Assignment 3

Exercise 1.

If $f : X \to Y$ is a homotopy equivalence, then show that $\Pi(f) : \Pi(X) \to \Pi(Y)$ is an equivalence of groupoids.

Exercise 2.

For any subspace Z of a space Y, let $\Pi(Y, Z)$ denote the full subcategory of $\Pi(Y)$ consisting of points of Z as objects. Let U and V be subspaces of a space X such that the interiors of U and V cover X. Let A be a subspace of $U \cap V$ such that A intersects every connected component of U, V and $U \cap V$. Show that the diagram

$$\begin{array}{ccc} \Pi(U \cap V, A) & \longrightarrow & \Pi(U, A) \\ \downarrow & & \downarrow \\ \Pi(V, A) & \longrightarrow & \Pi(X, A) \end{array}$$

is a pushout square in the category of groupoids.

Exercise 3.

Use Exercise 2 to give another proof of the fact that $\pi_1(S^1, 1) = \mathbb{Z}$.

Exercise 4.

The *Klein bottle* is the quotient space of $S^1 \times I$ under the equivalence relation generated by $(z, 0) \sim (z^{-1}, 1)$, for all $z \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Compute the fundamental group of the Klein bottle.