

# KATO COMPLEXES OF RECIPROCITY SHEAVES AND APPLICATIONS

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ABSTRACT. We show that every reciprocity sheaf gives rise to a cycle (pre)module in the sense of Rost over a perfect field, under mild additional hypotheses. Over a perfect field of positive characteristic, we show that the first cohomology group of a logarithmic de Rham-Witt sheaf has a partial cycle module structure. As a consequence, we show that Kato complexes of logarithmic de Rham-Witt sheaves satisfy functoriality properties similar to Rost's cycle complexes.

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## 1. INTRODUCTION

The notion of homotopy invariance is the cornerstone of Voevodsky's construction of the triangulated category of motives [Voe00] in the sense that the category of homotopy invariant sheaves with transfers on smooth schemes a field is used as an essential building block. However, especially in positive characteristic, many sheaves of interest are not homotopy invariant, but satisfy a weaker condition. This led to the study of reciprocity sheaves started in [KSY16], which in turn, has led to the development of the theory of motives with modulus, extending Voevodsky's theory of motives. A certain special class of homotopy invariant sheaves with transfers, called *homotopy modules*, was identified with *cycle modules* in the sense of [Ros96] in the Ph.D. thesis of Déglise [Dég02] (see also [Dég08]). Rost's theory of cycle modules gives an alternate approach and a generalization of classical intersection theory and can be seen as an axiomatization of fundamental properties of Milnor  $K$ -theory.

One of the aims of this article is to investigate to what extent this special property of homotopy modules extends to reciprocity sheaves. Let  $k$  be a perfect field and let  $S_{m/k}$  denote the big Nisnevich site of smooth, separated finite type schemes over  $k$ . The first main result of this article is to show that every reciprocity sheaf gives rise to a cycle (pre)module under mild hypotheses (see Theorem 3.9).

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**Theorem 1.** *Let  $k$  be a perfect field. Let  $\mathcal{F}$  be a good reciprocity sheaf on  $\text{Sm}/k$  in the sense of Definition 2.3. Then for any finitely generated field extension  $K$  of  $k$ , the association*

$$\mathcal{F}(K) := \operatorname{colim}_{\phi \neq U \subset X} \mathcal{F}(U),$$

where  $U$  varies over all the open subsets of a model  $X$  of  $K$ , defines a cycle premodule. Moreover, this cycle premodule structure satisfies the cycle module axioms.

The key difference from the work of Déglise is the construction of the residue map in the absence of  $\mathbb{A}^1$ -invariance and purity. This is done in Theorem 3.5 for *good* reciprocity sheaves (see Definition 2.3) by using the recent work of Binda, Rülling and Saito [BRS22] on the cohomology of reciprocity sheaves and especially, on the Gysin triangle.

Let us now assume that  $k$  is a perfect field of characteristic  $p > 0$ . One of the main examples of an interesting good reciprocity sheaf that is not  $\mathbb{A}^1$ -invariant is the logarithmic de Rham-Witt sheaf of Illusie [Ill79]. The importance of this example is that over the étale site, the logarithmic de Rham-Witt sheaf  $\nu_r(q)$  in weight  $q$  can be identified up to a shift with the étale motivic complex  $\mathbb{Z}/p^r\mathbb{Z}(q)$  in weight  $q$ , by the work of Geisser and Levine [GL01].

In [Kat86], Kato defined a family of complexes for  $q \in \mathbb{Z}$  when  $n \in \mathbb{Z} \setminus \{1\}$  and  $q \geq 0$  when  $n = 1$  given (in homological conventions) by:

$$C(X, \mathbb{Z}/p^r\mathbb{Z}(q), n): \\ 0 \rightarrow \bigoplus_{x \in X_{(d)}} H^{d+q+n}(k(x), \mathbb{Z}/p^r\mathbb{Z}(d+q)) \rightarrow \cdots \rightarrow \bigoplus_{x \in X_{(0)}} H^{q+n}(k(x), \mathbb{Z}/p^r\mathbb{Z}(q)) \rightarrow 0$$

Under the identification  $\mathbb{Z}/p^r\mathbb{Z}(q)[q] = \nu_r(q)$ , this takes the form (in cohomological conventions)

$$C^\bullet(X, \mathbb{Z}/p^r\mathbb{Z}(q), n): 0 \rightarrow \bigoplus_{x \in X^{(0)}} H^n(k(x), \nu_r(q)) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(d)}} H^n(k(x), \nu_r(q-d)) \rightarrow 0.$$

The complex  $C(X, \mathbb{Z}/p^r\mathbb{Z}(q), n)$  is nonzero only for  $n = 0$  or  $n = 1$ . In the case  $n = 0$ , it can be identified with Rost's cycle complex for the cycle module corresponding to mod- $p^r$  Milnor  $K$ -theory under the isomorphism  $H_{\text{ét}}^n(F, \mathbb{Z}/p^r\mathbb{Z}(n)) \simeq K_n^M(F)/p^r$  for any field  $F$  obtained by Bloch-Gabber-Kato (see [BK86]). In fact, this observation was the main motivation behind investigating whether reciprocity sheaves under mild hypotheses give rise to cycle premodules, leading to Theorem 1 above.

In the case  $n = 1$ , it is known that the groups  $H^1(-, \nu_r(q))$  do not form a cycle module as the residue map is not defined for all valuations. However, we verify in Section 4.2 that this *partial cycle premodule data* does satisfy the cycle premodule and cycle module axioms. We also verify that the classical Gysin maps for logarithmic de Rham-Witt sheaves constructed by Gros [Gro85] agree with the ones given by Binda, Rülling and Saito. As a consequence, we show the following functoriality properties for Kato complexes analogous to cycle modules in Section 5 (see Definition 5.1, Definition 5.8 and Proposition 5.11).

**Theorem 2.** *Let  $k$  be a perfect field of characteristic  $p > 0$  and let  $r \geq 0$  be an integer. The assignment of the Kato complex  $C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1)$  to a smooth  $k$ -scheme  $X$  admits the following functoriality properties:*

- a proper morphism  $f : X \rightarrow Y$  induces a push-forward morphism

$$f_* : C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \rightarrow C(Y, \mathbb{Z}/p^r\mathbb{Z}(q), 1),$$

where  $\delta = \dim Y - \dim X$ ;

- an arbitrary morphism  $g : X \rightarrow Y$  induces a pullback morphism

$$g_\tau^* : C(Y, \mathbb{Z}/p^r\mathbb{Z}(q + \delta), 1) \rightarrow C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[\delta],$$

which depends on the choice of a coordination  $\tau$  of the tangent bundle of  $Y$ .

Consequently, for any proper schemes  $X, Y, Z$  over  $k$  with  $X, Z$  smooth and a correspondence in  $\mathrm{CH}^{\dim Z}(Y \times Z)$  represented by a cycle  $z \in \mathcal{Z}^{\dim Z}(Y \times Z)$ , there exists an action

$$z_* : C(X \times Y, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \rightarrow C(X \times Z, \mathbb{Z}/p^r\mathbb{Z}(q), 1)$$

such that the induced action on the cohomology groups passes through rational equivalence and agrees with the usual action of correspondences.

Although we follow the approach outlined in the works of Rost [Ros96] and Déglise [Dég02], the key difference here is that the classical argument using the deformation to the normal cone to construct general pullbacks has to be slightly modified in the absence of homotopy invariance. This has been alluded to in [KS12, Section 4, page 147].

One of the motivations for this work is to develop some tools needed to attack the question of Rost nilpotence for cycles having torsion primary to the characteristic of the base field, using a combination of the methods in [RS18], [Dia19] and [Gil14]. A precise obstruction to the Rost nilpotence principle for smooth projective varieties of dimension  $\geq 3$  can be explicitly written down in terms of actions of correspondences on certain cohomology groups of étale motivic complexes  $\mathbb{Q}_\ell/\mathbb{Z}_\ell(q)$  (see [RS18, Remark 4.7] and [Dia19, Theorem 2.4]), where  $\ell$  runs through all the primes. Theorem 2 above gives an action of a correspondence at the level of Kato complexes that is compatible with the correspondence action on the cohomology groups. Applications to Rost nilpotence using the methods developed in this article will be explored elsewhere.

**Conventions.** We work over a perfect field  $k$ . We assume that every scheme is equidimensional, separated and of finite type over  $k$ .

All fields will be assumed to be finitely generated over  $k$ . Let  $\mathcal{F}_k$  denote the category of finitely generated field extensions of  $k$ . All valuations on a field are assumed to be of rank 1 and of geometric type over  $k$ , which means that the local ring of the valuation is a regular local ring which is the localization of a height 1 prime ideal of an integral domain finitely generated over  $k$ .

For a field  $F$ , we will denote its Henselization by  $F^h$  and its strict Henselization by  $F^{sh}$ , with respect to a separable closure  $F^{sep}$ . The absolute Galois group of  $F$  will be denoted by  $\Gamma_F := \mathrm{Gal}(F^{sep}/F)$ . The  $i$ th Milnor  $K$ -group of  $F$  will be denoted by  $K_i^M(F)$ . For any  $\Gamma_F$ -module  $M$ , the Galois cohomology groups  $H^i(\Gamma_F, M)$  will be denoted by  $H^i(F, M)$ , which is also the notation for the corresponding étale cohomology groups. We will abuse the notation and denote the Galois cohomology classes and cocycles representing them by the same symbol as long as there is no confusion.

For a scheme  $X$  over  $k$ , we write  $X^{(i)}$  for the set of points of codimension  $i$  on  $X$  and  $X_{(i)}$  for the set of points of dimension  $i$  on  $X$ . We will write  $\mathcal{Z}^i(X)$  for the group of algebraic cycles of codimension  $i$  on  $X$  and  $\mathrm{CH}^i(X)$  for the Chow group of algebraic cycles of codimension  $i$  on  $X$  (that is, the quotient of  $\mathcal{Z}^i(X)$  modulo rational equivalence).

## 2. PRELIMINARIES ON RECIPROCITY SHEAVES

In this section, we briefly recall the notions regarding reciprocity sheaves required for our purposes from [BRS22].

A *modulus pair* is a pair  $(X, D)$  where  $X$  a separated scheme of finite type and  $D$  is an effective (or empty) Cartier divisor on  $X$  such that  $X \setminus D$  is a smooth. A modulus pair  $(X, D)$  is said to be *proper* if  $X$  is proper over  $k$ . Let  $(X', D')$  be another modulus pair. A *proper prime correspondence* from  $(X, D)$  to  $(X', D')$  is defined to be a prime correspondence  $Z \subset X \times X'$  between  $X' \setminus D'$  and  $X \setminus D$  such that the normalization of its closure  $\overline{Z}^N$  is proper over  $X$  and  $D|_{\overline{Z}^N} \geq D'|_{\overline{Z}^N}$ . We write the free abelian group generated by such proper prime correspondences as  $\underline{\mathbf{MCor}}((X, D), (X', D'))$ . The category of modulus pairs along with these as morphisms will be denoted by  $\underline{\mathbf{MCor}}$  and its full subcategory consisting of proper modulus pairs will be denoted by  $\mathbf{MCor}$ .

Let  $\underline{\mathbf{MPST}}$  (respectively,  $\mathbf{MPST}$ ) denote the category of presheaves on  $\underline{\mathbf{MCor}}$  (respectively,  $\mathbf{MCor}$ ). We have a functor  $\tau^*: \underline{\mathbf{MPST}} \rightarrow \mathbf{MPST}$  given by restriction; this has a left adjoint  $\tau_!$ . We also have functors

$$\underline{\omega}_!: \underline{\mathbf{MPST}} \rightleftarrows \mathbf{PST} : \underline{\omega}^*,$$

where  $\underline{\omega}_!$  is left adjoint to  $\underline{\omega}^*$ . For  $F \in \underline{\mathbf{MPST}}$  and  $G \in \mathbf{PST}$ , we have

$$\underline{\omega}_!F(X) = F(X, \emptyset)$$

and

$$\underline{\omega}^*G(X, D) = G(X \setminus D).$$

For  $F \in \underline{\mathbf{MPST}}$  and  $\mathcal{X} = (X, D)$ , we have a presheaf  $F_{\mathcal{X}}$  on the small étale site of  $X$  given by  $F_{\mathcal{X}}(U) = F(U, D|_U)$ . If  $F_{\mathcal{X}}$  is a Nisnevich sheaf for all modulus pairs  $\mathcal{X}$ , then we say that  $F$  is a Nisnevich sheaf. We denote the category of such sheaves by  $\underline{\mathbf{MNST}}$ .

For two modulus pairs  $(X, D)$  and  $(X', D')$ , we set

$$(X, D) \otimes (X', D') := (X \times X', p^*D + q^*D'),$$

where  $p$  and  $q$  are the projection maps from  $X \times X'$  to  $X$  and  $X'$  respectively. We set  $\overline{\square} := (\mathbb{P}^1, \infty)$ .

**Definition 2.1.** For  $\mathcal{F} \in \underline{\mathbf{MPST}}$ , we say that

- (1)  $\mathcal{F}$  is *cube-invariant* if for each  $\mathcal{X} \in \underline{\mathbf{MCor}}$ , the map  $\mathcal{F}(\mathcal{X}) \rightarrow \mathcal{F}(\mathcal{X} \otimes \overline{\square})$  induced by the projection  $\mathcal{X} \otimes \overline{\square} \rightarrow \mathcal{X}$  is an isomorphism. We denote the category of cube-invariant presheaves by  $\mathbf{CI}$
- (2)  $\mathcal{F}$  has *M-reciprocity* if the map  $\tau_! \tau^* \mathcal{F} \rightarrow \mathcal{F}$  is an isomorphism.
- (3)  $\mathcal{F}$  is *semipure* if the map  $\mathcal{F} \rightarrow \underline{\omega}^* \underline{\omega}_! \mathcal{F}$  is injective.

We denote the category of the cube-invariant, semipure presheaves having M-reciprocity by  $\mathbf{CI}^{\tau, sp}$  and set  $\mathbf{CI}_{\text{Nis}}^{\tau, sp} := \underline{\mathbf{MNST}} \cap \mathbf{CI}^{\tau, sp}$ .

**Definition 2.2.** We say that a Nisnevich presheaf (respectively, sheaf) with transfers  $F$  is a *reciprocity presheaf* (respectively, *reciprocity sheaf*) if there exists some  $G \in \mathbf{CI}^{\tau, sp}$  (respectively,  $\mathbf{CI}_{\text{Nis}}^{\tau, sp}$ ) such that  $\underline{\omega}_! G = F$ . Note that a reciprocity sheaf is a reciprocity presheaf, which is a Nisnevich sheaf.

**Definition 2.3.** We say that a reciprocity sheaf  $\mathcal{F}$  is *good* if it satisfies the following conditions:

- (1) for two finite correspondences  $\alpha, \beta: S \rightarrow T$ , we have  $(\alpha + \beta)^* = \alpha^* + \beta^*$ ;
- (2) there exists an integer  $N$  such that  $\mathcal{F}(S)$  has no  $N$ -torsion for every  $S$ .

