CORRIGENDUM: STRONG $\mathbb{A}^1$-INVARIANCE OF $\mathbb{A}^1$-CONNECTED COMPONENTS OF REDUCTIVE ALGEBRAIC GROUPS (J. TOPOL. 16 (2023), NO. 2, 634–649.)

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Abstract. The proof of [2, Lemma 5.1] is incomplete as it relies on some results in [4], the proof of which contains a gap. The goal of this note is to give a complete and self-contained proof of [2, Lemma 5.1].

1. Introduction

The main theorem of [2] is as follows:

Theorem 1.1. [2, Theorem 1.1, Remark 1.2] *If G is an algebraic group over a perfect field k, then $\pi_0^{\mathbb{A}^1}(G)$ is a strongly $\mathbb{A}^1$-invariant sheaf. Consequently, for any Nisnevich locally trivial G-torsor $\mathcal{P} \to \mathcal{X}$ in $\mathcal{H}(k)$,*

$$P \to X \to BG$$

is an $\mathbb{A}^1$-fiber sequence.

One of the key steps in the proof of Theorem 1.1 is [2, Lemma 5.1]. The proof of [2, Lemma 5.1] given in [2] is incomplete for two reasons -

(1) it relies on a result in [4], the proof of which contains a gap, and
(2) the existence of the key exact sequence at the bottom of [2, page 646] (in particular, the application of [7, Theorem 6.50]) is not properly justified.

In this note, we give a complete and self-contained proof of [2, Lemma 5.1], thereby completing the proof of Theorem 1.1.

We briefly outline the contents of this note. Section 2 contains preliminaries on long exact sequences of simplicial homotopy groups associated with homotopy principal fibrations of simplicial sheaves. In Section 3, we establish the key exact sequence mentioned in (2) above. In Section 4, we use the results of Section 3 along with an argument circumventing the use of results from [4] to give a proof of [2, Lemma 5.1] (see Lemma 4.2).

Notations and conventions. We will follow the notation used in [8], [7] and [2]. The letter k will always denote a perfect field. Let $\mathcal{H}_s(k)$ and $\mathcal{H}(k)$ denote the (pointed) unstable simplicial and $\mathbb{A}^1$-homotopy categories introduced by Morel and Voevodsky in [8]. These are obtained as homotopy categories associated with the locally injective model structrure on the category $\Delta^{op}Shv(Sm/k)_{\rm Nis}$ of simplicial Nisnevich sheaves of sets on the category

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Sm/k of essentially smooth, separated schemes on k and its Bousfield localization at the class of projection maps $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$, respectively. For each $n \in \mathbb{N}$ and every pointed object $\mathcal{X} \in \Delta^{op}Shv(Sm/k)_{\text{Nis}}$, the simplicial homotopy sheaf $\pi_n(\mathcal{X})$ is defined to be the Nisnevich sheafification of the presheaf on Sm/k

$$U \mapsto \mathrm{Hom}_{\mathcal{H}_s(k)}(\Sigma^n U_+, \mathcal{X})$$

and the $\mathbb{A}^1$-homotopy sheaf $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is defined to be the Nisnevich sheafification of the presheaf on Sm/k

$$U \mapsto \mathrm{Hom}_{\mathcal{H}(k)}(\Sigma^n U_+, \mathcal{X}),$$

where we suppress the basepoint whenever it is clear from the context.

2. Principal fibrations of simplicial sheaves on a Grothendieck site

In this section, we collect preliminaries regarding the long exact sequence of homotopy groups associated with principal fibrations of simplicial sheaves that will be used in the proof of the main result. The key result here is Proposition 2.4, which allows us to compare homotopy quotients in the categories of Nisnevich and étale simplicial sheaves. The rest of the material in this section is expected to be well-known to experts and is included only for the convenience of the reader.

Let T be a Grothendieck site with enough points. We work with the category $\Delta^{op}Shv(T)$ of simplicial sheaves on T endowed with the locally injective model structure. We will denote by $*$ the final object of T.

Proposition 2.1. *Let $\mathcal{P} \to \mathcal{X}$ be a local fibration and $\mathcal{Q} \to \mathcal{X}$ be any morphism in $\Delta^{op}Shv(T)$. Then we have a weak equivalence*

$$\mathcal{P} \times_{\mathcal{X}} \mathcal{Q} \cong \mathcal{P} \times^h_{\mathcal{X}} \mathcal{Q}$$

between the fiber product and the homotopy fiber product of $\mathcal{P}$ and $\mathcal{Q}$ over $\mathcal{X}$.

