Assignment 11

(Submission deadline: 24.12.2020)

Exercise 1.
Let $X = T^{\# n}$, where $T = S^1 \times S^1$. Compute the (singular) homology groups of X.

Exercise 2.
Give an example of a space X with a finite covering $\{U_1, \dots, U_n\}$ by open subsets such that $H_*(U_i)$ is finitely generated for every i, but $H_*(X)$ is not finitely generated.

Exercise 3.
Give an example of a continuous map $f : X \to Y$ which is not a homotopy equivalence but induces an isomorphism $f_* : H_*(X) \to H_*(Y)$.

Exercise 4.
Let G be a finitely generated abelian group and let $n \geq 1$ be an integer. Show that there exists a finite connected CW-complex X of dimension $\leq n+1$ such that $H_n(X) = G$ and $H_i(X) = 0$ for all $i > 0, i \neq n$.
Hint: Reduce to the case where G is cyclic.

Exercise 5.
Let $p : E \to B$ be an r-sheeted covering space.

(a) Show that if $\sigma : \Delta^n \to B$ is a singular n-simplex in X, then there are precisely r singular simplices $\sigma_i : \Delta^n \to E$, $1 \leq i \leq r$, such that $f \circ \sigma_i = \sigma$ for all i.

(b) Show that $[\sigma] \mapsto [\sigma_1] + \cdots + [\sigma_r]$ defines a chain map $Lf^\bullet : S_\bullet(B) \to S_\bullet(E)$ and hence, a homomorphimsm $Lf^* : H_*(B) \to H_*(E)$.

(c) Compute the composition $f_* \circ Lf^* : H_*(B) \to H_*(B)$.

(d) Suppose moreover that $G = \mathrm{Aut}(p)$ is Galois. Show that every $g \in G$ determines an automorphism $g_* : H_*(E) \to H_*(E)$ and that the composition

$$L(f)^* \circ f_* : H_*(E) \to H_*(E)$$

coincides with $\sum_{g \in G} g_*$.

Exercise 6.
Let $f : S^n \to S^n$ be a map and define its (homological) *degree* to be the unique integer $\deg(f)$ such that $f_* : H_n(S^n) \to H_n(S^n)$ is the multiplication by $\deg(f)$. Show that the antipode map $A : S^n \to S^n$ has degree $(-1)^{n+1}$.