**Examples 2.4.**

- (1) For every integer  $n$ , the  $n$ th Milnor  $K$ -theory sheaf  $K_n^M$  is an example of a reciprocity sheaf.
- (2) The  $n$ -th logarithmic de Rham-Witt sheaf  $\nu_r(n) = W_r\Omega_{X,log}^n$  (defined in [Mil76], [Ill79]) for any integer  $n \geq 0$  is an example of a good reciprocity sheaf in the sense of Definition 2.3. There is a quasi-isomorphism

$$\nu_r(n) = W_r\Omega_{X,log}^n \simeq \mathbb{Z}/p^r\mathbb{Z}(n)[n]$$

of étale motivic complexes, due to [GL01], [GL00], where  $p > 0$  is the characteristic of the base field.

There are inclusions  $i_0, i_1: (\text{Spec } k, ) \rightarrow \overline{\square}$  corresponding to the  $k$ -rational points 0 and 1 of  $\mathbb{P}^1$ . For a modulus pair  $\mathcal{X}$ , set

$$h_0(\mathcal{X}) := \text{Coker}(\mathbf{MCor}(- \otimes \overline{\square}, \mathcal{X}) \xrightarrow{i_0^* - i_1^*} \mathbf{MCor}(-, \mathcal{X})) \in \mathbf{MPST}.$$

There is a canonical surjection  $\mathbb{Z}_{tr}(X \setminus D) \rightarrow \omega_1 h_0(X, D)$ .

**Definition 2.5.** Let  $\mathcal{F}$  be a presheaf with transfers and let  $\alpha \in F(U)$ , for a smooth  $k$ -scheme  $U$ . We say that  $\alpha$  has modulus  $(X, D) \in \mathbf{MCor}$  if  $X \setminus D = U$  and the morphism  $\mathbb{Z}_{tr}(U) \xrightarrow{\alpha} \mathcal{F}$  corresponding to  $\alpha$  factors through  $\mathbb{Z}_{tr}(U) \rightarrow \omega_1 h_0(X, D)$ .

**Theorem 2.6.** [KSY22, Theorem 3.2.1, Corollary 3.2.3] *A presheaf with transfers  $F$  is a reciprocity presheaf if and only if for each smooth separated scheme  $U$ , every element  $\alpha \in F(U)$  has modulus  $\mathcal{X}$  for some proper modulus pair  $\mathcal{X}$ .*

**Notation 2.7.** For an integral scheme  $\overline{C}$  a closed subscheme  $D$  of  $\overline{C}$  defined by an ideal  $\mathcal{I}$ , set

$$G(\overline{C}, D) := \bigcap_{x \in D} \text{Ker}(\mathcal{O}_{\overline{C},x}^\times \rightarrow \mathcal{O}_{D,x}^\times) = \bigcap_{x \in D} \mathcal{I}_x^\times.$$

With this notation, for  $f \in G(\overline{C}, D)$ , we have  $f^n \in G(\overline{C}, nD)$ . Also, for two closed subschemes  $Y, Y'$  of  $\overline{C}$  such that  $\mathcal{I}_Y \subset \mathcal{I}_{Y'}$ , we have  $G(\overline{C}, Y) \subset G(\overline{C}, Y')$ .

**Remark 2.8.** One gets an equivalent characterization of Definition 2.5 by [KSY16, Theorem 2.1.5] and [KSY22, Theorem 3.2.1], which is often helpful in practical applications.

Suppose that for a smooth separated scheme  $S$ , an integral normal scheme  $\overline{C}$  and a proper modulus pair  $(X, D)$  with  $U = X \setminus D$  quasi affine, we are given a commutative diagram

$$(2.1) \quad \begin{array}{ccccc} & & \overline{C} & & \\ & p_\phi \swarrow & \downarrow \phi & \searrow \gamma_\phi & \\ S & \longleftarrow & X \times S & \longrightarrow & X \end{array}$$

satisfying the following conditions:

- (1)  $\phi$  is finite;

- (2) For some generic point  $\eta$  of  $S$ ,  $\dim \overline{C} \times_S \eta = 1$ ;
- (3) The image of  $\gamma_\phi$  is not contained in  $D$ .

Then for any  $f \in G(\overline{C}, \gamma_\phi^* D)$ , we have  $\phi_* \text{div}_{\overline{C}}(f) \in \mathbf{Cor}(S, X)$ . With this setting, the element  $\alpha \in \mathcal{F}(U)$  has modulus  $(X, D)$  if and only if for every diagram as above, and each  $f \in G(\overline{C}, \gamma_\phi^* D)$ , we have  $(\phi_* \text{div}_{\overline{C}}(f))^*(a) = 0$ .

### 3. CYCLE MODULE STRUCTURE ASSOCIATED WITH A GOOD RECIPROCITY SHEAF

In [Dég02], a cycle module in the sense of Rost [Ros96] is associated with every *homotopy module*, which is a homotopy invariant presheaf with transfers satisfying an additional condition. In this section, we associate a cycle premodule with every good reciprocity sheaf in the sense of Definition 2.3. The construction goes exactly analogous to [Dég02], except for the definition of the residue map, which in the case of homotopy modules relies on homotopy invariance. The key point of our work is to bypass this use of homotopy invariance by using appropriate results from [BRS22].

#### 3.1. The cycle premodule data.

**Definition 3.1.** Let  $R$  be an essentially smooth local  $k$ -algebra. Let  $X$  be an integral separated smooth  $k$ -scheme with a dominant morphism  $x: \text{Spec } R \rightarrow X$  inducing an isomorphism between  $R$  and  $\mathcal{O}_{X,x}$ , where we denote by  $x$  the image of the closed point of  $\text{Spec } R$  as well by abuse of notation. We call such a pair  $(X, x)$  a *model* for  $R$  and a compatible morphism of schemes a morphism of models. Existence of models is guaranteed by [Dég02, Lemma 2.1.39].

**Definition 3.2.** Let  $\mathcal{F}$  be a reciprocity sheaf. For a finitely generated field  $K$  over  $k$ , define  $\mathcal{F}(K) = H^0(K, \mathcal{F})$  as follows. Choose a model  $X$  for  $K$  and set

$$\mathcal{F}(K) = H^0(K, \mathcal{F}) := \text{colim}_{\phi \neq U \subset X} \mathcal{F}(U),$$

where  $U$  varies over all the open subsets of  $X$ .

**Definition 3.3.** For every reciprocity sheaf  $\mathcal{F}$ , we define its *contraction*  $\mathcal{F}_{-1}$  to be the internal Hom

$$\mathcal{F}_{-1} := \text{Hom}_{\mathbf{PST}}(\mathbb{K}_1^{\mathbf{M}}, \mathcal{F}).$$

By induction, we define

$$\mathcal{F}_{-n} := (\mathcal{F}_{-n+1})_{-1},$$

for all positive integers  $n$ . We have  $\mathcal{F}_{-n} = \text{Hom}_{\mathbf{PST}}(\mathbb{K}_n^{\mathbf{M}}, \mathcal{F})$ , for all positive integers  $n$ .

In the following data regarding the cycle premodule structure, items **(D1)**, **(D2)** and **(D3)** are given exactly as in [Dég02] and hold for all presheaves with transfers, which we restate for the convenience of readers.

- (D1)** [Dég02, Definition 5.2.1] For every field extension  $\phi: K \rightarrow L$  in  $\mathcal{F}_k$ , define the map  $\phi_*: \mathcal{F}(K) \rightarrow \mathcal{F}(L)$  as follows: there exist models  $X$  and  $Y$  of  $L$  and  $K$ , respectively and a morphism of models  $f: X \rightarrow Y$ . We define  $\phi_*$  to be the induced map

$$\text{colim}_{\phi \neq V \subset Y} \mathcal{F}(V) \rightarrow \text{colim}_{\phi \neq U \subset X} \mathcal{F}(U)$$

given by the restrictions along  $f$  and  $\mathcal{F}(V) \rightarrow \mathcal{F}(f^{-1}V)$ .

(D2) For each finite field extension  $\phi: K \rightarrow L$  in  $\mathcal{F}_k$ , the map  $\phi^*: \mathcal{F}(L) \rightarrow \mathcal{F}(K)$  defined as follows: there exist models  $X, Y$  of  $L, K$  respectively and a finite dominant morphism of models  $f: X \rightarrow Y$  [Dég02, Lemma 5.3.16]. Since the graph of  $f$  is finite and surjective over both  $X$  and  $Y$ , it may be considered as a correspondence  $f^t \in \text{Cor}(Y, X)$ . Open subsets of the form  $f^{-1}V$  where  $V$  is an open subset of  $Y$  are cofinal among the open subsets of  $X$ . So we define

$$\phi^*: \mathcal{F}(L) \cong \text{colim}_{\phi \neq U \subset X} \mathcal{F}(U) \cong \text{colim}_{\phi \neq V \subset Y} \mathcal{F}(f^{-1}V) \rightarrow \text{colim}_{\phi \neq V \subset Y} \mathcal{F}(V)$$

as the one given by the restrictions along  $f^t$  and  $\mathcal{F}(f^{-1}V) \rightarrow \mathcal{F}(V)$ .

(D3) We have a natural pairing  $\mathbb{K}_n^M \times \text{Hom}_{\mathbf{PST}}(\mathbb{K}_n^M, \mathcal{F}) \rightarrow \mathcal{F}$ . This defines an action  $\mathbb{K}_n^M(K) \times \mathcal{F}_{-n}(K) \rightarrow \mathcal{F}(K)$  for every  $K \in \mathcal{F}_k$  by taking the colimit of the above pairing over the sections over the open subsets of some model of  $K$ .

**Remark 3.4.** Let  $\mathcal{F}$  be a reciprocity sheaf and let  $m, n$  be integers. We note that every  $\alpha \in \mathbb{K}_m^M(K)$  induces a morphism

$$\mathcal{F}_{-n-m} \xrightarrow{\alpha \cdot -} \mathcal{F}_{-n}$$

defined as follows. The element  $\alpha$  defines a map  $\mathbb{Z}_{tr}(X) \rightarrow \mathbb{K}_m^M$ , which induces a morphism

$$\mathbb{K}_n^M \otimes^{tr} \mathbb{Z}_{tr}(X) \rightarrow \mathbb{K}_m^M \otimes^{tr} \mathbb{K}_n^M \otimes^{tr} \mathbb{Z}_{tr}(X) \rightarrow \mathbb{K}_{m+n}^M \otimes^{tr} \mathbb{Z}_{tr}(X).$$

For any  $\mathcal{G} \in \mathbf{PST}$ , since we have  $\text{Hom}_{\mathbf{PST}}(\mathcal{F}, \mathcal{G})(X) = \text{Hom}_{\mathbf{PST}}(\mathcal{F} \otimes^{tr} \mathbb{Z}_{tr}(X), \mathcal{G})$ , we obtain a morphism

$$\mathcal{F}_{-n-m} \simeq \text{Hom}_{\mathbf{PST}}(\mathbb{K}_{n+m}^M, \mathcal{F}) \rightarrow \text{Hom}_{\mathbf{PST}}(\mathbb{K}_n^M, \mathcal{F}) \simeq \mathcal{F}_{-n},$$

which we denote by  $\alpha \cdot -$ .

Henceforth, we assume that  $\mathcal{F}$  is a good reciprocity sheaf in the sense of Definition 2.3. In this case, the data (D4) is given by Theorem 3.5 below.

**Theorem 3.5.** *Let  $\mathcal{F}$  be a good reciprocity sheaf. For every valuation  $v$  of  $K \in \mathcal{F}_k$  with residue field  $k(v)$ , there is a residue map  $\partial_v: \mathcal{F}(K) \rightarrow \mathcal{F}_{-1}(k(v))$ .*

*Proof.* By [Dég02, Lemma 5.4.53], we have a model  $X$  for  $\mathcal{O}_v$  such that the closed point of  $\text{Spec } \mathcal{O}_v$  maps to a codimension 1 point  $z$  of  $X$  and the reduced subscheme  $Z = \overline{\{z\}}$  is smooth. Note that this also gives a model for  $K$  and that  $k(v) \cong k(z)$ .

We first define a map

$$\delta: \mathcal{F}(K) \rightarrow H_z^1(X, \mathcal{F}) := \text{colim}_{z \in V} H_{Z \cap V}^1(V, \mathcal{F})$$

as follows. Abusing notation, let  $\alpha \in \mathcal{F}(K)$  be represented by  $\alpha \in \mathcal{F}(V)$  for some open subset  $V$  of  $X$ . If  $X \setminus V \not\supseteq Z$ , define  $\delta_V(\alpha) = 0$ . Otherwise, define  $\delta_V(\alpha)$  to be the image of  $\alpha$  under the composition

$$\mathcal{F}(V) \rightarrow H_{X \setminus V}^1(X, \mathcal{F}) \rightarrow H_{Z \cap U}^1(U, \mathcal{F}) \rightarrow H_z^1(X, \mathcal{F}),$$

where  $U$  is the complement of the union of the other irreducible components of  $X \setminus V$ . We next show that  $H^0(k(z), \mathcal{F}_{-1})$  injects inside  $H_z^1(X, \mathcal{F})$  and that the image of  $\delta$  falls inside it, thus giving the desired residue map.

By [BRS22, Theorem 7.16], for a smooth scheme  $X$  and a smooth subvariety  $i: Z \rightarrow X$  of codimension 1 and a modulus presheaf with transfers  $\mathcal{G} \in \mathbf{CI}_{Nis}^{T,sp}$  we have an exact triangle

$$i_* \mathcal{G}_{(Z, \emptyset)_{-1}}[-1] \xrightarrow{g_{Z/X}} \mathcal{G}_{(X, \emptyset)} \rightarrow \mathcal{G}_{(X, Z)}.$$



Applying  $\mathbf{R}\Gamma_Z(X, -)$  to the above triangle and taking cohomology, we get the exact sequence:

$$H_Z^0(X, \mathcal{G}_{(X,Z)}) \rightarrow H^0(Z, \mathcal{G}_{(Z,\emptyset)_{-1}}) \rightarrow H_Z^1(X, \mathcal{G}_{(X,\emptyset)}) \rightarrow H_Z^1(X, \mathcal{G}_{(X,Z)}).$$

Note that  $H_Z^0(X, \mathcal{G}_{(X,Z)}) = 0$  since  $\mathcal{G}(X, Z) \rightarrow \mathcal{G}(X \setminus Z, \emptyset)$  is injective by semipurity.

