Assignment 4

(Submission deadline: 22.10.2020)

Exercise 1.

(a) Give an example of a connected covering map $p : E \to B$ of locally path connected spaces for which there exists a continuous map $f : E \to E$ such that
- $p \circ f = p$; and
- f is *not* a homeomorphism.

(Thus, in the definition of deck transformations, we need to assume that f is a homeomorphism; it is not automatically true.)

(b) Let B be locally path connected, let $p : E \to B$, $p' : E' \to B$ be covering maps and let $f : E \to E'$ be a morphism of coverings $p \to p'$. Show that f is also a covering map.

Exercise 2.

Let $p : E \to B$ be a covering map, which is a left principal G-covering for a discrete group G. Let $\ell : G \to \mathrm{Aut}(p)$ denote the group homomorphism that takes $g \in G$ to the deck transformation

$$\ell_g : E \to E; \quad x \mapsto gx$$

of p given by left translation by g. Suppose that E is connected.

(1) Show that ℓ is an isomorphism of groups.
(2) Show that the action of each subgroup of $\mathrm{Aut}(p)$ on E is properly discontinuous.
(3) Assume that B is locally path connected and that H is a subgroup of $\mathrm{Aut}(p)$. Show that p induces a covering map $E/H \to B$, where E/H denotes the quotient of E by the equivalence relation $x \sim hx$, for all $h \in H$.

Exercise 3.

Consider the subspace of $\mathbb{R}^2$ defined by

$$B = \partial(I \times I) \cup \left(\bigcup_{n \in \mathbb{N}} \{1/n\} \times I \right).$$

Show that B cannot admit a covering $p : E \to B$, where E is simply connected.

Exercise 4.

Let $k \leq n$ be positive integers and let $Gr_{\mathbb{C}}(k, n)$ denote the set of k-dimensional $\mathbb{C}$-vector subspaces of $\mathbb{C}^n$. Let $S(k, n) \subset (\mathbb{C}^n)^k$ denote the subset consisting of linearly independent k-tuples of elements of $\mathbb{C}^n$ endowed with the subspace topology. There is a surjective map

$$\pi : S(k, n) \to Gr_{\mathbb{C}}(k, n),$$

taking a k-tuple to its span; endow $Gr_{\mathbb{C}}(k, n)$ with the quotient topology. Show that $Gr_{\mathbb{C}}(k, n)$ is path connected and compute its fundamental group using the van Kampen theorem. (The space $Gr_{\mathbb{C}}(k, n)$ is called the *complex Grassmannian* of k-dimensional subspaces of $\mathbb{C}^n$.)