By $\mathbb{A}^{\sigma^i}$, we mean $\mathbb{A}^1$ endowed with the G-action corresponding to multiplication by α^i. Morphisms between any G-scheme and $\mathbb{A}^{\sigma^i}$ will be assumed to be G-equivariant.

In this article, we extend the results of [1] to the G-equivariant setting, where G is a finite abelian group under suitable hypotheses.

Theorem 1.2. *Let k be an infinite field and let X be a smooth k-scheme. Let G be a finite abelian group of order ℓ with a G action on X. Let $Z \subset X$ be a closed invariant subscheme of positive codimension and let $z \in Z$ be a (not necessarily closed) point. In addition, assume that ℓ is invertible in k and that k has primitive ℓth roots of unity. Then there exist*

- *a G-equivariant étale neighbourhood $X' \to X$ of Gz;*
- *a smooth G-scheme W;*
- *a G-equivariant étale neighbourhood $X' \to \mathbb{A}^1_W$ of $Z_{X'}$, for some action of G on $\mathbb{A}^1$.*

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These satisfy the following conditions:

- *The composite $Z_{X'} \to \mathbb{A}^1_W \to W$ is finite.*
- *In the case where the action on $\mathbb{A}^1_W$ is non-trivial, then $X' \to W$ admits a section.*

In Section 3, we prove Theorem 1.2 in the case where G is a cyclic group of prime power order. Our proof follows along the same lines as the proof in [1]. More precisely, we generalize the preparatory result [1, Lemma 2.5] and treat the two cases, where $z \in X^H$ for some subgroup $H \subset G$ and where $z \notin X^H$ for any subgroup H, separately. The proof of the first case follows along the same lines as Bachmann's proof, using our generalized version of the preparatory lemma (Lemma 2.2). In the second case, the reduction to the set-theoretically fixed case in [1] has to be generalized to an inductive argument on the size of the group. The case of a general finite abelian group is handled in Section 4.

2. Preliminaries

In this section, we prove some preliminary results about obtaining nice equivariant morphisms to affine spaces, which will be used in the proof of the main theorem. We work in the generality of an arbitrary finite group, keeping future use in mind. We begin with an extension of [1, Lemma 2.4].

Lemma 2.1. *Let X be an affine finite type k-scheme, k an infinite field. Let $S \subset X$ a finite set of closed points. There exists a function $f : X \to \mathbb{A}^d$ inducing a universal injection $S \hookrightarrow \mathbb{A}^d$.*

More generally, suppose given $T \subset X$ another finite set of closed points, disjoint from S. Let $\rho_1, \dots, \rho_m$ be linear automorphisms of $\mathbb{A}^d$, where $m = |S|$. Let $X \hookrightarrow \mathbb{A}^N$ be an embedding. Suppose that T consists of n points after geometric base change. Among the set of polynomial morphisms $X \to \mathbb{A}^d$ of degree $\leq n+m-1$ (with respect to the embedding $X \hookrightarrow \mathbb{A}^N$) vanishing on T, those which induce a morphism $f : S \hookrightarrow \mathbb{A}^d \setminus 0$ satisfying the following conditions forma dense subset:

1. *The morphism f is universally injective.*
2. *For each permutation $\sigma \in S_m$, the property $\sum_{1 \leq i \leq m} \rho_i \cdot f(s_{\sigma.i}) \neq 0$ holds universally.*

Proof. The first claim follows directly from the second by taking $T = \emptyset$. To prove the second claim, we may assume that k is algebraically closed. We need to find a d-tuple of polynomials $P := (P_j)_{1 \leq j \leq d}$ of degree $n+1$ such that for every $s \in S$, we have $P(s) \neq 0$; for $s \neq s'$, we have $P(s) \neq P(s')$ and for each permutation $\sigma \in S_m$, the property $\sum_{1 \leq i \leq m} \rho_i \cdot f(s_{\sigma.i}) \neq 0$. Since each of these conditions are open, we may satisfy them separately. So, without loss of generality, take $\sigma = id$. Clearly, it will suffice to show that there exists a P of degree $\leq n+m-1$ vanishing at T and $s_2, \dots, s_m$, but not at s_1. Taking a product over terms for each $t \in T$ and s_i for $2 \leq i \leq m$, it suffices to find a linear polynomial vanishing at some specific point of $T \cup \{s_2, \dots, s_m\}$, which is clearly possible since this set is disjoint from S. $\square$

We now prove a generalization of [1, Lemma 2.5].