Proof. We factorize $\mathcal{Q} \to \mathcal{X}$ as $\mathcal{Q} \to \mathcal{Q}' \to \mathcal{X}$ where $\mathcal{Q} \to \mathcal{Q}'$ is a weak equivalence and $\mathcal{Q}' \to \mathcal{X}$ is a fibration. We then have the diagram of cartesian squares

$$\begin{array}{ccccc} \mathcal{P} \times_{\mathcal{X}} \mathcal{Q} & \longrightarrow & \mathcal{P} \times_{\mathcal{X}} \mathcal{Q}' & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Q} & \longrightarrow & \mathcal{Q}' & \longrightarrow & \mathcal{X} \end{array}$$

where $\mathcal{P} \times_{\mathcal{X}} \mathcal{Q}'$ is (weakly equivalent to) the homotopy fibre product $\mathcal{P} \times^h_{\mathcal{X}} \mathcal{Q}$. We wish to prove that the top left arrow is a weak equivalence. It is enough to prove this on stalks.

Let x denote a point of the site. We wish to prove that

$$(\mathcal{P} \times_{\mathcal{X}} \mathcal{Q})_x \to (\mathcal{P} \times_{\mathcal{X}} \mathcal{Q}')_x$$

is a weak equivalence. Since filtered colimits commute with finite limits in the category of sheaves of sets (and hence also in the category of sheaves of simplicial sets), this is equivalent to proving that

$$\mathcal{P}_x \times_{\mathcal{X}_x} \mathcal{Q}_x \to \mathcal{P}_x \times_{\mathcal{X}_x} \mathcal{Q}'_x$$

is a weak equivalence. This is true since $P_x \to X_x$ is a fibration, $\mathcal{Q}_x \to \mathcal{Q}'_x$ is a weak equivalence and since the model category of simplicial sets is right proper. $\square$

We briefly recall the terminology related to group actions (see [8, page 128] and [7, page 170] for details):

Let $\mathcal{G}$ be a simplicial group sheaf. Recall that a *principal $\mathcal{G}$-fibration* (or a *principal $\mathcal{G}$-bundle*) is a morphism $\mathcal{P} \to \mathcal{X}$ along with a free $\mathcal{G}$-action on $\mathcal{P}$ over $\mathcal{X}$ such that the induced morphism $\mathcal{P}/\mathcal{G} \to \mathcal{X}$ is an isomorphism.

More generally, if $\mathcal{P}$ is a simplicial sheaf with a $\mathcal{G}$-action, we consider the diagonal action of $\mathcal{G}$ on $E\mathcal{G} \times \mathcal{P}$, which is easily seen to be free. We will refer to the quotient $(E\mathcal{G} \times \mathcal{P})/\mathcal{G}$ as the *homotopy quotient* of the $\mathcal{G}$-action on $\mathcal{P}$. A morphism $\mathcal{P} \to \mathcal{X}$, along with a $\mathcal{G}$-action on $\mathcal{P}$ over $\mathcal{X}$, is said to be a *homotopy principal $\mathcal{G}$-fibration* if it induces a weak equivalence between $\mathcal{X}$ and the homotopy quotient of the $\mathcal{G}$-action on $\mathcal{P}$.

Corollary 2.2. *Let $\mathcal{G}$ be a simplicial group sheaf and let $\mathcal{P} \to \mathcal{X}$ be principal $\mathcal{G}$-fibration in $\Delta^{op}Shv(T)$. Then, for any global point $x : * \to \mathcal{X}$, we have a weak equivalence*

$$\mathcal{P} \times_{\mathcal{X},x} * \cong \mathcal{P} \times^h_{\mathcal{X},x} *.$$

Proof. Any principal $\mathcal{G}$-fibration is a local fibration by [6, Chapter 5, Corollary 2.6]. Thus, the result follows from Proposition 2.1. □

