Assignment 12

(Submission deadline: 05.01.2021)

Exercise 1.

Let $n > 1$ and let $f : S^n \to S^n$ be a map.

(a) For every $x \in S^n$, show that the natural map $H_n(S^n) \to H_n(S^n, S^n \setminus \{x\})$ is an isomorphism.

(b) The image of the standard generator of $H_n(S^n)$ defines a *local homology generator* $\xi_x \in H_n(S^n, S^n \setminus \{x\})$. Assume moreover that $f^{-1}(x) = \{x_1, \dots, x_r\}$. For every $1 \le i \le r$, define the *local degree* of f at x_i to be the unique integer $\deg_{x_i}(f)$ such that the homomorphism

$$\mathbb{Z}\xi_{x_i} = H_n(S^n, S^n \setminus \{x_i\}) \xrightarrow{f_*} H_n(S^n, S^n \setminus \{x\}) = \mathbb{Z}\xi_x.$$

is multiplication by $\deg_{x_i}(f)$. Show that the (homological) degree of f satisfies the formula

$$\deg(f) = \sum_{i=1}^{r} \deg_{x_i}(f).$$

Exercise 2.

Let X be a space and let R be a commutative ring. Let $\alpha \in S^p(X; R)$, $\beta \in S^q(X; R)$ and $x \in S_n(X; R)$. Show that

(a) $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^p \alpha \cup \delta\beta$;

(b) $(-1)^p \partial(\alpha \cap x) = \alpha \cap \partial x - \delta\alpha \cap x$.

Exercise 3.

Determine the cohomology ring of $T = S^1 \times S^1$.

Exercise 4.

The *Euler characteristic* of a space X is defined to be

$$\chi(X) := \sum_{i \ge 0} (-1)^i b_i(X),$$

where $b_i(X)$ denotes the ith Betti number of X.

(a) If X is a finite CW-complex with N_i cells in dimension i, then show that

$$\chi(X) = \sum_i (-1)^i N_i.$$

(b) If X and Y are finite CW-complexes, show that

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

(c) If X is a finite CW-complex and Y is such that there is an n-sheeted covering map $p : Y \to X$, then show that Y has a natrual CW-structure such that p is cellular and we have

$$\chi(Y) = n\chi(X).$$