Lemma 2.2. *Let G be a finite group and let X be an affine finite type k-scheme with a G-action, k an infinite field of characteristic relatively prime to l. Let $S \subset X$ be a finite set of closed points. Let ρ be a d-dimensional G-representation.*

1. *For $x \in X^G \setminus S$, there exists an equivariant map $f : X \to \mathbb{A}^d$ (where $\mathbb{A}^d$ has the trivial action) with $f(x) \notin f(S)$.*

(2) *If $S \subset X \setminus X^H$ for any subgroup $H \subset G$, then there exists an equivariant map $f : X \to \mathbb{A}^d$, with $f(s) \neq 0$ for $s \in S$, where $\mathbb{A}^d$ is endowed with the G-action induced by ρ.*

(3) *Suppose $S \subset X \setminus X^H$ for any subgroup $H \subset G$ and $T \subset X$ is another finite set of closed points disjoint from $\cup_{g \in G} gS$. There exists a map $f : X \to \mathbb{A}^d$ vanishing on T and non-vanishing on S.*

Proof.

(1) On applying Lemma 2.1 to X/G and $S/G \cup \{x\}$, we get a morphism $f' : X/G \to \mathbb{A}^d$ injective on $S/G \cup \{x\}$, which in turn gives a G-equivariant morphism $f : X \to \mathbb{A}^d$ such that $f(x) \notin f(S)$.

(2) On applying Lemma 2.1 to X and $G \cdot S$, we get a morphism $f_0 : X \to \mathbb{A}^\sigma$. Put $f = \sum_{g \in G} \rho_{g^{-1}} f_0 g$; this is clearly equivariant with respect to the action of α on $\mathbb{A}^1$. Fix an element $s \in S$. Taking $\{\rho_i\}_i = \{\rho_g\}_{g \in G}$ and $T = \emptyset$ and $\{s_i\}_i = Gs$ in Lemma 2.1 (note that $gs \neq g's$ for $g \neq g'$ by our assumption), we have that the set of f_0 such that f is non-vanishing on S is dense. Therefore, we may choose such an f_0.

(3) Choose an embedding $X \hookrightarrow \mathbb{A}^N$ and let n be the sum of the cardinalities of S and T after geometric base change. Consider the space of polynomials P vanishing on T such that $\deg P \leq n$. Since the set of polynomials P such that $\sum_{g \in G} \rho_{g^{-1}} P g$ does not vanish on S is open, it will suffice us to show that it is non-empty, which we may show after geometric base change.

The conditions on S imply that $\cup_{g \in G} gS = \coprod_{g \in G} gS'$ for some subset $S' \subset S$. Indeed, if $s = gs'$ for some $s, s' \in S$, put $S' = S \setminus s$; the conditions on S imply that the orbit of S' is the same as that of S. We conclude by induction. Put $T' = \cup_{g \neq 1} gS' \cup T$ and apply Lemma 2.1 to yield a P vanishing on T' and non-vanishing on S'. This gives the desired map.

□

Remark 2.3. When G is a cyclic group of order ℓ and k has a primitive ℓth root of unity α, we get an irreducible one-dimensional representation of G given by multiplication by α. In this case, the $\mathbb{A}^d$ in Lemma 2.1 becomes $\mathbb{A}^\sigma$. This case of Lemma 2.1 will be used in the following section.

3. The case of a cyclic group

In this section, we treat the case where G is a cyclic group of order equal to a prime power. The proof is split into two cases which will be treated in the next two sections. The main theorem (Theorem 1.2) in this case follows from Propositions 3.3 and 3.7 below.

3.1. Proof of Theorem 1.2 in the case where $z \in X^H$ for some subgroup $H \subset G$.

In this section, we assume that $z \in X^H$ for some subgroup $H \subset G$. We first prove some preliminary lemmas.

