MODULAR REPRESENTATIONS OF $\mathrm{GL}_2(\mathbb{F}_q)$ USING CALCULUS

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Abstract. We show that certain modular induced representations of $\mathrm{GL}_2(\mathbb{F}_q)$ can be written as cokernels of operators acting on symmetric power representations of $\mathrm{GL}_2(\mathbb{F}_q)$. When the induction is from the Borel subgroup, respectively the anisotropic torus, the operators involve multiplication by newly defined twisted Dickson polynomials, respectively, twisted Serre operators. Our isomorphisms are explicitly defined using differential operators. As a corollary, we improve some periodicity results for quotients in the theta filtration.

1. Introduction

Let G be the general linear group GL_2 and let p be a prime. Let V_r for $r \geq 0$ be the r-th symmetric power representation of the standard two-dimensional representation of $G(\mathbb{F}_p)$. It is modeled on homogeneous polynomials of degree r over $\mathbb{F}_p$ in two variables X and Y with the usual action of $G(\mathbb{F}_p)$. Let $\theta = X^pY - XY^p$ be the theta or Dickson polynomial on which $G(\mathbb{F}_p)$ acts by determinant. Let $V_r^{(m+1)}$ for $m \geq 0$ be the subrepresentation of V_r consisting of polynomials divisible by $m+1$ copies of θ. The sequence $V_r^{(m+1)}$ is called the theta filtration of V_r.

1.1. **Principal series.** It is a classical fact going back to Glover [Glo78, (4.2)] that $\frac{V_r}{V_r^*}$ is periodic in r with period $p-1$ where $V_r^* = V_r^{(1)}$. This is proved nowadays by noting that

$$\frac{V_r}{V_r^*} \simeq \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} d^r \tag{1.1}$$

is a principal series representation of $G(\mathbb{F}_p)$ obtained by inducing the character d^r of the Borel subgroup $B(\mathbb{F}_p) = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\}$ to $G(\mathbb{F}_p)$, and by noting that the character depends only on r modulo $(p-1)$. Similar periodicity results have been investigated for higher quotients in the theta filtration of V_r. Indeed, it was shown in [GV22, Lemma 4.1] that for $0 \leq m \leq p-1$, the quotient $\frac{V_r}{V_r^{(m+1)}}$ is periodic in r modulo $p(p-1)$ by constructing an embedding

$$\frac{V_r}{V_r^{(m+1)}} \hookrightarrow \mathrm{ind}_{B(\mathbb{F}_p[\epsilon])}^{G(\mathbb{F}_p[\epsilon])} d^r, \tag{1.2}$$

where $\mathbb{F}_p[\epsilon]$ is the ring of generalized dual numbers (with $\epsilon^{m+1} = 0$), noting that d^r only depends on r modulo $p(p-1)$, and by showing that the image of (1.2) is independent of r modulo $p(p-1)$.

The map (1.2) is no longer surjective when $m > 0$. In this paper, instead of working with generalized dual numbers and *characters* of the inducing subgroup, we work with the induction of *higher dimensional representations* of the inducing subgroup and obtain *isomorphisms* between $\frac{V_r}{V_r^{(m+1)}}$ and induced spaces. We have:

Theorem 1.1. *Let $0 \leq m \leq p-1$. Then, we have the following explicit isomorphisms:*

(1) *If $p \nmid \binom{r}{m}$, then*

$$\frac{V_r}{V_r^{(m+1)}} \simeq \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} (V_m \otimes d^{r-m}), \tag{1.3}$$

where V_m is the representation of $B(\mathbb{F}_p)$ obtained by restriction from $G(\mathbb{F}_p)$.

(2) *If $p \mid \binom{r}{m}$ and $m = 1$, then*

$$\frac{V_r}{V_r^{(2)}} \simeq \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} (V_1^{\mathrm{ss}} \otimes d^{r-1}),$$

where V_1^{ss} is the split representation of $B(\mathbb{F}_p)$ obtained as the semi-simplification of V_1.

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The map in (1.3) generalizes the 'evaluation of the polynomial at the second row of a matrix' map that is used to prove (1.1) and (1.2). Additionally, it involves the use of a differential operator ∇. Such operators have found sporadic use in the literature (see [Glo78] and [BG15]), but are used systematically throughout this paper. We also need to divide by constants which do not vanish if $p \nmid \binom{r}{m}$.

Consider now the case $p|\binom{r}{m}$, that is, r is in one of the congruence classes $0, 1, \ldots, m-1$ modulo p. While it is well known that the individual principal series $\frac{V_r^{(i)}}{V_r^{(i+1)}}$ occurring as subquotients in the theta filtration are extensions of two Jordan-Hölder factors (which split exactly when $r \equiv 2i \mod (p-1)$), it is not as clear whether the extensions between consecutive principal series

$$0 \to \frac{V_r^{(i+1)}}{V_r^{(i+2)}} \to \frac{V_r^{(i)}}{V_r^{(i+2)}} \to \frac{V_r^{(i)}}{V_r^{(i+1)}} \to 0 \tag{1.4}$$

for $0 \leq i \leq m-1$ occurring in the theta filtration split. We show that if $p|\binom{r}{m}$, then exactly one of the extensions (1.4) split. Indeed, since $\frac{V_r^{(i)}}{V_r^{(i+2)}} \simeq \frac{V_{r'}}{V_{r'}^{(2)}} \otimes \det^i$ with $r' = r - i(p+1) \equiv r - i \mod p$ we are reduced to analyzing the case $m = 1$. If $p \nmid r$, then $\frac{V_r}{V_r^{(2)}}$ does not split by (1.3) and [Alp86, Lemma 6 (5)]. Modifying the above mentioned differential operator ∇ by dropping the constants mentioned above, in the second part of the theorem we show that if $p|r$, then

$$\frac{V_r}{V_r^{(2)}} \simeq \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} ad^{r-1} \oplus \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} d^r$$

splits. In other words, if $p|r$, then the two-dimensional standard representation V_1 in (1.3) gets replaced by the split representation $V_1^{\mathrm{ss}} = a \oplus d$ of $B(\mathbb{F}_p)$. We deduce that the extensions (1.4) split exactly when $r \equiv i \mod p$; moreover, under this condition, we have:

$$\frac{V_r}{V_r^{(m+1)}} \simeq \frac{V_r}{V_r^{(i+1)}} \oplus \frac{V_r^{(i+1)}}{V_r^{(m+1)}}. \tag{1.5}$$

As a corollary of Theorem 1.1, we obtain a strengthening of the afore-mentioned periodicity result from [GV22] in the case that $p \nmid \binom{r}{m}$ since again the right hand side of (1.3) only depends on r modulo $(p-1)$ and not on r modulo p. Thus, to obtain periodicity in this case, we no longer need to restrict to r in a fixed congruence class modulo p, only to those r that avoid collectively the congruence classes $0, 1, \ldots, m-1$ mod p. We obtain:

Corollary 1.2. *Let $0 \leq m \leq p-1$ and $r \equiv s \mod (p-1)$. If $p \nmid \binom{r}{m}, \binom{s}{m}$, then*

$$\frac{V_r}{V_r^{(m+1)}} \simeq \frac{V_s}{V_s^{(m+1)}}. \tag{1.6}$$

We also remark that the results above (and just below) are clearly false for very small values of r and s for dimension reasons. So in all the results in the principal series case, we assume that r and s are sufficiently large (but do not mention explicit lower bounds on them to keep the statements simple).

With future applications in mind, we equally treat the case of $G(\mathbb{F}_q) = \mathrm{GL}_2(\mathbb{F}_q)$ for an arbitrary finite field $\mathbb{F}_q$ with $q = p^f$ elements for $f \geq 1$. Indeed, we prove the following twisted version of the isomorphism (1.3):

Theorem 1.3. *Let $V_r = \otimes_{i=0}^{f-1}(V_{r_i} \circ \mathrm{Fr}^i)$, $V_m = \otimes_{i=0}^{f-1}(V_{m_i} \circ \mathrm{Fr}^i)$ with $0 \leq m_i \leq p-1$ and d^{r-m} be the character $\otimes_{i=0}^{f-1} d^{(r_i - m_i)p^i}$ of $B(\mathbb{F}_q)$. If $p \nmid \binom{r}{m} \equiv \prod_{i=0}^{f-1} \binom{r_i}{m_i} \mod p$, then*

$$\frac{V_r}{\langle \theta_0^{m_0+1}, \theta_1^{m_1+1}, \cdots, \theta_{f-1}^{m_{f-1}+1} \rangle} \simeq \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \left(V_m \otimes d^{r-m} \right).$$

Here the V_{r_i} are modeled on homogeneous polynomials of degree r_i over $\mathbb{F}_q$ in the variables X_i and Y_i and $V_{r_i} \circ \mathrm{Fr}^i$ means that we twist the standard action of $G(\mathbb{F}_q)$ on V_{r_i} by the i-th power of Frobenius. The polynomials

$$\theta_i = X_i Y_{i-1}^p - Y_i X_{i-1}^p$$

for $0 \leq i \leq f-1$ are what we call *twisted Dickson polynomials* (we adopt the convention that $-1 = f-1$); they do not seem to appear in the literature.

If $p \nmid \binom{r}{m}$, then the periodicity of the quotient on the left hand side again follows since the right hand side only depends on the sum $r = \sum r_i p^i$, which is periodic modulo $(q-1)$. We obtain:

Corollary 1.4. *Let $0 \le m \le q-1$ and $r \equiv s \mod (q-1)$. If $p \nmid \binom{r}{m}, \binom{s}{m}$, then*

$$\frac{V_r}{\langle \theta_0^{m_0+1}, \theta_1^{m_1+1}, \cdots, \theta_{f-1}^{m_{f-1}+1} \rangle} \simeq \frac{V_s}{\langle \theta_0^{m_0+1}, \theta_1^{m_1+1}, \cdots, \theta_{f-1}^{m_{f-1}+1} \rangle}.$$

We end this discussion of the case of principal series by noting that the isomorphism in Theorem 1.3 is expected to play an important role in future investigations into the reduction problem of two-dimensional crystalline and semi-stable representations over arbitrary p-adic fields F with residue field $\mathbb{F}_q$ using the compatibility with respect to reduction of the (yet to be discovered) p-adic and mod p local Langlands correspondences for $G(F)$ for an arbitrary finite extension of $\mathbb{Q}_p$.

1.2. **Cuspidal case.** Let $T(\mathbb{F}_p) = \mathbb{F}_{p^2}^\times \hookrightarrow G(\mathbb{F}_p)$ be the anisotropic torus. The theme of writing (generalized) principal series representations as cokernels of theta operators raises the question (asked by Khare) as to whether one may similarly write representations induced from the anisotropic torus as cokernels of symmetric power representations.

Let D be the differential operator $X^p \frac{\partial}{\partial X} + Y^p \frac{\partial}{\partial Y}$ and let ω_2 be the identity character $T(\mathbb{F}_p) = \mathbb{F}_{p^2}^\times \to \mathbb{F}_{p^2}^\times$. We prove the following analog of (1.1):

Theorem 1.5. *Let $2 \le r \le p-1$. Then there is an explicit isomorphism*

$$\frac{V_{r+p-1}}{D(V_r)} \otimes V_{p-1} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2^r$$

defined over $\mathbb{F}_{p^2}$.

The theorem is also true for $r = 1$ (see Remark 2). We prove similar isomorphisms for other values of r by twisting (Corollary 4.2) using the fact that D preserves the theta filtration in a strong sense (Lemma 4.3). We also prove similar isomorphisms when D is replaced by a higher power $D^{(m+1)}$ (Corollary 4.4, note the analogy with (1.3)).

A non-explicit version of the isomorphism in Theorem 1.5 can be deduced from the work of Reduzzi [Red10]. Let us provide some background and explain our contribution. In the discussion that follows, we sometimes think of ω_2 as a character taking values in a characteristic zero field (by taking its Teichmüller lift). Recall that for each complex character χ of $T(\mathbb{F}_p) = \mathbb{F}_{p^2}^\times$ (with χ not self-conjugate) there is an irreducible cuspidal complex representation $\Theta(\chi)$ of $G(\mathbb{F}_p)$. Moreover, $\Theta(\chi)$ is a factor of an induced representation: we have

$$\Theta(\chi) \otimes \mathrm{St} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \chi, \tag{1.7}$$

where St is the p-dimensional complex irreducible Steinberg representation of $G(\mathbb{F}_p)$ with reduction $\overline{\mathrm{St}} \simeq V_{p-1}$. While the group $G(\mathbb{F}_p)$ has no mod p cuspidal representations (since, for instance, the Jacquet functor is never 0 because there are always invariant elements under the upper unipotent subgroup of $G(\mathbb{F}_p)$), one may still study the mod p reductions of $\Theta(\omega_2^r)$. Following a suggestion of Serre to use the operator D, Reduzzi [Red10] proved that the mod p reduction $\overline{\Theta(\omega_2^r)}$ is isomorphic to the cokernel of D on an appropriate symmetric power representation, namely:

$$\frac{V_{r+p-1}}{D(V_r)} \simeq \overline{\Theta(\omega_2^r)} \tag{1.8}$$

for $2 \le r \le p-1$. The proof uses a specific integral model of $\Theta(\omega_2^r)$ arising from the action of $G(\mathbb{F}_p)$ on the crystalline cohomology of the Deligne-Lusztig variety $XY^p - X^pY = Z^{p+1}$ (see Haastert-Jantzen [HJ90]). Thus, Reduzzi's isomorphism (1.8) is not at all explicit given that the right hand side involves crystalline cohomology. However, by tensoring (1.8) with V_{p-1} and using the mod p reduction of (1.7) for $\chi = \omega_2^r$, one sees that the isomorphism in Theorem 1.5 must hold, at least abstractly. An immediate question that arises is whether one can make this isomorphism explicit, given that the right hand side of this isomorphism no longer involves crystalline cohomology. Thus the point of Theorem 1.5 is that it contains an *explicit* isomorphism (which was found after much computation with special cases). Again, the map involves a differential operator ∇_α, where α is an element of $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$, which generalizes the operator ∇ used in the principal series case.[1]

[1] It also involves the difference of a polynomial evaluated at 2 points reminding one of the evaluation of a direct integral in calculus.

In fact, Reduzzi [Red10] proved[2] that, more generally, for $G(\mathbb{F}_q)$ with $q = p^f$, and for $\omega_{2f} : \mathbb{F}_{q^2}^\times \to \mathbb{F}_{q^2}^\times$ the fundamental (identity) character of level $2f$, one similarly has

$$\frac{V_{r+q-1}}{D(V_r)} \simeq \overline{\Theta(\omega_{2f}^r)},$$

for $2 \le r \le p-1$, where now $D = X^q \frac{\partial}{\partial X} + Y^q \frac{\partial}{\partial Y}$. We end this paper by proving the following twisted version of this result, which extends Theorem 1.5 to $G(\mathbb{F}_q)$:

Theorem 1.6. *Let $r = r_0 + r_1 p + \cdots + r_{f-1}p^{f-1}$, where $2 \le r_0 \le p-1$ and $r_j = 0$ for all $1 \le j \le f-1$. Then there is an explicit isomorphism over $\mathbb{F}_{q^2}$:*

$$\frac{\bigotimes_{j=0}^{f-1} V_{r_j+p-1}^{\mathrm{Fr}^j}}{\langle D_0, \ldots, D_{f-1}\rangle} \otimes \otimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j} \simeq \operatorname{ind}_{T(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \omega_{2f}^r.$$

We remark that $\otimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j} \simeq \overline{\mathrm{St}}$ where St is now the q-dimensional Steinberg representation of $G(\mathbb{F}_q)$. Also in the statement of the theorem we need the following twisted versions of Serre's differential operator D, namely:

$$D_0 = X_0^p X_1^{p-1} \cdots X_{f-1}^{p-1} \frac{\partial}{\partial X_0} + Y_0^p Y_1^{p-1} \cdots Y_{f-1}^{p-1} \frac{\partial}{\partial Y_0},$$

and

$$D_j = X_0^p X_1^{p-1} \cdots X_{j-1}^{p-1} \frac{\partial}{\partial X_j} + Y_0^p Y_1^{p-1} \cdots Y_{j-1}^{p-1} \frac{\partial}{\partial Y_j},$$

for all $1 \le j \le f-1$. Interestingly, these operators are only $G(\mathbb{F}_q)$-linear modulo the images of the previous ones (with the convention that D_0 is to be thought of as D_f). Again, they do not seem to appear in the literature and one might refer to them as *twisted Serre operators*.

2. Principal series case

2.1. **The case of** $\mathrm{GL}_2(\mathbb{F}_p)$**.** Recall $G(\mathbb{F}_p) = \mathrm{GL}_2(\mathbb{F}_p)$ and $B(\mathbb{F}_p)$ is the subgroup of upper triangular matrices of $G(\mathbb{F}_p)$. For $r \ge 0$, let $V_r := \mathrm{Sym}^r(\mathbb{F}_p^2)$ denote the r-th symmetric power of the standard representation of $G(\mathbb{F}_p)$ over $\mathbb{F}_p$. We identify V_r with homogeneous polynomials $P(X,Y)$ of degree r in two variables X and Y with coefficients in $\mathbb{F}_p$, with action $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_p)$ given by

$$g \cdot P(X,Y) = P(aX + cY, bX + dY).$$

Consider the Dickson polynomial

$$\theta(X,Y) := X^p Y - XY^p.$$

Note that $G(\mathbb{F}_p)$ acts on $\theta(X,Y)$ by the determinant character. So for each $m \ge 0$, we have

$$V_r^{(m+1)} := \left\{ f(X,Y) \in V_r \mid f(X,Y) \text{ is divisible by } \theta(X,Y)^{m+1} \right\}$$

is a $G(\mathbb{F}_p)$-stable subspace of V_r. These spaces give a decreasing filtration of submodules of V_r:

$$V_r \supset V_r^{(1)} \supset \cdots \supset V_r^{(m+1)} \supset \cdots \supset (0).$$

Let $d^r : B(\mathbb{F}_p) \to \mathbb{F}_p^\times$ denote the character given by $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto d^r$. For $n \ge 0$, $m \in \mathbb{Z}$, define

$$[n]_m = \begin{cases} 1, & \text{if } m = 0, \\ n(n-1)\cdots(n-(m-1)), & \text{if } m > 0, \\ 0, & \text{if } m < 0. \end{cases}$$

Lemma 2.1. *Let $k \ge 0$. We have*

$$\sum_{m=0}^{k} \binom{k}{m} [r-j]_{t-l-m} [j]_{l+m} = [r-t+k]_k [r-j]_{t-k-l} [j]_l. \tag{2.1}$$

Proof. Follows by induction on k. $\square$

[2]Technically speaking, Reduzzi does not treat the case $r = \frac{p+1}{2}$, though it is covered by Theorems 1.5 and 1.6.

