Assignment 2

(Submission deadline: 08.10.2020)

Exercise 1.

Let X be a pointed topological space and let $n \geq 1$ be an integer. Let $f, g : X \to S^n$ be non-antipodal maps; that is, $f(x) \neq -g(x)$, for all $x \in X$. Show that f and g are homotopic.

Exercise 2.

Let (X, x_0) be a pointed space and let $\alpha : I \to X$ be a path from x_0 to x_1. Show that α induces a well-defined group homomorphism

$$\gamma_{[\alpha]} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

taking the class of a loop $[\beta]$ to $[\alpha\beta\alpha^{-1}]$. Show that for a path β from x_1 to x_2, we have

$$\gamma_{[\beta \cdot \alpha]} = \gamma_{[\beta]} \circ \gamma_{[\alpha]}.$$

Conclude that if $\pi_1(X, x_0)$ is abelian, then $\gamma_{[\alpha]}$ is independent of $[\alpha]$. Consequently, $\pi_1(X, x_0)$ is canonically isomorphic to $\pi_1(X, x_1)$.

Exercise 3.

Let (X, x_0) and (Y, y_0) be pointed topological spaces. Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ as groups.

Exercise 4.

Show the fundamental theorem of algebra by proceeding in the following steps.

(1) Given a continuous map $f : S^1 \to S^1$, choose a path α in S^1 taking $f(1)$ to 1. The composite

$$\pi_1(S^1, 1) \xrightarrow{f_*} \pi_1(S^1, f(1)) \xrightarrow{\gamma_\alpha} \pi_1(S^1, 1)$$

sends a generator, say a, of $\pi_1(S^1, 1) \simeq \mathbb{Z}$ to na, for an integer n. We define the *degree* of f by $\deg(f) = n$. Show that the degree is well-defined and is independent of the choice of α.

(2) What is the degree of the constant map? What is the degree of the map $x \mapsto x^n$?

(3) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$ be a polynomial. Suppose, if possible, that f has no root in $\mathbb{C}$. Define $\hat{f} : S^1 \to S^1$ by $x \mapsto f(x)/|f(x)|$.

(a) If $f(x) \neq 0$ for all $x \in \mathbb{C}$ with $|x| \leq 1$, then show that $\hat{f}$ is homotopic to the constant map at $f(0)/|f(0)|$.

(b) If $f(x) \neq 0$ for all $x \in \mathbb{C}$ with $|x| \geq 1$, then show that $\hat{f}$ is homotopic to the map $x \mapsto x^n$.

(4) Get a contradiction and conclude.