Lemma 3.1. *Let X be a smooth affine scheme over k and let G act on X. Let $Z \subset X$ be a closed invariant subscheme of positive codimension, $z \in Z$ a point such that $z \in X^H$ for some subgroup $H \subset G$ and $\overline{z} \in X^H$ a closed specialisation of z. Moreover, assume that G acts non-trivially on every neighbourhood of z. Then there exist G-equivariant morphisms $f_1, \ldots, f_n : X \to \mathbb{A}^1$ and $g_{i,1}, \ldots, g_{i,m_i} : X \to \mathbb{A}^{\sigma_i}$ for primitive roots of unity $\sigma_1, \ldots, \sigma_t$, such that if $\varphi : X \to \mathbb{A}^{n + \sum_i m_i \sigma_i}$ is the induced morphism, $\dim \varphi^{-1}(\varphi(\overline{z})) \cap Z = 0$.*

Proof. First, note that the action of G on $T_{\overline{z}}X$ is non-trivial. To see this, let $X = \operatorname{Spec} A$ and let $\overline{z}$ correspond to the maximal ideal $\mathfrak{m}$. Since the action of G is nontrivial on every neighbourhood of z, we may assume that the action of σ is non-trivial. Let A_0 be the subring of A such that $\sigma.a = a$ and let A_1 be the eigen-space of σ, which is non-trivial. If σ acted trivially on $T_{\overline{z}}X$, it would act trivially on its dual, which is isomorphic $\mathfrak{m}/\mathfrak{m}^2$. This would imply that $\mathfrak{m}_1/\mathfrak{m}_0\mathfrak{m}_1$=0, where $\mathfrak{m}_i = \mathfrak{m} \cap (A_{\mathfrak{m}})_i$. By considering $\mathfrak{m}_1$ as a finite $(A_0)_{\mathfrak{m}}$-module and using the Nakayama Lemma, we get $\mathfrak{m}_1 = 0$ so that $A_1 = 0$, which contradicts our assumption that σ acts non-trivially on all neighbourhoods of $\overline{z}$.

The group action of G induces a representation on $T_{\overline{z}}X$, which splits into irreducible representations as $k(\overline{z})^n \oplus k(\overline{z})^{\sum(m_i+1)\sigma_i}$, where $m_i \geq 0$.

By Lemma 2.2(1), we have $f_1 : X \to \mathbb{A}^1$ such that $f_1^{-1}(f_1(\overline{z}))$ does not contain any component of positive dimension of Z or Z^G. This implies that $\dim f_1^{-1}(f_1(\overline{z})) \cap Z < \dim Z$ and $f_1^{-1}(f_1(\overline{z})) \cap Z^G < \dim Z^G$. By induction, using the fact that $(f,g)^{-1}((f,g)(\overline{z})) \subset f^{-1}(f(\overline{z}) \cap g^{-1}(g(\overline{z}))$, we get $f_1, \ldots, f_n$ such that $\dim(f_1, \ldots, f_n)^{-1}(f_1, \ldots, f_n)(\overline{z}) \cap Z \leq \dim Z - n$ and $\dim(f_1, \ldots, f_n)^{-1}(f_1, \ldots, f_n)(\overline{z}) \cap Z^G = 0$ because $\dim Z^G \leq \dim X^G = n$. Therefore, each positive dimensional component of $f^{-1}(f(\overline{z})) \cap Z$ has some point not lying in Z^G. Now, $\dim Z \leq \dim X - 1 = n + t - 1 + \sum_i m_i$, so that $\dim(f_1, \ldots, f_n)^{-1}(f_1, \ldots, f_n)(\overline{z}) \cap Z \leq t - 1 + \sum_i m_i$. We apply Lemma 2.2(2) to get an equivariant morphism $g_{i,1} : X \to \mathbb{A}^{\sigma_i}$ such that $\dim(f, g_{i,1})^{-1}((f, g_{i,1})(\overline{z}) \cap Z \leq t - 2 + \sum_i m_i$. Repeating this process for each (i,j), $1 \leq j \leq m_i$, $1 \leq i \leq t$, we get our result by taking $\varphi = (f_1, \ldots, f_n, g_{1,1}, \ldots, g_{1,m_1}, \ldots, g_{t,1}, \ldots, g_{t,m_t})$. □