Lemma 2.2. *Let $z \in \mathbb{F}_p$ and let $P(X,Y) = \sum_{j=0}^{r} a_j X^{r-j}Y^j \in V_r$ with $a_j \in \mathbb{F}_p$ for all $0 \le j \le r$. Define the differential operators*

$$\nabla = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y} \quad \text{and} \quad \nabla' = c\frac{\partial}{\partial X} + d\frac{\partial}{\partial Y},$$

for $a,b,c,d \in \mathbb{F}_p$. Then, for all $0 \le k \le t$, we have

$$\nabla^{t-k}\nabla'^k(P)\Big|_{(zc,zd)} = z^{r-t}[r-t+k]_k \left(a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}\right)^{t-k}(P)\Big|_{(c,d)}.$$

Proof. Without loss of generality assume that $P = X^{r-j}Y^j$. We note that

$$\frac{\partial^n}{\partial X^{n-i}\partial Y^i}(P) = [r-j]_{n-i}[j]_i X^{r-j-(n-i)}Y^{j-i}. \tag{2.2}$$

Now,

$$\begin{aligned}
&\left(a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}\right)^{t-k}\left(c\frac{\partial}{\partial X} + d\frac{\partial}{\partial Y}\right)^{k}(P)\Big|_{(zc,zd)} \\
&= \sum_{l=0}^{t-k}\sum_{m=0}^{k}\binom{t-k}{l}\binom{k}{m}a^{t-k-l}b^l c^{k-m}d^m \frac{\partial^t}{\partial X^{t-l-m}\partial Y^{l+m}}(P)\Big|_{(zc,zd)} \\
&= \sum_{l=0}^{t-k}\sum_{m=0}^{k}\binom{t-k}{l}\binom{k}{m} z^{r-t}a^{t-k-l}b^l c^{k-m}d^m[r-j]_{t-l-m}[j]_{l+m}c^{r-j-(t-l-m)}d^{j-l-m} \\
&= \sum_{l=0}^{t-k} z^{r-t}a^{t-k-l}b^l c^{r-j-(t-l-k)}d^{j-l}\binom{t-k}{l}\left(\sum_{m=0}^{k}\binom{k}{m}[r-j]_{t-l-m}[j]_{l+m}\right) \\
&= z^{r-t}[r-t+k]_k\sum_{l=0}^{t-k}\binom{t-k}{l}a^{t-k-l}b^l c^{r-j-(t-l-k)}d^{j-l}[r-j]_{t-k-l}[j]_l \\
&= z^{r-t}[r-t+k]_k\left(\sum_{l=0}^{t-k}\binom{t-k}{l}a^{t-k-l}b^l\frac{\partial^{t-k}}{\partial X^{t-k-l}\partial Y^l}\right)(P)\Big|_{(c,d)} \\
&= z^{r-t}[r-t+k]_k\left(a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}\right)^{t-k}(P)\Big|_{(c,d)}.
\end{aligned}$$

The second and last but one equalities hold by (2.2). The fourth equality follows from (2.1). □

Lemma 2.3. *Let $a,b,c,d,u,v,w,z \in \mathbb{F}_p$ and $P(X,Y) \in V_r$. Let $P_1 = P(U,V)$, with $U = uX + wY$ and $V = vX + zY$. Then for $k \ge 0$ we have*

$$\left(a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}\right)^k(P_1)\Big|_{(c,d)} = \left((ua+wb)\frac{\partial}{\partial X} + (va+zb)\frac{\partial}{\partial Y}\right)^k(P)\Big|_{(uc+wd,vc+zd)}.$$

Proof. This is just the chain rule. □

Lemma 2.4. *Let $a,b \in \mathbb{F}_p$ and $\nabla = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$. Let $f := f(X,Y), g := g(X,Y) \in \mathbb{F}_p[X,Y]$. Then, for all $m \ge 1$, we have*

$$\nabla^m(fg) = \sum_{i=0}^{m}\binom{m}{i}\nabla^{m-i}(f)\nabla^i(g).$$

Proof. This is just Leibnitz rule. □

Lemma 2.5. *Let $a,b,c,d \in \mathbb{F}_p$. We let $\nabla = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$ and $\theta(X,Y) = X^pY - XY^p$. Then, for $l,k \ge 0$, we have*

$$\nabla^l(\theta(X,Y)^k)\big|_{(c,d)} = \begin{cases} l!\left(\nabla\theta(X,Y)\big|_{(c,d)}\right)^l, & \text{if } k = l, \\ 0, & \text{otherwise.}\end{cases}$$

Proof. We first show that

$$\nabla^l(\theta^k) = l!\binom{k}{l}\theta^{k-l}(\nabla\theta)^l, \ \forall\, k \ge l. \tag{2.3}$$

We prove the result by induction on l. The $l=0$ case is trivial. Suppose $l=1$. Then, we have

$$\nabla(\theta^k) = \left(a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}\right)(\theta^k) = ak\theta^{k-1}\frac{\partial\theta}{\partial X} + bk\theta^{k-1}\frac{\partial\theta}{\partial Y} = k\theta^{k-1}(\nabla\theta),$$

as desired. Assume that the result is true for l. If $k \geq l+1$, then

$$\begin{aligned}
\nabla^{l+1}(\theta^k) &= \nabla\left(l!\binom{k}{l}\theta^{k-l}(\nabla\theta)^l\right) = l!\binom{k}{l}\left(a\frac{\partial}{\partial X}\left(\theta^{k-l}(\nabla\theta)^l\right) + b\frac{\partial}{\partial Y}\left(\theta^{k-l}(\nabla\theta)^l\right)\right) \\
&= l!\binom{k}{l}a\left(\theta^{k-l}\frac{\partial}{\partial X}(\nabla\theta)^l + (k-l)\theta^{k-l-1}\frac{\partial\theta}{\partial X}(\nabla\theta)^l\right) \\
&\quad + l!\binom{k}{l}b\left(\theta^{k-l}\frac{\partial}{\partial Y}(\nabla\theta)^l + (k-l)\theta^{k-l-1}\frac{\partial\theta}{\partial Y}(\nabla\theta)^l\right) \\
&= l!\binom{k}{l}\theta^{k-l}\nabla((\nabla\theta)^l) + l!\binom{k}{l}(k-l)\theta^{k-l-1}(\nabla\theta)^l\left(a\frac{\partial\theta}{\partial X} + b\frac{\partial\theta}{\partial Y}\right) \\
&= (l+1)!\binom{k}{l+1}\theta^{k-(l+1)}(\nabla\theta)^{l+1}.
\end{aligned}$$

The first equality holds by the induction hypothesis. Note that $\nabla\theta = bX^p - aY^p$, so $\nabla((\nabla\theta)^l) = 0$, and hence the last equality follows. Thus the identity (2.3) follows by induction.

Now, suppose $k < l$. Then

$$\nabla^l(\theta^k) = \nabla^{l-k}(\nabla^k(\theta^k)) = k!\nabla^{l-k}((\nabla\theta)^k) = k!\nabla^{l-k-1}\nabla((\nabla\theta)^k) = 0. \tag{2.4}$$

The second equality follows by taking $l = k$ in (2.3). The last equality follows because $\nabla((\nabla\theta)^k) = 0$.

Combining (2.3) and (2.4), we have

$$\nabla^l(\theta^k)\big|_{(c,d)} = \begin{cases} l!\binom{k}{l}\theta^{k-l}\big|_{(c,d)}\left(\nabla\theta\big|_{(c,d)}\right)^l, & \text{if } k \geq l, \\ 0, & \text{if } k < l. \end{cases} = \begin{cases} l!\left(\nabla\theta\big|_{(c,d)}\right)^l, & \text{if } k = l, \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

2.1.1. *Non-split case.* We prove Theorem 1.1 (1) from the introduction.

Theorem 2.6. *Let $0 \leq m \leq p-1$ and $p \nmid \binom{r}{m}$. Then we have*

$$\frac{V_r}{V_r^{(m+1)}} \simeq \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}\left(V_m \otimes d^{r-m}\right).$$

Proof. We show that there is a $G(\mathbb{F}_p)$-equivariant isomorphism

$$\psi : \frac{V_r}{V_r^{(m+1)}} \to \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}\left(V_m \otimes d^{r-m}\right)$$

given by $\psi(P(X,Y)) = \psi_P$ for all $P = P(X,Y) \in V_r$, where $\psi_P : G(\mathbb{F}_p) \to V_m \otimes d^{r-m}$ is defined by

$$\psi_P\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \left(\frac{\binom{m}{j}}{[r]_{m-j}}\left(a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}\right)^{m-j}(P)\bigg|_{(c,d)}\right)_{0\leq j\leq m}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in G(\mathbb{F}_p)$. Note that, by hypothesis the constant is a well-defined non-zero element of $\mathbb{F}_p$.

Recall that we denote $\nabla = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$ and $\nabla' = c\frac{\partial}{\partial X} + d\frac{\partial}{\partial Y}$.

$B(\mathbb{F}_p)$**-linearity:** We first show that ψ_P is $B(\mathbb{F}_p)$-linear. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in G(\mathbb{F}_p)$, $b = \left(\begin{smallmatrix} u & v \\ 0 & z\end{smallmatrix}\right) \in B(\mathbb{F}_p)$ and $\underline{x} := (x_j)_{0\leq j\leq m} \in V_m \otimes d^{r-m}$. Then the action of b on $\underline{x}$ is given by

$$b \cdot \underline{x} = z^{r-m} \cdot \left(\sum_{j=i}^{m}\binom{j}{i}u^{m-j}v^{j-i}z^i x_j\right)_{0\leq i\leq m}.$$

We have

$$\begin{aligned}\psi_P(b\cdot\gamma) &= \psi_P\left(\begin{pmatrix} ua+vc & ub+vd \\ zc & zd\end{pmatrix}\right) = \left(\frac{\binom{m}{j}}{[r]_{m-j}}\left(u\nabla+v\nabla'\right)^{m-j}(P)\Big|_{(zc,zd)}\right)_{0\le j\le m} \\ &= \left(\frac{\binom{m}{j}}{[r]_{m-j}}\left(\sum_{k=0}^{m-j}\binom{m-j}{k}u^{m-j-k}v^k\nabla^{m-j-k}\nabla'^k\right)(P)\Big|_{(zc,zd)}\right)_{0\le j\le m},\end{aligned}$$

which by taking $t=m-j$ in Lemma 2.2 and by using the fact $[r]_{m-j}=[r]_{m-j-k}[r-(m-j-k)]_k$ equals

$$\left(\sum_{k=0}^{m-j}\frac{\binom{m}{j}\binom{m-j}{k}z^{r-(m-j)}}{[r]_{m-j-k}}\left(u^{m-j-k}v^k\nabla^{m-j-k}\right)(P)\Big|_{(c,d)}\right)_{0\le j\le m}. \tag{2.5}$$

Now,

$$b\cdot\psi_P(\gamma)=\begin{pmatrix} u & v\\ 0 & z\end{pmatrix}\cdot\left(\frac{\binom{m}{j}}{[r]_{m-j}}\nabla^{m-j}(P)\Big|_{(c,d)}\right)_{0\le j\le m} = \left(\sum_{j=i}^{m}\frac{\binom{j}{i}\binom{m}{j}z^{r-(m-i)}}{[r]_{m-j}}u^{m-j}v^{j-i}\nabla^{m-j}(P)\Big|_{(c,d)}\right)_{0\le i\le m},$$

which, by relabeling j as l and i as j, and by further replacing l by $j+k$, equals

$$\left(\sum_{k=0}^{m-j}\frac{\binom{j+k}{j}\binom{m}{j+k}z^{r-(m-j)}}{[r]_{m-j-k}}u^{m-j-k}v^k\nabla^{m-j-k}(P)\Big|_{(c,d)}\right)_{0\le j\le m}. \tag{2.6}$$

Observing $\binom{m}{j}\binom{m-j}{k}=\binom{j+k}{j}\binom{m}{j+k}$ and comparing (2.5) and (2.6), we have $\psi_P(b\cdot\gamma)=b\cdot\psi_P(\gamma)$. So ψ_P is $B(\mathbb{F}_p)$-linear and hence ψ is well defined.

$G(\mathbb{F}_p)$**-linearity:** Now, we show that ψ is $G(\mathbb{F}_p)$-linear. Let $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, $g=\left(\begin{smallmatrix} u & v\\ w & z\end{smallmatrix}\right)\in G(\mathbb{F}_p)$. Then $g\cdot P(X,Y)=P(U,V)=:P_1$, where $U=uX+wY$ and $V=vX+zY$. We have

$$\psi(g\cdot(P(X,Y)))(\gamma) = \psi(P_1)\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right) = \left(\frac{\binom{m}{j}}{[r]_{m-j}}\left(a\frac{\partial}{\partial X}+b\frac{\partial}{\partial Y}\right)^{m-j}(P_1)\Big|_{(c,d)}\right)_{0\le j\le m},$$

which, by Lemma 2.3, equals

$$\left(\frac{\binom{m}{j}}{[r]_{m-j}}\left((ua+wb)\frac{\partial}{\partial X}+(va+zb)\frac{\partial}{\partial Y}\right)^{m-j}(P)\Big|_{(uc+wd,vc+zd)}\right)_{0\le j\le m} = \psi(P(X,Y))(\gamma g) = (g\cdot\psi(P(X,Y)))(\gamma)$$

for all $\gamma\in G(\mathbb{F}_p)$. So $\psi(g\cdot P(X,Y))=g\cdot\psi(P(X,Y))$ for all $g\in G(\mathbb{F}_p)$. Hence ψ is $G(\mathbb{F}_p)$-linear.

Kernel: Next we show that $\ker\psi=V_r^{(m+1)}$ by induction on m. If $m=0$, it is well known that $\ker\psi=V_r^{(1)}$ (e.g., use [GV22, Lemma 2.7] or Lemma 2.11 with $f=1$). Let $P(X,Y)\in\ker\psi$. By definition of ψ, we have $\nabla^{m-j}(P)\big|_{(c,d)}=0$ for all $0\le j\le m$, $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in G(\mathbb{F}_p)$. In particular, this is true for all $1\le j\le m$. So by the induction hypothesis, we have $P(X,Y)\in V_r^{(m)}$, which gives $P(X,Y)=\theta(X,Y)^mQ(X,Y)$ for some $Q(X,Y)\in V_{r-m(p+1)}$. Now, taking $j=0$ and using Lemma 2.4,

$$0=\nabla^m(P)\Big|_{(c,d)} = \left(\sum_{i=0}^{m}\binom{m}{i}\nabla^{m-i}(\theta^m)\nabla^i(Q(X,Y))\right)\Big|_{(c,d)}.$$

This implies $Q(c,d)=0$ since, by Lemma 2.5, all terms above die except for the $i=0$ term. Then by the $m=0$ case, we have $\theta\mid Q(X,Y)$, so $P(X,Y)\in V_r^{(m+1)}$. Thus $\ker\psi\subset V_r^{(m+1)}$. On the other hand if $P(X,Y)=\theta^{m+1}Q'(X,Y)$, it is easy to check using Lemmas 2.4, 2.5 that $\nabla^j(P(X,Y))\big|_{(c,d)}=0$ for all $0\le j\le m$. By the definition of ψ, we have $P(X,Y)\in\ker\psi$. Thus $\ker\psi=V_r^{(m+1)}$.

Isomorphism: This follows since the dimension of both sides of ψ is $(m+1)(p+1)$. $\square$

2.1.2. *Split case.* We turn to the case $p \mid \binom{r}{m}$. As explained in the introduction we may assume that $m = 1$. In the next theorem, we prove Theorem 1.1 (2). Recall that $V_1^{\mathrm{ss}} := a \oplus d$ denotes the two-dimensional split representation of $B(\mathbb{F}_p)$.

Theorem 2.7. *If $p \mid r$, then we have*

$$\frac{V_r}{V_r^{(2)}} \simeq \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \left(V_1^{\mathrm{ss}} \otimes d^{r-1}\right) \simeq \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} a d^{r-1} \oplus \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} d^{r}.$$

Proof. Define

$$\psi^{\mathrm{ss}} : \frac{V_r}{V_r^{(2)}} \to \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} (V_1^{\mathrm{ss}} \otimes d^{r-1})$$

by $\psi^{\mathrm{ss}}(P) = \psi_P^{\mathrm{ss}}$, where $\psi_P^{\mathrm{ss}} : G(\mathbb{F}_p) \to V_1^{\mathrm{ss}} \otimes d^{r-1}$ is defined by

$$\psi_P^{\mathrm{ss}}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\nabla P\Big|_{(c,d)},\, P(c,d)\right),$$

where $\nabla = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$.

We check the $B(\mathbb{F}_p)$-linearity of ψ_P^{ss}. Let $b = \left(\begin{smallmatrix} u & v \\ 0 & z \end{smallmatrix}\right) \in B(\mathbb{F}_p)$ and $\gamma \in \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_p)$. Then,

$$\begin{aligned}
\psi_P^{\mathrm{ss}}(b \cdot \gamma) &= \psi_P^{\mathrm{ss}}\left(\begin{pmatrix} ua+vc & ub+vd \\ zc & zd \end{pmatrix}\right) = \left(uz^{r-1}\nabla(P)\Big|_{(c,d)} + vz^{r-1}\nabla'(P)\Big|_{(c,d)},\, z^r P(c,d)\right) \\
&= \left(uz^{r-1}\nabla(P)\Big|_{(c,d)} + vz^{r-1} rP(c,d),\, z^r P(c,d)\right) = \left(uz^{r-1}\nabla(P)\Big|_{(c,d)},\, z^r P(c,d)\right) \\
&= \begin{pmatrix} u & v \\ 0 & z \end{pmatrix} \cdot \left(\nabla(P)\Big|_{(c,d)},\, P(c,d)\right) = b \cdot \psi_P^{\mathrm{ss}}(\gamma).
\end{aligned}$$

Here $\nabla' = c\frac{\partial}{\partial X} + d\frac{\partial}{\partial Y}$ and the fourth equality follows because $p \mid r$. Hence ψ_P^{ss} is $B(\mathbb{F}_p)$-linear.

The proof of the $G(\mathbb{F}_p)$-linearity of ψ^{ss} and the fact that $\ker \psi^{\mathrm{ss}} = V_r^{(2)}$ follows as in the proof of Theorem 2.6. We conclude as in the proof of Theorem 2.6 by comparing dimensions. □

As discussed in the introduction, we obtain the splitting (1.5):

Corollary 2.8. *Let $0 \le m \le p-1$. If $p \mid \binom{r}{m}$, that is, $p \mid r-i$ for some $0 \le i \le m-1$, then we have*

$$\frac{V_r}{V_r^{(m+1)}} \simeq \frac{V_r}{V_r^{(i+1)}} \oplus \frac{V_r^{(i+1)}}{V_r^{(m+1)}}.$$

2.2. **The case of** $\mathrm{GL}_2(\mathbb{F}_q)$**.** Let $G(\mathbb{F}_q) := \mathrm{GL}_2(\mathbb{F}_q)$ with $q = p^f$ for $f \ge 1$. Let $B(\mathbb{F}_q)$ denote the subgroup of upper triangular matrices of $G(\mathbb{F}_q)$. Let $r \ge 0$ and let $r = r_0 + r_1 p + \cdots + r_{f-1}p^{f-1}$ be the p-adic expansion of r with $0 \le r_j \le p-1$ and $0 \le j \le f-1$. Let $V_{r_j}^{\mathrm{Fr}^j} := \mathrm{Sym}^{r_j}(\mathbb{F}_p^2) \circ \mathrm{Fr}^j$ for all $0 \le j \le f-1$, where Fr denotes the Frobenius morphism. Let $V_{r_j}^{\mathrm{Fr}^j}$ be modeled on polynomials in X_j and Y_j over $\mathbb{F}_q$ of degree r_j for all $0 \le j \le f-1$. Let

$$\theta_0 := X_0 Y_{f-1}^p - Y_0 X_{f-1}^p \quad \text{and} \quad \theta_k := X_k Y_{k-1}^p - Y_k X_{k-1}^p \tag{2.7}$$

for all $1 \le k \le f-1$ denote the twisted Dickson polynomials.

Lemma 2.9. *Let $a, b, c, d, z \in \mathbb{F}_q$. We write*

$$\nabla_j = a^{p^j}\frac{\partial}{\partial X_j} + b^{p^j}\frac{\partial}{\partial Y_j} \quad \textit{and} \quad \nabla'_j = c^{p^j}\frac{\partial}{\partial X_j} + d^{p^j}\frac{\partial}{\partial Y_j}.$$

Let $P_j(X_j, Y_j) = \sum_{i_j=0}^{r_j} a_{i_j} X_j^{r_j - i_j} Y^{i_j} \in V_{r_j}^{\mathrm{Fr}^j}$ with $a_{i_j} \in \mathbb{F}_q$ for all $0 \le i_j \le r_j$. Then, for all $0 \le k_j \le t_j$, we have

$$\nabla_j^{t_j - k_j} \nabla_j'^{k_j}(P_j)\Big|_{\left((zc)^{p^j},(zd)^{p^j}\right)} = z^{(r_j - t_j)p^j}[r_j - t_j + k_j]_{k_j} \nabla_j^{t_j-k_j}(P_j)\Big|_{(c^{p^j}, d^{p^j})}.$$

Proof. Similar to Lemma 2.2. □

Lemma 2.10. *Let $a,b,c,d,u,v,w,z \in \mathbb{F}_q$ and $P_j(X_j,Y_j) \in V_{r_j}^{\mathrm{Fr}^j}$. Let $P_j' := P_j(U_j,V_j)$, where $U_j = u^{p^j}X_j + w^{p^j}Y_j$ and $V_j = v^{p^j}X_j + z^{p^j}Y_j$. Then for $k_j \geq 0$, we have*

$$\left(a^{p^j}\frac{\partial}{\partial X_j} + b^{p^j}\frac{\partial}{\partial Y_j}\right)^{k_j}(P_j')\Big|_{\left(c^{p^j},d^{p^j}\right)} = \left((ua+wb)^{p^j}\frac{\partial}{\partial X_j} + (va+zb)^{p^j}\frac{\partial}{\partial Y_j}\right)^{k_j}(P_j)\Big|_{\left((uc+wd)^{p^j},(vc+zd)^{p^j}\right)}.$$

Proof. Similar to Lemma 2.3. □

Lemma 2.11. *Let $\theta' = X^qY - Y^qX$. Let $P(X,Y) = \sum_{i=0}^{r} a_iX^{r-i}Y^i \in \mathbb{F}_q[X,Y]$ be such that $P(c,d) = 0$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{F}_q$. Then $P(X,Y) \in \langle\theta'\rangle$.*

Proof. Exercise. □

Lemma 2.12. *Let $a,b,c,d \in \mathbb{F}_q$. We let $\nabla_j = \left(a^{p^j}\frac{\partial}{\partial X_j} + b^{p^j}\frac{\partial}{\partial Y_j}\right)$ and $\theta_j' = X_j^qY_j - X_jY_j^q$. Then we have*

$$\nabla_j^l(\theta_j'^k)\Big|_{(c^{p^j},d^{p^j})} = \begin{cases} l!\left(\nabla_j\theta_j'\Big|_{(c^{p^j},d^{p^j})}\right)^l, & \text{if } k = l, \\ 0, & \text{otherwise.}\end{cases}$$

Proof. Similar to Lemma 2.5. □

Lemma 2.13. *For $0 \leq j \leq f-1$ and $0 \leq m_j \leq p-1$, let*

$$\psi^j : V_{r_j}^{\mathrm{Fr}^j} \to \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\left(V_{m_j}^{\mathrm{Fr}^j} \otimes d^{(r_j-m_j)p^j}\right)$$

be defined by $\psi^j(P_j(X_j,Y_j)) = \psi^j_{P_j(X_j,Y_j)}$, where $\psi^j_{P_j(X_j,Y_j)} : G(\mathbb{F}_q) \to V_{m_j}^{\mathrm{Fr}^j} \otimes d^{(r_j-m_j)p^j}$ is given by

$$\psi^j_{P_j(X_j,Y_j)}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\frac{\binom{m_j}{n_j}}{[r_j]_{m_j-n_j}}\nabla_j^{m_j-n_j}(P_j)\Big|_{(c^{p^j},d^{p^j})}\right)_{0\leq n_j\leq m_j},$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{F}_q)$ and $\nabla_j = a^{p^j}\frac{\partial}{\partial X_j} + b^{p^j}\frac{\partial}{\partial Y_j}$. Then

(i) *$\psi^j_{P_j(X_j,Y_j)}$ is $B(\mathbb{F}_q)$-linear,*
(ii) *ψ^j is $G(\mathbb{F}_q)$-linear,*
(iii) *ψ^j is an isomorphism.*

Proof. (i) $B(\mathbb{F}_q)$**-linearity:** Let $b = \left(\begin{smallmatrix} u & v \\ 0 & z \end{smallmatrix}\right) \in B(\mathbb{F}_q)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_q)$. Let $\underline{x}' := (x'_{n_j})_{0\leq n_j\leq m_j} \in V_{m_j}^{\mathrm{Fr}^j} \otimes d^{(r_j-m_j)p^j}$. Then the action of b on $\underline{x}'$ is given by

$$b \cdot \underline{x}' = z^{(r_j-m_j)p^j} \cdot \left(\sum_{n_j=i_j}^{m_j}\binom{n_j}{i_j}u^{(m_j-n_j)p^j}v^{(n_j-i_j)p^j}z^{i_jp^j}x'_{n_j}\right)_{0\leq i_j\leq m_j}.$$

As in the proof of Theorem 2.6, but using Lemma 2.9 instead, $\psi^j_{P_j}(b\cdot\gamma) = \psi^j_{P_j}(b\cdot\gamma)$. Thus $\psi^j_{P_j}$ is $B(\mathbb{F}_q)$-linear.

