Assignment 1

(Submission deadline: 30.03.2021)

Exercise 1.

Let $M_g = T^{\#g}$ denote the compact oriented surface of genus g and regard it as the quotient of a regular $4g$-gon. Obtain a presentation for the singular cohomology ring $H^*(M_g, \mathbb{Z})$. Assume the result in the case $g = 1$ (which was asked in the first semester exam) and argue by induction using the following: collapse a suitable diagonal in the regular $4g$-gon to to show that there is a quotient map $M_g \to M_1 \vee M_{g-1}$, which induces an isomorphism on H^1 and a surjection on H^2.

Exercise 2.

Determine the integral singular cohomology ring of $\mathbb{RP}^n$ for n odd.

Exercise 3.

Show that the integral singular cohomology ring of the product of n copies of S^1 is isomorphic to the exterior algebra on n generators. You can assume the result for $S^1 \times S^1$ from the first semester.

Exercise 4.

Let $f : M \to N$ be a map of compact connected oriented n-manifolds such that $f_*([M]) = [N]$. Show that the maps $\pi_1(M) \to \pi_1(N)$ and $H_1(M; \mathbb{Z}) \to H_1(N; \mathbb{Z})$ are surjective.

Exercise 5.

Let M be a compact connected 3-manifold and write $H_1(M; \mathbb{Z})$ as $\mathbb{Z}^r \oplus F$, where F is a finite abelian group. Show that $H_2(M; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^r$ when M is orientable and $\mathbb{Z}^r \oplus \mathbb{Z}/2\mathbb{Z}$ otherwise.

Exercise 6.

- (a) Let $A \in \mathrm{GL}_n(\mathbb{C})$ and consider the map $f_A : \mathbb{CP}^{n-1} \to \mathbb{CP}^{n-1}$ given by $[v] \mapsto [Av]$ for $v \in \mathbb{C}^n$. Show that the Lefschetz number of f_A equals the Euler characteristic of $\mathbb{CP}^{n-1}$. Does f_A have a fixed point?
- (b) Prove the fundamental theorem of algebra (that is, every monic polynomial in one variable with coefficients in $\mathbb{C}$ has a root) using part (a).