Lemma 3.2. *With the same notation as in Lemma 3.1, there exists a G-equivariant closed immersion $X \hookrightarrow \mathbb{A}^{N+\sum_i M_i\sigma_i}$ where N, M_i are integers such that $N \geq n$ and $M_i \geq m_i$ satisfying the following property. For a general linear projection $\varphi : \mathbb{A}^{N+\sum_i M_i\sigma_i} \to \mathbb{A}^{n+\sum_i m_i\sigma_i} \times \mathbb{A}^{\sum_i \sigma_i}$, we have:*

1. *The composite morphism $Z \to \mathbb{A}^{n+\sum_i m_i\sigma_i} \times \mathbb{A}^{\sum_i \sigma_i} \to \mathbb{A}^{n+\sum_i m_i\sigma_i}$ is quasi-finite at $\overline{z}$.*
2. *The composite $X \hookrightarrow \mathbb{A}^{N+\sum_i M_i\sigma_i} \to \mathbb{A}^{n+\sum_i m_i\sigma_i} \times \mathbb{A}^{\sum_i \sigma_i}$ is étale at $\overline{z}$.*

Proof. Let $p_1, \ldots, p_N$ be the generators of $\mathcal{O}_X(X)$. We claim that we may choose the p_i such that $\sigma \cdot p_i = \alpha' \cdot p_i$ for some primitive root of unity α. If this holds, the morphism induced by $p_i : X \to \mathbb{A}^{\sigma'}$ is equivariant, where σ' is the element of G inducing the representation α on k. Combining these with the morphisms of Lemma 3.1, we get our closed immersion $X \hookrightarrow \mathbb{A}^{N+\sum_i M_i}$.

To prove the claim, observe that if $p \in \mathcal{O}_X(X)$, then

$$l^{-1} \cdot \sum_{i=0}^{l-1} \alpha^{ij} \cdot \sigma^i p$$

for $0 \leq j \leq l-1$ satisfy our conditions. Moreover, p can be recovered by adding up these expressions over j. So applying this construction to each p_i, the claim is proved.

To prove that our immersion satisfies conditions (1) and (2), observe that both conditions are open and therefore, we need only check that they are non-empty. Therefore, if we take the projection ϕ such that they give the f_is and $g_{i,j}$s of Lemma 3.1 when composed with X, we are done by Lemma 3.1.

To show the condition (2), we may base change to the algebraic closure of k. It suffices to show that the condition is non-empty for each point lying over $\overline{z}$. Therefore, we may assume that $\overline{z}$ is a rational point, so that $k(\overline{z}) \cong k$. Also, the inclusion $T_{\overline{z}}X \to k^{N+\sum_i M_i}$ splits equivariantly, giving an isomorphism with the tangent space of $\mathbb{A}^{n+\sum_i m_i\sigma_i} \times \mathbb{A}^{\sum_i \sigma_i}$ which is $k^{n+\sum_i(1+m_i)\sigma_i}$. Therefore, by [5, Theorem 17.11.1 (d)], we are done.

□

Now we are ready to the presentation lemma under the conditions of Lemma 3.1.

Proposition 3.3. *Theorem 1.2 holds in the case where $z \in X^H$ for some subgroup $H \subset G$.*

Proof. We may assume that the conditions of Lemma 3.1 is satisfied. Indeed, if not, there is a neighbourhood of z on which G acts trivially, in which case we reduce to the usual Gabber lemma.

First we produce étale maps $X \leftarrow X_3 \to W_z^h \times A^\sigma$ such that the preimage of Z is finite over W_z^h and maps via a closed immersion to $W_z^h \times \mathbb{A}^\sigma$. By Lemma 3.2, we have a G-equivariant closed immersion $X \hookrightarrow \mathbb{A}^{N+\sum_i M_i\sigma_i}$ and a linear projection $\varphi : \mathbb{A}^{N+\sum_i M_i\sigma_i} \to \mathbb{A}^{n+\sum_i m_i\sigma_i} \times \mathbb{A}^{\sum_i \sigma_i}$, such that the maps $Z \to \mathbb{A}^{n+\sum_i m_i\sigma_i}$ is quasi-finite at $\overline{z}$ and $X \to \mathbb{A}^{n+\sum_i m_i\sigma_i} \times \mathbb{A}^{\sum_i \sigma_i}$ is étale at $\overline{z}$. After passing to a smaller open subset if necessary, we may assume that these two maps are quasi-finite and étale respectively. Choose some σ_i, which we take to be $\sigma_1 \in H$ without loss of generality. Let $W \subset X$ denote the vanishing locus of the projection $X \to \mathbb{A}^{\sigma_1}$. Since W is étale over an affine space, it is smooth. Also $z \in W$ since $X^H \subset W$; indeed $x \in X^H$ implies that $g(x) = g(\sigma_1 \cdot x) = \alpha \cdot g(x)$ so that $g(x) = 0$. We have the following pullback squares.