(ii) $G(\mathbb{F}_q)$**-linearity:** This follows as in the proof of Theorem 2.6 using Lemma 2.10 instead.

(iii) **Isomorphism:** We now show by induction on m_j that $\ker\psi^j = \langle\theta_j'^{(m_j+1)}\rangle$, where $\theta_j' = X_j^qY_j - X_jY_j^q$. Suppose $m_j = 0$. Then $\psi^j(P_j) = \psi^j_{P_j}$ is defined by $\psi^j_{P_j}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = P_j(c^{p^j},d^{p^j})$. Clearly $\langle\theta_j'\rangle \subset \ker\psi^j$. Let $P_j \in \ker\psi^j$. Then $P_j(c^{p^j},d^{p^j}) = 0$ for all $(c,d) \in \mathbb{F}_q \times \mathbb{F}_q \setminus (0,0)$. Thus, $P_j(c,d) = 0$ for all $(c,d) \in \mathbb{F}_q \times \mathbb{F}_q \setminus (0,0)$. Then by Lemma 2.11, we have $P_j \in \langle\theta_j'\rangle$. So $\ker\psi^j \subset \langle\theta_j'\rangle$. Thus $\ker\psi^j = \langle\theta_j'\rangle$.

Assume that the result is true for $m_j - 1$. By Lemma 2.12, we have

$$\nabla_j^{a_j}(\theta_j'^{m_j+1})\big|_{(c^{p^j},d^{p^j})} = 0$$

for all $0 \leq a_j \leq m_j$. By definition of ψ^j, we have $\theta_j'^{m_j+1} \in \ker\psi^j$. So $\langle \theta_j'^{m_j+1} \rangle \subset \ker\psi^j$. Let $P_j \in \ker\psi^j$. Then $\nabla_j^{a_j}(P_j)\big|_{(c^{p^j}, d^{p^j})} = 0$ for all $0 \leq a_j \leq m_j$. In particular, $\nabla_j^{a_j}(P_j)\big|_{(c^{p^j}, d^{p^j})} = 0$ for all $0 \leq a_j \leq m_j - 1$. By the induction hypothesis $P_j \in \langle \theta_j'^{m_j} \rangle$. Write $P_j = Q_j \theta_j'^{m_j}$ with $Q_j \in \mathbb{F}_q[X_j, Y_j]$. Now, taking $a_j = m_j$ and proceeding exactly as in the proof of Theorem 2.6, but using Lemma 2.12 instead, we see θ_j' divides Q_j. So $\ker\psi^j \subset \langle \theta_j'^{m_j+1} \rangle$. Thus $\ker\psi^j = \langle \theta_j'^{m_j+1} \rangle$.

So we have an injective map $\psi^j : \frac{V_{r_j}^{\mathrm{Fr}^j}}{\langle \theta_j'^{m_j+1} \rangle} \to \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \left(V_{m_j}^{\mathrm{Fr}^j} \otimes d^{(r_j - m_j)p^j} \right)$. Note that the dimension of both sides of ψ^j is $(m_j+1)(q+1)$. So ψ^j is also surjective and an isomorphism. □

Remark 1. It is also possible to give a direct proof of the surjectivity of ψ^j, at least when $m_j = 0$. Let σ be a representation of $B(\mathbb{F}_q)$. For $g \in G(\mathbb{F}_q)$, $v \in \sigma$, let $[g, v] \in \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \sigma$ denote the map defined by

$$[g, v](g') = \begin{cases} \sigma(g'g)v, & \text{if } g'g \in B(\mathbb{F}_q), \\ 0, & \text{otherwise.} \end{cases}$$

A basis of $\mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} d^{(r_j - m_j)p^j}$ is given by (cf. [Bre07, Lemma 7.2], [BP12, Lemma 2.5 (2)])

$$\left\{ f_i := \sum_{\lambda \in \mathbb{F}_q} \lambda^i \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} [1,1],\ \phi := [1,1] \right\}_{0 \leq i \leq q-1}, \tag{2.8}$$

where $[1,1]$ denotes the function supported on $B(\mathbb{F}_q)$ and $[1,1](u) = 1$ for all $u \in \left\{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \mid x \in \mathbb{F}_q \right\}$. When, e.g., $j = 0$ and $m_0 = 0$, then one may check ψ^0 maps $(-1)^i X_0^{r_0 - i} Y_0^i$ to f_i for $0 \leq i \leq q-1$, and maps $Y^{r_0} - X_0^{q-1} Y_0^{r_0 - (q-1)}$ to ϕ.

Similarly for j, m_j arbitrary, a basis of $V_{m_j} \otimes \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} d^{(r_j - m_j)p^j} \simeq \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} (V_{m_j} \otimes d^{(r_j - m_j)p^j})$ is given by

$$\left\{ S_j^{m_j - l} T_j^l \otimes f_i,\ S_j^{m_j - l} T_j^l \otimes \phi \,\middle|\, 0 \leq l \leq m_j,\ 0 \leq i \leq q-1 \right\},$$

where V_{m_j} is modeled on polynomials of degree m_j over $\mathbb{F}_q$ in S_j, T_j and f_i, ϕ are as in (2.8). One should similarly be able to write down polynomials mapping to each of these basis elements under ψ^j.

Theorem 1.3 is a twisted version of the Lemma 2.13. However the proof is more involved. To prove it we need a few more lemmas. Recall the twisted Dickson polynomials θ_j for $0 \leq j \leq f-1$ were defined in (2.7).

Lemma 2.14. *Let $a, b, c, d \in \mathbb{F}_q$. For $0 \leq j \leq f-1$ and $0 \leq l_j, k_j \leq m_j$, let $\nabla_j = a^{p^j} \frac{\partial}{\partial X_j} + b^{p^j} \frac{\partial}{\partial Y_j}$. Then*

$$\left(\prod_{j=0}^{f-1} \nabla_j^{l_j} \right) \left(\prod_{j=0}^{f-1} \theta_j^{k_j} \right) \Big|_{(c,d;\ldots;c^{p^{f-1}}, d^{p^{f-1}})} = \begin{cases} l_0! \cdots l_{f-1}! \prod_{j=0}^{f-1} \left(\nabla_j(\theta_j)\Big|_{(c,d;\ldots;c^{p^{f-1}}, d^{p^{f-1}})} \right)^{l_j}, & \textit{if } (l_0, \ldots, l_{f-1}) = (k_0, \ldots, k_{f-1}), \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We induct on f. Lemma 2.5 is the case $f = 1$. Assume the result for $f-1$. Now, consider

$$\begin{aligned}
&\left(\prod_{j=0}^{f-1} \nabla_j^{l_j} \right) \left(\prod_{j=0}^{f-1} \theta_j^{k_j} \right) \Big|_{(c,d;\ldots;c^{p^{f-1}}, d^{p^{f-1}})} \\
&= \left(\prod_{j=1}^{f-1} \nabla_j^{l_j} \right) \left(\left(\prod_{j=2}^{f-1} \theta_j^{k_j} \right) \theta_1^{k_1} \nabla_0^{l_0} \left(\theta_0^{k_0} \right) \right) \Big|_{(c,d;\ldots;c^{p^{f-1}}, d^{p^{f-1}})} \\
&= \left(\prod_{j=1}^{f-1} \nabla_j^{l_j} \right) \left(\prod_{j=1}^{f-1} \theta_j^{k_j} \right) \Big|_{(c,d;\ldots;c^{p^{f-1}}, d^{p^{f-1}})} \nabla_0^{l_0} \left(\theta_0^{k_0} \right) \Big|_{(c,d;\ldots;c^{p^{f-1}}, d^{p^{f-1}})} \\
&= \begin{cases} l_0! \cdots l_{f-1}! \prod_{j=0}^{f-1} \left(\nabla_j(\theta_j)\Big|_{(c,d;\ldots;c^{p^{f-1}}, d^{p^{f-1}})} \right)^{l_j}, & \textit{if } (l_0, \ldots, l_{f-1}) = (k_0, \ldots, k_{f-1}), \\ 0, & \textit{otherwise.} \end{cases}
\end{aligned}$$

The first equality follows since θ_j for $2 \le j \le f-1$ is independent of X_0, Y_0, and since $\nabla_0(\theta_1) = 0$, and the last from the induction hypothesis and a twisted analogue of the $j = 0$ version of Lemma 2.12. □

For $0 \le j \le f-1$, assume that $r_j \ge p^{f-j}$. Then $r = \sum_{j=0}^{f-1} r_j p^j \ge fq$. Let $0 \le i_j \le r_j$ for $0 \le j \le f-1$. Set $i = \sum_{j=0}^{f-1} i_j p^j$ and $\vec{i} = (i_0, \dots, i_{f-1})$. We write $\sum_{\vec{i}=\vec{0}}^{\vec{r}} := \sum_{i_0=0}^{r_0} \cdots \sum_{i_{f-1}=0}^{r_{f-1}}$.

Lemma 2.15. *For $0 \le j \le f-1$, let $0 \le i_j \le r_j$ and $0 \le k_j \le p-1$. Let $k = \sum_{j=0}^{f-1} k_j p^j$. Let $P = \sum_{\vec{i}=\vec{0}}^{\vec{r}} a_{\vec{i}} \prod_{j=0}^{f-1} X_j^{r_j - i_j} Y_j^{i_j} \in \mathbb{F}_q[X_0, Y_0, \dots, X_{f-1}, Y_{f-1}]$ be such that $P\left(c, d; \dots; c^{p^{f-1}}, d^{p^{f-1}}\right) = 0$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_q)$. Then $a_{\vec{0}} = 0 = a_{\vec{r}}$, and the polynomial P is of the form*

$$P = \sum_{k=1}^{q-1} \left(\sum_{\substack{\vec{i}=\vec{0} \\ \vec{i} \neq \vec{k} \\ i \equiv k \mod (q-1)}}^{\vec{r}} a_{\vec{i}} \left(\prod_{j=0}^{f-1} X_j^{r_j - i_j} Y_j^{i_j} - \prod_{j=0}^{f-1} X_j^{r_j - k_j} Y_j^{k_j} \right) \right).$$

Proof. From the given condition we have $P\left(c, d; \dots; c^{p^{f-1}}, d^{p^{f-1}}\right) = \sum_{\vec{i}=\vec{0}}^{\vec{r}} a_{\vec{i}} c^{r-i} d^i = 0$, for all $(c, d) \in \mathbb{F}_q \times \mathbb{F}_q \setminus \{(0,0)\}$. In particular, choosing $(c, d) = (1, 0)$, we have $a_{\vec{0}} = 0$ and choosing $(c, d) = (0, 1)$, we have $a_{\vec{r}} = 0$. Let $\mathbb{F}_q^\times = \{\lambda_l \mid 1 \le l \le q-1\}$. Then taking $(c, d) = (1, \lambda_l)$, we have

$$P(1, \lambda_l; \dots; 1, \lambda_l^{p^{f-1}}) = \sum_{k=1}^{q-1} \left(\left(\sum_{\substack{\vec{i}=\vec{0} \\ i \equiv k \mod (q-1)}}^{\vec{r}} a_{\vec{i}} \right) \lambda_l^k \right) = 0,$$

which, by writing $A_k = \sum_{\substack{\vec{i}=\vec{0} \\ i \equiv k \mod (q-1)}}^{\vec{r}} a_{\vec{i}}$, gives $\sum_{k=1}^{q-1} A_k \lambda_l^k = 0$. Since the (essentially) Vandermonde matrix (λ_l^k) is invertible, we have

$$A_k = 0 \tag{2.9}$$

for all $1 \le k \le q-1$. Now, we have

$$\begin{aligned} P &= \sum_{\vec{i}=\vec{0}}^{\vec{r}} a_{\vec{i}} \prod_{j=0}^{f-1} X_j^{r_j - i_j} Y_j^{i_j} = \sum_{k=1}^{q-1} \left(\sum_{\substack{\vec{i}=\vec{0} \\ i \equiv k \mod (q-1)}}^{\vec{r}} a_{\vec{i}} \prod_{j=0}^{f-1} X_j^{r_j - i_j} Y_j^{i_j} \right) \\ &= \sum_{k=1}^{q-1} \left(\left(\sum_{\substack{\vec{i}=\vec{0} \\ \vec{i} \neq \vec{k} \\ i \equiv k \mod (q-1)}}^{\vec{r}} a_{\vec{i}} \prod_{j=0}^{f-1} X_j^{r_j - i_j} Y_j^{i_j} \right) + a_{\vec{k}} \prod_{j=0}^{f-1} X_j^{r_j - k_j} Y_j^{k_j} \right) = \sum_{k=1}^{q-1} \left(\sum_{\substack{\vec{i}=\vec{0} \\ \vec{i} \neq \vec{k} \\ i \equiv k \mod (q-1)}}^{\vec{r}} a_{\vec{i}} \left(\prod_{j=0}^{f-1} X_j^{r_j - i_j} Y_j^{i_j} - \prod_{j=0}^{f-1} X_j^{r_j - k_j} Y_j^{k_j} \right) \right). \end{aligned}$$

The last equality follows from (2.9). □

Lemma 2.16. *Let $\theta_0 := X_0 Y_{f-1}^p - Y_0 X_{f-1}^p$ and $\theta_k := X_k Y_{k-1}^p - Y_k X_{k-1}^p$ for all $1 \le k \le f-1$. For $0 \le j \le f-1$, let $c_j, d_j, g_j, h_j \in \mathbb{N} \cup \{0\}$ be such that $c_j + d_j = r_j$ with $r_j \ge p^{f-j}$. Let*

$$P(X_0, Y_0; \dots; X_{f-1}, Y_{f-1}) = \prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} - \prod_{j=0}^{f-1} X_j^{g_j} Y_j^{h_j},$$

where $\sum_{j=0}^{f-1} d_j p^j = \sum_{j=0}^{f-1} h_j p^j + t(q-1)$ *for all* $t \geq 1$. *Assume that* P *has no pure term which involves either only* X_j *or only* Y_j. *Then, for* $0 \leq j \leq f-1$ *there exists* $c'_j, d'_j \in \mathbb{N} \cup \{0\}$ *such that*

$$P(X_0, Y_0; \ldots; X_{f-1}, Y_{f-1}) = \left(\prod_{j=0}^{f-1} X_j^{c'_j} Y_j^{d'_j} - \prod_{j=0}^{f-1} X_j^{g_j} Y_j^{h_j} \right) \mod \langle \theta_0, \ldots, \theta_{f-1} \rangle$$

with $\sum_{j=0}^{f-1} d'_j p^j = \sum_{j=0}^{f-1} h_j p^j$.

Proof. We first make two observations. If $c_0 \geq 1, d_{f-1} \geq p$, we have

$$\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} = \left(\prod_{j=1}^{f-2} X_j^{c_j} Y_j^{d_j} \right) X_0^{c_0-1} Y_0^{d_0+1} X_{f-1}^{c_{f-1}+p} Y_{f-1}^{d_{f-1}-p} \mod \theta_0. \tag{2.10}$$

For $1 \leq k \leq f-1$, if $c_k \geq 1, d_{k-1} \geq p$, then

$$\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} = \left(\prod_{\substack{j=0 \\ j \neq k, k-1}}^{f-1} X_j^{c_j} Y_j^{d_j} \right) X_{k-1}^{c_{k-1}+p} Y_{k-1}^{d_{k-1}-p} X_k^{c_k-1} Y_k^{d_k+1} \mod \theta_k. \tag{2.11}$$

Note that the first operation involving θ_0 decreases $\sum_{j=0}^{f-1} d_j p^j$ in (2.10) by $q-1$. However, the second operation involving θ_k does not change $\sum_{j=0}^{f-1} d_j p^j$ in (2.11).

Now, to prove the result we may assume that $d_{f-1} \geq h_{f-1}$ (replacing P by -P). We also assume that $t = 1$. For $t > 1$ we reiterate the proof below till $t = 1$.

Case 1: Suppose $d_{f-1} \geq p$.

First assume $c_0 \geq 1$. Since $c_0 \geq 1$ and $d_{f-1} \geq p$, by (2.10), we have

$$\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} = \left(\prod_{j=1}^{f-2} X_j^{c_j} Y_j^{d_j} \right) X_0^{c_0-1} Y_0^{d_0+1} X_{f-1}^{c_{f-1}+p} Y_{f-1}^{d_{f-1}-p} \mod \theta_0.$$

Note that,

$$(d_0+1) + \sum_{j=1}^{f-2} d_j p^j + (d_{f-1}-p) p^{f-1} = \sum_{j=0}^{f-1} d_j p^j - (q-1) = \sum_{j=0}^{f-1} h_j p^j.$$

Thus the result follows by taking $c'_0 = c_0 - 1, d'_0 = d_0 + 1; c'_{f-1} = c_{f-1} + p, d'_{f-1} = d_{f-1} - p$ and $c'_k = c_k, d'_k = d_k$ for all $1 \leq k \leq f-2$.

Next suppose $c_0 = 0$. Since there is no pure term in Y_j in P, we choose k to be the least index for which $c_j \neq 0$ in $\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j}$. Then we have $c_j = 0$ for all $j < k$ and $c_k \neq 0$. This implies that $d_j = r_j \geq p$ for all $j < k$ and $c_k \geq 1$. Then, using (2.11), we have

$$\begin{aligned}
\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} &= \left(\prod_{\substack{j=0 \\ j \neq k, k-1}}^{f-1} X_j^{c_j} Y_j^{d_j} \right) X_{k-1}^{c_{k-1}+p} Y_{k-1}^{d_{k-1}-p} X_k^{c_k-1} Y_k^{d_k+1} \mod \theta_k \\
&= \left(\prod_{\substack{j=0 \\ j \neq k, k-1, k-2}}^{f-1} X_j^{c_j} Y_j^{d_j} \right) X_k^{c_k-1} Y_k^{d_k+1} X_{k-1}^{c_{k-1}+p-1} Y_{k-1}^{d_{k-1}-p+1} X_{k-2}^{c_{k-2}+p} Y_{k-2}^{d_{k-2}-p} \mod \theta_{k-1} \\
&\;\;\vdots \\
&= \left(\prod_{j=k+1}^{f-1} X_j^{c_j} Y_j^{d_j} \right) X_k^{c_k-1} Y_k^{d_k+1} \left(\prod_{j=1}^{k-1} X_j^{c_j+p-1} Y_j^{d_j-p+1} \right) X_0^{c_0+p} Y_0^{d_0-p} \mod \theta_1.
\end{aligned}$$

Clearly $c_0 + p \geq 1$ and we are reduced to the previous paragraph ($c_0 \geq 1$).