We take  $\mathcal{G}$  to be such that  $\mathcal{F} = \omega_! \mathcal{G}$ . Then the above exact sequence takes the form

$$0 \rightarrow H^0(Z, \mathcal{F}_{-1}) \rightarrow H_Z^1(X, \mathcal{F}) \rightarrow H_Z^1(X, \mathcal{F}'),$$

where  $\mathcal{F}' := \mathcal{G}_{(X,Z)}$ . Taking the colimit over affine open sets containing  $z$ , we get an exact sequence

$$0 \rightarrow H^1(k(z), \mathcal{F}_{-1}) \rightarrow H_z^1(X, \mathcal{F}) \rightarrow H_z^1(X, \mathcal{F}').$$

We claim that if  $X \setminus Z$  is quasi-affine, then the composite map

$$H^0(X \setminus Z, \mathcal{F}) \rightarrow H_Z^1(X, \mathcal{F}) \rightarrow H_Z^1(X, \mathcal{F}')$$

vanishes. This implies that the image of  $\delta$  falls inside  $H^1(k(z), \mathcal{F})$ , proving Theorem 3.5.

We obtain the following commutative diagram with exact rows by taking the localization sequences for  $\mathcal{F}$  and  $\mathcal{F}'$ .

$$\begin{array}{ccccccc} & & H^0(X \setminus Z, \mathcal{F}) & \xrightarrow{t} & H_Z^1(X, \mathcal{F}) & & \\ & & \downarrow & & \downarrow s & & \\ H^0(X, \mathcal{F}') & \xrightarrow{r} & H^0(X \setminus Z, \mathcal{F}') & \xrightarrow{q} & H_Z^1(X, \mathcal{F}') & \xrightarrow{p} & H^1(X \setminus Z, \mathcal{F}') \end{array}$$

It is clear from the diagram that  $p \circ s \circ t = 0$ . We need to show that  $s \circ t = 0$ , so it suffices to show that  $p$  is injective. This, in turn, follows if we show that  $r$  is surjective.

We have  $H^0(X, \mathcal{F}') = \mathcal{G}(X, Z)$  and  $H^0(X \setminus Z, \mathcal{F}') = \mathcal{G}(X \setminus Z, \emptyset)$ . These groups can be calculated as follows (see [OR22, Definition 1.11]): let  $(\overline{X}, \overline{Z} + B)$  be a proper modulus pair such that  $X = \overline{X} \setminus B$  and  $Z = \overline{Z} \cap X$ . Then

$$\mathcal{G}(X, Z) = \{\alpha \in \mathcal{F}(X \setminus Z) \mid \alpha \text{ has modulus } (\overline{X}, \overline{Z} + NB) \text{ for some } N \geq 0\},$$

and

$$\mathcal{G}(X \setminus Z, \emptyset) = \{\alpha \in \mathcal{F}(X \setminus Z) \mid \alpha \text{ has modulus } (\overline{X}, N\overline{Z} + NB) \text{ for some } N \geq 0\}.$$

Now,  $X \setminus Z$  is quasi-affine, so we can use the characterization mentioned in Remark 2.8. Suppose that an integral scheme  $\overline{C}$  and a diagram as in (2.1) has been given for the proper modulus pair  $(\overline{X}, \overline{Z} + B)$ . Let  $\alpha \in \mathcal{G}(X \setminus Z, \emptyset)$ , so that  $\alpha$  has modulus  $N\overline{Z} + NB$ , for some integer  $N \geq 0$  and let  $f \in G(\overline{C}, \gamma_\phi^*(\overline{Z} + NB))$ . Then for every  $M \geq N$ , we have

$$f^M \in G(\overline{C}, \gamma_\phi^*(M\overline{Z} + MNB)) \subset G(\overline{C}, \gamma_\phi^*(N\overline{Z} + NB)).$$

Therefore,  $(\phi_*(\text{div}_{\overline{C}}(f^M)))^*(\alpha) = 0$ , for every  $M \geq N$ . However, since  $\mathcal{F}$  is good, for every  $M \geq N$  we have

$$(\phi_*(\text{div}_{\overline{C}}(f^M)))^*(\alpha) = (\phi_*(M \cdot \text{div}_{\overline{C}}(f)))^*(\alpha) = (M \cdot \phi_*(\text{div}_{\overline{C}}(f)))^*(\alpha) = M \cdot (\phi_*(\text{div}_{\overline{C}}(f)))^*(\alpha).$$

Since there is some integer  $m$  such that each  $\mathcal{F}(S)$  is  $m$ -torsion free, we may take  $M$  to be a large enough power of  $m$  that is greater than  $N$  to conclude that  $\phi_*(\text{div}_{\overline{C}}(f))^* \alpha = 0$ . Thus,  $\alpha \in \mathcal{G}(X, Z)$ . This shows that  $r$  is surjective, as desired. This completes the proof.  $\square$

This allows us to associate the data of a cycle premodule to every good reciprocity sheaf.



**3.2. The cycle premodule axioms.** In this subsection, we verify that the cycle premodule data associated with a good reciprocity sheaf  $\mathcal{F}$  satisfies the cycle premodule axioms of [Ros96]. The axioms not involving the residue map have been proved in [Dég02]; we list them here for the sake of completeness.

**R1a.** For field extensions  $\phi: K \rightarrow L$  and  $\psi: L \rightarrow M$ , we have  $(\psi \circ \phi)_* = \psi_* \circ \phi_*$  [Dég02, 5.2.2].

**R1b.** For finite field extensions  $\phi: K \rightarrow L$  and  $\psi: L \rightarrow M$ , we have  $(\psi \circ \phi)^* = \phi^* \circ \psi^*$  [Dég02, Corollary 5.3.22].

**R1c.** For field extensions  $\phi: K \rightarrow L$  and  $\psi: K \rightarrow M$ , where  $\phi$  is finite and  $R = L \otimes_K M$ , we have

$$\psi_* \circ \phi^* = \sum_{p \in \text{Spec } R} l(R_p) \phi_p^*(\psi_z)_*,$$

where  $\phi_p$  is the extension  $K \rightarrow L \rightarrow R \rightarrow R/p$  and  $\psi_z$  is similarly defined [Dég02, Corollary 5.3.22].

**R2a.** For a field extension  $\phi: K \rightarrow L$ ,  $\alpha \in K_n^M(K)$  and  $\rho \in H^0(K, \mathcal{F}_{-n})$ , we have  $\phi_*(\alpha \cdot \rho) = \phi_*(\alpha) \cdot \phi_*(\rho)$  [Dég02, 5.5.18].

**R2b.** If  $\phi: K \rightarrow L$  is a finite extension and  $\mu \in H^0(L, \mathcal{F}_{-n})$ , then  $\phi^*((\phi_*\alpha) \cdot \mu) = \alpha \cdot \phi^*(\mu)$  [Dég02, Corollary 5.5.19(1)].

**R2c.** If  $\phi: K \rightarrow L$  is a finite extension and  $\beta \in K_n^M(L)$ , then  $\phi^*(\beta \cdot \phi_*(\rho)) = \phi^*(\beta) \cdot (\rho)$  [Dég02, Corollary 5.5.19(2)].

**R3a.** Let  $\phi: K \rightarrow L$  be a field extension and  $v$  be a valuation on  $L$  restricting to a nontrivial valuation  $w$  on  $K$ . Let  $\bar{\phi}: k(w) \rightarrow k(v)$  be the induced map on residue fields and  $e$  be the ramification index of the extension. Then  $\partial_v \phi_* = e \bar{\phi}_* \partial_w$ .

*Proof.* There exist models  $X, X'$  of  $v, w$  respectively with codimension 1 points  $z, z'$  such that  $\mathcal{O}_{X,z} \cong \mathcal{O}_w$  and  $\mathcal{O}_{X',z'} \cong \mathcal{O}_v$ . Let  $\{z\} = Z$  and  $\{z'\} = Z'$ . We can also assume that there exists a morphism  $f: X' \rightarrow X$  compatible with  $v$  and  $w$  such that  $f(z') = z$  and  $Z'$  is an  $e$ th order thickening of  $Y := f^{-1}Z$ . For this, it suffices to show the commutativity of the following diagram.

$$\begin{array}{ccccc} H^0(X \setminus Z, \mathcal{F}) & \longrightarrow & H^1_Z(X, \mathcal{F}) & \xleftarrow{g_{Z/X}} & H^0(Z, \mathcal{F}_{-1}) \\ \downarrow & & \downarrow & & \downarrow e \\ H^0(X' \setminus Z', \mathcal{F}) & \longrightarrow & H^1_{Z'}(X', \mathcal{F}) & \xleftarrow{g_{Z'/X'}} & H^0(Z', \mathcal{F}_{-1}) \end{array}$$

The left square is commutative by functoriality. It remains to prove the commutativity of the right square. A special case of [BRS22, Proposition 7.9] states that when  $e = 1$ , that is,  $Z' = f^{-1}Z$ , the right square commutes. We shall modify the proof of [BRS22, Proposition 7.9] for the special case we need:  $Z$  and  $Z' = (f^{-1}Z)^{red}$  are of codimension 1. Therefore, the excess intersection in [BRS22, Proof of Proposition 7.9] becomes trivial. The only modification we need to make is after [BRS22, (7.9.4)]. We still have a cartesian square as follows

$$\begin{array}{ccc} Z' & \xrightarrow{f_Z} & Z \\ s' \downarrow & & \downarrow s \\ E'_1 & \xrightarrow{\tilde{f}_{E_1}} & E_1 \end{array}$$

in which  $E_1 = \mathbb{P}(\mathcal{N}_{Z/X}^\vee \oplus \mathcal{O}_Z)$ ,  $E'_1 = \mathbb{P}(\mathcal{N}_{Z'/X'}^\vee \oplus \mathcal{O}_Z)$  and  $s$  and  $s'$  are the zero sections of the bundles. Let  $\zeta = c_1(\mathcal{O}_{E'_1}(1)) \in \mathrm{CH}^1(E'_1)$  and  $\xi = c_1(\mathcal{O}_{E_1}(1)) \in \mathrm{CH}^1(E_1)$ . We need to show that  $\tilde{f}_{E_1}^* \xi = e \cdot \zeta$ . We have  $\zeta = s'_* Z'$  and  $\xi = s_* Z$ . Hence,

$$\tilde{f}_{E_1}^* \xi = \tilde{f}_{E_1}^* s_* Z = s'_* f_Z^* = s'_* [f^{-1} Z] = e \cdot s'_* Z' = e \cdot \zeta.$$

□

**R3b.** Let  $\phi: K \rightarrow L$  be a finite extension and  $v$  a valuation on  $K$  and let  $w$  be extensions of  $v$  to  $L$ . Let  $\phi_w: k(v) \rightarrow k(w)$  be the induced extensions. Then

$$\partial_v \circ \phi^* = \sum_w \phi_w^* \circ \partial_w.$$

*Proof.* We may assume that  $v$  is a complete valuation that extends to a unique complete valuation  $w$  on  $L$ . We have models  $Y$  and  $X$  of  $w$  and  $v$  respectively, with the closed points of  $\mathcal{O}_v$  and  $\mathcal{O}_w$  mapping to codimension one points  $z \in X$  and  $t \in Y$  with reduced closures  $Z$  and  $T$  respectively. We also have a dominant finite map  $f: Y \rightarrow X$  such that  $T$  is the reduced subscheme associated with  $f^{-1}Z$ . We have the following diagram, where the left square is commutative by functoriality and we need to show that the right square is commutative.

$$\begin{array}{ccccc} H^0(Y \setminus T, \mathcal{F}) & \longrightarrow & H_T^1(Y, \mathcal{F}) & \xleftarrow{g_{T/Y}} & H^0(T, \mathcal{F}_{-1}) \\ \downarrow (f^t)^* & & \downarrow (f^t)^* & & \downarrow (f^t|_T)^* \\ H^0(X \setminus Z, \mathcal{F}) & \longrightarrow & H_Z^1(X, \mathcal{F}) & \xleftarrow{g_{Z/X}} & H^0(Z, \mathcal{F}_{-1}) \end{array}$$

By [BRS22, Theorem 8.8(3)], we have  $(f^t)^* = f_*$  and  $(f^t|_T)^* = (f|_T)_*$ , where  $f_*$  is the pushforward defined in [BRS22, Sections 8.7 and 9.5]. Also,  $g_{Z/X} = i_*$  and  $g_{T/Y} = j_*$ , where  $i$  and  $j$  are the inclusion maps of  $Z$  and  $T$  respectively. By [BRS22, Theorem 9.7], we obtain  $f_* \circ j_* = (f \circ j)_* = (i \circ f|_T)_* = i_* \circ (f|_T)_*$ , as desired. □

**R3c.** Let  $\phi: K \rightarrow L$  be a field extension  $v$  a valuation on  $L$  that becomes trivial on  $K$ . Then  $\partial_v \circ \phi_* = 0$ .

*Proof.* This is exactly analogous to [Dég02, Proposition 5.4.58] with the appropriate replacement of the residue map. □

**R3d.** With the same notation as above, let  $\bar{\phi}: K \rightarrow k(v)$  be the induced extension of residue fields,  $\pi$  be a uniformizer of  $v$  and  $\rho \in H^0(K, \mathcal{F}_{-1})$ . Then  $\partial_v(\{-\pi\} \cdot \phi_*(\rho)) = \bar{\phi}_*$ .

*Proof.* We use the notation of the proof of **R3c**. We may assume that  $f^{-1}(U) = X$  and that  $\pi \in \mathcal{O}_V^\times$ , where  $V = X \setminus Z$ . Restricting further, we may assume that  $\pi \in \mathcal{O}_X(X)$  and that  $Z$  is cut out by  $\pi$ . Then  $\bar{\phi}_*(\rho)$  is the class of the image of  $\rho$  under the composition

$$\mathcal{F}_{-1}(U) \xrightarrow{f^*} \mathcal{F}_{-1}(X) \rightarrow \mathcal{F}_{-1}(Z) \xrightarrow{g_{Z/X}} H_Z^1(X, \mathcal{F})$$

and  $\partial_v(\{-\pi\} \cdot \phi_*(\rho))$  is the class of the image of  $\rho$  under the composition

$$\mathcal{F}_{-1}(U) \xrightarrow{f^*} \mathcal{F}_{-1}(X) \rightarrow \mathcal{F}_{-1}(V) \xrightarrow{\{-\pi\}} \mathcal{F}(V) \xrightarrow{\partial_Z} H_Z^1(X, \mathcal{F}).$$

Thus, it suffices to show that the following diagram is commutative for a sufficiently small open subset  $W$  of  $X$ .

$$\begin{array}{ccccc} \mathcal{F}_{-1}(W) & \xrightarrow{i^*} & \mathcal{F}_{-1}(W \cap Z) & & \\ j^* \downarrow & & \searrow^{g_{Z/X}} & & \\ \mathcal{F}_{-1}(V \cap W) & \xrightarrow{\{-\pi\}} & \mathcal{F}(V \cap W) & \xrightarrow{\partial_Z} & H_Z^1(W, \mathcal{F}) \end{array}$$

Let  $\Gamma_Z: Sh(X) \rightarrow Sh(X)$  be the functor given by  $\Gamma_Z(\mathcal{F})(W) = \Gamma_{Z \cap W}(W, \mathcal{F})$  for  $W \in X_{\text{Nis}}$ . Consider the localization sequence for  $W \in X_{\text{Nis}}$ :

$$0 \rightarrow H_{W \cap Z}^0(W, \mathcal{F}) \rightarrow H^0(W, \mathcal{F}) \rightarrow H_Z^0(V \cap W, \mathcal{F}) \rightarrow H_{W \cap Z}^1(W, \mathcal{F}) \rightarrow H^1(W, \mathcal{F}).$$

Since the Nisnevich sheafification of the presheaf  $W \mapsto H^1(W, \mathcal{F})$  is zero and  $R^1\Gamma_Z\mathcal{F}_X$  is the sheafification of the presheaf  $W \mapsto H_{W \cap Z}^1(W, \mathcal{F})$ , there exists an isomorphism  $R^1\Gamma_Z\mathcal{F}_X \cong j_*\mathcal{F}_V/\mathcal{F}_X$ . Consequently, for a sufficiently small open subset  $W$  of  $X$ , the residue morphism  $\partial_Z: \mathcal{F}(V \cap W) \rightarrow H_{Z \cap W}^1(W, \mathcal{F})$  is given by the map induced by the restriction  $\mathcal{F}(V \cap W) \rightarrow \mathcal{F}(V \cap W)/\mathcal{F}(W)$ . Set  $f^*(\rho) = \alpha$ . Then  $\partial_Z(\{-\pi\} \cdot \alpha|_{V \cap W})$  is given by the class of  $\{-\pi|_{V \cap W}\} \cdot \alpha|_{V \cap W}$ .