$$\begin{array}{ccccc} X_1 & \longrightarrow & W \times \mathbb{A}^{\sigma_1} & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{A}^{n+\sum m_i\sigma_i} \times \mathbb{A}^{\sum_i \sigma_i} & \longrightarrow & \mathbb{A}^{n+\sum m_i\sigma_i} \times \mathbb{A}^{\sum_{i\neq 1} \sigma_i} \end{array}$$

Since W is a closed subscheme of X, $W \times_{\mathbb{A}^{n+\sum m_i\sigma_i} \times \mathbb{A}^{\sum_{i\neq 1}\sigma_i}} W$ is a closed subscheme of X_1. Since both schemes are étale over $\mathbb{A}^{n+\sum m_i\sigma_i} \times \mathbb{A}^{\sum_i \sigma_i}$ and separated, the diagonal $W \to W \times_{\mathbb{A}^{n+\sum m_i\sigma_i} \times \mathbb{A}^{\sum_{i\neq 1}\sigma_i}} W$ is both open and closed. Therefore, we may write the scheme $W \times_{\mathbb{A}^{n+\sum m_i\sigma_i} \times \mathbb{A}^{\sum_{i\neq 1}\sigma_i}} W$ as a disjoint union $W \coprod W'$ and put $X_1' = X_1 \setminus W'$. We have the following pullback squares.

$$\begin{array}{ccccccc} W_z^h & \longrightarrow & X_2 & \longrightarrow & W_z^h \times \mathbb{A}^{\sigma_1} & \longrightarrow & W_z^h \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W & \longrightarrow & X_1' & \longrightarrow & W \times \mathbb{A}^{\sigma_1} & \longrightarrow & W \end{array}$$

Since W_z^h lifts to X_2 as the square indicates, z lifts to X_2. Since $Z \to \mathbb{A}^{n+\sum m_i\sigma_i}$ is quasi-finite at z, so is $Z_{X_2} \to W_z^h$. By [8, Theorem 10.153.4], we can write $Z_{X_3} = A_1 \coprod \cdots \coprod A_n \coprod B$, where A_i are local finite schemes over W_z^h and $B \to W_1$ is not quasi-finite over any point over z. Therefore, without loss of generality, we may take $z \in A_1$. Set $X_3 = X_2 \setminus (\cup_{i>1} A_i \cup B)$. Then $Z_{X_3} \to W_z^h$ is finite and the composite map $Z_{X_3} \to X_3 \to W_z^h \times \mathbb{A}^{\sigma_1}$ is finite and unramified. Since the fiber of Z_{X_3} over z consists solely of z, the fiber of $Z_{X_3} \to W_z^h \times \mathbb{A}^{\sigma_1}$ is a closed immersion. By Nakayama's lemma, the map $Z_{X_3} \to W_z^h \times \mathbb{A}^{\sigma_1}$ is itself a closed immersion.

Note that in Diagram 3.1, the composite of the bottom row $W \to X_1' \to W \times \mathbb{A}^{\sigma_1} \to W$ is the identity and therefore, so is its base change $W_z^h \to X_2 \to W_z^h \times \mathbb{A}^{\sigma_1} \to W_z^h$. This implies that $W_z^h \to X_3 \to W_z^h$ is the identity so that $X_3 \to W_z^h$ admits a section.