Lemma 2.3. *Let $\mathcal{G}$ be a simplicial group sheaf and let $p : \mathcal{P} \to \mathcal{X}$ be principal $\mathcal{G}$-fibration in $\Delta^{op}Shv(T)$. Let U be an object of T and let $u : U \to \mathcal{P}$ be a morphism. We have the long exact sequence*

$$\ldots \to \pi_i(\mathcal{G}|_U) \to \pi_i(\mathcal{P}, u) \to \pi_i(\mathcal{X}, p \circ u) \to \ldots \pi_0(\mathcal{G}|_U) \to \pi_0(\mathcal{P}, u) \to \pi_0(\mathcal{X}, p \circ u) \to *$$

of homotopy group sheaves over T/U. (As usual, this is an exact sequence of group sheaves for $i > 0$, we have an action of $\pi_1(\mathcal{X}, p \circ u)$ on $\pi_0(\mathcal{G})$ and for the terms involving π_0 it is an exact sequence of pointed sets.)

Proof. The restriction functor $\Delta^{op}Shv(T) \to \Delta^{op}Shv(T/U)$, $\mathcal{F} \mapsto \mathcal{F}|_U$ is exact. Thus, $\mathcal{P}|_U \to \mathcal{X}|_U$ is a principal $\mathcal{G}|_U$-fibration. Using Corollary 2.2, we get a homotopy fiber sequence

$$\mathcal{P}|_U \times_{\mathcal{X},p\circ u} U|_U \to \mathcal{P}|_U \to \mathcal{X}|_U.$$

The morphism $\mathcal{P}|_U \times_{\mathcal{X},p\circ u} U|_U$ is a trivial $\mathcal{G}|_U$-torsor over $U|_U$. Now, the long exact sequence of homotopy group sheaves corresponding to this homotopy fiber sequence gives the desired long exact sequence in the statement of the lemma. □

We now specialize to the situation to which we wish to apply the above results. Let k be a field and let Sm/k denote the category of smooth, essentially of finite type, separated schemes over k. Let $\pi : (Sm/k)_{\text{ét}} \to (Sm/k)_{\text{Nis}}$ be the obvious morphism of sites. We will work with the categories $\Delta^{op}Shv(Sm/k)_{\text{ét}}$ and $\Delta^{op}Shv(Sm/k)_{\text{Nis}}$ of simplicial étale and Nisnevich sheaves on Sm/k, respectively, equipped with the locally injective model structure.

Proposition 2.4. *Let H be an étale (simplicially discrete) group sheaf on Sm/k. Let P be a fibrant object in $\Delta^{op}Shv(Sm/k)_{\acute{e}t}$ with an H-action. Let $p : P \to Q$ be the homotopy quotient of this action in $\Delta^{op}Shv(Sm/k)_{\text{Nis}}$ and let $\widetilde{p} : P \to \widetilde{Q}$ be the homotopy quotient in $\Delta^{op}Shv(Sm/k)_{\acute{e}t}$. Let $Q_{\acute{e}t} = \mathbf{R}\pi_*(Q)$. Let $U \in Obj(Sm/k)$ and let $u : U \to \mathcal{P}$ be any morphism. Then, for any $i \geq 1$, the homomorphism $\pi_i(Q, p \circ u) \to \pi_i(Q_{\text{et}}, \widetilde{p} \circ u)$ is an isomorphism.*

Proof. We have the simplicial fibration sequence

$$H|_U \to P|_U \to Q|_U$$

in $\Delta^{op}Shv(Sm/U)_{\mathrm{Nis}}$. Using Lemma 2.3, this gives us the long exact sequence

$$\begin{aligned} \cdots \to \pi_i(H|_U) \to \pi_i(P,u) \to \pi_i(Q, p\circ u) \to \pi_{i-1}(H|_U) \to \cdots \\ \cdots \to \pi_0(H|_U) \to \pi_0(P,u) \to \pi_0(Q, p\circ u) \to *. \end{aligned} \tag{2.1}$$

Similarly, We have the simplicial fibration sequence

$$H|_U \to P|_U \to \widetilde{Q}|_U$$

in $\Delta^{op}Shv(Sm/U)_{\acute{e}t}$. Applying $\mathbf{R}\pi_*$, we get the simplicial fibration sequence

$$H|_U \to P|_U \to Q_{\mathrm{et}}|_U.$$

Note that we have used here the fact that H and P are fibrant. We obtain the long exact sequence

$$\begin{aligned} \cdots \to \pi_i(H|_U) \to \pi_i(P|_U,u) \to \pi_i(Q_{\acute{e}t}|_U, p\circ u) \to \pi_{i-1}(H|_U) \to \cdots \\ \cdots \to \pi_0(H|_U) \to \pi_0(P,u) \to \pi_0(Q_{\acute{e}t}|_U, p\circ u). \end{aligned} \tag{2.2}$$