Let  $\mathcal{G} \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$  be such that  $\mathcal{F} = \omega_!\mathcal{G}$ . By [BRS22, Theorem 7.12],  $g_{Z/X}i^* = H^1(X, c_Z)$ , where  $c_Z$  is the cupping action defined in [BRS22, Section 5.8] for the class of  $Z$  in  $\text{CH}_Z^1(X)$ . Now we apply [BRS22, Lemma 5.10] for  $\mathcal{G} \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$ , taking  $E = Z$  and  $D = \emptyset$ ,  $\mathcal{U} = (V, \emptyset)$  and  $e = -\pi$ . It states that the map

$$H^1(c_Z): (\mathcal{G}_{-1})_X \rightarrow R^1\Gamma_Z\mathcal{G}_X \cong j_*\mathcal{G}_V/\mathcal{G}_X$$

factors through the natural injection  $\mathcal{G}_{(X, Z)}/\mathcal{G}_X \hookrightarrow j_*\mathcal{G}_V/\mathcal{G}_X$ . The map  $H^1(c_Z)(\alpha)$  is given as follows: take a representative of  $\alpha \in \mathcal{F}((\mathbb{A}^1, 0) \otimes X)$  and pull it back to  $\mathcal{G}(X, Z)$  under the morphism  $\gamma \otimes id: (X, Z) \rightarrow (\mathbb{A}^1, 0) \otimes (X, Z)$ , where  $\gamma$  is induced by the morphism  $X \rightarrow \mathbb{A}^1$  corresponding to  $-\pi \in \mathcal{O}_X(X)$ . Now, for a sufficiently small open subset  $W$  of  $X$ , we have

$$H^0(W, R^1\Gamma_Z\mathcal{G}_X) \cong H_Z^1(W, \mathcal{G}_X) \cong \mathcal{G}(W \cap V)/\mathcal{G}(W).$$

Therefore, by [BRS22, Lemma 5.6], we get

$$H^1(W, c_Z)(\alpha) = \{-\pi|_{W \cap V}\} \cdot \alpha|_{W \cap V} \in \mathcal{G}(W \cap V)/\mathcal{G}(W).$$

Since the images of  $\alpha$  under  $g_{Z/X} \circ i^*$  and  $\partial_Z\{-\pi\} \cdot \circ j^*$  agree on  $W$ , we are done.  $\square$

**R3e.** For an arbitrary unit  $u$  for the valuation  $v$  and  $\rho \in \mathcal{F}_{-1}(K)$ , we have  $\partial_v(\{u\} \cdot \rho) = -\{\bar{u}\} \cdot \partial_v(\rho)$ .

*Proof.* Let  $X$  be a model for  $v$  with the closed point of  $\mathcal{O}_v$  mapping to  $z$ . Let  $Z = \{\bar{z}\}$  and  $V = X \setminus Z$ . By passing to a small enough open subset, we may assume that  $u \in \mathcal{O}_X(X)^*$  so that we have a map  $u: X \rightarrow \mathbb{G}_m$ . We may assume that  $\rho \in H^0(V, \mathcal{F}_{-1})$ . We have the following commutative diagram, where  $r$  is the morphism induced by

$$\{\bar{u}\}: (\mathcal{F}_{-1})_{-1} \rightarrow \mathcal{F}_{-1}.$$

$$\begin{array}{ccc} H^0(V, \mathcal{F}_{-1}) & \xrightarrow{\{u\}} & H^0(V, \mathcal{F}) \\ \downarrow & & \downarrow \\ H_Z^1(X, \mathcal{F}_{-1}) & \xrightarrow{\{u\}} & H_Z^1(X, \mathcal{F}) \\ \uparrow & & \uparrow \\ H^0(Z, (\mathcal{F}_{-1})_{-1}) & \xrightarrow{r} & H^0(Z, \mathcal{F}_{-1}) \end{array}$$

We claim that  $r = -\{\bar{u}\}$ . We make the following convention: the adjunction map  $\text{Hom}(\mathbb{K}_n^M \otimes^{tr} \mathcal{F}, \mathcal{G}) \rightarrow \text{Hom}(\mathbb{K}_{n-1}^M \otimes^{tr} \mathcal{F}, \text{Hom}(\mathbb{K}_1^M, G))$  corresponds to the map  $\mathbb{K}_1^M \times \mathbb{K}_{n-1}^M \rightarrow \mathbb{K}_n^M$ . Consider the following diagram

$$\begin{array}{ccc} \text{Hom}(\mathbb{K}_1^M \otimes^{tr} \mathbb{Z}_{tr}(X), \mathcal{F}_{-1}) & \xrightarrow{r} & \text{Hom}(\mathbb{Z}_{tr}(X), \mathcal{F}_{-1}) \\ \downarrow \sim & & \downarrow \sim \\ \text{Hom}(\mathbb{K}_2^M \otimes^{tr} \mathbb{Z}_{tr}(X), \mathcal{F}) & \xrightarrow{r'} & \text{Hom}(\mathbb{K}_1^M \otimes^{tr} \mathbb{Z}_{tr}(X), \mathcal{F}) \\ \downarrow \epsilon & & \downarrow 1 \\ \text{Hom}(\mathbb{K}_2^M \otimes^{tr} \mathbb{Z}_{tr}(X), \mathcal{F}) & \xrightarrow{\{u\}} & \text{Hom}(\mathbb{K}_1^M \otimes^{tr} \mathbb{Z}_{tr}(X), \mathcal{F}) \end{array}$$

in which  $\epsilon$  is induced by the map  $\mathbb{K}_2^M \rightarrow \mathbb{K}_2^M$  given by  $\{a, b\} \rightarrow \{b, a\}$  and therefore, multiplication by  $-1$ . The top square is commutative since  $r'$  is merely the adjunction isomorphism applied to  $r$ . For the commutativity of the bottom square, observe that  $r'$  is given by the map

$$\mathbb{K}_1^M \otimes^{tr} \mathbb{Z}_{tr}(X) \rightarrow \mathbb{K}_1^M \otimes^{tr} \mathbb{K}_1^M \otimes^{tr} \mathbb{Z}_{tr}(X) \rightarrow \mathbb{K}_2^M \otimes^{tr} \mathbb{Z}_{tr}(X)$$

given by  $a \otimes b \rightarrow \bar{u}(b) \otimes a \otimes b$ , while  $\{\bar{u}\}$  is given by  $a \otimes b \rightarrow a \otimes \bar{u}(b) \otimes b$  and  $\epsilon$  interchanges the first two factors.  $\square$

**3.3. The cycle module structure associated with a good reciprocity sheaf.** We are now set to show that the cycle premodule structure associated with a good reciprocity sheaf in Sections 3.1 and 3.2 is in fact a cycle module structure in the sense of Rost [Ros96]. Throughout the section,  $\mathcal{F}$  will denote a good reciprocity sheaf in the sense of Definition 2.3. The following is the first cycle module axiom.

**Proposition 3.6** (Finite support). *Let  $X$  be a normal integral scheme of finite type over  $k$  with fraction field  $K$  and  $\rho \in H^0(K, \mathcal{F})$ . Then for all but finitely many  $x \in X^{(1)}$ ,  $\partial_x(\rho) = 0$ , where by  $\partial_x$ , we mean  $\partial_v$ , where  $v$  is the valuation of  $K$  corresponding to  $x$ .*

*Proof.* Since  $X$  is normal,  $X^{(1)}$  lies inside the smooth locus of  $X$ ; so we may assume that  $X$  is smooth. Let  $\rho$  be represented by an element of  $\mathcal{F}(U)$ . Suppose  $x \in U^{(1)}$ ; then  $\overline{\{x\}}$  and  $U$  intersect non-trivially and therefore  $\partial_x(\rho) = 0$  by definition. Since  $X^{(1)} \setminus U^{(1)}$  is a finite set, we are done.  $\square$

Let  $q$  be an integer. Let  $x \in X_{(i+1)}$  and  $y \in X_{(i)}$  for a scheme  $X$ ; put  $Z = \overline{\{x\}}$ . We define a map

$$\partial_y^x: H^0(k(x), \mathcal{F}) \rightarrow H^0(k(y), \mathcal{F}_{-1})$$

as follows: if  $y \notin Z^{(1)}$ , set  $\partial_y^x = 0$ . Otherwise, let  $\tilde{Z} \rightarrow Z$  be the normalization of  $Z$ . For any point  $z$  of  $\tilde{Z}$  lying over  $y$ , let  $\phi_z : k(y) \rightarrow k(z)$  denote the induced finite extension and  $\partial_z : H^0(k(\tilde{Z}), \mathcal{F}) \rightarrow H^0(k(z), \mathcal{F}_{-1})$  denote the residue map. Define

$$\partial_y^x := \sum_{z \in \tilde{Z}; z \rightarrow y} \phi_z^* \partial_z.$$

By Proposition 3.6, there are only finitely many points  $y$  for a given  $x \in X$  such that  $\partial_y^x \neq 0$ . Therefore, for a scheme  $X$ , the  $\partial_y^x$  give a differential graded module

$$C(X, \mathcal{F}(q)) : \\ 0 \rightarrow \bigoplus_{x \in X_{(d)}} H^0(k(x), \mathcal{F}_{d+q}) \rightarrow \cdots \bigoplus_{x \in X_{(i)}} H^0(k(x), \mathcal{F}_{i+q}) \rightarrow \cdots \bigoplus_{x \in X_{(0)}} H^0(k(x), \mathcal{F}_q) \rightarrow 0.$$

We also have the cousin complex (introduced in [Har66, Chapter IV])

$$C'(X, \mathcal{F}(q)) : \\ 0 \rightarrow \bigoplus_{x \in X_{(d)}} H_x^0(X, \mathcal{F}_{d+q}) \rightarrow \cdots \bigoplus_{x \in X_{(i)}} H_x^{d-i}(X, \mathcal{F}_{d+q}) \rightarrow \cdots \bigoplus_{x \in X_{(0)}} H_x^d(X, \mathcal{F}_{d+q}) \rightarrow 0.$$

Suppose that  $\mathcal{G} \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$  is such that  $\mathcal{F} = \underline{\omega}_! \mathcal{G}$ . We now define a differential graded map  $g : C(X, \mathcal{F}(q)) \rightarrow C(X', \mathcal{F}(q))$ . Let  $z \in X_{(i)}$  and set  $Z := \overline{\{z\}}$ . For open subsets  $U$  such that  $U \cap Z$  is smooth, we have the Gysin morphism  $g_{Z \cap U/U} : H^0(Z \cap U, \mathcal{F}_{i+q}) \rightarrow H_{Z \cap U}^{d-i}(U, \mathcal{F}_{d+q})$ . Taking the colimit over such open subsets, we get a map  $H^0(k(z), \mathcal{F}_{i+q}) \rightarrow H^{d-i}(U, \mathcal{F}_{d+q})$ . This gives a map  $g : C(X, \mathcal{F}(q)) \rightarrow C'(X, \mathcal{F}(q))$ . It can be verified that this morphism commutes with the differentials.

**Definition 3.7.** We say that  $\mathcal{F}$  satisfies weak purity if for each smooth schemes  $X$  and a smooth closed subscheme  $Z$  of codimension  $r$ , the Gysin map  $g_{Z/X} : H^0(Z, \mathcal{F}_{-r}) \rightarrow H_Z^r(X, \mathcal{F})$  is injective.

Now if  $\mathcal{F}$  satisfies weak purity, then the morphism of complexes

$$g : C(X, \mathcal{F}(q)) \rightarrow C(X', \mathcal{F}(q))$$

is injective in each degree. Since  $C'(X, \mathcal{F}(q))$  is a complex, this implies that  $C(X, \mathcal{F}(q))$  is also a complex. By the work of Saito [Sai20], every reciprocity sheaf  $\mathcal{F}$  satisfies weak purity *Zariski locally*, in the sense made clear at the end of the proof of Proposition 3.8. The second cycle module axiom can be deduced from this.

**Proposition 3.8** (Closedness). *For any good reciprocity sheaf  $\mathcal{F}$  on  $\text{Sm}/k$ , the differential graded module  $C(X, \mathcal{F}(q))$  is a complex.*

*Proof.* Since  $C'(X, \mathcal{F}(q))$  is a complex, it suffices to show that  $g$  is injective. On each direct summand  $\bigoplus_{z \in X_{(d-r)}} H^0(k(z), \mathcal{F}_{d-r+q})$ ,  $g$  is defined by

$$\text{colim}_{z \in U} H^0(Z \cap U, \mathcal{F}_{d-r+q}) \xrightarrow{g_{Z \cap U/U}} \text{colim}_{z \in U} H_{Z \cap U}^r(U, \mathcal{F}_{d+q}),$$

where  $Z = \overline{\{z\}}$ . Let  $\mathcal{G} \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$  be such that  $\mathcal{F}_{d+q} = \underline{\omega}_! \mathcal{G}$ . By [BRS22, Section 7.4], we have an exact triangle

$$i_*(\mathcal{G}_{-r})_Z[-r] \rightarrow \mathcal{G}_X \rightarrow \mathbf{R}\rho_* \mathcal{G}_{(\tilde{X}, E)} \xrightarrow{+1},$$

where  $\rho: \tilde{X} \rightarrow X$  is the blow-up of  $X$  at  $Z$  with exceptional divisor  $E$ . Applying  $\mathbf{R}\Gamma_Z(X, -)$  and taking the long exact sequence, we get an exact sequence

$$H_E^{r-1}(\tilde{X}, \mathcal{G}_{(\tilde{X}, E)}) \rightarrow H^0(Z, (\mathcal{G}_{-r})_Z) \rightarrow H^r(X, \mathcal{G}_X).$$

By [BRS22, Theorem 2.12], we have  $H_E^{r-1}(\tilde{X}, \mathcal{G}_{(\tilde{X}, E)}) \cong H_Z^{r-1}(X, \mathcal{G}_{(X, Z)})$ . So to prove the injectivity of  $g$ , we only need to show that  $\operatorname{colim}_{z \in U} H_{U \cap Z}^{r-1}(U, \mathcal{G}_{(X, Z)}) \cong H_z^{r-1}(X, \mathcal{G}_{(X, Z)}) = 0$ . But this is true by [Sai20, Corollary 8.3(2)].  $\square$

We end this section by summarizing the results.

