Assignment 3

(Submission deadline: 07.05.2021)

Exercise 1.

Let M be a smooth manifold. Show that any sheaf homomorphism $\mathcal{C}_M^\infty \to \mathcal{C}_M^\infty$, which is $\mathbb{R}$-linear and satisfies the Leibniz rule is given by $f \mapsto Xf$, for a unique vector field X on M.

Exercise 2.

Determine the tangent space $T_{\mathrm{Id}}(SO_n(\mathbb{R}))$ as a subspace of $T_{\mathrm{Id}}(GL_n(\mathbb{R})) = M_n(\mathbb{R})$.

Exercise 3.

Show that for a real vector space V of dimension n, the real Grassmannian $Gr(k,V)$ of k-dimensional subspaces of V has the structure of a smooth manifold.

Hint: You may use the following atlas $\{U_W\}_W$. For $W \in Gr(k,V)$ define a chart U_W containing W as follows: fix an inner product on V and consider the orthogonal decomposition $V = W \oplus W^\perp$ with natural projections $\pi_W : V \to W$ and $\pi_{W^\perp} : V \to W^\perp$. Set $U_W := \{E \mid E \cap W^\perp = 0\}$ and appropriately define $\varphi_W : U_W \to \mathrm{Hom}(W, W^\perp) = \mathbb{R}^{k(n-k)}$.

Exercise 4.

We use the same notation as in Exercise 3. Let

$$E := \coprod_{W \in Gr(k,V)} W$$

with the natural map $E \to Gr(k,V)$.

(*a*) Show that E has a natural vector bundle structure and is a subbundle of the trivial bundle $\widetilde{E} := V \times Gr(k,V)$ (the vector bundle E is called the *tautological bundle* of $Gr(k,V)$). Compute the transition functions of $E \to M$ with respect to the atlas given in Exercise 3.

(*b*) Show that the tangent space to $Gr(k,V)$ at W can be canonically identified with $\mathrm{Hom}(W, V/W)$. Conclude that there is a natural vector bundle isomorphism between the tangent bundle $TGr(k,V)$ with $\mathrm{Hom}(E, \widetilde{E}/E)$.

Exercise 5.

Let G be a Lie group.

(*a*) Show that $dexp_0 : Lie(G) \to Lie(G)$ is the identity map. (Hence, *exp* is a diffeomorphism around 0.)

(*b*) Show that every Lie group homomorphism $\mathbb{R} \to G$ is of the form $t \mapsto exp(tX)$, for some $X \in Lie(G)$.

Exercise 6.

(*a*) Show that any smooth manifold can be given a Riemannian metric.

(*b*) Fix a Riemannian metric on a smooth manifold M. The *unit sphere bundle* of M is the subset of the tangent bundle TM defined by

$$SM := \{(x, v_x) \in TM \mid \langle v_x, v_x \rangle_x = 1\}.$$

Show that SM is a smooth manifold is independent of the Riemannian metric up to diffeomorphism.

(*c*) Show that the unit sphere bundle of S^2 is diffeomorphic to $SO_3(\mathbb{R})$.