(Note that unlike the sequence (2.1), the above long exact sequence does not necessarily end with an epimorphism.) The canonical morphism $Q \to Q_{\acute{e}t} := \mathbf{R}\pi_*\pi^*Q$ induces a commutative diagram of simplicial fibration sequences

$$\begin{array}{ccccc} H|_U & \longrightarrow & P|_U & \longrightarrow & Q|_U \\ \downarrow & & \downarrow & & \downarrow \\ H|_U & \longrightarrow & P|_U & \longrightarrow & Q_{\acute{e}t}|_U \end{array}$$

which gives us a map between the corresponding long exact sequences (2.2) and (2.1). The proposition now follows by an application of the five lemma. □

Remark 2.5. Actually, the proof of Proposition 2.4 shows that for $i > 0$, the morphism $Q \to Q_{\acute{e}t}$ induces an isomorphism of groups

$$Hom_{\mathcal{H}_s(k)}(\Sigma^i U_+, Q) \xrightarrow{\sim} Hom_{\mathcal{H}_s(k)}(\Sigma^i U_+, Q_{\acute{e}t})$$

for any $U \in Obj(Sm/k)$. In other words, the canonical morphism $Q \to Q_{\acute{e}t}$ induces an isomorphism of *presheaves* of i-th homotopy groups for $i > 0$. We have only stated the result in terms of the associated homotopy group sheaves because that is what we will require in Section 3.

If we consider the special case $P = *$, the homotopy quotient Q is just the classifying space BH_{Nis} (where $H_{\mathrm{Nis}} := \pi_*(H)$) and $Q_{\acute{e}t}$ is the étale classifying space $B_{\acute{e}t}H$. We recall the following assertion from [8, page 130]:

$$\mathrm{Hom}_{\mathcal{H}_s(k)}(\Sigma^i U_+, B_{\acute{e}t}H) = \begin{cases} H^1_{\acute{e}t}(U,H), & \text{if } i = 0; \\ H(U), & \text{if } i = 1; \\ 0, & \text{if } i > 1 \end{cases}$$

For $i > 0$, this is just the statement that the morphism $BH_{\mathrm{Nis}} \to B_{\acute{e}t}H$ induces an isomorphism of n-th homotopy group presheaves. Thus, Proposition 2.4 can be viewed as a generalization of this statement to homotopy quotients of actions of H on other spaces.

3. A key long exact sequence

Let $\rho : \widetilde{G} \to G$ be a central isogeny of semisimple algebraic groups with kernel a group μ of multiplicative type. The morphism $\widetilde{G} \to G$ is an étale-locally trivial μ-torsor. Let $\widetilde{Q}$ and Q denote the étale and Nisnevich homotopy quotients of the action of $\widetilde{G}$ on G, respectively. Let $\pi : \Delta^{op}Shv(Sm/k)_{\text{ét}} \to \Delta^{op}Shv(Sm/k)_{\text{Nis}}$ denote the canonical morphism. Observe that G is a fibrant object in $\Delta^{op}Shv(Sm/k)_{\text{ét}}$ and that $\widetilde{Q}$ and Q can be explicitly described as the quotients

$$Q = (G \times E\widetilde{G})/^{\text{Nis}}\widetilde{G}$$

and

$$\widetilde{Q} = (G \times E\widetilde{G})/^{\text{ét}}\widetilde{G} = B\mu$$

up to weak equivalence. Note also that

$$Q_{\text{ét}} := \mathbf{R}\pi_*\widetilde{Q} = B_{\text{ét}}\mu.$$

Thus, there is a canonical morphism $Q \to B_{\text{ét}}\mu$.