The rest of the proof proceeds as in [1, Page 592, Step 5]. Note that $W_z^h \subset X_3$. Indeed, for $i > 1$, we have that $W_z^h \cup A_i \subset W_z^h \subset X_2$ is closed, but does not contain z and is therefore empty and by the same argument, so is $W_z^h \cup B$. The map $Z_{X_3} \to \varphi^{-1}(\varphi(Z_{X_3}))$ is both open and closed and so we may write $\varphi^{-1}(\varphi(Z_{X_3})) = Z_{X_3} \coprod Z'$. We have $z \notin Z'$ and therefore, $Z' \cap W_z^h = \emptyset$ and $W_z^h \subset X_4 := X_3 \setminus Z'$. By definition, $X_4 \to W_z^h \times \mathbb{A}^{\sigma_1}$ is an étale neighbourhood of Z_{X_4}. The composite $W_z^h \to X_4 \to W_z^h \times \mathbb{A}^{\sigma_1}$ consists of finitely presented W_z^h schemes. Writing W_z^h as the cofiltered limit of étale neighbourhoods of z in W, we conclude by continuity that there exists such an étale neighbourhood $\widetilde{W} \to W$ together with a model $\widetilde{W} \to \widetilde{X_4} \to \widetilde{W} \times \mathbb{A}^{\sigma_1}$ of the previous composite, satisfying all the properties as above. In particular, $\widetilde{X_4} \to \widetilde{W} \times \mathbb{A}^{\sigma_1}$ is an étale neighbourhood of $Z_{\widetilde{X_4}}$ and $\widetilde{X_4}$ is an open subscheme of $X_1' \times_W \widetilde{W}$. Since $\widetilde{W} \to W$ and $X_1' \to X$ are open neighbourhoods of z, so is $\widetilde{X_4} \to X$.

Note that since $X_3 \to W_z^h$ admits a section, so does $X_4 \to W_z^h$ and therefore, so does $\widetilde{X_4} \to \widetilde{W}$. Setting $X' := \widetilde{X_4}$ and $W := \widetilde{W}$ in the statement of Theorem 1.2, we get the desired result. □

3.2. **Proof of Theorem 1.2 in the case where $z \notin X^H$ for any subgroup $H \subset G$.** Now, it remains for us to treat the case where $z \notin X^H$ for any subgroup $H \subset G$. After replacing X by $X \setminus (\cup_{H \subset G} X^H)$, we may assume that G acts freely on X.

Lemma 3.4. *We may assume without loss of generality that $\sigma z = z$, where σ is a generator of G.*

Proof. If $\sigma^i z \neq z$ for all $i \neq 0$, we have $Gz = G \times z$ and therefore, the projection map $G \times X \to X$ is an equivariant étale neighbourhood of z. Therefore, we may replace X by $G \times X$. Now, $G \times X \to G$ is a smooth map to a zero-dimensional scheme on which G acts freely. Thus, we may reduce to the non-equivariant Gabber lemma over $(G \times X)/G \to \operatorname{Spec} k$.

On the other hand if $\sigma z \neq z$ and $\sigma^i z = z$ for some $i > 1$. Let H be the subgroup generated by σ^i. Since G is cyclic, G/H is isomorphic to some subgroup $H' \subset G$ such that $G/H' \cong H$. Note that $Gz = G/H \times z \cong H' \times z$, so that $H' \times X \to X$ is an equivariant étale neighbourhood of Gz. The projection map $H' \times X \to H'$ is a G-equivariant smooth map between schemes on which H' acts freely. Now, the G-equivariant case of the map $H' \times X \to H'$ is equivalent to the $G/H' \cong H$ equivariant case of the map $(H' \times X)/H' \to H'/H' \cong \operatorname{Spec} k$ which we may assume by induction on the size of G since H' is a cyclic group with a prime power order strictly less than the order of G. □

The proof of the next lemma is identical to the corresponding section in [1, Page 593] and therefore, we omit it.

Lemma 3.5. *We may assume without loss of generality that* $\dim X > 1$.

Because of this lemma, we may assume that $\dim X = n + 1$ for some $n > 0$. The next Lemma corresponds to [1, Page 593, Step 1].