**Theorem 3.9.** *Let  $\mathcal{F}$  be a good reciprocity sheaf on  $Sm/k$ . Then for any finitely generated field extension  $K$  of  $k$ , the association*

$$\mathcal{F}(K) := \operatorname{colim}_{\phi \neq U \subset X} \mathcal{F}(U),$$

where  $U$  varies over all the open subsets of a model  $X$  of  $K$ , defines a cycle premodule. Moreover, this cycle premodule structure satisfies the cycle module axioms.

**Remark 3.10.** We have the following classes of Nisnevich sheaves on  $Sm/k$ :

Homotopy modules  $\subset$  Homotopy invariant sheaves with transfers  $\subset$  Reciprocity sheaves.

Following the work of Déglise [Dég02], the categories of cycle modules and homotopy modules are equivalent. Moreover, the first inclusion above admits a left adjoint (see [Dég02, Proposition 3.1.7, Remarque 3.1.8]). The cycle module associated with a good reciprocity sheaf is defined by the formula analogous to the one used by Déglise. One can introduce a notion of a *good homotopy module* analogous to Definition 2.3. A consequence of Theorem 3.9 is that the inclusion

Good homotopy modules  $\subset$  Good reciprocity sheaves

admits a left adjoint. The proof is exactly analogous to [Dég02, Proof of Proposition 3.1.7, Remarque 3.1.8]

**Remark 3.11.** Let  $\mathcal{F}$  be a good reciprocity sheaf on  $Sm/k$  and let  $X \in Sm/k$ . The cycle complex  $C(X, \mathcal{F})$  is in general only a subcomplex of the cousin complex  $C'(X, \mathcal{F})$ . If  $\mathcal{F}$  is a homotopy invariant sheaf with transfers, then the complexes  $C(X, \mathcal{F})$  and  $C'(X, \mathcal{F})$  are isomorphic. This difference can be attributed to semi-purity of reciprocity sheaves; more specifically, the lack of purity in general. In the situation where purity is known (for example, in the case of logarithmic de Rham-Witt sheaves following the work of Gros [Gro85]), one can conclude that cycle complex and the cousin complex are isomorphic.

## 4. LOGARITHMIC DE RHAM-WITT SHEAVES AND THEIR KATO COMPLEXES

**4.1. Gysin maps for Logarithmic de Rham-Witt sheaves.** Let  $k$  be a perfect field of characteristic  $p > 0$ . Let  $X$  be a scheme of dimension  $d$  over  $k$ . For any integer  $r > 0$ , let  $W_r \Omega_X^\bullet$  denote the de Rham-Witt complex of  $X$  defined in [Ill79]. For any integer  $q \geq 0$ , we denote by  $\nu_r(q) := W_r \Omega_{X, \log}^q$  the logarithmic de Rham-Witt sheaf of  $X$  defined in [Shi07, Definition 2.6] to be the étale sheaf on  $X$  defined to be the image of

$$(\mathcal{O}_X^\times)^{\otimes q} \rightarrow W_r \Omega_X^q; \quad x_1 \otimes \cdots \otimes x_q \mapsto \operatorname{dlog}[x_1] \wedge \cdots \wedge \operatorname{dlog}[x_q],$$

where  $[x_i] \in W_r \mathcal{O}_X$  is the Teichmüller representative of  $x_i$ , for each  $i$ .

In [Kat86], Kato defined a family of complexes for  $q \in \mathbb{Z}$  when  $n \in \mathbb{Z} \setminus \{1\}$  and  $q \geq 0$  when  $n = 1$  given by:

$$(4.1) \quad C^\bullet(X, \mathbb{Z}/p^r\mathbb{Z}(q), n): 0 \rightarrow \bigoplus_{x \in X^{(0)}} H^n(k(x), \nu_r(q)) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(d)}} H^n(k(x), \nu_r(q-d)) \rightarrow 0,$$

under the identification  $\mathbb{Z}/p^r\mathbb{Z}(q)[q] = \nu_r(q)$ . The complex  $C^\bullet(X, \mathbb{Z}/p^r\mathbb{Z}(q), n)$  is nonzero only for  $n = 0$  or  $n = 1$ . In the case  $n = 0$ , it can be identified with Rost's cycle complex for the cycle module corresponding to mod- $p^r$  Milnor  $K$ -theory under the isomorphism  $H_{\text{ét}}^n(F, \mathbb{Z}/p^r\mathbb{Z}(n)) \simeq K_n^M(F)/p^r$  for any field  $F$  obtained by Bloch-Gabber-Kato (see [BK86]).

Our aim is to show functoriality properties analogous to those for Rost's cycle complexes in the case  $n = 1$  above. It is known that the functor  $F \mapsto H^1(F, \mu_r(q))$  does not give rise to a cycle module [Tot22]. In Section 4.2, we will exhibit that although this data comprises a slightly weaker structure than that of a cycle module, it is good enough to define the required functoriality properties for the associated Kato complexes, thanks to the recent purity results obtained in [BRS22]. We begin with a comparison of the Gysin maps for logarithmic de Rham-Witt sheaves constructed by Gros [Gro85] and by Binda-Rülling-Saito [BRS22].

Let  $i: Z \rightarrow X$  be a codimension  $r$  closed immersion of smooth schemes,  $\mathcal{F}$  a reciprocity sheaf and  $\mathcal{G} \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$  such that  $\mathcal{F} = \omega_! \mathcal{G}$ . In [BRS22, Section 7.4], a Gysin map

$$g_{Z/X}: i_*(\mathcal{G}_{-r})_Z[-r] \rightarrow \mathcal{G}_X$$

is defined. Applying  $H_Z^r(X, -)$ , we get a map

$$H^0(Z, \mathcal{F}_{-r}) \rightarrow H_Z^r(X, \mathcal{F}).$$

We take  $\mathcal{F} = \mathbf{R}\tau_* W_r \Omega_{\log}^q$ , where  $\tau: X_{\text{ét}} \rightarrow X_{\text{Nis}}$  is the canonical morphism of sites.

$$(4.2) \quad g_{X/Z}: H^0(Z, W_n \Omega_{X, \log}^{q-r}) \rightarrow H_Z^r(X_{\text{ét}}, W_n \Omega_{X, \log}^q),$$

which we also denote by  $g_{Z/X}$  abusing notation. On the other hand, a Gysin map

$$(4.3) \quad g'_{X/Z}: H^0(Z, W_n \Omega_{X, \log}^{q-r}) \rightarrow H_Z^r(X_{\text{ét}}, W_n \Omega_{X, \log}^q)$$

was constructed in [Gro85, Chapitre II, Définition 1.2.1], which we denote by  $g'_{Z/X}$ .

**Proposition 4.1.** *With the above notation, we have the equality  $g_{Z/X} = g'_{Z/X}$  of morphisms mentioned in (4.2) and (4.3).*

*Proof.* Let  $\mathcal{F}$  be a reciprocity sheaf and  $\mathcal{G} \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$  such that  $\mathcal{F} = \omega_! \mathcal{G}$ . In [BRS22, Section 5.8], for each  $\alpha \in \text{CH}_Z^r(X)$ , a map  $c_\alpha: (\mathcal{G}_{-r})_X[-r] \rightarrow \mathbf{R}\Gamma_Z \mathcal{G}_X$  is defined. By [BRS22, Theorem 7.12], we have the equality  $g_{Z/X} \circ i^* = H_Z^r(X, c_Z)$  of morphisms

$$H_Z^0(X, \mathcal{F}_{-r}) \xrightarrow{i^*} H^0(Z, \mathcal{F}_{-r}) \xrightarrow{g_{Z/X}} H_Z^r(X, \mathcal{F}).$$

On the other hand, [Gro85, Chapitre II, Corollaire 2.2.8] states that  $g'_{Z/X} i^*$  is the multiplication by  $cl(Z/X)$ , where  $cl(Z/X)$  is the image of 1 under the map

$$g'_{Z/X}: H^0(Z, W_n \Omega_{\log}^0) = \mathbb{Z}/p^n \mathbb{Z} \rightarrow H^r(X_{\text{ét}}, W_n \Omega_{\log}^r).$$



By [BRS22, Theorem 7.14],  $i$  has a retraction Nisnevich locally. Therefore,  $i^*$  is an epimorphism and in order to show that  $g'_{Z/X} = g_{Z/X}$ , it suffices to show that  $c_Z$  coincides with multiplication by  $cl(Z/X)$ . Note that

$$H^r(X_{et}, W_n \Omega_{log}^r) \cong H^{2r}(X_{et}, \mathbb{Z}/p^n \mathbb{Z}(r)) \cong \mathrm{CH}^r(X)/p^n.$$

Under the above isomorphism,  $cl(Z/X)$  is sent to the class of  $Z$  in  $\mathrm{CH}^r(X, \mathbb{Z}/p^n \mathbb{Z})$  and multiplication by  $cl(Z/X)$  is same as the action of the class of  $Z$ , which is exactly the map  $c_Z$ . This completes the proof.  $\square$

**4.2. Weak cycle module structure.** Let  $X$  be a variety of dimension  $d$  over  $k$ , which is assumed to be perfect of characteristic  $> 0$ . Fix a positive integer  $r$ . Let

$$\nu_r(m) = W_r \Omega_{X, log}^m = \mathbb{Z}/p^r \mathbb{Z}(m)[m]$$

be the logarithmic de Rham-Witt sheaf on  $X_{\acute{e}t}$  defined in [III79] (see also [Shi07, Section 2]).

**Notation 4.2.** Fix a positive integer  $r$ . For any any field extension  $F$  of  $k$  and any integer  $i$ , we write  $M_i(F) := H^1(F, \nu_r(i))$ .

It is known that the family of functors  $M_*$  from the category of field extensions of  $k$  to abelian groups do not form a cycle module since the homology of the Kato complex is not  $\mathbb{A}^1$ -invariant (see [Ota23], for instance). However, they are still endowed with the following data analogous to the definition of cycle premodules [Ros96, Definition 1.1].

- (D1) For a field extension  $\varphi: E \rightarrow F$ , there are restriction maps  $\varphi_*: M_i(E) \rightarrow M_i(F)$ .
- (D2) For a finite field extension  $\varphi: E \rightarrow F$ , there are corestriction maps  $\varphi^*: M_i(F) \rightarrow M_i(E)$  defined as in [JSS14, Section 0.7]: if  $\pi$  is the induced map on schemes, the norm map in Milnor  $K$ -theory induces a map of étale sheaves  $\pi_* \nu_{r,F}(i) \rightarrow \nu_{r,E}(i)$ . Taking cohomology and using the isomorphism  $H^1(F, \nu_{r,F}(i)) \cong H^1(E, \pi_* \nu_{r,F}(i))$ , we get the desired map. This agrees with Kato's transfer map defined in [Kat78, p.658].
- (D3) There is an action  $K_i^M(F) \times M_j(F) \rightarrow M_{i+j}(F)$  induced by the cup product in Galois cohomology and an isomorphism  $H^0(F, \nu_r(i)) \cong K_i^M(F)/p^r$  compatible with the cup product. We will use  $\cdot$  to denote the product as well as the action of Milnor  $K$ -theory groups.
- (D4) For a valuation  $v$  on  $F$  such that  $(k(v) : k(v)^p) \leq p^i$ , there is a residue map  $\partial_v: M_{i+1}(F) \rightarrow M_i(k(v))$  defined in [Kat86] as the composite:

$$\begin{aligned} H^1(F, \nu_r(i+1)) &\rightarrow H^1(F^h, \nu_r(i+1)) \xrightarrow{\cong} H^1(k(v), H^0(F^{sh}, \nu_r(i+1))) \\ &\xrightarrow{H^1(k(v), \partial_v)} H^1(k(v), H^0(k(v)^{sep}, \nu_r(i))) \cong H^1(k(v), \nu_r(i)). \end{aligned}$$

Here the  $\partial_v: H^0(F^{sh}, \nu_r(i+1)) \rightarrow H^0(k(v)^{sep}, \nu_r(i))$  is defined in [Kat86] through the isomorphism with Milnor  $K$ -theory.

The key difference above from the cycle module axioms is that the data (D4) is defined only for valuations satisfying an additional condition.

**Remark 4.3.** We will freely use the description of elements of  $M_i(F) = H^1(F, \nu_r(i))$  in terms of 1-cocycles. If  $F$  is Henselian, then  $H^1(F^h, \nu_r(i+1)) \xrightarrow{\cong} H^1(k(v), H^0(F^{sh}, \nu_r(i+1)))$  and the residue map in terms of 1-cocycles representing  $H^1(k(v), H^0(F^{sh}, \nu_r(i+1)))$  is given by:

$$\partial_v(\alpha)(\sigma) = \partial_v^M(\alpha(q^{-1}\sigma)),$$

where  $\sigma \in \Gamma_{k(v)}$ ,  $q$  is the quotient map  $\Gamma_F \rightarrow \text{Gal}(F^{sh}/F) \cong \Gamma_{k(v)}$  and the  $\partial_v^M$  is the residue map on Milnor  $K$ -theory under the identification  $H^0(F, \nu_r(i+1)) \cong K_{i+1}^M(F)/p^r$ .

We will use the explicit description of the corestriction map in **(D2)** in terms of cocycles [NSY08, Chapter 1, Section 5.4]. If  $H$  is an open subgroup of a profinite group  $G$  and  $A$  is a  $G$ -module, the corestriction map  $H^1(H, A) \rightarrow H^1(G, A)$  is as follows: let  $\alpha: H \rightarrow A$  be a cocycle; then the corestriction of  $\alpha$  to  $G$  sends  $\sigma \in G$  to

$$\sum_{\tau \in H \backslash G} s(\tau)^{-1} \alpha(\overline{s(\tau)\sigma s(\tau\sigma)^{-1}}),$$

where  $s$  is a set-theoretic splitting of  $G \rightarrow H \backslash G$ .