Lemma 3.1. *The morphism $\pi_0(Q) \to \pi_0(B_{\acute{e}t}\mu)$ is a monomorphism.*

Proof. The assertion of the lemma can be verified stalkwise. Let U be an essentially smooth Henselian local scheme. We will show that $\pi_0(Q)(U) \to \pi_0(B_{\text{ét}}\mu)(U) \cong H^1_{\text{ét}}(U, \mu)$ is an injective map. It is easy to check that $\pi_0(Q)(U)$ is the quotient of the action of $\widetilde{G}(U)$ on $G(U)$. Let H denote the Galois group of $U^{sh} \to U$, where U^{sh} denotes the strict henselization of U. Consider the short exact sequence

$$1 \to \mu(U^{sh}) \to \widetilde{G}(U^{sh}) \to G(U^{sh}) \to 1$$

of H-modules. Since $\mu(U^{sh})$ is central in $\widetilde{G}(U^{sh})$, the long exact sequence of Galois cohomology groups (see [9, Chapter I, Section 5.7, Proposition 43], for example) takes the form

$$1 \to \mu(U) \to \widetilde{G}(U) \to G(U) \to H^1_{\text{ét}}(U, \mu) \to H^1_{\text{ét}}(U, \widetilde{G}) \to \cdots ,$$

where the connecting map $G(U) = G(U^{sh})^H \to H^1(H, \mu(U^{sh})) = H^1_{\text{ét}}(U, \mu)$ is a group homomorphism. Since $\pi_0(B_{\text{ét}}\mu)(U) = H^1_{\text{ét}}(U, \mu)$, the exactness of the above long exact sequence proves the lemma. □

Lemma 3.2. *Let $\mathcal{X}$ be an $\mathbb{A}^1$-local object in $\Delta^{op}(Sm/k)_{\text{Nis}}$. Let $\mathcal{F}$ be a subsheaf of $\pi_0(\mathcal{X})$. Then, the sheaf $\mathcal{X} \times_{\pi_0(\mathcal{X})} \mathcal{F}$ is $\mathbb{A}^1$-local.*

Proof. We factor $\mathcal{X} \to \pi_0(\mathcal{X})$ as $\mathcal{X} \to \widehat{\mathcal{X}} \to \pi_0(\mathcal{X})$ such that $\widehat{X} \to \pi_0(\mathcal{X})$ is simplicially fibrant and $\mathcal{X} \to \widehat{\mathcal{X}}$ is a weak equivalence. Then, as $\pi_0(\mathcal{X})$ is fibrant, so is $\widehat{\mathcal{X}}$. Since $\mathcal{F} \to \pi_0(\mathcal{X})$ is a fibration, the morphism

$$\mathcal{F} \times_{\pi_0(\mathcal{X})} \mathcal{X} \to \mathcal{F} \times_{\pi_0(\widehat{\mathcal{X}})} \widehat{\mathcal{X}}$$

is a weak equivalence (since $\Delta^{op}Shv(Sm/k)_{\text{Nis}}$ is right proper). Thus, it suffices to show that $\mathcal{F} \times_{\pi_0(\widehat{\mathcal{X}})} \widehat{\mathcal{X}}$ is $\mathbb{A}^1$-local.

Let $h : \mathcal{Z} \times \mathbb{A}^1 \to \mathcal{F} \times_{\pi_0(\widehat{\mathcal{X}})} \widehat{\mathcal{X}}$ be a morphism in the simplicial homotopy category.

$$\begin{array}{ccccc} \mathcal{Z} \times \mathbb{A}^1 & \xrightarrow{h} & \mathcal{F} \times_{\pi_0(\widehat{\mathcal{X}})} \widehat{\mathcal{X}} & \longrightarrow & \widehat{\mathcal{X}} \\ & & \downarrow & & \downarrow \\ & & \mathcal{F} & \longrightarrow & \pi_0(\widehat{\mathcal{X}}) \end{array}$$