Case 2: Suppose $d_{f-1} < p$.

Then $d_{f-1} = p - i_1 = i_0 p - i_1$, where $i_0 = 1$ and $0 < i_1 \le p$. Since $r_{f-1} \ge p$, we have $c_{f-1} = r_{f-1} - i_0 p + i_1 \ge i_1$. Now, if $d_{f-2} \ge i_1 p$, taking $k = f-1$, by (2.11) we have

$$\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} = \left(\prod_{j=0}^{f-3} X_j^{c_j} Y_j^{d_j}\right) X_{f-2}^{c_{f-2}+i_1 p} Y_{f-2}^{d_{f-2}-i_1 p} X_{f-1}^{c_{f-1}-i_1} Y_{f-1}^{d_{f-1}+i_1} \mod \theta_{f-1}^{i_1}.$$

Note that $d_{f-1} + i_1 = p$, so we are done by Case 1.

Suppose, $d_{f-2} < i_1 p$. We write $d_{f-2} = i_1 p - i_2$ with $0 < i_2 \le i_1 p$. Since by assumption $r_{f-2} \ge p^2$ and $0 < i_1 \le p$, we have $r_{f-2} \ge p^2 \ge i_1 p$. This shows that $c_{f-2} = r_{f-2} - i_1 p + i_2 \ge i_2$. Now, if $d_{f-3} \ge i_2 p$, taking $k = f-2$, by (2.11) we have

$$\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} = \left(\prod_{\substack{j=0 \\ j \ne f-2, f-3}}^{f-1} X_j^{c_j} Y_j^{d_j}\right) X_{f-3}^{c_{f-3}+i_2 p} Y_{f-3}^{d_{f-3}-i_2 p} X_{f-2}^{c_{f-2}-i_2} Y_{f-2}^{d_{f-2}+i_2} \mod \theta_{f-2}^{i_2}.$$

Note that $d_{f-2} + i_2 = i_1 p$. So we are done by the '$d_{f-2} \ge i_1 p$' case above. And so on.

Thus it is enough to show that this process stops. Suppose not. Then for $0 \le j \le f-1$, we have $0 < i_{j+1} \le i_j p$ such that $d_{f-1-j} = i_j p - i_{j+1}$. In particular, $d_0 < i_{f-1} p$. Now, note that

$$\sum_{j=0}^{f-1} d_j p^j < i_{f-1} p + \sum_{j=0}^{f-2} d_{f-1-j} p^{f-1-j} = i_{f-1} p + \sum_{j=0}^{f-2} (i_j p - i_{j+1}) p^{f-1-j} = i_0 q.$$

Since $i_0 = 1$, we have $\sum\limits_{j=0}^{f-1} d_j p^j < q$. Also, note that $\sum\limits_{j=0}^{f-1} d_j p^j \ge q-1$ by assumption. Hence,

$$q - 1 \le \sum_{j=0}^{f-1} d_j p^j \le q-1 \implies \sum_{j=0}^{f-1} d_j p^j = q-1 \implies \sum_{j=0}^{f-1} h_j p^j = 0 \implies h_j = 0,$$

for all $0 \le j \le f-1$. This is a contradiction because P does not contain any pure term in X_j. Thus modulo $\langle \theta_1, \ldots, \theta_{f-1} \rangle$, we can always assume that $d_{f-1} \ge p$, hence by Case 1 the result follows. ☐

Lemma 2.17. *Let $\theta_k := X_k Y_{k-1}^p - Y_k X_{k-1}^p$ for all $1 \le k \le f-1$. For $0 \le j \le f-1$, let $c_j, d_j, g_j, h_j \in \mathbb{N} \cup \{0\}$ be such that $c_j + d_j = r_j$ with $r_j \ge p^{f-j}$. Let*

$$P(X_0, Y_0; \ldots; X_{f-1}, Y_{f-1}) = \prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} - \prod_{j=0}^{f-1} X_j^{g_j} Y_j^{h_j},$$

where $\sum\limits_{j=0}^{f-1} d_j p^j = \sum\limits_{j=0}^{f-1} h_j p^j$. Then $P(X_0, Y_0; \ldots; X_{f-1}, Y_{f-1}) \in \langle \theta_1, \ldots, \theta_{f-1} \rangle$.

Proof. We make the following observation. For $1 \le k \le f-1$, if $d_k \ge 1, c_{k-1} \ge p$, then

$$\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} = \left(\prod_{\substack{j=0 \\ j \ne k, k-1}}^{f-1} X_j^{c_j} Y_j^{d_j}\right) X_{k-1}^{c_{k-1}-p} Y_{k-1}^{d_{k-1}+p} X_k^{c_k+1} Y_k^{d_k-1} \mod \theta_k. \tag{2.12}$$

Note that the operation involving θ_k in (2.12) does not change the sum $\sum_{j=0}^{f-1} d_j p^j$.

Now, we prove the result by induction on f. If $f = 1$, then $P = 0$, and hence, the result follows. Assume that the result is true for $f-1$. Without loss of generality we assume that $d_{f-1} \ge h_{f-1}$. If $d_{f-1} = h_{f-1}$, we are done by the induction hypothesis.

Suppose $d_{f-1} > h_{f-1}$. We assume that $d_{f-1} - h_{f-1} = 1$. If this difference is bigger than 1, then we reiterate the proof below until it is 1. Clearly $d_{f-1} \ge 1$. If $c_{f-2} \ge p$, taking $k = f-1$ in (2.12), we have

$$\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} = \left(\prod_{j=0}^{f-3} X_j^{c_j} Y_j^{d_j}\right) X_{f-2}^{c_{f-2}-p} Y_{f-2}^{d_{f-2}+p} X_{f-1}^{c_{f-1}+1} Y_{f-1}^{d_{f-1}-1} \mod \theta_{f-1}.$$

Since the operation involving θ_{f-1} does not change the sum $\sum_{j=0}^{f-1} d_j p^j$ and $d_{f-1} - 1 = h_{f-1}$, we are done by the case '$d_{f-1} = h_{f-1}$'.

If $c_{f-2} < p$, we write $c_{f-2} = p - i_1 = i_0 p - i_1$, where $i_0 = 1$ and $0 < i_1 \leq p$. Since by assumption $r_{f-2} \geq p$, we have $d_{f-2} = r_{f-2} - i_0 p + i_1 \geq i_1$. Now, if $c_{f-3} \geq i_1 p$, taking $k = f-2$ in (2.12), we have

$$\prod_{j=0}^{f-1} X_j^{c_j} Y_j^{d_j} = \left(\prod_{\substack{j=0 \\ j \neq f-2, f-3}}^{f-1} X_j^{c_j} Y_j^{d_j} \right) X_{f-3}^{c_{f-3} - i_1 p} Y_{f-3}^{d_{f-3} + i_1 p} X_{f-2}^{c_{f-2} + i_1} Y_{f-2}^{d_{f-2} - i_1} \mod \theta_{f-2}^{i_1}.$$

Since the operation involving θ_{f-2} does not change the sum $\sum_{j=0}^{f-1} d_j p^j$ and $c_{f-2} + i_1 = p$, we are done by the '$c_{f-2} \geq p$' case. And so on.

Thus it is enough to show that this process stops. Suppose not. Then for $0 \leq j \leq f-2$, we have $0 < i_{j+1} \leq i_j p$ such that $c_{f-2-j} = i_j p - i_{j+1}$. In particular, $c_0 < i_{f-2} p$. By hypothesis, we have

$$\sum_{j=0}^{f-1} d_j p^j = \sum_{j=0}^{f-1} h_j p^j \implies \sum_{j=0}^{f-2} d_j p^j + (d_{f-1} - h_{f-1}) p^{f-1} = \sum_{j=0}^{f-2} h_j p^j,$$

which by substituting $d_{f-1} - h_{f-1} = 1, d_j = r_j - c_j$ and $h_j \leq r_j$ for all $0 \leq j \leq f-2$ gives

$$\sum_{j=0}^{f-2} (r_j - c_j) p^j + p^{f-1} \leq \sum_{j=0}^{f-2} r_j p^j \implies -\sum_{j=0}^{f-2} c_j p^j + p^{f-1} \leq 0,$$

which further by substituting $c_j = i_{f-2-j} p - i_{f-1-j}$ implies that

$$-\sum_{j=0}^{f-2} \left(i_{f-2-j} p - i_{f-1-j} \right) p^j + p^{f-1} \leq 0 \implies -i_0 p^{f-1} + i_{f-1} + p^{f-1} \leq 0.$$

Since $i_0 = 1$, we conclude $i_{f-1} \leq 0$. But we had $0 < i_{f-1} \leq i_{f-2} p$. Thus we arrive at a contradiction. So modulo $\langle \theta_1, \ldots, \theta_{f-1} \rangle$ we are always reduced to the '$d_{f-1} = h_{f-1}$' case, and so we are done. □

We finally prove Theorem 1.3 from the introduction.

Theorem 2.18. *Let $r = r_0 + r_1 p + \cdots + r_{f-1} p^{f-1}$ with $r_j \geq p^{f-j}$ for all $0 \leq j \leq f-1$. Let $m = m_0 + m_1 p + \cdots + m_{f-1} p^{f-1}$ be the p-adic expansion of m with $0 \leq m_j \leq p-1$ and $p \nmid \binom{r_j}{m_j}$ for all $0 \leq j \leq f-1$. Then we have*

$$\frac{\bigotimes_{j=0}^{f-1} V_{r_j}^{\mathrm{Fr}^j}}{\langle \theta_0^{m_0+1}, \ldots, \theta_{f-1}^{m_{f-1}+1} \rangle} \simeq \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \bigotimes_{j=0}^{f-1} \left(V_{m_j}^{\mathrm{Fr}^j} \otimes d^{(r_j - m_j) p^j} \right).$$

Proof. We show that there is a $G(\mathbb{F}_q)$-equivariant isomorphism

$$\psi : \frac{\bigotimes_{j=0}^{f-1} V_{r_j}^{\mathrm{Fr}^j}}{\langle \theta_0^{m_0+1}, \ldots, \theta_{f-1}^{m_{f-1}+1} \rangle} \to \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \bigotimes_{j=0}^{f-1} \left(V_{m_j}^{\mathrm{Fr}^j} \otimes d^{(r_j - m_j) p^j} \right)$$

defined by $\psi(\bigotimes_{j=0}^{f-1} P_j(X_j, Y_j)) = \bigotimes_{j=0}^{f-1} \psi^j_{P_j(X_j, Y_j)}$, where $P_j(X_j, Y_j) \in V_{r_j}^{\mathrm{Fr}^j}$ for all $0 \leq j \leq f-1$, and

$$\bigotimes_{j=0}^{f-1} \psi^j_{P_j(X_j, Y_j)} : G(\mathbb{F}_q) \to \bigotimes_{j=0}^{f-1} \left(V_{m_j}^{\mathrm{Fr}^j} \otimes d^{(r_j - m_j) p^j} \right)$$

is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \bigotimes_{j=0}^{f-1} \left(\frac{\binom{m_j}{n_j}}{[r_j]_{m_j - n_j}} \nabla_j^{m_j - n_j} (P_j) \Big|_{(c^{p^j}, d^{p^j})} \right)_{0 \leq n_j \leq m_j}.$$

We first show that $\bigotimes_{j=0}^{f-1} \psi^j_{P_j(X_j, Y_j)}$ is $B(\mathbb{F}_q)$-linear. This follows from the $B(\mathbb{F}_q)$-linearity of each $\psi^j_{P_j(X_j, Y_j)}$. Let $b \in B(\mathbb{F}_q)$ and $g \in G(\mathbb{F}_q)$. Then

$$\bigotimes_{j=0}^{f-1} \psi^j_{P_j(X_j, Y_j)}(b \cdot g) = \bigotimes_{j=0}^{f-1} \psi^j_{P_j(X_j, Y_j)}(b \cdot g) = \bigotimes_{j=0}^{f-1} b \cdot \psi^j_{P_j(X_j, Y_j)}(g) = b \cdot \left(\bigotimes_{j=0}^{f-1} \psi^j_{P_j(X_j, Y_j)}(g) \right).$$

The second equality holds by Lemma 2.13 (i).

Similarly, for the $G(\mathbb{F}_q)$-linearity of ψ, we note that $\psi(g\cdot\bigotimes_{j=0}^{f-1}P_j(X_j,Y_j)) = \psi(\bigotimes_{j=0}^{f-1}g\cdot P_j(X_j,Y_j)) = \bigotimes_{j=0}^{f-1}\psi^j_{g\cdot P_j(X_j,Y_j)} = \bigotimes_{j=0}^{f-1}\psi^j(g\cdot P_j(X_j,Y_j))$, which by Lemma 2.13 (ii), equals $\bigotimes_{j=0}^{f-1}g\cdot\psi^j(P_j(X_j,Y_j)) = g\cdot\bigotimes_{j=0}^{f-1}\psi^j(P_j(X_j,Y_j)) = g\cdot\bigotimes_{j=0}^{f-1}\psi^j_{P_j(X_j,Y_j)} = g\cdot\psi(\bigotimes_{j=0}^{f-1}P_j(X_j,Y_j))$.

Now there is a natural surjection

$$\pi : \bigotimes_{j=0}^{f-1}\mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\left(V_{m_j}^{\mathrm{Fr}^j}\otimes d^{(r_j-m_j)p^j}\right)\twoheadrightarrow \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\left(\bigotimes_{j=0}^{f-1}V_{m_j}^{\mathrm{Fr}^j}\otimes d^{(r_j-m_j)p^j}\right)$$

given by $\pi(\otimes_{j=0}^{f-1}F_j) = F$ with $F(g) = \otimes_{j=0}^{f-1}F_j(g)$ for all $g\in G(\mathbb{F}_q)$. By definition of ψ, we have that $\psi(\otimes_{j=0}^{f-1}P_j(X_j,Y_j))(g) = \otimes_{j=0}^{f-1}\psi^j_{P_j(X_j,Y_j)}(g) = \pi(\otimes_{j=0}^{f-1}\psi^j_{P_j(X_j,Y_j)})(g)$ for all $g\in G(\mathbb{F}_q)$. Hence we have that $\psi(\otimes_{j=0}^{f-1}P_j(X_j,Y_j)) = \pi(\otimes_{j=0}^{f-1}\psi^j_{P_j(X_j,Y_j)})$. But the last map is equal to $\pi((\otimes_{j=0}^{f-1}\psi^j)(\otimes_{j=0}^{f-1}P_j(X_j,Y_j))) = (\pi\circ\otimes_{j=0}^{f-1}\psi^j)(\otimes_{j=0}^{f-1}P_j(X_j,Y_j))$. Thus

$$\psi = \pi\circ(\otimes_{j=0}^{f-1}\psi^j).$$

By Lemma 2.13 (iii), each ψ^j is an isomorphism, hence so is $\otimes_{j=0}^{f-1}\psi^j$. Since π is surjective, so is ψ.

Note that π is not necessarily injective (take $f>1$ and compare dimensions on both sides of π), so ψ is not necessarily injective. It remains to compute $\ker\psi$. We show that $\ker\psi_m = \langle\theta_0^{m_0+1},\dots,\theta_{f-1}^{m_{f-1}+1}\rangle$, where we write ψ_m instead of ψ for emphasis. By Lemma 2.14, for $0\le a_j\le m_j$,

$$\left(\prod_{j=0}^{f-1}\nabla_j^{a_j}\right)(\theta_s^{m_s+1})\Big|_{(c,d;\dots;c^{p^{f-1}},d^{p^{f-1}})} = 0,$$

for all $0\le s\le f-1$. By the definition of ψ_m, we see that $\langle\theta_0^{m_0+1},\dots,\theta_t^{m_t+1},\dots,\theta_{f-1}^{m_{f-1}+1}\rangle\subset\ker\psi_m$. We now prove the other containment

$$\ker\psi_m\subset\langle\theta_0^{m_0+1},\dots,\theta_t^{m_t+1},\dots,\theta_{f-1}^{m_{f-1}+1}\rangle. \tag{2.13}$$

This is the trickiest part of the proof of the theorem. We need to make use of the three lemmas proved just before the theorem.

The proof is by induction on $\sum_p(m)$, where $\sum_p(m) = m_0+\cdots+m_{f-1}$ denotes the sum of the p-adic digits in the base p expansion of m. If $\sum_p(m)=0$, then we have $m_j=0$ for all $0\le j\le f-1$. Then (2.13) follows immediately from Lemmas 2.15, 2.16 and 2.17.

Now, suppose $\sum_p(m)\ge 1$. Assume that (2.13) holds for m' with $\sum_p(m')\le\sum_p(m)-1$. Pick t such that $m_t\ge 1$. Let $m' = m_0+\cdots+(m_t-1)p^t+\cdots+m_{f-1}p^{f-1}$. Then $\sum_p(m') = \sum_p(m)-1$ and so

$$\ker\psi_{m'}\subset\langle\theta_0^{m_0+1},\dots,\theta_t^{m_t},\dots,\theta_{f-1}^{m_{f-1}+1}\rangle, \tag{2.14}$$

by the induction hypothesis. Now, let $P\in\ker\psi_m$. So $\left(\prod_{j=0}^{f-1}\nabla_j^{a_j}\right)(P)\big|_{(c,d;\dots;c^{p^{f-1}},d^{p^{f-1}})} = 0$ for all $0\le a_j\le m_j$. In particular, $\left(\prod_{j=0}^{f-1}\nabla_j^{a_j}\right)(P)\big|_{(c,d;\dots;c^{p^{f-1}},d^{p^{f-1}})} = 0$ for all $0\le a_j\le m_j$ with $j\ne t$ and for all $0\le a_t\le m_t-1$. This shows that $P\in\ker\psi_{m'}$. By (2.14), we may write

$$P = Q_0\theta_0^{m_0+1}+\cdots+Q_t\theta_t^{m_t}+\cdots+Q_{f-1}\theta_{f-1}^{m_{f-1}+1},$$

with $Q_j\in\mathbb{F}_q[X_0,Y_0;\dots;X_{f-1},Y_{f-1}]$ for $0\le j\le f-1$. Clearly,

$$P\in\langle\theta_0^{m_0+1},\dots,\theta_t^{m_t+1},\dots,\theta_{f-1}^{m_{f-1}+1}\rangle \iff Q_t\theta_t^{m_t}\in\langle\theta_0^{m_0+1},\dots,\theta_t^{m_t+1},\dots,\theta_{f-1}^{m_{f-1}+1}\rangle.$$

So without loss of generality, let $P = Q_t\theta_t^{m_t}$. Since $P \in \ker \psi_m$, for $0 \le a_j \le m_j$ and $j \neq t$, we have

$$\begin{aligned}
&\left(\prod_{j=0, j\neq t}^{f-1} \nabla_j^{a_j}\right) \nabla_t^{m_t}(Q_t\theta_t^{m_t})\Big|_{(c,d;\ldots;c^{p^{f-1}},d^{p^{f-1}})} = 0\\
\implies &\left(\prod_{j=0, j\neq t}^{f-1} \nabla_j^{a_j}\right) \left(\sum_{l=0}^{m_t}\binom{m_t}{l}\nabla_t^{m_t-l}(Q_t)\nabla_t^{l}(\theta_t^{m_t})\right)\Big|_{(c,d;\ldots;c^{p^{f-1}},d^{p^{f-1}})} = 0\\
\implies &\sum_{l=0}^{m_t}\binom{m_t}{l}\nabla_{t-1}^{a_{t-1}}\left(\left(\prod_{j=0, j\neq t,t-1}^{f-1} \nabla_j^{a_j}\right)\nabla_t^{m_t-l}(Q_t)\nabla_t^{l}(\theta_t^{m_t})\right)\Big|_{(c,d;\ldots;c^{p^{f-1}},d^{p^{f-1}})} = 0\\
\implies &\sum_{l=0}^{m_t}\binom{m_t}{l}\left(\sum_{k=0}^{a_{t-1}}\binom{a_{t-1}}{k}\nabla_{t-1}^{a_{t-1}-k}\left(\left(\prod_{j=0, j\neq t,t-1}^{f-1} \nabla_j^{a_j}\right)\nabla_t^{m_t-l}(Q_t)\right)\nabla_{t-1}^{k}\left(\nabla_t^{l}\left(\theta_t^{m_t}\right)\right)\right)\Big|_{(c,d;\ldots;c^{p^{f-1}},d^{p^{f-1}})} = 0\\
\implies &\left(\prod_{j=0, j\neq t}^{f-1} \nabla_j^{a_j}\right)\nabla_t^{0}(Q_t)\Big|_{(c,d;\ldots;c^{p^{f-1}},d^{p^{f-1}})} = 0\\
\implies &Q_t \in \langle \theta_0^{m_0+1},\ldots,\theta_t^{1},\ldots,\theta_{f-1}^{m_{f-1}+1}\rangle \implies Q_t\theta_t^{m_t} \in \langle \theta_0^{m_0+1},\ldots,\theta_t^{m_t+1},\ldots,\theta_{f-1}^{m_{f-1}+1}\rangle.
\end{aligned}$$

The fourth implication follows from Lemma 2.14: if $(k,l) = (0, m_t)$, then

$$\nabla_{t-1}^{k}\nabla_t^{l}(\theta_t^{m_t})\Big|_{(c,d;\ldots;c^{p^{f-1}},d^{p^{f-1}})} = m_t!(ad-bc)^{m_t p^t} \neq 0,$$

and is 0 for all other (k,l). The penultimate implication holds by the induction hypothesis as the sum of the p-adic digits is $\Sigma_p(m) - m_t \le \Sigma_p(m) - 1$. So $P \in \langle \theta_0^{m_0+1},\ldots,\theta_t^{m_t+1},\ldots,\theta_{f-1}^{m_{f-1}+1}\rangle$, proving (2.13). □

3. Dual numbers

This section is an aside. The ring of generalized dual numbers is defined by $\mathbb{F}_p[\epsilon] = \frac{\mathbb{F}_p[X]}{\langle X^{m+1}\rangle}$. We make some remarks on two questions that arise in the context of the lack of surjectivity when $m > 0$ of the map (1.2) which involves dual numbers (introduced in [GV22, Lemma 4.1]). Firstly, can one possibly replace the inducing subgroup $B(\mathbb{F}_p[\epsilon])$ in (1.2) by another subgroup B' of index $p+1$ in $G(\mathbb{F}_p[\epsilon])$ and d^r by a surjective character $\chi_r : B' \to \mathbb{F}_p[\epsilon]^\times$ such that there is an isomorphism

$$\frac{V_r}{V_r^{(m+1)}} \overset{?}{\simeq} \operatorname{ind}_{B'}^{G(\mathbb{F}_p[\epsilon])}\chi_r$$

which might then be used to study periodicity results? The answer is no, and explains why in this paper we turned towards proving the isomorphism in Theorem 1.1. Secondly, can one at least describe the image of (1.2) in a more conceptual way than is done in [GV22, Lemma 4.1]? The answer in some cases is yes (see Proposition 3.1, whose proof we omit).

