Assignment 2

(Submission deadline: 15.04.2021)

Exercise 1.
Let $f : M \to \mathbb{R}^n$ be a map, where M is a smooth compact n-manifold. Show that df_x must be singular at some $x \in M$. Conclude that you cannot embed a smoth compact n-manifold in $\mathbb{R}^n$.

Exercise 2.
Let M be a smooth manifold and let G be a finite group acting on M by diffeomorphisms. Show that the set of fixed points $M^G := \{m \in M \mid g \cdot m = m\}$ is a smooth manifold.

Exercise 3.

(a) Let M be a smooth manifold and let G be a finite group acting freely on M by diffeomorphisms. Show that the quotient space M/G is a smooth manifold.

(b) Give an example of a smooth action of the group $\mathbb{R}$ on a manifold M (that is, the action map $\mathbb{R} \times M \to M$ is smooth) such that the quotient $M/\mathbb{R}$ with the quotient topology is not a manifold.

Exercise 4.
Let $f : M \to N$ be a smooth map between smooth manifolds and let $q \in N$ be such that f is a subersion at every $p \in f^{-1}(q)$ (In this case, q is said to be a *regular value* of f). Show that $f^{-1}(q)$ is an embedded submanifold of M of dimension $\dim M - \dim N$. Also show that $T_p f^{-1}(q)$ is isomorphic to Ker df_p.

Exercise 5.
Let V be the $\frac{n(n+1)}{2}$-dimensional real vector space of $n \times n$ symmetric real matrices. Define $f : GL_n(\mathbb{R}) \to V$ by $A \mapsto {}^tA \cdot A$ and show that it is a smooth map. Show that the identity matrix $\mathrm{Id} \in V$ is a *regular value* of f; that is, f is a submersion at every $A \in f^{-1}(\mathrm{Id})$. What is $f^{-1}(\mathrm{Id})$?

Exercise 6.
Let $n > 1$ be an integer and set $N = \frac{n(n+1)}{2}$. Let $F : \mathbb{R}^n - \{0\} \to \mathbb{R}^N$ be defined by $(x_i)_i \mapsto (x_i x_j)_{i \le j}$, where the $x_i x_j$ are ordered lexicographically.

(a) Show that F is an immersion.

(b) Show that $F(v) = F(w)$ if and only if $v = \pm w$. Show that F induces an embedding $f : \mathbb{RP}^{n-1} \to S^{N-1}$.

(c) Show that f cannot be surjective. Conclude that f induces an embedding of $\mathbb{RP}^{n-1}$ in $\mathbb{R}^{N-1}$.

Note: For $n = 3$, this gives an embedding of $\mathbb{RP}^2$ in $\mathbb{R}^5$. This can be improved to $\mathbb{R}^4$ using the so-called Whitney embedding theorem but not to $\mathbb{R}^3$ (as we have already seen in class).