Since $\widehat{\mathcal{X}}$ is $\mathbb{A}^1$-local, the two composites

$$\mathcal{Z} \rightrightarrows \mathcal{Z} \times \mathbb{A}^1 \to \mathcal{F} \times_{\pi_0(\mathcal{X})} \widehat{\mathcal{X}} \to \widehat{\mathcal{X}} \to \pi_0(\mathcal{X})$$

are equal, where the leftmost maps are the 0- and 1-sections. Now, any simplicial homotopy $\mathcal{Z} \times \Delta^1 \to \widehat{\mathcal{X}}$ connecting the two maps

$$\mathcal{Z} \rightrightarrows \mathcal{Z} \times \mathbb{A}^1 \to \mathcal{F} \times_{\pi_0(\mathcal{X})} \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$$

must factor through $\mathcal{F} \times_{\pi_0(\mathcal{X})} \widehat{\mathcal{X}}$, as can be checked stalkwise. Thus, it follows that h factors through a morphism $\mathcal{Z} \times \mathbb{A}^1 \to \mathcal{F} \times_{\pi_0(\mathcal{X})} \widehat{\mathcal{X}}$ in the simplicial homotopy category. This proves that $\mathcal{F} \times_{\pi_0(\widehat{\mathcal{X}})} \widehat{\mathcal{X}}$ is $\mathbb{A}^1$-local, as desired. □

Proposition 3.3. *Let $\widehat{Q}$ denote the fibre product $\pi_0(Q) \times_{\pi_0(B_{\text{et}}\mu)} B_{\text{ét}}\mu$. Then $\widehat{Q}$ is $\mathbb{A}^1$-local and the canonical morphism $Q \to \widehat{Q}$ is a simplicial weak equivalence.*

Proof. By [8, page 137, Proposition 3.1], $B_{\text{ét}}\mu$ is $\mathbb{A}^1$-local. So, by Lemma 3.2, it follows that $\widehat{Q}$ is $\mathbb{A}^1$-local. The morphism $Q \to \widehat{Q}$ induces an isomorphism on π_0 by construction. Since the morphism $\widehat{Q} \to B_{\text{ét}}\mu$ is a pullback of the morphism $\pi_0(Q) \to \pi_0(B_{\text{ét}}\mu)$, it induces an isomorphism on π_i for all $i > 0$, and for every basepoint $u : U \to \widehat{Q}$ where $U \in Sm/k$. The morphism $Q \to B_{\text{ét}}\mu$ induces an isomorphism on π_i for all $i > 0$ due to Proposition 2.4. Also, the morphism $Q \to Q_{\text{ét}}$ factors through $Q \to \widehat{Q}$. Thus, we see that $Q \to \widehat{Q}$ is a simplicial weak equivalence. □

Corollary 3.4. *Let the notation be as above. If $\pi_0^{\mathbb{A}^1}(\widetilde{G})$ is strongly $\mathbb{A}^1$-invariant, then we have a long exact sequence*

$$\cdots \to \pi_1^{\mathbb{A}^1}(B_{\text{ét}}\mu) \to \pi_0^{\mathbb{A}^1}(\widetilde{G}) \to \pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu).$$

Proof. We have a simplicial homotopy principal $\widetilde{G}$-fibration

$$\widetilde{G} \to G \to Q,$$

which is an $\mathbb{A}^1$-fiber sequence by [7, Theorem 6.50]. This gives rise to an exact sequence of sheaves of groups/sets

$$\cdots \to \pi_1^{\mathbb{A}^1}(Q) \to \pi_0^{\mathbb{A}^1}(\widetilde{G}) \to \pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(Q). \tag{3.1}$$

In this sequence, the terms $\pi_i^{\mathbb{A}^1}(Q)$ for $i > 1$ may be replaced by $\pi_i^{\mathbb{A}^1}(B_{\text{ét}}\mu)$ due to Propositions 2.4 and 3.3. The term $\pi_0^{\mathbb{A}^1}(Q)$ may be replaced by $\pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)$ due to Lemma 3.1 and Proposition 3.3. This gives us the desired long exact sequence. □

4. Proof of the main result

The proof of [2, Lemma 5.1] depends upon [4, Theorem 1.3 and Theorem 1.5]. However, the proof of [4, Lemma 2.9] contains a gap, which renders the proofs of [4, Theorem 1.3 and Theorem 1.5] incomplete in their generality, as of now. In this note, we give a complete proof of [2, Lemma 5.1] bypassing the use of [4, Lemma 2.9]. Our proof is based on the following weaker version of [4, Theorem 1.5].