Lemma 3.6. *Let $\overline{z}$ be a closed specialisation of z. There exists a G-equivariant map $\varphi : X \to \mathbb{A}^{n\sigma}$ which is smooth at $\overline{z}$, and whose restriction to Z is quasi-finite at $\overline{z}$ and which is non-vanishing at $\overline{z}$.*

Proof. After choosing an open embedding into some G-representation and considering the set of linear projections from it to $\mathbb{A}^{n\sigma}$, we can take the conditions of quasi-finiteness, smoothness and non-vanishing at $\overline{z}$ to be open conditions. Therefore, it suffices to show that each of the conditions may be satisfied separately.

The non-vanishing condition is already satisfied because of Lemma 2.2(2).

For quasi-finiteness, the proof is very similar to that of Lemma 3.1. Given a closed subset Z' of X of dimension $\leq d$, by Lemma 2.2(3) we can find a map $f : X \to \mathbb{A}^\sigma$ vanishing at $\overline{z}$ with $\dim Z' \cap Z(f) < d$. Indeed, let S be a set of points in each $\dim d$ component of Z such that none of them are conjugate to $\overline{z}$; let T be the set of conjugates of $\overline{z}$. Then Lemma 2.2(3) gives a map satisfying our conditions. By the same arguments as in the proof of Lemma 3.1, we can construct the required map $X \to \mathbb{A}^{n\sigma}$.

For smoothness, choose finitely many generators of $\mathfrak{m}_z$, the maximal ideal at z and let V be the affine space spanned by them. For general elements $f_1, \dots, f_n$ of V, we show that

$$\varphi = \Big(\sum_{0\leq i\leq l-1} \alpha^{-i}\cdot f_1\sigma^i, \sum_{0\leq i\leq l-1} \alpha^{-i}\cdot f_2\sigma^i, \dots, \sum_{0\leq i\leq l-1} \alpha^{-i}\cdot f_n\sigma^i\Big). \tag{3.1}$$

will satisfty the desired conditions. By the same argument as in the proof of Lemma 2.2, we can see that φ is G-equivariant. For the smoothness, we may base change to the algebraic closure, where $\overline{z}$ splits into finitely many G orbits $Gz_1, \dots, Gz_r$ and V is the affine space generated by a set of generators of the maximal ideal at each point in $\cup_i Gz_i$. It suffices to show that φ is smooth at Gz_1 without loss of generality. The map $V \to \bigoplus_{\sigma\in G} \Omega_{\sigma z_1} X$ is surjective as the right hand side is a quotient of $\mathfrak{m}_{\overline{z}}/\mathfrak{m}_{\overline{z}}^2 \otimes \overline{k}$. Let $e_1, \dots, e_n$ be a basis of $\Omega_{z_1} X$; if we pick each f_i to be a preimage of $(e_i, 0, \dots, 0)$, we are done. □

Now we proceed with the final part of the proof.

Proposition 3.7. *Theorem 1.2 holds in the case where $z \notin X^H$ for any subgroup $H \subset G$.*

Proof. We first construct étale maps $X \leftarrow X_3 \to W \times \mathbb{A}^1$ such that the induced map $Z_{X_3} \to W \times \mathbb{A}^1$ is finite over the Henselian local scheme W and maps via a closed immersion to $W \times \mathbb{A}^\sigma$. By Lemma 3.6, and passing to an open subset if necessary, we may assume that we have a smooth map $\varphi : X \to \mathbb{A}^{n\sigma}$ which is smooth and quasi-finite when restricted to Z and such that $w := \varphi(\overline{z}) \neq 0$. Since $\overline{z}$ is set-theoretically fixed, so is w, but it has a non-trivial G-action scheme-theoretically. Set $W = (\mathbb{A}^{n\sigma})^h_w$ and $X_1 = X \times_{\mathbb{A}^{n\sigma}} W$. Since $Z_{X_1} \to W$ is quasi-finite, just as in the proof of Proposition 3.3, we can define X_2 by removing finitely many connected components of X_1 such that $z \in X_2$ and $Z_{X_2} \to W$ is finite.

As in [1, Page 594], we claim that there exists a map $X \to \mathbb{A}^1_W$ such that the special fibre $X_w \to \mathbb{A}^1_w$ is quasi-finite, étale and geometrically injective at z. Indeed, it is shown in [1, Page 594, Proof of claim] that there exists a map $f_w : X_w \to \mathbb{A}^1_w$ with the desired properties. We take any (not-necessarily equivariant) lift $f_0 : X \to \mathbb{A}^1_W$ of f_w. The map $f := \sum_i f_0\sigma^i$ will satisy the required conditions.