It would be interesting to see how the material in this section connected to announced work of Schein and his coauthors on the modular representation theory of $\mathrm{GL}_2(R)$ where R is a finite quotient ring of $\mathcal{O}_F$ for F a p-adic field, and, e.g., to work of Avni, Onn, Prasad, Vaserstein [AOPV].

3.1. **Isomorphisms using dual numbers.** There are two notions of projective space over the generalized dual numbers. The first is standard projective space

$$\mathbb{P}^1(\mathbb{F}_p[\epsilon]) = \{[x:y] \mid (x,y) = 1\}.$$

It has cardinality $p(p+1)$ when $m = 1$ and is the cylinder obtained by glueing the line at ∞, namely $[1 : d\epsilon]$, to the plane $[c : 1]$. The second is $\tilde{\mathbb{P}}^1(\mathbb{F}_p[\epsilon]) = \{[x:y] \mid (x,y) \neq 0\}$. It has cardinality $(p+1)^2$ when $m = 1$. The group $G(\mathbb{F}_p[\epsilon])$ acts on the left on both these spaces via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot [x:y] = [ax+by : cx+dy]$.

Let B' be the ϵ-Iwahori subgroup of $G(\mathbb{F}_p[\epsilon])$ obtained as the pre-image of the usual Borel $B(\mathbb{F}_p)$ under the reduction modulo ϵ map $G(\mathbb{F}_p[\epsilon]) \to G(\mathbb{F}_p)$. Then B' is the stabilizer of $[\epsilon : 0]$ under the action of $G(\mathbb{F}_p[\epsilon])$ on $\tilde{\mathbb{P}}^1(\mathbb{F}_p[\epsilon])$ (whereas $B(\mathbb{F}_p[\epsilon])$ is the stabilizer of $[1:0]$ in $\mathbb{P}^1(\mathbb{F}_p[\epsilon])$). Clearly B' has index $p+1$.

One may ask if there is a surjective character $\chi : B' \to \mathbb{F}_p[\epsilon]^\times$ which induces a $G(\mathbb{F}_p)$-isomorphism

$$\frac{V_r}{V_r^{(m+1)}} \simeq \operatorname{ind}_{B'}^{G(\mathbb{F}_p[\epsilon])}\chi_r$$

from which the periodicity of the left side would follow if one knew χ_r only depended on r modulo $p(p-1)$.

The answer is no. Indeed, one quickly sees that B' has abelianization $\mathbb{F}_p[\epsilon]^\times \times \mathbb{F}_p^\times$ and so the only surjective characters $B' \to \mathbb{F}_p[\epsilon]^\times$ it supports are powers of the determinant character. These characters are not genuine characters of B', since they are obtained by restricting from $G(\mathbb{F}_p[\epsilon])$, so the induction is not so well-behaved.

Moreover, one checks that every other subgroup B'' of index $p+1$ in $G(\mathbb{F}_p[\epsilon])$ is conjugate to B' so this line of reasoning does not bear fruit.

3.2. **Image of** (1.2)**.** Thus, the best one can hope to do is to characterize the image of the (non-surjective) map (if $m > 0$)

$$\frac{V_r}{V_r^{(m+1)}} \hookrightarrow \mathrm{ind}_{B(\mathbb{F}_p[\epsilon])}^{G(\mathbb{F}_p[\epsilon])} d^r$$

mentioned in the introduction (cf. [GV22, Lemma 4.1]).

To this end, we consider the right action of $G(\mathbb{F}_p[\epsilon])$ on standard projective space $\mathbb{P}^1(\mathbb{F}_p[\epsilon])$ defined via $[x:y] \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = [ax+cy : bx+dy]$. This action is transitive and the stabilizer of $[0:1]$ under this action is $B(\mathbb{F}_p[\epsilon])$. We have the following decomposition

$$G(\mathbb{F}_p[\epsilon]) = B(\mathbb{F}_p[\epsilon]) \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \bigsqcup B(\mathbb{F}_p[\epsilon]) \begin{pmatrix} 0 & 1 \\ 1 & d\epsilon \end{pmatrix},$$

where $c, d \in \mathbb{F}_p[\epsilon]$. There is a bijection between $B(\mathbb{F}_p[\epsilon])\backslash G(\mathbb{F}_p[\epsilon])$ and $\mathbb{P}^1(\mathbb{F}_p[\epsilon])$ by sending $B(\mathbb{F}_p[\epsilon])g$ to $[0:1]g$ for $g \in G(\mathbb{F}_p[\epsilon])$.

We say that $f : \mathbb{P}^1(\mathbb{F}_p[\epsilon]) \to \mathbb{F}_p[\epsilon]$ is *smooth* if for all $z_0 + z'\epsilon \in \mathbb{F}_p[\epsilon]$ with $z_0 \in \mathbb{F}_p$ and $z' = z_1 + z_2\epsilon + \cdots + z_{m-1}\epsilon^{m-1}$ with $z_1, \ldots, z_{m-1} \in \mathbb{F}_p$ and all $0 \le j \le m$, there exist constants $f^{(j)}([z_0:1])$ and $f^{(j)}([1:0])$ in $\mathbb{F}_p$ such that

$$f\left([z_0 + z'\epsilon : 1]\right) = \sum_{j=0}^{m} \frac{(z'\epsilon)^j}{j!} f^{(j)}\left([z_0:1]\right) \quad \text{and} \quad f\left([1 : z'\epsilon]\right) = \sum_{j=0}^{m} \frac{(z'\epsilon)^j}{j!} f^{(j)}\left([1:0]\right).$$

Proposition 3.1. *Let $\psi : V_r \to \mathrm{ind}_{B(\mathbb{F}_p[\epsilon])}^{G(\mathbb{F}_p[\epsilon])} d^r$ be given by $\psi(P(X,Y)) = \psi_{P(X,Y)}$ for all $P(X,Y) \in V_r$, where $\psi_{P(X,Y)} : G(\mathbb{F}_p[\epsilon]) \to \mathbb{F}_p[\epsilon]$ is defined by $\psi_P\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = P(c,d)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_p[\epsilon])$. If $r \equiv 0$ modulo $p(p-1)$, then*

$$\mathrm{Im}\,\psi = \left\{ f : \mathbb{P}^1(\mathbb{F}_p[\epsilon]) \to \mathbb{F}_p[\epsilon] \mid f(\alpha) \in \mathbb{F}_p \text{ if } \alpha \in \mathbb{P}^1(\mathbb{F}_p) \text{ and } f \text{ is smooth} \right\}.$$

4. Cuspidal case

In this section, we prove Theorems 1.5 and 1.6 which, as explained in the introduction, are the cuspidal analogs of Theorems 1.1 and 1.3.

4.1. **The case of** $\mathrm{GL}_2(\mathbb{F}_p)$**.** Let $\alpha \in \mathbb{F}_{p^2}$ be such that $\alpha^2 \in \mathbb{F}_p$ and $\alpha \notin \mathbb{F}_p$. Fix an identification $i : \mathbb{F}_{p^2}^\times \simeq T(\mathbb{F}_p) \subset \mathrm{GL}_2(\mathbb{F}_p)$ given by $u + v\alpha \mapsto \left(\begin{smallmatrix} u & v\alpha^2 \\ v & u \end{smallmatrix}\right)$ for $u, v \in \mathbb{F}_p$ not both zero.

We define some functions in induced spaces. For $r \ge 0$ and $0 \le i \le r + p^2 + 1$, let $f_i : G(\mathbb{F}_p) \to \mathbb{F}_{p^2}$ be

$$f_i\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a + c\alpha)^{(r+p^2+1)-i}(b + d\alpha)^i,$$

for all $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_p)$. Then f_i is $T(\mathbb{F}_p)$-linear and hence $f_i \in \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2^{r+2}$. Indeed, for $t = \left(\begin{smallmatrix} u & v\alpha^2 \\ v & u \end{smallmatrix}\right) \in T(\mathbb{F}_p)$, we have

$$f_i(t \cdot g) = f_i\left(\begin{pmatrix} ua + v\alpha^2 c & ub + v\alpha^2 d \\ va + uc & vb + ud \end{pmatrix}\right) = (u + v\alpha)^{r+p^2+1}(a + c\alpha)^{(r+p^2+1)-i}(b + d\alpha)^i$$

which equals

$$(u + v\alpha)^{(r+2)+p^2-1} \cdot f_i(g) = (u + v\alpha)^{r+2} \cdot f_i(g) = \omega_2^{r+2}(t) \cdot f_i(g) = t \cdot f_i(g).$$

One can check that the functions in $\mathcal{B} = \left\{ f_i \mid 0 \le i \le p^2 - p - 1 \right\}$ are linearly independent. Also, $T(\mathbb{F}_p)$ has index $p^2 - p$ in $G(\mathbb{F}_p)$. So $\mathcal{B}$ forms a basis of $\mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2^{r+2}$. We fix this basis in the computations to follow.

For $p^2 - 1 \le i \le r + p^2 + 1$, we observe that

$$f_i = f_{p^2-1+j} = f_j \tag{4.1}$$

for some $0 \leq j \leq r+2$. In this case, we say that f_i is a *flip*. We shall soon assume that $r \leq p-3 \leq p^2-p-3$ in which case f_i lies in $\mathcal{B}$.

On the other hand, for $p^2-p \leq i \leq p^2-2$, we have $f_i = f_{p^2-p+j}$ for some $0 \leq j \leq p-2$. Then we say that f_i is a *flop*. It is the last term in the following relation:

$$f_j + f_{j+(p-1)} + f_{j+2(p-1)} + \cdots + f_{j+(p-1)(p-1)} + f_{j+p^2-p} = 0 \tag{4.2}$$

where all but the last term lie in $\mathcal{B}$. Indeed, we have

$$X^{p^2-1} - 1 = \left(X^{p-1}-1\right)\left(X^{p(p-1)} + X^{(p-1)(p-1)} + \cdots + X^{p-1} + 1\right).$$

Then for $A \in \mathbb{F}_{p^2}^\times \setminus \mathbb{F}_p^\times$, we have $A^{p(p-1)} + A^{(p-1)(p-1)} + \cdots + A^{p-1} + 1 = 0$. So for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_p)$, we have

$$\left(\frac{a+c\alpha}{b+d\alpha}\right)^{p(p-1)} + \left(\frac{a+c\alpha}{b+d\alpha}\right)^{(p-1)(p-1)} + \cdots + \left(\frac{a+c\alpha}{b+d\alpha}\right)^{p-1} + 1 = 0,$$

which, after multiplying by $(a+c\alpha)^{(r+p+1)-j}(b+d\alpha)^{j+p^2-p}$ on both sides, gives

$$(a+c\alpha)^{(r+p^2+1)-j}(b+d\alpha)^j + (a+c\alpha)^{(r+p^2-p+2)-j}(b+d\alpha)^{j+(p-1)} + \cdots + (a+c\alpha)^{(r+p+1)-j}(b+d\alpha)^{j+p^2-p} = 0,$$

which shows that (4.2) holds for $0 \leq j \leq p-2$.

Thus any flip or flop can be changed to a linear combination of vectors in $\mathcal{B}$.

For any polynomial $P(X,Y)$ and $A, B, C, D \in \mathbb{F}_{p^2}$, we set

$$P(X,Y)\Big|_{(A,B)}^{(C,D)} := P(C,D) - P(A,B).$$

The following theorem is Theorem 1.5 from the introduction (replacing r by $r+2$).

Theorem 4.1. *Let $0 \leq r \leq p-3$. Then there is an explicit isomorphism defined over $\mathbb{F}_{p^2}$:*

$$\frac{V_{r+p+1}}{D(V_{r+2})} \otimes V_{p-1} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2^{r+2},$$

where $D := X^p \frac{\partial}{\partial X} + Y^p \frac{\partial}{\partial Y}$.

Proof. Let $P \in V_{r+p+1}$ and $Q \in V_{p-1}$. Define

$$\psi : \frac{V_{r+p+1}}{D(V_{r+2})} \otimes V_{p-1} \to \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2^{r+2}$$

by $\psi(P \otimes Q) = \psi_{P\otimes Q}$, where $\psi_{P\otimes Q} : G(\mathbb{F}_p) \to \mathbb{F}_{p^2}$ is defined by

$$\psi_{P\otimes Q}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \nabla_\alpha^r(P)\Big|_{(a+c\alpha,\, b+d\alpha)}^{((a+c\alpha)^p,\, (b+d\alpha)^p)} \cdot Q\Big|_{(0,0)}^{((a+c\alpha)^p,\, (b+d\alpha)^p)}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_p)$, and where

$$\nabla_\alpha = (a+c\alpha)\frac{\partial}{\partial X} + (b+d\alpha)\frac{\partial}{\partial Y}.$$

For convenience, we set

$$A_\alpha = a + c\alpha \quad \text{and} \quad B_\alpha = b + d\alpha.$$

$T(\mathbb{F}_p)$**-linearity:** We show that $\psi_{P\otimes Q}$ is $T(\mathbb{F}_p)$-linear. Since $P(X,Y)$ is a homogeneous polynomial of degree $r+p+1$ and $Q(S,T)$ is a homogeneous polynomial of degree $p-1$, we have $\nabla_\alpha^r(P(X,Y)) \cdot Q(S,T)$ is a linear combination of terms of the form

$$A_\alpha^{r-j} B_\alpha^j \cdot X^{p+1-k} Y^k \cdot S^{p-1-l} T^l$$

for $0 \leq k \leq p+1, 0 \leq l \leq p-1$, and $0 \leq j \leq r$. Now,

$$A_\alpha^{r-j} B_\alpha^j \cdot X^{p+1-k} Y^k \Big|_{(A_\alpha, B_\alpha)}^{(A_\alpha^p, B_\alpha^p)} \cdot S^{p-1-l} T^l \Big|_{(0,0)}^{(A_\alpha^p, B_\alpha^p)} = A_\alpha^{r-j} B_\alpha^j \left(A_\alpha^{p^2-kp-lp+1} B_\alpha^{kp+lp} - A_\alpha^{p^2-lp-k+1} B_\alpha^{k+lp}\right),$$

which shows that $\psi_{P\otimes Q}$ is a linear combination of the functions f_i defined above. Since these functions are $T(\mathbb{F}_p)$-linear, $\psi_{P\otimes Q}$ is also $T(\mathbb{F}_p)$-linear.

$G(\mathbb{F}_p)$**-linearity:** We show that ψ is $G(\mathbb{F}_p)$-linear. Let $g = \left(\begin{smallmatrix} u & v \\ w & z \end{smallmatrix}\right) \in G(\mathbb{F}_p)$. Then we have

$$g \cdot (P \otimes Q) = P(U,V) \otimes Q(U',V') = P_1 \otimes Q_1 (\text{say}),$$

where $U = uX + wY, V = vX + zY$ and $U' = uS + wT, V' = vS + zT$. Now,

$$\begin{aligned} \psi(g \cdot (P \otimes Q)) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \psi(P_1 \otimes Q_1) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \\ &= \left((a + c\alpha) \frac{\partial}{\partial X} + (b + d\alpha) \frac{\partial}{\partial Y} \right)^r (P_1) \Big|_{((a+c\alpha),(b+d\alpha))}^{((a+c\alpha)^p,(b+d\alpha)^p)} \cdot \\ &\qquad Q\left(U'\left((a+c\alpha)^p, (b+d\alpha)^p\right), V'\left((a+c\alpha)^p, (b+d\alpha)^p\right)\right), \end{aligned}$$

which, by Lemma 2.10 applied twice

$$\begin{aligned} &= \left((uA_\alpha + wB_\alpha) \frac{\partial}{\partial X} + (vA_\alpha + zB_\alpha) \frac{\partial}{\partial Y} \right)^r (P) \Big|_{(uA_\alpha + wB_\alpha, vA_\alpha + zB_\alpha)}^{((uA_\alpha+wB_\alpha)^p,(vA_\alpha+zB_\alpha)^p)} \cdot \\ &\qquad Q\left((uA_\alpha + wB_\alpha)^p, (vA_\alpha + zB_\alpha)^p\right) \\ &= \psi_{P\otimes Q}\left(\begin{pmatrix} au+bw & av+bz \\ cu+dw & cv+dz \end{pmatrix} \right) = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \cdot \psi_{P\otimes Q}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = g \cdot \psi_{P\otimes Q}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right). \end{aligned}$$

Thus we have $\psi(g \cdot (P \otimes Q)) = g \cdot \psi_{P\otimes Q}$. Hence ψ is $G(\mathbb{F}_p)$-linear.

Kernel of ψ: Next we show that $\ker\psi = D(V_{r+2}) \otimes V_{p-1}$. We first show that $D(V_{r+2}) \otimes V_{p-1} \subset \ker\psi$. Let $P = \sum_{i=0}^{r+2} a_i X^{r+2-i} Y^i \in V_{r+2}$ and $Q \in V_{p-1}$. Then $D(P) = \sum_{i=0}^{r+2} a_i \left((r+2-i)X^{r+p+1-i}Y^i + iX^{r+2-i}Y^{i+p-1}\right)$.

Claim: for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_p)$, we have $\nabla_\alpha^r(D(P))\Big|_{(a+c\alpha,b+d\alpha)}^{((a+c\alpha)^p,(b+d\alpha)^p)} = 0$.