Lemma 4.1. *Let k be a perfect field. Let G be a strongly $\mathbb{A}^1$-invariant sheaf of groups on Sm/k and $G \xrightarrow{\phi} H$ an epimorphism of sheaves. Further assume that* $\mathrm{Ker}(\phi)$ *is central in G. Then H is strongly $\mathbb{A}^1$-invariant if and only if it is $\mathbb{A}^1$-invariant.*

Proof. We only need to show that if H is $\mathbb{A}^1$-invariant, then it is strongly $\mathbb{A}^1$-invariant as the reverse implication is trivial. Let $K := \mathrm{Ker}(\phi)$. For every smooth k-scheme U, the short exact sequence of Nisnevich sheaves of groups

$$1 \to K \to G \xrightarrow{\phi} H \to 1$$

gives us an exact sequence of pointed cohomology sets

$$1 \to H^0(U,K) \to H^0(U,G) \to H^0(U,H) \to H^1(U,K) \to H^1(U,G) \to H^1(U,H),$$

by [5, Ch. III, Proposition 3.3.1]. Since this exact sequence is functorial in U and since G is strongly $\mathbb{A}^1$-invariant, it follows that $\mathbb{A}^1$-invariance of H is equivalent to strong $\mathbb{A}^1$-invariance of K. Since K is a Nisnevich sheaf of abelian groups and H is $\mathbb{A}^1$-invariant by hypothesis, it follows from [7, Theorem 5.46] that K is strictly $\mathbb{A}^1$-invariant.

In order to show strong $\mathbb{A}^1$-invariance of H, it is enough to show that for every essentially smooth k-scheme U, the projection map $U \times \mathbb{A}^1 \to U$ induces an isomorphism $H^1(U,H) \to H^1(U \times \mathbb{A}^1_k, H)$. Since K is central, by functoriality and [5, Ch. IV, Remarque 4.2.10], we have a commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^1(U,G) & \longrightarrow & H^1(U,H) & \longrightarrow & H^2(U,K) \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & H^1(U \times \mathbb{A}^1_k, G) & \longrightarrow & H^1(U \times \mathbb{A}^1_k, H) & \longrightarrow & H^2(U \times \mathbb{A}^1_k, K)
\end{array}$$

whose rows are exact sequences (of pointed sets). Since G is strongly $\mathbb{A}^1$-invariant and K is strictly $\mathbb{A}^1$-invariant, the desired assertion follows. □

We are now set to give a proof of [2, Lemma 5.1]. We restate it here for the convenience of the reader.

Lemma 4.2. [2, Lemma 5.1] *Let k be a field. Let $\widetilde{G} \to G$ be a central isogeny of semisimple algebraic groups, the kernel of which is a group μ of multiplicative type. If $\pi_0^{\mathbb{A}^1}(\widetilde{G})$ is strongly $\mathbb{A}^1$-invariant, then so is $\pi_0^{\mathbb{A}^1}(G)$.*

Proof. Applying Corollary 3.4, we obtain an exact sequence of sheaves of sets

$$\cdots \to \pi_1^{\mathbb{A}^1}(B_{\text{ét}}\mu) \to \pi_0^{\mathbb{A}^1}(\widetilde{G}) \to \pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu). \tag{4.1}$$

From the above exact sequence, we extract the short exact sequence

$$1 \to \mathrm{Im}\left(\pi_0^{\mathbb{A}^1}(\widetilde{G}) \to \pi_0^{\mathbb{A}^1}(G)\right) \to \pi_0^{\mathbb{A}^1}(G) \to \mathrm{Im}\left(\pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)\right) \to 1.$$

It thus suffices to show that $\mathrm{Im}\left(\pi_0^{\mathbb{A}^1}(\widetilde{G}) \to \pi_0^{\mathbb{A}^1}(G)\right)$ and $\mathrm{Im}\left(\pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)\right)$ are both strongly $\mathbb{A}^1$-invariant.

Proof of strong $\mathbb{A}^1$-invariance of $\mathrm{Im}\left(\pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(B_{\acute{e}t}\mu)\right)$:

