An example to illustrate Corollary 1.2 of the paper on the theta operator

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This is concerning the remark a few lines after display (5) in the paper of Ghate and Kumar on 'Reductions of Galois representations and the theta operator' (to appear in IJNT):

Finally, we remark that while the forms f and f_k satisfying the congruence conditions on the weight k in Theorem 1.1 and Corollary 1.2 may not be typical, examples of such forms are not hard to write down under some plausible assumptions on the size of M_{f_k}.

The eigenforms satisfying the hypotheses and conclusion of Corollary 1.2 are probably not typical since there is a congruence condition on the weight of f that needs to be satisfied.

However, it is easy to give examples of forms satisfy Corollary 1.2, modulo some intelligent guesses about the radii of the various families occurring in the proof (there does not seem to be an effective method to compute these radii).

Here is an example of forms f and g satisfying the conclusion (3) of Corollary 1.2, with the former living in a family of slope $\alpha = 1$ and the latter in a family of slope $\alpha + 1 = 2$.

Let $p = 5$. Let Δ_k be the unique eigenform of slope $\alpha = 1$ in $S_k(1,1)$ of weight $k \in \{20, 40, 60, 80, 100\}$ and level $N = 1$. Using SAGE, one sees: the space $S_{20}(1,1)$ is 1-dimensional, spanned by the newform

$$\Delta_{20}(z) = q + 456q^2 + 50652q^3 - 316352q^4 - 2377410q^5 + O(q^6),$$

and the space $S_{40}(1,1)$ is 3-dimensional, spanned by the newforms

$$h_a(z) = q + aq^2 + \left(\frac{1}{168}a^2 - \frac{6501}{7}a - \frac{22907296044}{7}\right)q^3 + (a^2 - 549755813888)q^4 + \left(-\frac{1053}{14}a^2 + \frac{481748300}{7}a + \frac{289722287396034}{7}\right)q^5 + O(q^6),$$

where a is a root of the polynomial

$$x^3 - 548856x^2 - 810051757056x + 213542160549543936.$$

The ideal (5) factors in the ring of integers of $\mathbb{Q}(a)$, the coefficient field of h_a, as $(5) = \lambda_1\lambda_2\lambda_3$, where

$$\lambda_1 = \left(5, \frac{1}{37739520}a^2 - \frac{523}{104832}a - \frac{7305104}{455}\right),$$
$$\lambda_2 = \left(5, \frac{1}{37739520}a^2 - \frac{523}{104832}a - \frac{7306924}{455}\right),$$
$$\lambda_3 = \left(5, \frac{1}{72}a - 2542\right)$$

are the prime ideals in the ring of integers of $\mathbb{Q}(a)$ lying over (5). Note that $\Delta_{40}(z) = h_a(z)$, for some a. One checks

$$\Delta_{20}(z) \equiv h_a(z) \pmod{\lambda_1},$$

so that we have

$$\bar{\rho}_{\Delta_{20}} \simeq \bar{\rho}_{\Delta_{40}}.$$

Similarly, one checks

$$\bar{\rho}_{\Delta_{20}} \simeq \bar{\rho}_{\Delta_{40}} \simeq \bar{\rho}_{\Delta_{60}} \simeq \bar{\rho}_{\Delta_{80}} \simeq \bar{\rho}_{\Delta_{100}}.$$

Now let us focus on $f = \Delta_k$ with $k = 100$. Let f_k be the p-stabilization of f of slope 1. Recall that M_{f_k} is the smallest non-negative integer such that f_k lives in a Coleman family of radius $p^{-M_{f_k}}$. The above computation suggests that $M_{f_k} = 1$. In fact, the computation only show that $M_{f_k} \neq 0$, since there are no such congruences of $\Delta_{20}(z)$ with forms of weights 24, 28, 32, etc. so technically we may only conclude that $M_{f_k} \geq 1$. Let us assume that $M_{f_k} = 1$.

Under this (plausible) assumption, f_k satisfies the congruence condition in the hypothesis of Corollary 1.2: $k = 100 \equiv 0 \mod p^{M_{f_k}+\delta_{f_k}} = 5$ or

25 noting that $\delta_{f_k} = 0$ or 1. The corollary predicts the existence of an eigenform g of slope $\alpha+1 = 2$ such that (3) holds. Indeed, some computation shows that if $g \in S_{22}(1,1)$ denotes the unique eigenform of slope 2, with q-expansion

$$g(z) = q - 288q^2 - 128844q^3 - 2014208q^4 + 21640950q^5 + O(q^6),$$

then we have $\theta f \equiv g \pmod 5$, so that

$$\bar{\rho}_f \otimes \omega \simeq \bar{\rho}_g.$$

The proof of Theorem 1.1 shows that there should be a Coleman family of slope 2 passing through the p-stabilized form g_{22} of g of slope 2. Computations with SAGE show that there is indeed such a family (with $M_{g_{22}}$ most likely 1: in fact, for any $l \in \{42, 62, 82\}$, there is a unique eigenform $g_l \in S_l(1,1)$ of slope 2 with $g_l(z) \equiv g(z) \pmod p$).

To summarize, while forms f (and f_k) satisfying the congruence hypothesis of Corollary 1.2 may not be typical, examples satisfying it are not hard to find modulo some plausible assumptions on the size of the radius M_{f_k}.