Indeed,

$$\begin{aligned} \nabla_\alpha^r(D(P)) &= \left(A_\alpha \frac{\partial}{\partial X} + B_\alpha \frac{\partial}{\partial Y} \right)^r \left(\sum_{i=0}^{r+2} a_i \left((r+2-i)X^{r+p+1-i}Y^i + iX^{r+2-i}Y^{i+p-1}\right) \right) \\ &= \sum_{i=0}^{r+2} \sum_{k=0}^{r} \binom{r}{k} A_\alpha^{r-k} B_\alpha^k \frac{\partial^r}{\partial X^{r-k} \partial Y^k} a_i \left((r+2-i)X^{r+p+1-i}Y^i + iX^{r+2-i}Y^{i+p-1}\right) \\ &= \sum_{i=0}^{r+2} \sum_{k=0}^{r} \binom{r}{k} a_i (r+2-i) A_\alpha^{r-k} B_\alpha^k [r+p+1-i]_{r-k} [i]_k X^{p+1-(i-k)} Y^{i-k} \\ &\quad + \sum_{i=0}^{r+2} \sum_{k=0}^{r} \binom{r}{k} a_i i A_\alpha^{r-k} B_\alpha^k [r+2-i]_{r-k} [i+p-1]_k X^{k+2-i} Y^{i+p-1-k}, \end{aligned}$$

which, by observing $[i]_k$ equals 0 if $i < k$ and equals $i!$ if $i = k$,

$$\begin{aligned} &= a_0(r+2)A_\alpha^r [r+p+1]_r X^{p+1} + \sum_{i=1}^{r+1} \sum_{k=0}^{r} \binom{r}{k} a_i (r+2-i) A_\alpha^{r-k} B_\alpha^k [r+p+1-i]_{r-k} [i]_k X^{p+1-(i-k)} Y^{i-k} \\ &\quad + a_{r+2}(r+2) B_\alpha^r [r+p+1]_r Y^{p+1} + \sum_{i=1}^{r+1} \sum_{k=0}^{r} \binom{r}{k} a_i i A_\alpha^{r-k} B_\alpha^k [r+2-i]_{r-k} [i+p-1]_k X^{k+2-i} Y^{i+p-1-k}, \end{aligned}$$

which again using $[r+p+1-i]_{r-k} = 0 \mod p$ for $k < i-1$, $[i]_k = 0$ for $k > i$ and $[r+2-i]_{r-k} = 0$ for $k < i-2$ and $[i+p-1]_k = 0 \mod p$ for $k > i-1$,

$$\begin{aligned} &= a_0(r+2)A_\alpha^r [r+p+1]_r X^{p+1} + \left(\sum_{i=1}^{r+1} \binom{r}{i} a_i (r+2-i) A_\alpha^{r-i} B_\alpha^i [r+p+1-i]_{r-i} [i]_i \right) X^{p+1} \\ &\quad + \left(\sum_{i=1}^{r+1} \binom{r}{i-1} a_i (r+2-i) A_\alpha^{r-i+1} B_\alpha^{i-1} [r+p+1-i]_{r-i+1} [i]_{i-1} \right) X^p Y \\ &\quad + a_{r+2}(r+2) B_\alpha^r [r+p+1]_r Y^{p+1} + \left(\sum_{i=1}^{r+1} \binom{r}{i-2} a_i i A_\alpha^{r-i+2} B_\alpha^{i-2} [r+2-i]_{r+2-i} [i+p-1]_{i-2} \right) Y^{p+1} \\ &\quad + \left(\sum_{i=1}^{r+1} \binom{r}{i-1} a_i i A_\alpha^{r-i+1} B_\alpha^{i-1} [r+2-i]_{r-i+1} [i+p-1]_{i-1} \right) X Y^p. \end{aligned}$$

Since $[r+p+1-i]_{r-i+1} = (r-i+1)!$ modulo p and $[i+p-1]_{i-1} = (i-1)!$ modulo p, the above expression

$$\begin{aligned} = \quad & a_0(r+2)A_\alpha^r[r+p+1]_r X^{p+1} + \left(\sum_{i=1}^{r+1}\binom{r}{i}a_i(r+2-i)A_\alpha^{r-i}B_\alpha^i[r+p+1-i]_{r-i}[i]_i\right)X^{p+1} \\ & + a_{r+2}(r+2)B_\alpha^r[r+p+1]_r Y^{p+1} + \left(\sum_{i=1}^{r+1}\binom{r}{i-2}a_i i A_\alpha^{r-i+2}B_\alpha^{i-2}[r+2-i]_{r+2-i}[i+p-1]_{i-2}\right)Y^{p+1} \\ & + \left(\sum_{i=1}^{r+1}\binom{r}{i-1}a_i(r+2-i)!i!A_\alpha^{r-i+1}B_\alpha^{i-1}\right)(X^pY+XY^p). \end{aligned}$$

Note that since $A_\alpha, B_\alpha \in \mathbb{F}_{p^2}$, we have

$$X^{p+1}\Big|_{(A_\alpha,B_\alpha)}^{(A_\alpha^p,B_\alpha^p)} = A_\alpha^{p(p+1)} - A_\alpha^{p+1} = 0 \quad \text{and} \quad Y^{p+1}\Big|_{(A_\alpha,B_\alpha)}^{(A_\alpha^p,B_\alpha^p)} = B_\alpha^{p(p+1)} - B_\alpha^{p+1} = 0.$$

Also,

$$X^pY + XY^p\Big|_{(A_\alpha,B_\alpha)}^{(A_\alpha^p,B_\alpha^p)} = A_\alpha B_\alpha^p + A_\alpha^p B_\alpha - A_\alpha^p B_\alpha - A_\alpha B_\alpha^p = 0.$$

Thus we have,

$$\nabla_\alpha^r(D(P))\Big|_{(a+c\alpha,b+d\alpha)}^{((a+c\alpha)^p,(b+d\alpha)^p)} = \nabla_\alpha^r(D(P))\Big|_{(A_\alpha,B_\alpha)}^{(A_\alpha^p,B_\alpha^p)} = 0,$$

proving the claim, and so $P \otimes Q \in \ker\psi$. Thus $D(V_{r+2}) \otimes V_{p-1} \subset \ker\psi$.

Next, we show that $\ker\psi \subset D(V_{r+2}) \otimes V_{p-1}$. We prove this inclusion by changing r to $r-2$, i.e., we show

$$\ker\psi \subset D(V_r) \otimes V_{p-1},$$

for $2 \le r \le p-1$, where

$$\psi : V_{r+p-1} \otimes V_{p-1} \to \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)}\omega_2^r$$

such that $\psi(P\otimes Q) = \psi_{P\otimes Q}$,where $\psi_{P\otimes Q} : G(\mathbb{F}_p) \to \mathbb{F}_{p^2}$ is defined by

$$\psi_{P\otimes Q}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \nabla_\alpha^{r-2}(P)\Big|_{(a+c\alpha,b+d\alpha)}^{((a+c\alpha)^p,(b+d\alpha)^p)} \cdot Q\left((a+c\alpha)^p, ((b+d\alpha)^p\right)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_p)$.

Let

$$P \otimes Q = \sum_{i=0}^{r+p-1}\sum_{j=0}^{p-1} a_{i,j}X^{r+p-1-i}Y^i \otimes S^{p-1-j}T^j \in \ker\psi, \tag{4.3}$$

where V_{p-1} is modeled on polynomials in S,T. Then we have

$$\sum_{j=0}^{p-1}\left(\sum_{i=0}^{r+p-1} a_{i,j}\left(\left(A_\alpha\frac{\partial}{\partial X} + B_\alpha\frac{\partial}{\partial Y}\right)^{r-2}\left(X^{r+p-1-i}Y^i\right)\Big|_{(A_\alpha,B_\alpha)}^{(A_\alpha^p,B_\alpha^p)}\right)\right)A_\alpha^{p^2-p-jp}B_\alpha^{jp} = 0. \tag{4.4}$$

Now,

$$\begin{aligned} & \sum_{i=0}^{r+p-1} a_{i,j}\left(\left(A_\alpha\frac{\partial}{\partial X} + B_\alpha\frac{\partial}{\partial Y}\right)^{r-2}\left(X^{r+p-1-i}Y^i\right)\Big|_{(A_\alpha,B_\alpha)}^{(A_\alpha^p,B_\alpha^p)}\right) \\ & = \sum_{i=0}^{r+p-1} a_{i,j}\left(\sum_{k=0}^{r-2}\binom{r-2}{k}A_\alpha^{r-2-k}B_\alpha^k\frac{\partial^{r-2}}{\partial X^{r-2-k}\partial Y^k}\left(X^{r+p-1-i}Y^i\right)\Big|_{(A_\alpha,B_\alpha)}^{(A_\alpha^p,B_\alpha^p)}\right) \\ & = \sum_{i=0}^{r+p-1} a_{i,j}\left(\sum_{k=0}^{r-2}\binom{r-2}{k}A_\alpha^{r-2-k}B_\alpha^k[r+p-1-i]_{r-2-k}[i]_k\left(X^{p+1-i+k}Y^{i-k}\right)\Big|_{(A_\alpha,B_\alpha)}^{(A_\alpha^p,B_\alpha^p)}\right) \\ & = \sum_{i=0}^{r+p-1} a_{i,j}\left(\sum_{k=0}^{r-2}\binom{r-2}{k}[r+p-1-i]_{r-2-k}[i]_k\left(A_\alpha^{r-ip+kp+p-k-1}B_\alpha^{ip-kp+k} - A_\alpha^{r+p-1-i}B_\alpha^i\right)\right) \\ & = \sum_{i=1}^{r+p-2} a_{i,j}\left(\sum_{k=0}^{r-2}\binom{r-2}{k}[r+p-1-i]_{r-2-k}[i]_k\left(A_\alpha^{r-ip+kp+p-k-1}B_\alpha^{ip-kp+k} - A_\alpha^{r+p-1-i}B_\alpha^i\right)\right). \end{aligned}$$

In the last equality, we dropped the terms for $i = 0, r+p-1$ as these are zero. Indeed, for $i = 0$, since $[i]_k = 0$ for $k \neq 0$, only the $k = 0$ term survives in the sum over k for which

$$A_\alpha^{r-ip+kp+p-k-1} B_\alpha^{ip-kp+k} - A_\alpha^{r+p-1-i} B_\alpha^{i} = A_\alpha^{r+p-1} - A_\alpha^{r+p-1} = 0.$$

Similarly, one can check that the terms for $i = r+p-1$ are also zero. The last expression

$$= \sum_{i=1}^{r+p-2} a_{i,j} \left(\left(\sum_{k=0}^{r-2} \binom{r-2}{k} [r+p-1-i]_{r-2-k} [i]_k A_\alpha^{r-ip+kp+p-k-1} B_\alpha^{ip-kp+k} \right) - (r-1)! A_\alpha^{r+p-1-i} B_\alpha^{i} \right), \quad (4.5)$$

since $A_\alpha^{r+p-1-i} B_\alpha^{i}$ is independent of k and, by (2.1),

$$\sum_{k=0}^{r-2} \binom{r-2}{k} [r+p-1-i]_{r-2-k} [i]_k = [r+p-1]_{r-2} = (r-1)! \mod p.$$

For $1 \leq i \leq r-1$, we now compute the coefficients of $a_{i,j}$ and $a_{i+p-1,j}$ in (4.5).

Coefficient of $a_{i,j}$: Note that for $1 \leq i \leq r-1$ and $0 \leq k \leq r-2$, we have $[r+p-1-i]_{r-2-k} = 0$ for $k < i-1$, and $[i]_k = 0$ for $i < k$. Thus the coefficient of $a_{i,j}$ in (4.5)

$$\begin{aligned}
&= \binom{r-2}{i-1} [r+p-1-i]_{r-i-1} [i]_{i-1} A_\alpha^{r-i} B_\alpha^{i+p-1} + \binom{r-2}{i} [r+p-1-i]_{r-i-2} [i]_i A_\alpha^{r+p-1-i} B_\alpha^{i} - (r-1)! A_\alpha^{r+p-1-i} B_\alpha^{i} \\
&= \binom{r-2}{i-1} (r-i-1)! i! A_\alpha^{r-i} B_\alpha^{i+p-1} + \left(\binom{r-2}{i} (r-i-1)! i! - (r-1)! \right) A_\alpha^{r+p-1-i} B_\alpha^{i} \\
&= i \left((r-2)! A_\alpha^{r-i} B_\alpha^{i+p-1} - (r-2)! A_\alpha^{r-i+p-1} B_\alpha^{i} \right).
\end{aligned}$$

Coefficient of $a_{i+p-1,j}$: Similarly, in (4.5), the coefficient of $a_{i+p-1,j}$ is

$$\begin{aligned}
&\left(\sum_{k=0}^{r-2} \binom{r-2}{k} [r-i]_{r-2-k} [i+p-1]_k A_\alpha^{r-ip-p^2+p+kp+p-k-1} B_\alpha^{ip+p^2-p-kp+k} \right) - (r-1)! A_\alpha^{r-i} B_\alpha^{i+p-1} \\
&= \binom{r-2}{i-2} [r-i]_{r-i} [i+p-1]_{i-2} A_\alpha^{r-i} B_\alpha^{i+p-1} + \binom{r-2}{i-1} [r-i]_{r-i-1} [i+p-1]_{i-1} A_\alpha^{r-i+p-1} B_\alpha^{i} - (r-1)! A_\alpha^{r-i} B_\alpha^{i+p-1} \\
&= \left(\binom{r-2}{i-2} (r-i)! (i-1)! - (r-1)! \right) A_\alpha^{r-i} B_\alpha^{i+p-1} + \binom{r-2}{i-1} (r-i)! (i-1)! A_\alpha^{r-i+p-1} B_\alpha^{i} \\
&= -(r-i) \left((r-2)! A_\alpha^{r-i} B_\alpha^{i+p-1} - (r-2)! A_\alpha^{r-i+p-1} B_\alpha^{i} \right).
\end{aligned}$$

The first equality holds because $[r-i]_{r-2-k} = 0$ for $k < i-2$ and $[i+p-1]_k = 0$ for $k > i-1$. The second equality follows because $[i+p-1]_{i-2} = (i-1)! = [i+p-1]_{i-1}$.

For $1 \leq i \leq r-1$, substituting the coefficients of $a_{i,j}$ and $a_{i+p-1,j}$ in (4.5), we have

$$\begin{aligned}
&\sum_{i=0}^{r+p-1} a_{i,j} \left(\left(A_\alpha \frac{\partial}{\partial X} + B_\alpha \frac{\partial}{\partial Y} \right)^{r-2} \left(X^{r+p-1-i} Y^i \right) \Big|_{(A_\alpha, B_\alpha)}^{(A_\alpha^p, B_\alpha^p)} \right) \\
&= \sum_{i=1}^{r-1} \left(i a_{i,j} - (r-i) a_{i+p-1,j} \right) (r-2)! \left(A_\alpha^{r-i} B_\alpha^{i+p-1} - A_\alpha^{r-i+p-1} B_\alpha^{i} \right) \\
&\quad + \sum_{i=r}^{p-1} a_{i,j} \left(\left(\sum_{k=0}^{r-2} \binom{r-2}{k} [r+p-1-i]_{r-2-k} [i]_k A_\alpha^{r-ip+kp+p-k-1} B_\alpha^{ip-kp+k} \right) - (r-1)! A_\alpha^{r+p-1-i} B_\alpha^{i} \right) \\
&= \sum_{i=1}^{r-1} \left(i a_{i,j} - (r-i) a_{i+p-1,j} \right) (r-2)! \left(A_\alpha^{r-i} B_\alpha^{i+p-1} - A_\alpha^{r-i+p-1} B_\alpha^{i} \right) \\
&\quad + \sum_{i=r}^{p-1} a_{i,j} \left(\left(\sum_{k=0}^{r-2} (r-2)! \binom{r+p-1-i}{r-2-k} \binom{i}{k} A_\alpha^{r-ip+kp+p-k-1} B_\alpha^{ip-kp+k} \right) - (r-1)! A_\alpha^{r+p-1-i} B_\alpha^{i} \right).
\end{aligned}$$

Thus (4.4) shows that the following linear combination of functions in the induced space vanishes:

$$\begin{aligned}&\sum_{j=0}^{p-1}\left(\sum_{i=1}^{r-1}\left(ia_{i,j}-(r-i)a_{i+p-1,j}\right)\left(A_\alpha^{r-i}B_\alpha^{i+p-1}-A_\alpha^{r-i+p-1}B_\alpha^{i}\right)\right)A_\alpha^{p^2-jp-p}B_\alpha^{jp}+\\&\sum_{j=0}^{p-1}\left(\sum_{i=r}^{p-1}a_{i,j}\left(\left(\sum_{k=0}^{r-2}\binom{r+p-1-i}{r-2-k}\binom{i}{k}A_\alpha^{r-ip+kp+p-k-1}B_\alpha^{ip-kp+k}\right)-(r-1)A_\alpha^{r+p-1-i}B_\alpha^{i}\right)\right)\cdot A_\alpha^{p^2-jp-p}B_\alpha^{jp}\;=\;0,\end{aligned}\tag{4.6}$$

after dividing by $(r-2)!$.

For simplicity, we denote

$$X_{i,j}:=ia_{i,j}-(r-i)a_{i+p-1,j},$$

for $1\le i\le r-1$ and $0\le j\le p-1$, and, we denote

$$X_{i,j}:=a_{i,j}\;\text{ and }\;Z_{i,k}:=\binom{r+p-1-i}{r-2-k}\binom{i}{k},$$

for $r\le i\le p-1$, $0\le j\le p-1$, $0\le k\le r-2$. The $X_{i,j}$ are variables (and the $Z_{i,k}$ constants).

Claim: For $1\le i\le p-1$ and $0\le j\le p-1$, we have $X_{i,j}=0$.
We prove the claim. Write (4.6) as

$$\begin{aligned}&\sum_{j=0}^{p-1}\left(\sum_{i=1}^{r-1}X_{i,j}\left(A_\alpha^{r+p^2-i-jp-p}B_\alpha^{i+jp+p-1}-A_\alpha^{r+p^2-i-jp-1}B_\alpha^{i+jp}\right)\right)+\\&\sum_{j=0}^{p-1}\left(\sum_{i=r}^{p-1}X_{i,j}\left(\left(\sum_{k=0}^{r-2}Z_{i,k}A_\alpha^{r+p^2-ip-jp+kp-k-1}B_\alpha^{ip+jp-kp+k}\right)-(r-1)A_\alpha^{r+p^2-1-i-jp}B_\alpha^{i+jp}\right)\right)=0.\end{aligned}$$

Collecting terms in the same congruence class n of $i+j$ modulo $(p-1)$, we have

$$\begin{aligned}&\sum_{n=1}^{p-1}\sum_{\substack{1\le i\le r-1\\0\le j\le p-1\\i+j\equiv n\mod (p-1)}}X_{i,j}\left(A_\alpha^{r+p^2-i-jp-p}B_\alpha^{i+jp+p-1}-A_\alpha^{r+p^2-i-jp-1}B_\alpha^{i+jp}\right)+\\&\sum_{n=1}^{p-1}\sum_{\substack{r\le i\le p-1\\0\le j\le p-1\\i+j\equiv n\mod (p-1)}}X_{i,j}\left(\left(\sum_{k=0}^{r-2}Z_{i,k}A_\alpha^{r+p^2-ip-jp+kp-k-1}B_\alpha^{ip+jp-kp+k}\right)-(r-1)A_\alpha^{r+p^2-1-i-jp}B_\alpha^{i+jp}\right)\;=\;0.\end{aligned}$$

Write the n-th summand above as $\mathcal{B}_n$ for $1\le n\le p-1$. By inspection, each of the functions in $\mathcal{B}_n$ is of the form $A_\alpha^{r+p^2-1-l}B_\alpha^{l}$ for $l\equiv n$ modulo $(p-1)$. Using (4.1) and (4.2), we may assume each of these functions belong to the basis $\mathcal{B}$, noting that these operations preserve the congruence class n. By the linear independence of the basis $\mathcal{B}$ and the vanishing of the sum of the $\mathcal{B}_n$, we conclude that each $\mathcal{B}_n=0$. Thus fixing $1\le n\le p-1$ we have:

$$\begin{aligned}&\sum_{\substack{1\le i\le r-1\\0\le j\le p-1\\i+j\equiv n\mod (p-1)}}X_{i,j}\left(A_\alpha^{r+p^2-i-jp-p}B_\alpha^{i+jp+p-1}-A_\alpha^{r+p^2-i-jp-1}B_\alpha^{i+jp}\right)+\\&\sum_{\substack{r\le i\le p-1\\0\le j\le p-1\\i+j\equiv n\mod (p-1)}}X_{i,j}\left(\left(\sum_{k=0}^{r-2}Z_{i,k}A_\alpha^{r+p^2-ip-jp+kp-k-1}B_\alpha^{ip+jp-kp+k}\right)-(r-1)A_\alpha^{r+p^2-1-i-jp}B_\alpha^{i+jp}\right)=0.\end{aligned}\tag{4.7}$$

The possible pairs of (i,j) such that $i+j\equiv n\mod (p-1)$ for $1\le i\le p-1$ and $0\le j\le p-1$ are:

$$(i,j)=\begin{cases}(i,n-i), & \text{if } 1\le i\le n,\\(i,p-1+n-i), & \text{if } n\le i\le p-1.\end{cases}\tag{4.8}$$

We analyze three cases:

Case 1: Suppose $1 \leq n \leq r-1$. Using (4.8), equation (4.7) becomes

$$\begin{aligned}
&\sum_{i=1}^{n} X_{i,n-i} \left(A_\alpha^{r+p^2-i-(n-i)p-p} B_\alpha^{i+(n-i)p+p-1} - A_\alpha^{r+p^2-i-(n-i)p-1} B_\alpha^{i+(n-i)p} \right) \\
&\quad + \sum_{i=n}^{r-1} X_{i,p-1+n-i} \left(A_\alpha^{r-i-np+ip} B_\alpha^{i+p^2+np-ip-1} - A_\alpha^{r-i-np+ip+p-1} B_\alpha^{i+p^2-p+np-ip} \right) \\
&+ \sum_{i=r}^{p-1} X_{i,p-1+n-i} \left(\sum_{k=0}^{r-2} Z_{i,k} A_\alpha^{r+(p-1)k-np+p-1} B_\alpha^{p^2-p+np-(p-1)k} \right) \\
&\quad - \sum_{i=r}^{p-1} (r-1) X_{i,p-1+n-i} A_\alpha^{r-i-np+ip+p-1} B_\alpha^{i+p^2-p+np-ip} = 0.
\end{aligned} \tag{4.9}$$

Looking at the indices occurring in $X_{i,j}$ in the above equation, we note for each $i \neq n$, there is a unique $j \neq 0$. But for $i = n$, we get both $j = 0$ and $j = p-1$. For convenience, we drop the index j for $j \neq 0$: we write

$$X_{i,j} \text{ as } \begin{cases} X_i, & \text{if } j \neq 0, \\ X_0, & \text{if } j = 0. \end{cases} \tag{4.10}$$

Now, there is no flip or flop in the first summand of the above equation. If $i = n$, then there is one flip in the first component of the second summand and one flop in its second component. Also, for $i = n+1$, there is one flop in the first component of the second summand if $n \leq r-2$ (and the $i = n+1$ term is not there if $n = r-1$). Finally in the third summand, there are flips for $0 \leq k \leq n-1$ and there is one flop for $k = n$. Apart from these there are no flips and flops appearing in the above equation.