First note that the sheaf $\pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu) = \mathcal{H}^1_{\text{ét}}(\mu)$ is strongly $\mathbb{A}^1$-invariant by [1, Example 4.6]. The short exact sequence

$$1 \to \mathrm{Im}\left(\pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)\right) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu) \to \mathrm{Coker}\left(\pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)\right) \to 1$$

shows that it suffices to prove $\mathbb{A}^1$-invariance of $\mathrm{Coker}\left(\pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)\right)$. For every smooth henselian local k-scheme U, we have a commutative diagram

$$\begin{array}{ccccccc}
G(U) & \longrightarrow & H^1_{\text{ét}}(U,\mu) & \longrightarrow & H^1_{\text{ét}}(U,\widetilde{G}) & \longrightarrow & H^1_{\text{ét}}(U,G) \\
\downarrow & & \downarrow\simeq & & \downarrow\simeq & & \\
\pi_0^{\mathbb{A}^1}(G)(U) & \longrightarrow & \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)(U) & \longrightarrow & \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\widetilde{G})(U) & &
\end{array}$$

in which the top row is exact. Note that the cokernel of the map $\pi_0^{\mathbb{A}^1}(G)(U) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)(U)$ is contained in $\pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)(U) \simeq H^1_{\text{ét}}(U,\widetilde{G})$. By [2, Theorem 3.9], the sheaf $\pi_0^{\mathbb{A}^1}(B_{\text{ét}}\widetilde{G}) = \mathcal{H}^1_{\text{ét}}(\widetilde{G})$ is $\mathbb{A}^1$-invariant. Thus, the cokernel of $\pi_0^{\mathbb{A}^1}(G) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}\mu)$ is an $\mathbb{A}^1$-invariant sheaf as required.

Proof of strong $\mathbb{A}^1$-invariance of $\mathrm{Im}\left(\pi_0^{\mathbb{A}^1}(\widetilde{G}) \to \pi_0^{\mathbb{A}^1}(G)\right)$:

Note that $\pi_0^{\mathbb{A}^1}(G)$ is $\mathbb{A}^1$-invariant by [3, Corollary 5.2]. By the long exact sequence (4.1) and Lemma 4.1, the claim follows if the image of $\pi_1^{\mathbb{A}^1}(B_{\text{ét}}\mu) \to \pi_0^{\mathbb{A}^1}(\widetilde{G})$ is central. In order to show this, it suffices to show by [3, Corollary 4.17] that for all finitely generated field extensions L/k, the image of $\pi_1^{\mathbb{A}^1}(B_{\text{ét}}\mu)(L) \to \pi_0^{\mathbb{A}^1}(\widetilde{G})(L)$ is central. Without loss of generality, we may assume that $L = k$ by base change. Since $\widetilde{G}(k) \to \pi_0^{\mathbb{A}^1}(\widetilde{G})(k)$ is an epimorphism of Nisnevich sheaves, it suffices to show that for any element $x \in \widetilde{G}(k)$, the diagram

$$\begin{array}{ccc}
\pi_1^{\mathbb{A}^1}(B_{\text{ét}}\mu)(k) & \longrightarrow & \pi_0^{\mathbb{A}^1}(\widetilde{G})(k) \\
\| & & \downarrow \overline{c}_x \\
\pi_1^{\mathbb{A}^1}(B_{\text{ét}}\mu)(k) & \longrightarrow & \pi_0^{\mathbb{A}^1}(\widetilde{G})(k)
\end{array} \tag{4.2}$$

commutes, where $\overline{c}_x$ denotes the endomorphism of $\pi_0^{\mathbb{A}^1}(\widetilde{G})$ induced by the conjugation $\widetilde{G} \xrightarrow{c_x} \widetilde{G}$ by $x \in \widetilde{G}(k)$. If $\overline{x}$ denotes the image of x in $G(k)$, then the diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mu & \longrightarrow & \widetilde{G} & \longrightarrow & G \\
 & & \| & & \downarrow c_x & & \downarrow c_{\overline{x}} \\
1 & \longrightarrow & \mu & \longrightarrow & \widetilde{G} & \longrightarrow & G
\end{array}$$

commutes. By construction of the fiber sequence $\widetilde{G} \to G \to B_{\text{ét}}\mu$, it follows that the diagram

$$\begin{array}{ccccc} \widetilde{G} & \longrightarrow & G & \longrightarrow & B_{\text{ét}}\mu \\ \downarrow{\scriptstyle c_x} & & \downarrow{\scriptstyle c_{\overline{x}}} & & \| \\ \widetilde{G} & \longrightarrow & G & \longrightarrow & B_{\text{ét}}\mu \end{array}$$

commutes. The corresponding commutative diagram with the long exact sequence (4.1) as rows gives us the required commutativity of (4.2). □

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