Assignment 4

(Submission deadline: 03.06.2021)

Exercise 1.

Consider the map $p : S^{2n+1} \to \mathbb{CP}^n$ given by the restriction of the canonical quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Set $dz_j := dx_j + idy_j$ and $d\overline{z}_j := dx_j - idy_j$, where $i = \sqrt{-1}$, for each j.

(a) Show that the restriction η of $\frac{i}{2} \sum_{j=0}^{n} dz_j \wedge d\overline{z}_j$ to S^{2n+1} is a closed form.
(b) Show that there is a closed 2-form ω on $\mathbb{CP}^n$ such that $p^*(\omega) = \eta$.
(c) Show that ω^j is nonzero for any $j \leq n$.

Exercise 2.

Using de Rham cohomology, show that S^n admits a nowhere vanishing vector field if and only if n is odd as follows:

(a) Give a nowhere vanishing vector field on S^n, for n odd.
(b) If X is a nowhere vanishing vector field on S^n, define a smooth map $S^n \times \mathbb{R} \to S^n$ using X, which defines a smooth homotopy connecting the identity map of S^n and the antipodal map.
(c) Compute the action of the antipodal map on $H^n_{dR}(S^n)$ and obtain a contradiction in case (b) holds for S^n with n even.

Exercise 3.

Let X be a paracompact space and let $i : Z \to X$ be the inclusion of a closed subset. For any sheaf of sets $\mathcal{F}$ on X, show that $\Gamma(Z, i^{-1}\mathcal{F}) = \varinjlim_{U} \Gamma(U, \mathcal{F})$, where the direct limit is taken over open subsets U of X containing Z.

Exercise 4.

(a) Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on $X = \mathbb{C}^\times$. Show that the cokernel presheaf of the morphism $\mathcal{O} \to \mathcal{O}$ given by the operator $\frac{d}{dz}$ is not a sheaf.
(b) Let $X = \mathbb{CP}^1$ and let $x, y \in X$ be distinct points. Show that the natural epimorphism $\mathcal{O}_X \to i_{x*}\mathbb{C} \oplus i_{y*}\mathbb{C}$ does not induce a surjection on global sections. Describe the kernel of this epimorphims of sheaves.

Exercise 5.

Show that there exists a topological space X and a sheaf of abelian groups $\mathcal{F}$ on X, which admits no epimorphim from a projective object in the category of sheaves of abelian groups on X. Thus, category of sheaves has enough injectives but not enough projectives.

Exercise 6.

Let X be a topological space, let $i : Z \to X$ be the inclusion of a closed subset and let $j : U \to X$ be the inclusion of the complement $U := X \setminus Z$. For any sheaf of sets $\mathcal{F}$ on U, define its *extension by zero* to X to be sheaf $j_!\mathcal{F}$ associated to the presheaf

$$V \mapsto \begin{cases} \mathcal{F}(V), & \text{if } V \cap Z = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $j_! : Shv(U) \to Shv(X)$ is an exact functor.
(b) Show that there are natural transformations of functors $j_!j^{-1} \to Id$ and $i_*i^{-1} \to Id$.
(c) Show that for every $\mathcal{F} \in Shv(X)$, there is a short exact sequence of sheaves

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0.$$