Assignment 0

(Submission deadline: 24.09.2020)

Exercise 1.
Show that a topological space X is Hausdorff if and only if the diagonal $\Delta(X) = \{(x,x) \in X \times X \mid x \in X\}$ is closed in $X \times X$.

Exercise 2.
Let $f : X \to Y$ be a surjective map of topological spaces. Show that the follwing are equivalent:

(a) f is a quotient map.
(b) a function $g : Y \to Z$ is continuous if and only if $g \circ f : X \to Z$ is continuous.

Exercise 3.
Let X and Y be topological spaces endowed with equivalence relations $\sim_X$ and $\sim_Y$, respectively. Let $\sim$ denote the equivalence relation on $X \times Y$ defined by setting $(x,y) \sim (x',y')$ if and only if $x \sim_X x'$ and $y \sim_Y y'$.

(a) Show that there is a unique bijective continuous map $\phi : \dfrac{X \times Y}{\sim} \to \dfrac{X}{\sim_X} \times \dfrac{Y}{\sim_Y}$ making the following diagram

$$\begin{array}{ccc} X \times Y & & \\ \downarrow & \searrow & \\ \dfrac{X \times Y}{\sim} & \overset{\phi}{\dashrightarrow} & \dfrac{X}{\sim_X} \times \dfrac{Y}{\sim_Y} \end{array}$$

commute, in which the vertical map is the quotient map and the diagonal map is the product of obvious quotient maps.
(b) Give an example to show that ϕ need not be a homeomorphism.
(c) If Y is locally compact and $\sim_Y$ is the identity relation, then show that ϕ is a homeomorphism.

Exercise 4.
Let X and Y be topological spaces and let Y^X dentoe the set of all continuous functions $X \to Y$. Define a topology on Y^X by taking as a sub-basis all sets of the form

$$N_{K,U} = \{f : X \to Y \mid f(K) \subset U\},$$

where K runs through all compact subsets of X and U runs through all open subsets of Y. This topology is called the *compact-open topology*.

(a) Let X be locally compact and Hausdorff. Show that the *evaluation function* $e : Y^X \times X \to Y$ defined by $e(f,x) = f(x)$ is continuous.
(b) Let X, Z be Hausdorff and Z be locally compact. Show the *exponential law*: there is a natural function $Y^{Z \times X} \to (Y^Z)^X$, which is a homeomorphism.