**Remark 4.4.** The definition of the residue  $\partial_v$  given in **(D4)** above agrees with the following definition given in [GS88a]: Take a smooth variety  $X$  such that the valuation ring of  $v$  is the local ring at a codimension 1 point  $x$  so that  $\mathcal{O}_{X,\eta} = K$ , where  $\eta$  is the generic point and  $k(x) = k(v)$ . For a point  $z \in X$ , let  $H_z^p(X, G) = \text{colim}_{z \in U} H_{\{z\} \cap U}^p(U, G)$  for  $G \in \text{Sh}(X_{\acute{e}t})$ .

From the differential of the  $E_1$  page of the coniveau spectral sequence  $E_1^{p,q} = H_x^p(X, \nu_r(s+q))$ , we get a map  $H_\eta^0(X, \nu_r(i+1)) \rightarrow \bigoplus_{y \in X^{(1)}} H_y^1(X, \nu_r(i+1))$ . By purity [Gro85, Theorem 3.5.8], for a point  $z$  of codimension  $c$ , we have an isomorphism  $H^0(k(z), \nu_r(i)) \cong H_z^c(X, \nu_r(i+c))$ . So the differential becomes  $H^0(K, \nu_r(i+1)) \rightarrow \bigoplus_{y \in X^{(1)}} H^0(k(y), \nu_r(i))$ . Projecting to the factor of  $x$ , we get a morphism that is compatible with  $\partial_v$  under the isomorphism with mod- $p^r$  Milnor  $K$ -theory given by **(D3)**.

Following [Ros96], given a valuation  $v$  on  $F \in \mathcal{F}_k$  and a uniformizer  $\pi$  for  $v$ , we define the specialization homomorphism  $s_v^\pi: M_i(F) \rightarrow M_i(F)$  by

$$s_v^\pi(\alpha) := \partial_v(\{-\pi\} \cdot \alpha).$$

The data **(D1)**–**(D4)** given above for the logarithmic de Rham-Witt sheaves satisfies the cycle premodule axioms **(R1)**–**(R3)** of [Ros96, Definition 1.1]. The axioms **(R1)** and **(R2)** are easy to verify and are left to the reader.

**R1a.** For field extensions  $\varphi: E \rightarrow F$  and  $\psi: F \rightarrow K$ , we have  $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ .

**R1b.** For finite field extensions  $\varphi: E \rightarrow F$  and  $\psi: F \rightarrow K$ , we have  $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ .

**R1c.** For field extensions  $\varphi: E \rightarrow F$  and  $\psi: F \rightarrow K$ , where  $\varphi$  is finite and  $R = F \otimes_E K$ , we have

$$\psi_* \circ \varphi^* = \sum_{p \in \text{Spec } R} l(R_p) \varphi_p^*(\psi_z)_*,$$

where  $\varphi_p$  is the extension  $E \rightarrow F \rightarrow R \rightarrow R/p$  and  $\psi_z$  is similarly defined.

**R2a.** For a field extension  $\varphi: E \rightarrow F$ ,  $\alpha \in K_n^M(E)$  and  $\rho \in M_i(E)$ , we have  $\varphi_*(\alpha \cdot \rho) = \varphi_*(\alpha) \cdot \varphi_*(\rho)$ .

**R2b.** If  $\varphi: E \rightarrow F$  is a finite extension and  $\mu \in M_i(F)$ , then  $\varphi^*((\varphi_*\alpha) \cdot \mu) = \alpha \cdot \varphi^*(\mu)$ .

**R2c.** If  $\varphi: E \rightarrow F$  is a finite extension and  $\beta \in K_n^M(F)$ , then  $\varphi^*(\beta \cdot \varphi_*(\rho)) = \varphi^*(\beta) \cdot (\rho)$ .

We verify the axiom **R3** below. This may be well-known to experts; we include the verification here for the convenience of readers.

**Proposition 4.5.** *Let  $\varphi: E \rightarrow F$  be a field extension in  $\mathcal{F}_k$  and let  $v$  be a valuation on  $F$  restricting to a valuation  $w$  on  $E$ . Let  $\overline{\varphi}_v: k(w) \rightarrow k(v)$  be the induced extension of residue fields. The following relations hold.*

**R3a.** If  $w$  is a nontrivial valuation with ramification index  $e$ , then

$$\partial_w \circ \varphi_* = e \cdot \bar{\varphi}_* \circ \partial_w.$$

**R3b.** If  $\varphi$  is a finite extension, then

$$\partial_w \circ \varphi^* = \sum_{v'} \varphi_{v'}^* \circ \partial_{v'},$$

where  $v'$  runs through the extensions of  $w$  to  $F$  and  $\bar{\varphi}_{v'} : k(w) \rightarrow k(v')$  denotes the induced extension of residue fields.

**R3c.** If  $v$  is trivial on  $E$  (that is,  $w$  is trivial), then

$$\partial_v \circ \varphi_* = 0.$$

**R3d.** If  $v$  is trivial on  $E$  and  $\bar{\varphi} : E \rightarrow k(v)$  denotes the induced map, then for any uniformizer  $\pi$  for  $v$ , we have

$$s_v^\pi \circ \varphi_* = \bar{\varphi}_*.$$

**R3e.** For a unit  $u$  with respect to  $v$  and for any  $\rho \in M_i(F)$ , we have

$$\partial_v(\{u\} \cdot \alpha) = -\{\bar{u}\} \cdot \partial_v(\alpha).$$

*Proof.* The isomorphism  $H^0(F, \nu_r(n)) \cong K_n^M(F)/p^r$  in **(D3)** is compatible with corestriction and residue maps. Since mod- $p^r$  Milnor  $K$ -theory satisfies the cycle premodule axioms **(R1)**–**(R3)** of [Ros96, Definition 1.1], so do the functors  $F \mapsto H^0(F, \nu_r(*))$ . This fact will be used repeatedly.

We first prove **R3a**. Note that we have commutative diagrams

$$\begin{array}{ccccc} H^1(E, \nu_r(i+1)) & \longrightarrow & H^1(E^h, \nu_r(i+1)) & \longrightarrow & H^1(k(w), H^0(E^{sh}, \nu_r(i+1))) \\ \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow H^1(\varphi_*) \\ H^1(F, \nu_r(i+1)) & \longrightarrow & H^1(F^h, \nu_r(i+1)) & \longrightarrow & H^1(k(v), H^0(F^{sh}, \nu_r(i+1))) \end{array}$$

and

$$\begin{array}{ccc} H^1(k(w), H^0(E^{sh}, \nu_r(i+1))) & \longrightarrow & H^1(k(w), H^0(k(w)^{sep}, \nu_r(i))) \\ \downarrow \varphi_* & & \downarrow e \cdot \bar{\varphi}_* \\ H^1(k(v), H^0(F^{sh}, \nu_r(i+1))) & \longrightarrow & H^1(k(v), H^0(k(v)^{sep}, \nu_r(i))) \end{array}$$

where the commutativity of the latter diagram follows from that of the corresponding diagram for mod- $p^r$  Milnor  $K$ -theory as the horizontal arrows in the diagram are induced by the residue maps

$$H^0(E^{sh}, \nu_r(i+1)) \cong K_{i+1}^M(E^{sh})/p^r \xrightarrow{\partial_w} K_i^M(k(w)^{sep})/p^r \cong H^0(k(w)^{sep}, \nu_r(i))$$

and

$$H^0(F^{sh}, \nu_r(i+1)) \cong K_{i+1}^M(F^{sh})/p^r \xrightarrow{\partial_v} K_i^M(k(v)^{sep})/p^r \cong H^0(k(v)^{sep}, \nu_r(i)).$$

This implies **R3a**. We next prove **R3c**. By the explicit formula given in the datum **(D4)**, it suffices to assume that  $F$  is henselian. Let  $\alpha \in M_i(E)$  and consider a 1-cocycle representing  $\alpha$ . By Remark 4.3, we have

$$\partial_v \circ \varphi_*(\alpha)(\sigma) = \partial_v(\varphi_*\alpha(q^{-1}\sigma)) = \partial_v(\varphi_*(\alpha(\tilde{\varphi}(q^{-1}\sigma))) = 0,$$

by the corresponding result for Milnor K-theory. Now, suppose that  $v$  is trivial on  $E$  and let  $\alpha \in M_i(E)$ . We prove (d) by a similar computation involving cocycles as above. For any  $\sigma \in \Gamma_{k(v)}$ ,

$$\begin{aligned} s_v^\pi \circ \varphi_*(\alpha)(\sigma) &= \partial_v(\{\{-\pi\} \cdot \varphi_*\alpha\}(\pi^{-1}\sigma)) = \partial_v(\{\{-\pi\} \cdot \varphi_*(\alpha(\tilde{\varphi}(\pi^{-1}\sigma)))\}) \\ &= s_v^\pi \varphi_*(\alpha(\tilde{\varphi}(\pi^{-1}\sigma))) \\ &= \bar{\varphi}_*(\alpha(\tilde{\varphi}(\pi^{-1}\sigma))) = \bar{\varphi}_*(\alpha)(\sigma). \end{aligned}$$

For a unit  $u$  with respect to  $v$ , we have

$$\begin{aligned} \partial_v(\{u\} \cdot \alpha)(\sigma) &= \partial_v(\{u\} \cdot \alpha(\pi^{-1}\sigma)) = -\{\bar{u}\} \cdot \partial_v(\alpha(\pi^{-1}\sigma)) = -\{\bar{u}\} \cdot \partial_v(\alpha)(\sigma) \\ &= -\{\bar{u}\} \cdot \partial_v(\alpha)(\sigma). \end{aligned}$$

This proves **R3e**.

It remains to prove **R3b**. Let  $E_w$  and  $F_v$  denote the completions of  $E$  and  $F$  with respect to  $w$  and  $v$ , respectively. Consider the diagram

$$\begin{array}{ccccc} H^1(F, \nu_r(i+1)) & \longrightarrow & \bigoplus_{v'} H^1(F_{v'}, \nu_r(i+1)) & \longrightarrow & \bigoplus_{v'} H^1(k(v'), \nu_r(i+1)) \\ \varphi^* \downarrow & & \downarrow & & \downarrow \sum_{v'} \varphi_{v'}^* \\ H^1(E, \nu_r(i)) & \longrightarrow & H^1(E_w, \nu_r(i)) & \longrightarrow & H^1(k(w), \nu_r(i)) \end{array}$$

in which the commutativity of the outer square is the assertion of (b). The left square is commutative by **(R1)**, since  $F \otimes_E E_w \cong \bigoplus_{v'} F_{v'}$ . Thus, in order to prove **R3b**, we are reduced to proving it for the extensions  $E_w \rightarrow F_{v'}$ . Therefore, replacing  $E$  by  $E_w$  and  $F$  by  $F_{v'}$ , we may assume that  $w$  is a complete valuation on  $E$  that extends uniquely to  $v$  on  $F$ . By a standard argument (see [GS17, Proof of Proposition 7.4.1] for instance), we reduce to the case where  $\varphi$  and  $\bar{\varphi}$  are Galois extensions. We need to show the commutativity of the following diagram, in which all the vertical arrows are appropriate corestriction maps.

$$\begin{array}{ccccc} H^1(F, \nu_r(i+1)) & \xrightarrow{\cong} & H^1(k(v), H^0(F^{sh}, \nu_r(i+1))) & \longrightarrow & H^1(k(v), H^0(k(v)^{sep}, \nu_r(i))) \\ \varphi^* \downarrow & & \varphi^{sh*} \downarrow & & \downarrow \\ H^1(E, \nu_r(i+1)) & \xrightarrow{\cong} & H^1(k(w), H^0(E^{sh}, \nu_r(i+1))) & \longrightarrow & H^1(k(w), H^0(k(w)^{sep}, \nu_r(i))) \\ & & \psi^* \downarrow & & \downarrow \\ H^1(E, \nu_r(i+1)) & \xrightarrow{\cong} & H^1(k(w), H^0(E^{sh}, \nu_r(i+1))) & \longrightarrow & H^1(k(w), H^0(k(w)^{sep}, \nu_r(i))) \end{array}$$

Since the analogue of **R3b** holds for Milnor K-theory, applying it to  $F^{sh} \xrightarrow{\varphi^{sh}} E^{sh}$  and then applying  $H^1(k(w), -)$ , we conclude that the top right square is commutative. The bottom right square commutes because of the functoriality of corestriction. So, it suffices to show that the diagram on the left commutes.

We have the following commutative diagram of groups, in which  $H$ ,  $H'$  and  $H''$  are defined so as to have exact rows and columns.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H'' & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_F & \longrightarrow & \Gamma_E & \xrightarrow{p_1} & \text{Gal}(F/E) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow p_4 & & \downarrow p_2 & & \\
0 & \longrightarrow & \Gamma_{k(v)} & \longrightarrow & \Gamma_{k(w)} & \xrightarrow{p_3} & \text{Gal}(k(v)/k(w)) & \longrightarrow & 0
\end{array}$$

Choose (set-theoretic) splittings  $s_1, s_2, s_3, s_4$  of  $p_1, p_2, p_3, p_4$  respectively such that the lower right square in the above diagram commutes also when the maps  $p_i$  are replaced by the splittings  $s_i$ . The map  $\varphi^{sh*}$  is given by applying  $H^1(k(w), -)$  to the norm map  $N : H^0(F^{sh}, \nu_r(i+1)) \rightarrow H^0(E^{sh}, \nu_r(i+1))$ . Let  $\alpha \in H^1(F, \nu_r(i+1))$ . Then  $\varphi^*(\alpha)$  is represented by the 1-cocycle that sends  $\sigma \in \Gamma_E$  to

$$\begin{aligned}
\varphi^*(\alpha)(\sigma) &= \sum_{\tau \in \text{Gal}(E/F)} s(\tau)^{-1} \alpha(\overline{s_1(\tau)\sigma s_1(\tau\sigma)^{-1}}) \\
&= \sum_{\omega \in \text{Gal}(k(w)/k(v))} \sum_{h \in H} s_1(hs_2(\omega))^{-1} \alpha(\overline{s_1(hs_2(\omega))\sigma s_1(hs_2(\omega)\sigma)^{-1}}) \\
&= \sum_{\omega \in \text{Gal}(k(w)/k(v))} s_3(\omega)^{-1} \sum_{h \in H} s_1(h)^{-1} \alpha(s_3(\omega)\bar{\sigma}s_3(\omega\bar{\sigma})^{-1}) \\
&= \sum_{\omega \in \text{Gal}(k(w)/k(v))} s_3(\omega)^{-1} N(\alpha(s_3(\omega)\bar{\sigma}s_3(\omega\bar{\sigma})^{-1})) \\
&= \psi^* \circ \varphi^{sh*}(\alpha)(\sigma),
\end{aligned}$$

where we have used the equalities  $\overline{s_1(hs_2(\omega))} = \overline{s_1(\omega)}$ ,  $\overline{s_1(hs_2(\omega)\sigma)^{-1}} = \overline{s_1(s_2(\omega)\sigma)^{-1}}$  and  $s_1(s_2(\omega)) = s_4(s_3(\omega))$  coming from the above diagram of Galois groups and the fact that the action of  $s_4(s_3(\omega))$  on  $H^0(F^{sh}, \nu_r(i+1))$  is the same as that of  $s_3(\omega)$ . This proves **R3b**.  $\square$

**Remark 4.6.** The fact that the above weak cycle premodule structure on the first cohomology groups of logarithmic de Rham-Witt sheaves also satisfies the cycle module axioms follows exactly as in Section 3.3 by the work of Gros [Gro85].

