Assignment 8

(Submission deadline: 19.11.2020)

Exercise 1.
Is the map $p : \mathbb{C} \to \mathbb{C}$ given by $p(z) = z^2$ a fibration? Is it a Serre fibration? (*Hint: Does p have homotopy lifting property for I^0? What about I^1?*)

Exercise 2.
Let $p : E \to B$ be a fibration and let $b_0 \in B$ be the basepoint. Show that the inclusion $p^{-1}(b_0) \to \mathrm{hofib}(p)$ given by $e \mapsto (e, \mathrm{const}_{b_0})$ is a pointed homotopy equivalence.

Exercise 3.
Given $n \geq 1$, a topological space X, a subset $A \subset X$ and a basepoint $x_0 \in A$, we define $\pi_n(X, A, x_0)$ to be the set of homotopy classes of maps from $(D^n, S^{n-1}, *)$ (or, equivalently, from $(I^n, \partial I^n, J^{n-1})$) to (X, A, x_0). We will suppress basepoints. Show that $\pi_{n+1}(X, A)$ can be identified with $\pi_n(P(X, x_0, A))$.

Exercise 4.

We have shown in class that a trivial fiber bundle $p : E \to B$ with fiber F (that is, E is homeomorphic to $F \times B$) is a fibration. Show that an arbitrary fiber bundle $p : E \to B$ with fiber F is a Serre fibration by proving the existence of the lift in the diagram

$$\begin{tikzcd} I^n \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "H"] \arrow[ur, dashed] & B \end{tikzcd}$$

in the following steps:

(1) Using compactness of I and the Lebesgue number lemma, get a finite open covering $\{U_1, \dots, U_r\}$ of B trivializing p and a subdivision of I^n into finitely many cubes $\{C_1, \dots, C_s\}$ and a subdivision of I into finitely many subintervals $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for every $1 \leq i \leq s$ and every $0 \leq j \leq m-1$, the product $C_i \times [t_j, t_{j+1}]$ is contained in $H^{-1}(U_k)$ for some $1 \leq k \leq s$.
(2) Show that H can be lifted to $C_0 \times [t_0, t_1]$.
(3) Using the fact that trivial fiber bundles are Serre fibrations, show that H can be inductively lifted cube-by-cube to get a complete lift $\widetilde{H} : I^n \times I \to E$.