By Miracle Flatness ([8, Theorem 10.128.1]), we have that this map is flat at z and therefore, étale (see [8, 29.36.15]).

Just as in the proof of Proposition 3.3, we define $X_4 = X_3 \setminus \varphi^{-1}(\varphi(Z_{X_3}) \setminus Z_{X_3}$. We get an étale neighbourhood $X_4 \to \mathbb{A}^1_W$ of Z_{X_4}. These are all finitely presented schemes over W, which is the direct limit of the étale neighbourhoods of $w \in \mathbb{A}^{n\sigma}$. Similarly to the proof of Proposition 3.3, we obtain an étale neighbourhood $\widetilde{W} \to \mathbb{A}^{n\sigma}$ of w and a model $\widetilde{X_4} \to \mathbb{A}^1_{\widetilde{W}}$ having the same properties as before. Since $\widetilde{W} \to \mathbb{A}^{n\sigma}$ is an étale neighbourhood of W, so is $\widetilde{X_4} \to X$ an étale neighbourhood of z. Setting $X' := \widetilde{X}_4$ and $W := \widetilde{W}$ in the statement of Theorem 1.2, we get the desired result.

□

4. The case of a general finite abelian group

By the structure theorem for finite abelian groups, we may assume $G = \prod_{1\leq i\leq m} G_i$, where each G_i is a cyclic group of order l_i, where each l_i is a power of a prime. Assume that primitive l_ith roots of unity exist in k. Let $l = \prod_i l_i$. The proof of Theorem 1.2 in this case follows exactly the same lines as in the case of a cyclic group. We will need the following variation of Lemma 3.4.

Lemma 4.1. *Assume that the action of G on X is free and $z \notin X^H$ for any subgroup $H \subset G$. To prove the Gabber presentation lemma, we may assume without loss of generality that $(\sigma_i)_i z = z$, where each σ_i is a generator of G_i.*

Proof. If $\sigma_i^j z \neq z$ for all $i, j \neq 0$, we have $Gz = G \times z$ and therefore, the projection map $G \times X \to X$ is an equivariant étale neighbourhood of z. Therefore, we may replace X by $G \times X$. Now, $G \times X \to G$ is a smooth map to a zero-dimensional scheme on which G acts freely. Thus, we may reduce to the non-equivariant Gabber lemma over $(G\times X)/G \to \operatorname{Spec} k$.

Otherwise, for each i, let $H_i \subset G_i$ be the maximal subgroup that fixes z. Since each G_i is cyclic, there exists $H_i' \subset G_i$ such that $G/H_i \cong H_i'$. We have $Gz = G' \times z$, where $G' := \prod_i H_i'$, so that $G' \times X \to X$ is an equivariant étale neighbourhood of Gz. The projection morphism $G' \times X \to G'$ is a smooth G-equivariant morphism between schemes on which G' acts freely. Now, the G-equivariant case of the map $G' \times X \to G'$ is equivalent to the $G/G' \cong \prod_i H_i$ equivariant case of the map $(G'\times X)/G' \to G'/G' \cong \operatorname{Spec} k$ which we may assume by induction on the size of G as $\prod_i H_i$ is a product of cyclic groups of order strictly less than G. □

Theorem 4.2. *Theorem 1.2 holds for a general abelian group.*

Proof. The proof goes through exactly as in the case where G is a cyclic group, by using Lemma 4.1 in place of Lemma 3.4. The only other change that we need to replace the definition of φ in (3.1) with

$$\varphi = \Big(\sum\big(\prod_i \alpha_i^{-n_i} f_1.\prod_i \sigma_i^{n_i}\big), \sum\big(\prod_i \alpha_i^{-n_i} f_2.\prod_i \sigma_i^{n_i}\big), \ldots, \sum\big(\prod_i \alpha_i^{-n_i} f_n.\prod_i \sigma_i^{n_i}\big)\Big), \tag{4.1}$$

where each of the sums are taken over $(n_i)_i$ such that $0 \leq n_i < l_i$. □

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