Assume that $1 \leq n \leq r-2$. Changing the flips and flops appearing in (4.9) to functions in $\mathcal{B}$ and looking at the coefficient of $A_\alpha^{r+p^2-1-(n+l(p-1))} B_\alpha^{n+l(p-1)}$ for $0 \leq l \leq p-1$, we get the following system of equations:

$$\begin{aligned}
&2X_n - X_{n+1} + \sum_{i=r}^{p-1} (Z_{i,n-1} - Z_{i,n}) X_i - X_0 = 0, && \text{if } l = 0, \\
&X_n - X_{n+1} + \sum_{i=r}^{p-1} (Z_{i,n-2} - Z_{i,n}) X_i - X_{n-1} + X_0 = 0, && \text{if } l = 1, \\
&X_n - X_{n+1} + \sum_{i=r}^{p-1} (Z_{i,n-l-1} - Z_{i,n}) X_i + X_{n-l+1} - X_{n-l} = 0, && \text{if } 2 \leq l \leq n-1, \\
&X_n - X_{n+1} - \sum_{i=r}^{p-1} Z_{i,n} X_i + X_{n-l+1} = 0, && \text{if } l = n, \\
&X_n - X_{n+1} - \sum_{i=r}^{p-1} Z_{i,n} X_i - (r-1) X_{n+p-l} = 0, && \text{if } n+1 \leq l \leq p-r+n, \\
&X_n - X_{n+1} - \sum_{i=r}^{p-1} Z_{i,n} X_i - X_{n+p-l} = 0, && \text{if } l = p-r+n+1, \\
&X_n - X_{n+1} + \sum_{i=r}^{p-1} \left(Z_{i,p+n-l} - Z_{i,n} \right) X_i + X_{n+p-l+1} - X_{n+p-l} = 0, && \text{if } p-r+n+2 \leq l \leq p-1.
\end{aligned} \tag{4.11}$$

Case 2: Suppose $r \leq n \leq p-2$. In this case, by (4.7) we have

$$\sum_{i=1}^{r-1} X_{i,n-i} \left(A_\alpha^{r+p^2-i-(n-i)p-p} B_\alpha^{i+(n-i)p+p-1} - A_\alpha^{r+p^2-i-(n-i)p-1} B_\alpha^{i+(n-i)p} \right)$$
$$+ \sum_{i=r}^{n} X_{i,n-i} \left(\sum_{k=0}^{r-2} Z_{i,k} A_\alpha^{r+(p-1)k-np+p^2-1} B_\alpha^{np-(p-1)k} \right)$$
$$- \sum_{i=r}^{n} (r-1) X_{i,n-i} A_\alpha^{r+p^2-1-i-np+ip} B_\alpha^{i+np-ip}$$
$$+ \sum_{i=n}^{p-1} X_{i,p-1+n-i} \left(\sum_{k=0}^{r-2} Z_{i,k} A_\alpha^{r+(p-1)k-np+p-1} B_\alpha^{p^2-p+np-(p-1)k} \right)$$
$$- \sum_{i=n}^{p-1} (r-1) X_{i,p-1+n-i} A_\alpha^{r-i-np+ip+p-1} B_\alpha^{i+p^2-p+np-ip} = 0.$$

In the second but last sum above there are flips for all $0 \leq k \leq r-2$. Also, there is a flop in the last sum at $i = n$. As before, by changing the flips and flops appearing in the above equation to functions in $\mathcal{B}$ and then looking at the coefficient of $A_\alpha^{r+p^2-1-(n+l(p-1))} B_\alpha^{n+l(p-1)}$ for $0 \leq l \leq p-1$ and using the convention (4.10) for the variables, we get the following system of equations:

$$\begin{aligned}
&(r-1)X_n - (r-1)X_0 = 0, && \text{if } l = 0,\\
&(r-1)X_n - (r-1)X_{n-l} = 0, && \text{if } 1 \leq l \leq n-r,\\
&\left((r-1) + Z_{n,n-l-1}\right) X_n - X_{n-l} + \sum_{i=n+1}^{p-1} Z_{i,n-l-1} X_i = 0, && \text{if } l = n-r+1,\\
&\left((r-1) + Z_{n,n-l-1}\right) X_n + \sum_{i=r}^{n-1} Z_{i,n-l} X_i \\
&\quad - X_{n-l} + X_{n-l+1} + \sum_{i=n+1}^{p-1} Z_{i,n-l-1} X_i + Z_{n,n-l} X_0 = 0, && \text{if } n-r+2 \leq l \leq n-1,\\
&(r-1)X_n + X_{n-l+1} + \sum_{i=r}^{n-1} Z_{i,n-l} X_i + Z_{n,n-l} X_0 = 0, && \text{if } l = n,\\
&(r-1)X_n - (r-1)X_{n-l+p} = 0, && \text{if } n+1 \leq l \leq p-1.
\end{aligned} \tag{4.12}$$

Case 3: Suppose $n = p-1$. From (4.7) we have,

$$\sum_{i=1}^{r-1} X_{i,p-1-i} \left(A_\alpha^{r-i+ip} B_\alpha^{i-ip+p^2-1} - A_\alpha^{r+p-1-i+ip} B_\alpha^{i+p^2-p-ip} \right)$$
$$+ \sum_{i=r}^{p-1} X_{i,p-1-i} \left(\sum_{k=0}^{r-2} Z_{i,k} A_\alpha^{r+(p-1)k+p-1} B_\alpha^{p^2-p-(p-1)k} \right)$$
$$- \sum_{i=r}^{p-1} (r-1) X_{i,p-1-i} A_\alpha^{r-i+ip+p-1} B_\alpha^{i+p^2-p-ip}$$
$$+ X_{p-1,p-1} \left(\sum_{k=0}^{r-2} Z_{p-1,k} A_\alpha^{r+(p-1)k-(p-1)^2} B_\alpha^{2p^2-2p-(p-1)k} \right)$$
$$- (r-1) X_{p-1,p-1} A_\alpha^{r} B_\alpha^{p^2-1} = 0. \tag{4.13}$$

There are flops in the first component of the first sum for $i = 1$ and in the second sum for $k = 0$. Also, there are flips in the second last line above for all $0 \leq k \leq r-2$ and in the last line. Changing the flips and flops appearing in the above equation to functions in $\mathcal{B}$, looking at the coefficient of $A_\alpha^{r+p^2-1-l(p-1)} B_\alpha^{l(p-1)}$ for

$0 \leq l \leq p-1$ and using the convention (4.10) for the variables, we have the following system of equations:

$$\begin{aligned}
&-X_1 - \sum_{i=r}^{p-2} Z_{i,0}X_i - (r-1)X_{p-1} - Z_{p-1,0}X_0 = 0, && \text{if } l = 0,\\
&-X_1 - \sum_{i=r}^{p-2} Z_{i,0}X_i - (r-1)X_0 - Z_{p-1,0}X_0 = 0, && \text{if } l = 1,\\
&-X_1 - \sum_{i=r}^{p-2} Z_{i,0}X_i - (r-1)X_{p-l} - Z_{p-1,0}X_0 = 0, && \text{if } 2 \leq l \leq p-r,\\
&-X_1 - \sum_{i=r}^{p-2} Z_{i,0}X_i - X_{p-l} + Z_{p-1,r-2}X_{p-1} - Z_{p-1,0}X_0 = 0, && \text{if } l = p-r+1,\\
&-X_1 - \sum_{i=r}^{p-2} \left(Z_{i,0} - Z_{i,p-l}\right) X_i + X_{p-l+1} - X_{p-l}\\
&\quad + Z_{p-1,p-l-1}X_{p-1} + Z_{p-1,p-l}X_0 - Z_{p-1,0}X_0 = 0, && \text{if } p-r+2 \leq l \leq p-1.
\end{aligned} \tag{4.14}$$

Let M be the coefficient matrix of the above systems of equations in Cases 1, 2 and 3, respectively. A computation shows that $\det(M) \neq 0$. In fact, one can give a formula for $\det(M)$ in each case but we do not need it, so to keep this paper a reasonable size, we omit it (details about this and other omitted arguments in this current abridged version of the paper may be found in an earlier version of the paper on the arXiv at `https://arxiv.org/pdf/2308.10246.pdf`). It follows that in each case $X_i = 0$ for $0 \leq i \leq p-1$. Thus we have $X_{i,j} = 0$ for all $1 \leq i \leq p-1$ and $0 \leq j \leq p-1$, proving the Claim.

Thus, for all $0 \leq j \leq p-1$, we have

$$ia_{i,j} - (r-i)a_{i+p-1,j} = 0$$

for $1 \leq i \leq r-1$ and

$$a_{i,j} = 0$$

for $r \leq i \leq p-1$, where $a_{i,j}$ are the coefficients of $P \otimes Q$ in (4.3).

Using these relations, we have

$$\begin{aligned}
P \otimes Q &= \sum_{i=0}^{r+p-1} \sum_{j=0}^{p-1} a_{i,j} X^{r+p-1-i} Y^i \otimes S^{p-1-j} T^j\\
&= \sum_{j=0}^{p-1} a_{0,j} X^{r+p-1} \otimes S^{p-1-j} T^j + \sum_{j=0}^{p-1} a_{r+p-1,j} Y^{r+p-1} \otimes S^{p-1-j} T^j\\
&\quad + \sum_{i=1}^{r-1} \sum_{j=0}^{p-1} a_{i,j} X^{r+p-1-i} Y^i \otimes S^{p-1-j} T^j + \sum_{i=1}^{r-1} \sum_{j=0}^{p-1} a_{i+p-1,j} X^{r-i} Y^{i+p-1} \otimes S^{p-1-j} T^j\\
&= \sum_{j=0}^{p-1} a_{0,j} X^{r+p-1} \otimes S^{p-1-j} T^j + \sum_{j=0}^{p-1} a_{r+p-1,j} Y^{r+p-1} \otimes S^{p-1-j} T^j\\
&\quad + \sum_{j=0}^{p-1} \sum_{i=1}^{r-1} a_{i,j} \frac{1}{(r-i)} \left((r-i) X^{r+p-1-i} Y^i + i X^{r-i} Y^{i+p-1}\right) \otimes S^{p-1-j} T^j.
\end{aligned}$$

Each term in the last equality belongs to $D(V_r) \otimes V_{p-1}$, and hence, so does $P \otimes Q$. Thus $\ker \psi \subset D(V_r) \otimes V_{p-1}$ and so equality holds. Changing r back to $r+2$, we have $\ker \psi = D(V_{r+2}) \otimes V_{p-1}$. Since both sides of ψ have the same dimension $p^2 - p$, we conclude that ψ is an isomorphism. This finally proves Theorem 4.1. □

Remark 2. We have proved Theorem 1.5 for $2 \leq r \leq p-1$. One might wonder what happens for boundary values of r. Theorem 1.5 is also true if $r = 1$:

$$\frac{V_p}{D(V_1)} \otimes V_{p-1} \quad \simeq \quad \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2. \tag{4.15}$$

To prove this, define instead $\psi : V_p \otimes V_{p-1} \to \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2$ by

$$P \otimes Q \mapsto \left(\psi_{P\otimes Q} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto D(P)\Big|_{(0,0)}^{(A_\alpha^p, B_\alpha^p)} \cdot Q\Big|_{(0,0)}^{(A_\alpha^p, B_\alpha^p)} \right).$$

One checks that ψ is a well defined $G(\mathbb{F}_p)$-linear map and $\ker \psi = (\ker D \cap V_p) \otimes V_{p-1}$, which by [Red10, Proposition 3.3] (which is due to Fakhruddin), equals

$$(\mathbb{F}_p[X^p, Y^p, \theta] \cap V_p) \otimes V_{p-1} = (\mathbb{F}_p\text{-span of } X^p, Y^p) \otimes V_{p-1} = D(V_1) \otimes V_{p-1}.$$

By comparing dimensions on both sides, the isomorphism (4.15) follows. In fact, (4.15) is true 'without tensoring with V_{p-1}'. That is, Reduzzi's result (1.8) even holds for $r = 1$. Indeed, one has

$$\frac{V_p}{D(V_1)} \simeq V_{p-2} \otimes \det \simeq \overline{\Theta(\omega_2)},$$

by Diamond [Dia07, Proposition 1.3] (this does not use crystalline cohomology, see also the material around Prasad [Pra10, Lemma 4.2] for a survey: in fact, the reduction mod p of the complex cuspidal representation $\Theta(\omega_2^r)$ for $1 \le r \le p-1$ of $G(\mathbb{F}_p)$ is irreducible if and only if $r = 1$).

On the other hand, if $r = p$, then Theorem 1.5 is false for dimension reasons. Since $\mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2^p \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2$, the right hand side of the isomorphism in the theorem reduces to the case $r = 1$. As for the left side, one easily checks

$$\frac{V_{2p-1}}{D(V_p)} \simeq \mathrm{ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} d$$

is a principal series representation. Similarly, if $r = p+1$, then ω_2^{p+1} is self-conjugate and the induction on the right side is not as interesting, whereas on the left side one checks

$$\frac{V_{2p}}{D(V_{p+1})} \simeq V_{p-1} \otimes \det$$

is a twist of the mod p Steinberg representation.

In view of the above remark, Theorem 4.1 holds for $-1 \le r \le p-3$ but not for $r = p-2$, $p-1$. However, by twisting, the theorem may be extended to the following higher symmetric powers:

Corollary 4.2. *If $-1 \le r \le p-3-k$ for $0 \le k \le p-2$, then*

$$\frac{V_{r+(k+1)(p+1)}}{D(V_{r+2+k(p+1)})} \otimes V_{p-1} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2^{r+2+k(p+1)}.$$

Proof. We first show that

$$\frac{V_{r+p+1} \otimes \det^k}{D(V_{r+2}) \otimes \det^k} \simeq \frac{V^{(k)}_{r+(k+1)(p+1)}}{D(V^{(k)}_{r+2+k(p+1)})} \simeq \frac{V_{r+(k+1)(p+1)}}{D(V_{r+2+k(p+1)})}$$

for $0 \le k \le p-2$. Define $\pi : V_{r+p+1} \otimes \det^k \to \frac{V^{(k)}_{r+(k+1)(p+1)}}{D(V^{(k)}_{r+2+k(p+1)})}$ by sending P to $\theta^k P$ for $P \in V_{r+p+1}$. Let $Q \in V_{r+2}$. Then $\pi(D(Q)) = \theta^k D(Q) = D(\theta^k Q)$, where the last equality follows because $D(\theta) = -X^pY^p + X^pY^p = 0$. Thus we have $\pi(D(V_{r+2}) \otimes \det^k) \subset D(V^{(k)}_{r+2+k(p+1)})$, and hence,

$$\overline{\pi} : \frac{V_{r+p+1} \otimes \det^k}{D(V_{r+2}) \otimes \det^k} \to \frac{V^{(k)}_{r+(k+1)(p+1)}}{D(V^{(k)}_{r+2+k(p+1)})}$$

is a surjection. Also, both sides have dimension $p-1$, so $\overline{\pi}$ is an isomorphism.

For the second isomorphism, consider the composition

$$V^{(k)}_{r+(k+1)(p+1)} \hookrightarrow V_{r+(k+1)(p+1)} \twoheadrightarrow \frac{V_{r+(k+1)(p+1)}}{D(V_{r+2+k(p+1)})}. \tag{4.16}$$

In the above, the first map is the natural inclusion map and the second one is the natural surjection map. Note that, the kernel of the map (4.16) is $V^{(k)}_{r+(k+1)(p+1)} \cap D(V_{r+2+k(p+1)})$. We show that

$$V^{(k)}_{r+(k+1)(p+1)} \cap D(V_{r+2+k(p+1)}) = D(V^{(k)}_{r+2+k(p+1)}).$$

Clearly, $D(V^{(k)}_{r+2+k(p+1)}) \subset V^{(k)}_{r+(k+1)(p+1)} \cap D(V_{r+2+k(p+1)})$. The other containment is trivially true for $k = 0$. To establish it for $1 \leq k \leq p-2$, we need the following lemma which is easily proved by checking the two conditions in [GV22, Lemma 2.7].

Lemma 4.3. *If $0 \leq m \leq p-2$ and $p \nmid \binom{r}{m+1}$, then $\theta^{m+1} \mid D(Q) \iff \theta^{m+1} \mid Q$.*

Remark 3. By the lemma, D induces an inclusion

$$\frac{V_r}{V_r^{(m+1)}} \hookrightarrow \frac{V_{r+p-1}}{V_{r+p-1}^{(m+1)}},$$

which is an isomorphism for dimension reasons. This provides another proof of (1.6) using the D map, under the slightly stronger assumptions $0 \leq m \leq p-2$ and $p \nmid \binom{r}{m+1}$. The last condition is necessary (for instance, for $m = 0$, if $p \mid r$, then D maps X^r, Y^r to 0, so the map above is not injective).

Now, let $\theta^k P = D(Q)$, for some $P \in V_{r+p+1}$, $Q \in V_{r+2+k(p+1)}$ and $1 \leq k \leq p-2$. By Lemma 4.3, we have $\theta^k \mid D(Q)$ if and only if $\theta^k \mid Q$. Thus $\theta^k P = D(Q) \in D(V^{(k)}_{r+2+k(p+1)})$ and hence $V^{(k)}_{r+(k+1)(p+1)} \cap D(V_{r+2+k(p+1)}) \subset D(V^{(k)}_{r+2+k(p+1)})$. So the kernel of (4.16) is $D(V^{(k)}_{r+2+k(p+1)})$. Thus, there is an injection

$$\frac{V^{(k)}_{r+(k+1)(p+1)}}{D(V^{(k)}_{r+2+k(p+1)})} \hookrightarrow \frac{V_{r+(k+1)(p+1)}}{D(V_{r+2+k(p+1)})}.$$

If $-1 \leq r \leq p-3-k$ and $0 \leq k \leq p-2$, then D is injective on $V_{r+2+k(p+1)}$. Otherwise, by [Red10, Proposition 3.3] and comparing degrees, we would have a relation of the form $ap + bp + c(p+1) = r + 2 + k(p+1)$ for some $a, b, c \geq 0$. Comparing p-adic digits on both sides and noting they are in the range $[0, p-1]$, we have $c = r + 2 + k$ and $a + b + c = k$. The first equality implies $c \geq k+1$, whereas the second implies $c \leq k$, a contradiction. Thus the dimension of each side of the inclusion above is $p-1$. So it is an isomorphism.