**Remark 4.7.** Note that the residue map in data (D4) is constructed in [Kat86] by using the degeneration of the Hochschild-Serre spectral sequence and using the definition of the residue in the case of mod- $p^r$  Milnor  $K$ -theory. This is enabled by the vanishing of the group  $H^1(\widehat{K}^{sh}, \nu_r(i))$ , for any finitely generated field extension  $K$  of  $k$ . One can get a similar partial cycle module structure on the functor  $K \mapsto H^1(K, \mathcal{F})$  for a good reciprocity sheaf  $\mathcal{F}$  provided one has  $H^1(\widehat{K}^{sh}, \mathcal{F}) = 0$ , for any finitely generated field extension  $K$  of  $k$ .

## 5. FUNCTORIALITY OF KATO COMPLEXES OF LOGARITHMIC DE RHAM-WITT SHEAVES

**5.1. The Kato complex and the four basic maps at the level of complexes.** Let  $X$  be a variety of dimension  $d$  over a perfect field  $k$  of characteristic  $p > 0$ . We have the Kato

complexes for  $n = 0, 1$ :

(5.1)

$$C(X, \mathbb{Z}/p^r\mathbb{Z}(q), n) : 0 \rightarrow \bigoplus_{x \in X_{(d)}} H^n(k(x), \nu_r(d+q)) \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{x \in X_{(0)}} H^n(k(x), \nu_r(q)) \rightarrow 0.$$

One can pass from the cohomological conventions (4.1) to the homological conventions using the formula

$$C^i(X, \mathbb{Z}/p^r\mathbb{Z}(q), n) = C_{d-i}(X, \mathbb{Z}/p^r\mathbb{Z}(q-d), n).$$

Throughout this section, we will follow the homological convention for complexes. The differential of the Kato complex is defined as follows. Let  $x \in X_{(i+1)}$ ,  $y \in X_{(i)}$  and  $\alpha \in H^1(k(x), \nu_r(i+1+q))$ . If  $y \notin Z = \overline{\{x\}}$ , then we set the  $y$ -component of  $d(\alpha)$  to be zero. Suppose  $y \in Z = \overline{\{x\}}$  and consider  $Z$  with the reduced induced subscheme structure. Then  $R = \mathcal{O}_{Z,y}$  is a 1-dimensional  $k$ -algebra with residue field  $k(y)$  and fraction field  $k(x)$ . Let  $R'$  be its normalization; this is a 1-dimensional semilocal finite  $R$ -algebra. For each valuation  $w$  of  $k(x)$  corresponding to the maximal ideals of  $R'$ , we get a finite extension  $\varphi_w: k(y) \rightarrow k(w)$ . We define the  $y$ -component of  $d(\alpha)$  in this case to be  $\sum_w \varphi_w^* \circ \partial_w(\alpha)$ . Kato proved in [Kat86] that  $C(X, \mathbb{Z}/p^r\mathbb{Z}(q), n)$  defines a complex.

We will focus on the case  $n = 1$ , as in the case  $n = 0$ , the complex  $C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 0)$  can be identified with Rost's cycle complex associated with  $K_*^M/p^r$ . Now we define the four basic maps for the Kato complex in the same way as done in [Ros96] for cycle complexes.

**Definition 5.1** (Proper pushforward). For a morphism  $f: X \rightarrow Y$ , we define

$$f_*: C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \rightarrow C(Y, \mathbb{Z}/p^r\mathbb{Z}(q), 1)$$

as the map that sends  $\alpha \in M_{i+q}(k(x))$  with  $x \in X_{(i)}$  to 0 if  $k(y) \rightarrow k(x)$  is not finite and to  $f_{x,y}^*(\alpha)$  if  $k(y) \rightarrow k(x)$  is finite, where  $f_{x,y}$  denotes the induced map on the residue fields. We write the pointwise components as  $(f_*)_y^x$ . When  $f$  is proper,  $f_*$  is a morphism of complexes.

**Definition 5.2** (Flat pullback). For a flat morphism  $f: Y \rightarrow X$  of constant relative dimension  $n$ , we define a map of complexes

$$f^*: C(X, \mathbb{Z}/p^r\mathbb{Z}(q+n), 1) \rightarrow C(Y, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[n]$$

as follows: for  $\alpha \in M_{i+q+n}(k(x))$  and  $y \in Y_x^{(0)}$ , set  $(f^*(\alpha))_y = l(\mathcal{O}_{Y_x,y}) \cdot \phi_{y,x_*}(\alpha)$ . For  $y \notin Y_x^{(0)}$ , we set  $(f^*(\alpha))_y = 0$ . We denote the pointwise components of  $f^*$  by  $(f^*)_y^x$ .

**Definition 5.3** (Multiplication by a unit). For  $t \in \mathcal{O}_X(X)^*$ , define the map

$$\{t\}: C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \rightarrow C(X, \mathbb{Z}/p^r\mathbb{Z}(q+1), 1)$$

by  $\alpha \mapsto (\{t_x\} \cdot \alpha_x)_x$ . While this is not a map of complexes, it satisfies  $d \circ \{t\} = -\{t\} \circ d$  by the axioms **R2b** and **R3e**.

**Definition 5.4** (Boundary). Let  $U$  be an open subset of  $X$  and let  $Y$  be its complement in  $X$ . Define

$$\partial_Y^U: C(U, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \rightarrow C(Y, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[-1]$$

to be the composite of the canonical inclusion  $C(U, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \rightarrow C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1)$  and the projection composed with the boundary map  $C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \rightarrow C(Y, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[-1]$ . This map satisfies  $d_Y \circ \partial_Y^U + \partial_Y^U \circ d_U = d_X \circ d_X = 0$ .

**5.2. The Kato complex of a vector bundle.** One of the main constructions in [Ros96] is the homotopy property for affine bundles at the level of cycle modules. More precise, for every affine bundle  $\pi : V \rightarrow X$ , Rost shows in [Ros96, Section 9] that the pullback map  $\pi^*$  at the level of cycle complexes is a chain homotopy equivalence. This homotopy inverse is then used in the construction of the general pullback using the deformation to normal cone. For the Kato complex (5.1) in the case  $n = 1$ , we do not have the full homotopy property. However, we will show that the pullback  $\pi^*$  admits a retract, which enables us to define the general pullback in this case.

We first define the homotopy inverse for a trivial bundle  $X \times \mathbb{A}^n \rightarrow X$ . First consider the case  $n = 1$  and write  $\mathbb{A}^1 = \text{Spec } k[t]$ . Define  $r_X$  to be the composite

$$r_X : C(X \times \mathbb{A}^1, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[1] \rightarrow C(X \times \mathbb{G}_m, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[1] \xrightarrow{\{-1/t\}} \\ C(X \times \mathbb{G}_m, \mathbb{Z}/p^r\mathbb{Z}(q+1), 1)[1] \xrightarrow{\partial} C(X, \mathbb{Z}/p^r\mathbb{Z}(q+1), 1),$$

where the leftmost morphism is the flat pullback induced by the inclusion  $X \times \mathbb{G}_m \hookrightarrow X \times \mathbb{A}^1$ , the complement  $X \times 0$  of which we identify with  $X$ . Iterating this, we can define the map of complexes

$$(5.2) \quad r_X : C(X \times \mathbb{A}^n, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[n] \rightarrow C(X, \mathbb{Z}/p^r\mathbb{Z}(q+n), 1).$$

Let  $\pi : V \rightarrow X$  be a vector bundle of rank  $n$ . We will define a map of complexes

$$r(\tau) : C(V, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[n] \rightarrow C(X, \mathbb{Z}/p^r\mathbb{Z}(q+n), 1),$$

depending upon a coordination of  $V$  in the sense of [Ros96, page 371], which is a sequence of closed subsets

$$\emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_m = X$$

such that  $V_{X_i \setminus X_{i-1}}$  is a trivial bundle for each  $i$ . Such a sequence always exists because  $X$  is noetherian. Since

$$C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1) = C(X \setminus X_{m-1}, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \oplus C(X_{m-1}, \mathbb{Z}/p^r\mathbb{Z}(q), 1),$$

we define  $r(\tau)$  inductively as  $r_{X \setminus X_{m-1}} \oplus r(\tau|_{X_{m-1}})$ . Since  $X \setminus X_{m-1}$  is a trivial bundle and we have already defined  $r$  in that case in (5.2), we are done.

**Proposition 5.5.**  *$r(\tau)$  is a left inverse to  $\pi^*$ .*

*Proof.* It suffices to show this for a rank 1 trivial bundle. Let  $\mathbb{A}^1 = \text{Spec } k[t]$  and let  $V = X \times \mathbb{A}^1$  and view it as the open subscheme of  $X \times (\mathbb{P}^1 \setminus 0)$  of  $X \times \mathbb{P}^1$ . Let  $\tilde{\pi} : X \times \mathbb{P}^1 \rightarrow X$  denote the projection onto  $X$  and consider the section of  $\tilde{\pi}$  identifying  $X$  with  $X \times \infty \subset X \times \mathbb{P}^1$  cut out by the rational function  $-1/t$ . Let  $\pi' : X \times (\mathbb{P}^1 \setminus 0) \rightarrow X \times \infty$  denote the restriction of  $\tilde{\pi}$ . We need to show that the composition  $\partial \circ \{-1/t\} \circ \pi'^*$  is the identity map

$$C(X \times \infty, \mathbb{Z}/p^r\mathbb{Z}(q+1), 1) = C(X, \mathbb{Z}/p^r\mathbb{Z}(q+1), 1) \rightarrow C(X, \mathbb{Z}/p^r\mathbb{Z}(q+1), 1).$$

As in the above paragraph, we reduce to the case  $X = \text{Spec } E$ . With the same notation as in the above paragraph, the composition  $\partial \circ \{-1/t\} \circ \pi'^*$  takes the form  $s_v^{-1/t} \circ \varphi_*$ , which is the identity map by **R3d**.  $\square$

**Theorem 5.6.** *Let  $\tau, \tau'$  on  $X$  be two coordinations of  $V$  on  $X$ , then  $r(\tau)$  and  $r(\tau')$  induce the same map on homology.*



*Proof.* Let  $\tau = \emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_s = X$  and  $\tau' = \emptyset \subset X'_1 \subset X'_2 \subset \cdots \subset X'_t = X$ . We use double induction on the lengths of the coordinations. Let  $U = X \setminus X'_{t-1}$ . If we show that  $r(\tau) = r(\tau|_U) \oplus r(\tau|_{X'_{t-1}})$ , then by induction, we have  $r(\tau|_U) = r(\tau'|_U)$  and  $r(\tau|_{X'_{t-1}}) = r(\tau'|_{X'_{t-1}})$  and consequently,  $r(\tau) = r(\tau')$ .

Therefore, it suffices to show the following: if  $U$  is an open subset of  $X$  with complement  $Z$ , then  $r(\tau) = r(\tau|_U) \oplus r(\tau|_Z)$ . Since,  $r(\tau) = \bigoplus r_{X_i \setminus X_{i-1}}$  and we have analogous expressions for  $r(\tau|_U)$  and  $r(\tau|_Z)$ , it suffices to show this for each  $X_i \setminus X_{i-1}$ . Therefore, we may assume that  $V$  is a trivial bundle and that  $\tau$  is trivial. By induction, it suffices to consider the case  $V = X \times \mathbb{A}^1$ . So, it suffices to show that  $r_X = r_U \oplus r_Z$  at the level of homology for  $V = X \times \mathbb{A}^1 \xrightarrow{\pi} X$ .

Let  $\alpha \in C(X \times \mathbb{A}^1, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[1]$  such that  $d_{X \times \mathbb{A}^1}(\alpha) = 0$ . Suppose that  $\alpha$  is concentrated at a point  $P \in X \times \mathbb{G}_m$ . Let  $Q \in X \times \infty$  be such that  $Q \in \overline{\{P\}}$  and  $\dim P = \dim Q + 1$ . Let  $\phi$  be the projection to  $X \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$  and  $\eta$  be the generic point of  $\mathbb{A}^1$ . Then  $\phi(P) = \eta$ . We now claim that  $\pi(P) = \pi(Q)$ . We have  $\pi(Q) \in \overline{\{\pi(P)\}}$ . If  $\pi(P) \neq \pi(Q)$ , then we would have  $\dim \pi(Q) \leq \dim \pi(P) - 1$ , which would imply that  $\dim Q \leq \dim P - 2$ , contradicting our assumption. Therefore, it follows that if  $\alpha$  is concentrated in  $U \times \mathbb{A}^1$  (respectively,  $Z \times \mathbb{A}^1$ ), then  $r_X(\alpha)$  coincides with  $r_U(\alpha)$  (respectively,  $r_Z(\alpha)$ ). This proves the theorem.  $\square$

**5.3. The general pullback.** In order to define the action of correspondences at the level of Kato complexes, we need to show that any morphism of smooth schemes gives rise to a pullback morphism at the level of complexes. This is done using the deformation to the normal cone technique in [Ros96]. Our construction of the general pullback for Kato complexes of logarithmic de Rham-Witt sheaves is analogous, but with necessary modifications in absence of homotopy invariance for vector bundles.

We begin by constructing the deformation map at the level of complexes. Let  $\pi: X \times \mathbb{G}_m \rightarrow X$  be the projection onto  $X$  and consider  $t \in \mathcal{O}_{X \times \mathbb{G}_m}(X \times \mathbb{G}_m)^*$ . Although we only defined the complexes  $C(X, \mathbb{Z}/p^r\mathbb{Z}(1), 0)$  for  $q \geq 0$ , the construction of the Gysin map will involve graded groups of the form

$$C_i(X \times \mathbb{G}_m, \mathbb{Z}/p^r\mathbb{Z}(-1), 1) := \bigoplus_{y \in X \times \mathbb{G}_m(i)} H^i(k(y), \mathbb{Z}/p^r\mathbb{Z}(i-1)).$$

This does not form a complex as the differential is not defined in general. However, the definitions of the basic maps still give morphisms of graded groups

$$C(X, \mathbb{Z}/p^r\mathbb{Z}(0), 1) \xrightarrow{\pi^*} C(X \times \mathbb{G}_m, \mathbb{Z}/p^r\mathbb{Z}(-1), 1)[1] \xrightarrow{\{t\}} C(X \times \mathbb{G}_m, \mathbb{Z}/p^r\mathbb{Z}(0), 1)[1].$$

Although the composite morphism  $\{t\} \circ \pi^*$  is a priori only a morphism of graded groups, it is in fact an anti-morphism of complexes. We leave the verification to the reader.

