Assignment 10

(Submission deadline: 03.12.2020)

Fix a commutative ring R.

Exercise 1.

Consider the category of chain complexes of R-modules.

(a) Show that chain homotopy gives an equivalence relation on the morphisms of chain complexes.

(b) If $f, g : A_\bullet \to B_\bullet$ are chain homotopic morphisms of R-modules, then show that $H_n(f) = H_n(g)$, for all $n \in \mathbb{Z}$.

Exercise 2.

(a) Let

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

be a commutative diagram of R-modules with exact rows. If g is surjective and f' is injective, show that there is an exact sequence

$$\text{Ker}\,(\alpha) \to \text{Ker}\,(\beta) \to \text{Ker}\,(\gamma) \to \text{Coker}\,(\alpha) \to \text{Coker}\,(\beta) \to \text{Coker}\,(\gamma).$$

(b) Show that a short exact sequence of chain complexes of R-modules

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

induces a long exact sequence

$$\cdots \to H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet) \to H_{n-1}(A_\bullet) \to \cdots$$

of homology groups.

Exercise 3.

Let $(X_j, x_j)_{j\in J}$ be a family of pointed spaces and set $(X, x) = \bigvee_{j\in J}(X_j, x_j)$ with $i^j : (X_j, x_j) \to (X, x)$ denoting the inclusion. Suppose that $\overline{\{x\}}$ has a neighborhood U in X such that $H_*(U, \{x\}) \to H_*(X, \{x\}$ is the zero map. Show that

$$\sum i^j_* : \bigoplus_{j\in J} H_*(X_j, \{x_j\}) \to H_*(X, \{x\})$$

is an isomorphism.

Exercise 4.

Let $\{(C^{(i)}_\bullet, d^{(i)}_\bullet)\}_{i\in\Lambda}$ be a direct system of chain complexes of abelian groups.

(a) Let $C_n = \varinjlim_i C^{(i)}_n$. Show that the differentials $d^{(i)}_n$ induce differentials $d_n : C_n \to C_{n-1}$ making $(C_\bullet, d_\bullet)$ into a chain complex such that the natural maps $C^{(i)}_n \to C_n$ yield morphisms of chain complexes $f^{(i)}_\bullet : (C^{(i)}_\bullet, d^{(i)}_\bullet) \to (C_\bullet, d_\bullet)$.

(b) Formulate and prove the universal property of $(C_\bullet, d_\bullet)$ with respect to the above morphisms $f_\bullet^{(i)}$.

(c) Show that the natural maps

$$f_*^{(i)} : H_n(C_\bullet^{(i)}) \to H_n(C_\bullet)$$

induce an isomorphism

$$\varinjlim_{i \in \Lambda} H_n(C_\bullet^{(i)}) \to H_n(C_\bullet).$$