Now, let  $i: Y \rightarrow X$  be a closed immersion of codimension  $c$  with ideal sheaf  $I$ . Let  $N_Y X := \text{Spec} \bigoplus_{n \geq 0} I^n/I^{n+1}$  be the normal cone of  $Y$ . Let  $D(X, Y) := \text{Spec} \bigoplus_{n \in \mathbb{Z}} I^n t^{-n}$  be the deformation space, where  $I^n = \mathcal{O}_X$  for  $n \leq 0$ . Note that  $X \times \mathbb{G}_m$  is an open subset of  $D(X, Y)$  with complement  $N_Y X$ . Define the deformation morphism

$$J(i): C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \rightarrow C(N_Y X, \mathbb{Z}/p^r\mathbb{Z}(q), 1)$$

to be the composite

$$C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 1) \xrightarrow{\pi^*} C(X \times \mathbb{G}_m, \mathbb{Z}/p^r\mathbb{Z}(q-1), 1)[1] \xrightarrow{\{t\}} C(X \times \mathbb{G}_m, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[1] \\ \downarrow \partial \\ \xrightarrow{\partial} C(N_Y X, \mathbb{Z}/p^r\mathbb{Z}(q), 1).$$

Since  $\{t\} \circ \pi^*$  and  $\partial$  are anti-morphisms for complexes, it follows that  $J(i)$  is a morphism complexes.

**Proposition 5.7.** *Suppose  $g: X \rightarrow Z$  is a flat morphism of relative dimension  $d$  such that  $f: N_Y X \rightarrow Y \xrightarrow{i} X \xrightarrow{g} Z$  is flat. Then  $f^* = J(i) \circ g^*$ .*

*Proof.* The proof is exactly analogous to [EKM08, Proof of Lemma 51.9] and is hence, omitted.  $\square$

We are now set to define the pullback for a general morphism of schemes  $f: Y \rightarrow X$ , where  $X$  is smooth. We may factorize  $f$  as  $Y \xrightarrow{\Gamma} Y \times X \xrightarrow{p} X$ , where  $\Gamma$  denotes the graph of  $f$  and  $p$  is the projection. Since  $X$  is smooth,  $\Gamma$  is a regular closed immersion and the tangent cone  $TX := N_X(X \times X)$  is a vector bundle. We choose a coordination  $\tau$  of  $TX$ . Since  $N_Y(X \times Y) = f^*TX$ , this induces a coordination  $f^*\tau$  on  $N_Y(X \times Y)$ . Let  $d = \dim X$ ,  $d' = \dim Y$  and set  $m := d' - d$ .

**Definition 5.8.** Let the notation and setting be as above. Define  $f_\tau^*$  to be the composite

$$f_\tau^*: C(X, \mathbb{Z}/p^r\mathbb{Z}(q+m), 1) \xrightarrow{p^*} C(X \times Y, \mathbb{Z}/p^r\mathbb{Z}(q+m-d'), 1)[d'] \xrightarrow{J(\Gamma)} \\ C(N_Y(X \times Y), \mathbb{Z}/p^r\mathbb{Z}(q+m-d'), 1)[d'] \xrightarrow{r(f^*\tau)} C(Y, \mathbb{Z}/p^r\mathbb{Z}(q), 1)[m].$$

The map induced on homology by  $f_\tau^*$  is independent of the coordination  $\tau$  since the map induced by  $r(f^*\tau)$  on homology is independent of the choice of coordination.

**Proposition 5.9.** *If  $f: Y \rightarrow X$  is flat, the map of complexes  $f_\tau^*$  agrees with the flat pullback  $f^*$  defined in Definition 5.2 and is independent of the chosen coordination.*

*Proof.* Factorize  $f$  as  $Y \xrightarrow{\Gamma} Y \times X \xrightarrow{p} X$ , where  $\Gamma$  denotes the graph of  $f$  and  $p$  is the projection. We need to show that

$$f^* = r(f^*\tau) \circ J(\Gamma) \circ p^*.$$

Let  $\pi: N_Y(X \times Y) \rightarrow Y$  be the projection. Since  $r(\tau) \circ \pi^* = id$ , it suffices to show that  $\pi^* \circ f^* = J(\Gamma) \circ p^*$ . Applying Proposition 5.7 with  $i = \Gamma$  and  $g = p$ , we get  $J(\Gamma) \circ p^* = \pi^* \circ \Gamma^* \circ p^* = \pi^* \circ f^*$ , as desired.  $\square$

**5.4. Products and action of correspondences.** As a consequence of the functoriality properties of Kato complexes, we conclude that a correspondence induces a morphism of Kato complexes.

Let  $X, Y$  be schemes over  $k$ , which is assumed to be perfect of characteristic  $p > 0$ . Let  $q, q'$  be integers. Analogous to Rost's cycle complexes, there exists an *external product*

$$C_i(X, \mathbb{Z}/p^r\mathbb{Z}(q), 0) \times C_j(Y, \mathbb{Z}/p^r\mathbb{Z}(q'), 1) \rightarrow C_{i+j}(X \times Y, \mathbb{Z}/p^r\mathbb{Z}(q+q'), 1); \quad (\alpha, \beta) \mapsto \alpha \times \beta$$

defined as follows. For  $P \in (X \times Y)_{(i+j)}$ , we set  $(\alpha \times \beta)_P = 0$  unless  $P$  projects to a point  $x \in X_{(i)}$  and  $y \in Y_{(j)}$ . In the latter case, we set

$$(\alpha \times \beta)_P = \ell_P \cdot \text{res}_{k(P)/k(x)}(\alpha_x) \cdot \text{res}_{k(P)/k(y)}(\beta_y),$$

where  $\ell_P$  is the length of the local ring of  $P$  on  $\text{Spec } k(x) \times \text{Spec } k(y)$  and  $\text{res}$  denotes obvious restriction maps.

For  $(\alpha, \beta) \in C_i(X, \mathbb{Z}/p^r\mathbb{Z}(q), 0) \times C_j(Y, \mathbb{Z}/p^r\mathbb{Z}(q'), 1)$ , we define their *internal product* (or simply their *product*) by

$$(5.3) \quad \alpha \cdot \beta := \Delta_\tau^*(\alpha \times \beta) \in C_{i+j}(X \times Y, \mathbb{Z}/p^r\mathbb{Z}(q + q'), 1),$$

where  $\Delta : X \rightarrow X \times X$  is the diagonal and the pullback  $\Delta_\tau^*$  depends upon the chosen coordination  $\tau$  of the tangent bundle of  $X \times X$ .

The following proposition summarizes the basic properties of the internal and external products. Since the proofs are exactly analogous to the corresponding statements in Rost's theory, we leave them to the reader and give precise references to the corresponding statements in Rost's theory.

**Proposition 5.10.** *Let  $X$  and  $Y$  be schemes over  $k$  and let  $(\alpha, \beta) \in C_i(X, \mathbb{Z}/p^r\mathbb{Z}(q), 0) \times C_j(Y, \mathbb{Z}/p^r\mathbb{Z}(q'), 1)$ . Write  $\bar{\alpha} \in H_i(C(X, \mathbb{Z}/p^r\mathbb{Z}(q), 0))$  and  $\bar{\beta} \in H_j(C(Y, \mathbb{Z}/p^r\mathbb{Z}(q'), 1))$  for the classes of  $\alpha$  and  $\beta$  respectively in homology. Let  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$  be morphisms.*

- (a) *The internal and external product pairings factor through homology.*
- (b)  *$(f \times g)_*(\alpha \times \beta) = f_*(\alpha) \times g_*(\beta)$ .*
- (c) *If  $X, X', Y, Y'$  are smooth, then  $(f \times g)^*(\alpha \times \beta) = f^*(\alpha) \times g^*(\beta)$ , where we have suppressed the coordinations.*
- (d) *If  $X, Y$  are smooth and  $\phi : Y \rightarrow X$  is a morphism, then  $\phi^*(\bar{\alpha} \cdot \bar{\beta}) = \phi^*(\bar{\alpha}) \cdot \phi^*(\bar{\beta})$ .*
- (e) *(Projection formula) If  $X, Y$  are smooth and  $\phi : Y \rightarrow X$  is a proper morphism, then  $\phi_*(\bar{\alpha} \cdot \phi^*(\bar{\beta})) = \phi_*(\bar{\alpha}) \cdot \bar{\beta}$ .*
- (f) *Define  $[Y] := p_Y^*(1)$ , where  $p_Y : Y \rightarrow \text{Spec } k$  denotes the structure morphism. If  $X$  is smooth and  $\phi : Y \rightarrow X$  is a morphism, then  $\phi_*(\phi^*(\bar{\beta})) = (\phi_*[Y]) \cdot \bar{\beta}$ .*

*Proof.* The proofs are exactly analogous to those of [EKM08, 50.3, 50.4, 55.20, 56.8, 56.9 and 56.11].  $\square$

We are now set to construct the action of a correspondence at the level of Kato complexes. A similar action in the case of cycle modules has been considered in [Gil14, 1.11].

Let  $X, Y, Z$  be proper schemes over  $k$  with  $X, Z$  smooth and  $\dim Z = d$ . Assume

$$z = \sum_{i=1}^r n_i [W_i]$$

is a codimension  $d$  cycle on  $Y \times Z$ . For  $i = 1, \dots, r$  let  $\iota_i : W_i \hookrightarrow Y \times Z$  be the inclusion of the closed integral subscheme  $W_i$  of codimension  $d$ , and define morphisms  $f(i) : X \times W_i \rightarrow Y \times Z$

and  $g(i) : X \times W_i \rightarrow Y \times Z$  by the commutative diagram

$$\begin{array}{ccccc}
 & & X \times W_i & & \\
 & \swarrow g(i) & \downarrow \text{id}_X \times \iota_i & \searrow f(i) & \\
 & & X \times Y \times Z & & \\
 & \swarrow \pi_{XY} & & \searrow \pi_{XZ} & \\
 X \times Y & & & & X \times Z
 \end{array}$$

in which  $\pi_{XY}, \pi_{XZ}$  are the obvious projection maps. Note that the morphism  $f(i)$  is proper for each  $i$ . The morphism of Kato complexes induced by  $\alpha$  is given by

$$(5.4) \quad z_* = \sum_{i=1}^r n_i f(i)_* \circ g(i)_\tau^* : C(X \times Y, \mathbb{Z}/p^r \mathbb{Z}(q), 1) \rightarrow C(X \times Z, \mathbb{Z}/p^r \mathbb{Z}(q), 1)$$

for any coordination  $\tau$  of the tangent bundle of  $Y \times Z$  and such that the induced maps on cohomology groups

$$H^i(X \times Y, \mathbb{Z}/p^r \mathbb{Z}(q)) \rightarrow H^i(X \times Z, \mathbb{Z}/p^r \mathbb{Z}(q))$$

depend only on the class  $\bar{z}$  of  $z$  in the Chow group  $\text{CH}^d(X \times X)$  and are denoted by  $\bar{z}_*$ .

**Proposition 5.11.** *Let the setting and notation be as above and let  $\pi_{XY}, \pi_{YZ}$  and  $\pi_{XZ}$  denote the projection maps from  $X \times Y \times Z$  to  $X \times Y, Y \times Z$  and  $X \times Z$ , respectively. For any  $\alpha \in H^i(X \times Y, \mathbb{Z}/p^r \mathbb{Z}(q))$ , we have*

$$\bar{z}_*(\alpha) = \pi_{XZ*}(\pi_{YZ}^*(\bar{z}) \cdot \pi_{XY}^*(\alpha)).$$

*Proof.* Since  $z = \sum_{i=1}^r n_i [W_i]$ , by the formula (5.4) and the equalities  $f(i) = \pi_{XZ} \circ (\text{id}_X \times \iota_i)$  and  $g(i) = \pi_{XY} \circ (\text{id}_X \times \iota_i)$ , we have

$$\begin{aligned}
 \bar{z}_*(\alpha) &= \sum_{i=1}^r n_i f(i)_* \circ g(i)_\tau^*(\alpha) \\
 &= \sum_{i=1}^r n_i (\pi_{XZ} \circ (\text{id}_X \times \iota_i))_* \circ (\pi_{XY} \circ (\text{id}_X \times \iota_i))^*(\alpha) \\
 &= \sum_{i=1}^r n_i (\pi_{XZ} \circ (\text{id}_X \times \iota_i))_* (\text{id}_X \times \iota_i)^*(\alpha) \cdot \pi_{XY}^*(\alpha) \\
 &= \sum_{i=1}^r n_i \pi_{XZ*}((\text{id}_X \times \iota_i)_*[X \times W_i] \cdot \pi_{XY}^*(\alpha)) = \pi_{XZ*}(\pi_{YZ}^*(\bar{z}) \cdot \pi_{XY}^*(\alpha)),
 \end{aligned}$$

by Proposition 5.10. □

**Remark 5.12.** Let  $X$  be a smooth scheme over a perfect field  $k$  of characteristic  $p > 0$ . By the Gersten conjecture for logarithmic de Rham-Witt sheaves [GS88b], for integers  $i, r$ , and  $q$  the sheaf  $\mathcal{H}_{\text{ét}}^i(\mathbb{Z}/p^r \mathbb{Z}(q))$  on  $X_{\text{Zar}}$  has a resolution given by

$$\mathcal{C}(X, i, q) : \bigoplus_{x \in X^{(0)}} \mathcal{H}_{\text{ét}}^{i,q}(k(x)) \rightarrow \bigoplus_{x \in X^{(1)}} \mathcal{H}_{\text{ét}}^{i-1,q-1}(k(x)) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(d)}} \mathcal{H}_{\text{ét}}^{i-d,q-d}(k(x)),$$

where  $\mathcal{H}_{\text{ét}}^{m,n}(k(x)) := i_{x*} \mathcal{H}_{\text{ét}}^m(k(x), \mathbb{Z}/p^r \mathbb{Z}(n))$  for a point  $i_x : \{x\} \rightarrow X$ . Therefore, it follows that the cohomology  $H^j(X, \mathcal{H}_{\text{ét}}^i(\mathbb{Z}/p^r \mathbb{Z}(q)))$  of this complex agrees with the homology

$H_{d-j}(C(X, \mathbb{Z}/p^r\mathbb{Z}(q-d), i-j))$  of the Kato complex (5.1). The action of correspondences constructed above in Proposition 5.11 agrees with the action of correspondences considered in [Dia19, Section 1.4].

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