Now, by Theorem 4.1 and Remark 2, for $-1 \leq r \leq p-3$, we have

$$\frac{V_{r+p+1}}{D(V_{r+2})} \otimes V_{p-1} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2^{r+2}.$$

Twisting both sides by $\det^k$ with $0 \leq k \leq p-2$

$$\implies \frac{V_{r+p+1} \otimes \det^k}{D(V_{r+2}) \otimes \det^k} \otimes V_{p-1} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \left(\omega_2^{r+2} \otimes \det^k|_{T(\mathbb{F}_p)}\right)$$

$$\implies \frac{V^{(k)}_{r+(k+1)(p+1)}}{D(V^{(k)}_{r+2+(p+1)})} \otimes V_{p-1} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \left(\omega_2^{r+2} \otimes \omega_2^{k(p+1)}\right)$$

$$\implies \frac{V_{r+(k+1)(p+1)}}{D(V_{r+2+k(p+1)})} \otimes V_{p-1} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \omega_2^{r+2+k(p+1)}. \qquad \square$$

Corollary 4.4. *Let $m \geq 0$ and $2m-1 \leq r \leq p-3$. Then, we have*

$$\frac{V_{r+2+(m+1)(p-1)}}{D^{m+1}(V_{r+2})} \otimes V_{p-1} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \left(V_m \otimes \omega_2^{r+2-m}\right).$$

Proof. Note that $V_m|_{T(\mathbb{F}_p)} \simeq \bigoplus_{j=0}^m \omega_2^{m+j(p-1)}$. Thus to prove the corollary, it suffices to prove that for $2m-1 \leq r \leq p-3$, we have

$$\frac{V_{r+2+(m+1)(p-1)}}{D^{m+1}(V_{r+2})} \otimes V_{p-1} \simeq \mathrm{ind}_{T(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \left(\bigoplus_{j=0}^{m} \omega_2^{r+2+j(p-1)}\right).$$

For $m = 0$, this is Theorem 4.1 and Remark 2. The proof for $m > 0$ is by induction and is omitted. $\square$

4.2. **The case of** $\mathrm{GL}_2(\mathbb{F}_q)$**.** We now prove Theorem 1.6, which is a twisted version of Theorem 4.1. Recall that $V_{r_j}^{\mathrm{Fr}^j} := \mathrm{Sym}^{r_j}(\mathbb{F}_p^2) \circ \mathrm{Fr}^j$ and $\{X_j^{r_j-i_j}Y_j^{i_j}\}_{0\leq i_j \leq r_j}$ is a basis of $V_{r_j}^{\mathrm{Fr}^j}$, for all $0 \leq j \leq f-1$.

Lemma 4.5. *Let $r_0 \geq 1$. We define*

$$D_0 = X_0^p X_1^{p-1} \cdots X_{f-1}^{p-1} \frac{\partial}{\partial X_0} + Y_0^p Y_1^{p-1} \cdots Y_{f-1}^{p-1} \frac{\partial}{\partial Y_0}$$

and

$$D_j = X_0^p X_1^{p-1} \cdots X_{j-1}^{p-1} \frac{\partial}{\partial X_j} + Y_0^p Y_1^{p-1} \cdots Y_{j-1}^{p-1} \frac{\partial}{\partial Y_j},$$

for all $1 \leq j \leq f-1$. Then, the maps

(1)

$$D_0 : V_{r_0} \otimes V_0^{\mathrm{Fr}} \otimes \cdots \otimes V_0^{\mathrm{Fr}^{f-1}} \to \frac{V_{r_0+p-1} \otimes V_{p-1}^{\mathrm{Fr}} \otimes \cdots \otimes V_{p-1}^{\mathrm{Fr}^{f-1}}}{\langle D_1, \ldots, D_{f-1} \rangle} \text{ and}$$

(2)

$$D_j : V_{r_0-1} \otimes V_0^{\mathrm{Fr}} \otimes \cdots \otimes V_0^{\mathrm{Fr}^{j-1}} \otimes V_p^{\mathrm{Fr}^j} \otimes V_{p-1}^{\mathrm{Fr}^{j+1}} \otimes \cdots \otimes V_{p-1}^{\mathrm{Fr}^{f-1}} \to \frac{V_{r_0+p-1} \otimes V_{p-1}^{\mathrm{Fr}} \otimes \cdots \otimes V_{p-1}^{\mathrm{Fr}^{f-1}}}{\langle D_1, \ldots, D_{j-1} \rangle}$$

are $G(\mathbb{F}_q)$-linear.

Proof. (1). We show that D_0 is $G(\mathbb{F}_q)$-linear modulo $\langle D_1, \ldots, D_{f-1} \rangle$. By Bruhat decomposition, it is enough to check that $D_0(g \cdot X_0^{r_0-i_0}Y_0^{i_0}) = g \cdot D_0(X_0^{r_0-i_0}Y_0^{i_0})$ modulo $\langle D_1, \ldots, D_{f-1} \rangle$ for g either diagonal, the Weyl element $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or upper unipotent. The case when g diagonal is clear. If $g = w$, then

$$\begin{aligned}
D_0(w \cdot X_0^{r_0-i_0}Y_0^{i_0}) &= D_0(Y_0^{r_0-i_0}X_0^{i_0}) \\
&= i_0 X_0^{i_0+p-1} Y_0^{r_0-i_0} X_1^{p-1} \cdots X_{f-1}^{p-1} + (r_0-i_0) X_0^{i_0} Y_0^{r_0+p-1-i_0} Y_1^{p-1} \cdots Y_{f-1}^{p-1} \\
&= w \cdot \left(i_0 Y_0^{i_0+p-1} X_0^{r_0-i_0} Y_1^{p-1} \cdots Y_{f-1}^{p-1} + (r_0-i_0) Y_0^{i_0} X_0^{r_0+p-1-i_0} X_1^{p-1} \cdots X_{f-1}^{p-1} \right) \\
&= w \cdot D_0(X_0^{r_0-i_0}Y_0^{i_0}).
\end{aligned}$$

Finally, if $g = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$, for $a \in \mathbb{F}_q$, then

$$\begin{aligned}
D_0(g \cdot X_0^{r_0-i_0}Y_0^{i_0}) &= D_0\left(X_0^{r_0-i_0}(aX_0+Y_0)^{i_0} \right) \\
&= X_0^p \left((r_0-i_0) X_0^{r_0-i_0-1}(aX_0+Y_0)^{i_0} + X_0^{r_0-i_0} i_0 a (aX_0+Y_0)^{i_0-1} \right) X_1^{p-1} \cdots X_{f-1}^{p-1} \\
&\quad + Y_0^p X_0^{r_0-i_0} i_0 (aX_0+Y_0)^{i_0-1} Y_1^{p-1} \cdots Y_{f-1}^{p-1}
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
g \cdot D_0(X_0^{r_0-i_0}Y_0^{i_0}) &= g \cdot \left(X_0^p (r_0-i_0) X_0^{r_0-i_0-1} Y_0^{i_0} X_1^{p-1} \cdots X_{f-1}^{p-1} + Y_0^p X_0^{r_0-i_0} i_0 Y_0^{i_0-1} Y_1^{p-1} \cdots Y_{f-1}^{p-1} \right) \\
&= X_0^p (r_0-i_0) X_0^{r_0-i_0-1} (aX_0+Y_0)^{i_0} X_1^{p-1} \cdots X_{f-1}^{p-1} \\
&\quad + i_0 X_0^{r_0-i_0} (aX_0+Y_0)^{p+i_0-1} (a^p X_1 + Y_1)^{p-1} \cdots (a^{p^{f-1}} X_{f-1} + Y_{f-1})^{p-1},
\end{aligned} \tag{4.18}$$

so taking the difference of (4.17) and (4.18), we have

$$\begin{aligned}
&D_0(g \cdot X_0^{r_0-i_0}Y_0^{i_0}) - g \cdot D_0(X_0^{r_0-i_0}Y_0^{i_0}) \\
&= i_0 a X_0^p X_0^{r_0-i_0} (aX_0 + i_0 Y_0)^{i_0-1} X_1^{p-1} \cdots X_{f-1}^{p-1} + i_0 Y_0^p X_0^{r_0-i_0} (aX_0+Y_0)^{i_0-1} Y_1^{p-1} \cdots Y_{f-1}^{p-1} \\
&\quad - i_0 X_0^{r_0-i_0} (aX_0+Y_0)^{p+i_0-1} (a^p X_1 + Y_1)^{p-1} \cdots (a^{p^{f-1}} X_{f-1} + Y_{f-1})^{p-1} \\
&= i_0 X_0^{r_0-i_0} (aX_0+Y_0)^{i_0-1} \left(a X_0^p \prod_{j=1}^{f-1} X_j^{p-1} + Y_0^p \prod_{j=1}^{f-1} Y_j^{p-1} - (a^p X_0^p + Y_0^p) \prod_{j=1}^{f-1} (a^{p^j} X_j + Y_j)^{p-1} \right).
\end{aligned} \tag{4.19}$$

The term in the parentheses

$$
\begin{aligned}
&= \left(aX_0^p \prod_{j=1}^{f-1} X_j^{p-1} - a^p X_0^p \prod_{j=1}^{f-1} (a^{p^j} X_j + Y_j)^{p-1} \right) + \left(Y_0^p \prod_{j=1}^{f-1} Y_j^{p-1} - Y_0^p \prod_{j=1}^{f-1} (a^{p^j} X_j + Y_j)^{p-1} \right) \\
&= -\sum_{i_1=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1}{}' X_0^p a^p \prod_{j=1}^{f-1} \left(\binom{p-1}{i_j} a^{p^j(p-1-i_j)} X_j^{p-1-i_j} Y_j^{i_j} \right) \\
&\quad - \sum_{i_1=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1}{}'' Y_0^p \prod_{j=1}^{f-1} \left(\binom{p-1}{i_j} a^{p^j(p-1-i_j)} X_j^{p-1-i_j} Y_j^{i_j} \right) \\
&= -\sum_{i_1=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1}{}' a^{1-\sum\limits_{j=1}^{f-1} i_j p^j} (-1)^{\sum\limits_{j=1}^{f-1} i_j} X_0^p \prod_{j=1}^{f-1} X_j^{p-1-i_j} Y_j^{i_j} \\
&\quad - \sum_{i_1=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1}{}'' a^{1-p-\sum\limits_{j=1}^{f-1} i_j p^j} (-1)^{\sum\limits_{j=1}^{f-1} i_j} Y_0^p \prod_{j=1}^{f-1} X_j^{p-1-i_j} Y_j^{i_j},
\end{aligned}
$$

where $\sum \cdots \sum'$ means $(0, \ldots, 0)$ is omitted from the sum and $\sum \cdots \sum''$ means $(p-1, \ldots, p-1)$ is omitted from the sum. Writing $I_j = \sum\limits_{l=j}^{f-1} i_l p^l$ and $\sum_p I_j = \sum\limits_{l=j}^{f-1} i_l$, the above expression

$$
\begin{aligned}
&= -\sum_{j=1}^{f-1} \left(\sum_{i_1=0}^{0} \cdots \sum_{i_{j-1}=0}^{0} \sum_{i_j=1}^{p-1} \sum_{i_{j+1}=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1} (-1)^{\sum_p I_j} a^{1-I_j} X_0^p \left(\prod_{l=1}^{j-1} X_l^{p-1} \right) \prod_{l=j}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \\
&\quad - \sum_{j=1}^{f-1} \left(\sum_{i_1=p-1}^{p-1} \cdots \sum_{i_{j-1}=p-1}^{p-1} \sum_{i_j=0}^{p-2} \sum_{i_{j+1}=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1} (-1)^{\sum_p I_j} a^{1-p^j-I_j} Y_0^p \left(\prod_{l=1}^{j-1} Y_l^{p-1} \right) \left(\prod_{l=j}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \right) \\
&= -\sum_{j=1}^{f-1} \left(\sum_{i_j=1}^{p-1} \sum_{i_{j+1}=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1} (-1)^{\sum_p I_j} a^{1-I_j} X_0^p \left(\prod_{l=1}^{j-1} X_l^{p-1} \right) X_j^{p-1-i_j} Y_j^{i_j} \left(\prod_{l=j+1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \right) \\
&\quad - \sum_{j=1}^{f-1} \left(\sum_{i_j=1}^{p-1} \sum_{i_{j+1}=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1} (-1)^{\sum_p I_j - 1} a^{1-I_j} Y_0^p \left(\prod_{l=1}^{j-1} Y_l^{p-1} \right) X_j^{p-i_j} Y_j^{i_j-1} \left(\prod_{l=j+1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \right),
\end{aligned}
$$

where the fourth sum above is obtained by the transformation $i_j \mapsto i_j - 1$ in the second sum above, which, together with (4.19), gives

$$
\begin{aligned}
&D_0(g \cdot X_0^{r_0-i_0} Y_0^{i_0}) - g \cdot D_0(X_0^{r_0-i_0} Y_0^{i_0}) \\
&\quad = -\sum_{j=1}^{f-1} \left(\sum_{i_j=1}^{p-1} \sum_{i_{j+1}=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1} \frac{(-1)^{\sum_p I_j - 1}}{i_j} a^{1-I_j} D_j \left(i_0 X_0^{r_0-i_0} (aX_0 + Y_0)^{i_0-1} X_j^{p-i_j} Y_j^{i_j} \prod_{l=j+1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \right) \\
&\quad \in \langle D_1, \ldots, D_{f-1} \rangle.
\end{aligned}
$$

Thus D_0 is $G(\mathbb{F}_q)$-linear modulo $\langle D_1, \ldots, D_{f-1} \rangle$.

(2). The proof is similar and is omitted. □

Let $\alpha \in \mathbb{F}_{q^2}$ be such that $\alpha^2 \in \mathbb{F}_q$, $\alpha \notin \mathbb{F}_q$. Fix an identification $i : \mathbb{F}_q^\times \simeq T(\mathbb{F}_q) \subset \mathrm{GL}_2(\mathbb{F}_q)$ given by $u + v\alpha \mapsto \left(\begin{smallmatrix} u & v\alpha^2 \\ v & u \end{smallmatrix}\right)$.

Let $r \geq 0$ and $0 \leq i \leq r + q^2 - 1$. Let $f_i : G(\mathbb{F}_q) \to \mathbb{F}_{q^2}$ be a function such that

$$
f_i\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = (a + c\alpha)^{r+q^2-1-i} (b + d\alpha)^i,
$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_q)$. Then $f_i \in \mathrm{ind}_{T(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \omega_{2f}^r$ and the set $\mathcal{B}_q = \{f_i | 0 \leq i \leq q^2 - q - 1\}$ forms a basis of $\mathrm{ind}_{T(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \omega_{2f}^r$. Indeed, let $t = \left(\begin{smallmatrix} u & v\alpha^2 \\ v & u \end{smallmatrix}\right) \in T(\mathbb{F}_q)$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_q)$. Then we have

$$f_i\,(t \cdot g) = f_i\left(\begin{pmatrix} ua + v\alpha^2 c & ub + v\alpha^2 d \\ va + uc & vb + ud \end{pmatrix}\right) = (u + v\alpha)^{r+q^2-1}(a + c\alpha)^{(r+q^2-1)-i}(b + d\alpha)^i$$

which equals $(u + v\alpha)^r \cdot f_i(g) = \omega_{2f}^r(t) \cdot f_i(g) = t \cdot f_i(g)$. One can check that the functions in $\mathcal{B}_q$ are linearly independent. Also, $T(\mathbb{F}_q)$ has index $q^2 - q$ in $G(\mathbb{F}_q)$. So $\mathcal{B}_q$ forms a basis of $\mathrm{ind}_{T(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \omega_{2f}^r$.

For $q^2 - 1 \leq i \leq r + q^2 - 1$, we observe that

$$f_i = f_{q^2-1+j} = f_j \tag{4.20}$$

for some $0 \leq j \leq r$. We say that f_i is a *flip*. We soon assume that $r \leq p - 1 \leq q^2 - q - 1$, so $f_i \in \mathcal{B}_q$.

On the other hand, for $q^2 - q \leq i \leq q^2 - 2$, we have $f_i = f_{q^2-q+j}$ for some $0 \leq j \leq q - 2$. We say that f_i is a *flop* since it satisfies the following relation:

$$f_j + f_{j+(q-1)} + f_{j+2(q-1)} + \cdots + f_{j+(q-1)(q-1)} + f_{j+q^2-q} = 0, \tag{4.21}$$

where all terms but the last are in $\mathcal{B}_q$. Indeed, since $X^{q^2-1} - 1 = (X^{q-1} - 1)(X^{(q-1)q} + X^{(q-1)(q-1)} + \cdots + X^{q-1} + 1)$, for $A \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times$, we have $A^{(q-1)q} + A^{(q-1)(q-1)} + \cdots + A^{q-1} + 1 = 0$. Thus for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_q)$,

$$\left(\frac{a + c\alpha}{b + d\alpha}\right)^{(q-1)q} + \left(\frac{a + c\alpha}{b + d\alpha}\right)^{(q-1)(q-1)} + \cdots + \left(\frac{a + c\alpha}{b + d\alpha}\right)^{q-1} + 1 = 0,$$

which, after multiplying by $(a + c\alpha)^{(r+q-1)-j}(b + d\alpha)^{q^2-q+j}$, gives

$$(a + c\alpha)^{(r+q^2-1)-j}(b + d\alpha)^j + (a + c\alpha)^{(r+q^2-q)-j}(b + d\alpha)^{j+(q-1)} + \cdots + (a + c\alpha)^{(r+q-1)-j}(b + d\alpha)^{j+q^2-q} = 0,$$

which shows (4.21).

Thus any flip or flop can be changed to a linear combination of functions in $\mathcal{B}_q$. We fix the basis $\mathcal{B}_q$ in the computations to follow.

For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_q)$, we denote $A_\alpha = a + c\alpha$ and $B_\alpha = b + d\alpha$. For any polynomial $P(X, Y)$ and $A, B, C, D \in \mathbb{F}_q$, we write $P(X, Y)\Big|_{(A,B)}^{(C,D)} = P(C, D) - P(A, B)$.

Lemma 4.6. *Let* $r = r_0 + r_1 p + \cdots + r_{f-1} p^{f-1}$ *with* $2 \leq r_0 \leq p - 1$ *and* $r_j \geq 0$ *for all* $1 \leq j \leq f - 1$. *Let* $P \otimes Q := \bigotimes_{j=0}^{f-1} P_j \otimes \bigotimes_{j=0}^{f-1} Q_j \in \bigotimes_{j=0}^{f-1} V_{r_j+p-1}^{\mathrm{Fr}^j} \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j}$, *with* P_j *a homogeneous polynomial of degree* $r_j + p - 1$ *in* X_j, Y_j, *and* Q_j *homogeneous of degree* $p - 1$ *in* S_j, T_j. *Define* $\psi_{P \otimes Q} : G(\mathbb{F}_q) \to \mathbb{F}_{q^2}$ *by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \nabla_0^{r_0-2}(P_0)\nabla_1^{r_1}(P_1)\cdots\nabla_{f-1}^{r_{f-1}}(P_{f-1})\Big|_{(A_\alpha, B_\alpha, \ldots, A_\alpha^{p^{f-1}}, B_\alpha^{p^{f-1}})}^{(A_\alpha^{p^f}, B_\alpha^{p^f}, \ldots, A_\alpha^{p^{2f-1}}, B_\alpha^{p^{2f-1}})} \cdot \prod_{j=0}^{f-1} Q_j(A_\alpha^{p^{f+j}}, B_\alpha^{p^{f+j}}),$$

where

$$\nabla_j = A_\alpha^{p^j} \frac{\partial}{\partial X_j} + B_\alpha^{p^j} \frac{\partial}{\partial Y_j}$$

for all $0 \leq j \leq f - 1$. *Then the map*

(1) $\psi_{P \otimes Q}$ *is* $T(\mathbb{F}_q)$*-linear.*

(2) $\psi : \bigotimes_{j=0}^{f-1} V_{r_j+p-1}^{\mathrm{Fr}^j} \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j} \to \mathrm{ind}_{T(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \omega_{2f}^r$ *such that* $\psi(P \otimes Q) = \psi_{P \otimes Q}$ *is* $G(\mathbb{F}_q)$*-linear.*

Proof. $T(\mathbb{F}_q)$**-linearity:** Note $\nabla_0^{r_0-2}(P_0) \prod_{j=1}^{f-1} \nabla_j^{r_j}(P_j) \prod_{j=0}^{f-1} Q_j(S_j, T_j)$ is a linear combination of terms

$$A_\alpha^{r-i-2} B_\alpha^i X_0^{p+1-i_0} Y_0^{i_0} \prod_{j=1}^{f-1} X_j^{p-1-i_j} Y_j^{i_j} \cdot \prod_{j=0}^{f-1} S_j^{p-1-k_j} T_j^{k_j},$$

where $r=\sum_{j=0}^{f-1} r_j p^j$ and $i=\sum_{j=0}^{f-1} i_j p^j$. Now,

$$A_\alpha^{r-i-2}B_\alpha^i X_0^{p+1-i_0}Y_0^{i_0}\prod_{j=1}^{f-1} X_j^{p-1-i_j}Y_j^{i_j}\Big|_{(A_\alpha,B_\alpha,\dots,A_\alpha^{p^{f-1}},B_\alpha^{p^{f-1}})}^{(A_\alpha^{p^f},B_\alpha^{p^f},\dots,A_\alpha^{p^{2f-1}},B_\alpha^{p^{2f-1}})}\prod_{j=0}^{f-1} S_j^{p-1-k_j}T_j^{k_j}\Big|_{(0,0)}^{(A_\alpha^{p^{f+j}},B_\alpha^{p^{f+j}})}$$

$$= A_\alpha^{r-i-2}B_\alpha^i\left(A_\alpha^{p^f\left(p+1-i_0+\sum\limits_{j=1}^{f-1}(p^{j+1}-p^j-i_jp^j)\right)}B_\alpha^{p^f\left(i_0+\sum\limits_{j=1}^{f-1}i_jp^j\right)}-A_\alpha^{p+1-i_0+\sum\limits_{j=1}^{f-1}(p^{j+1}-p^j-i_jp^j)}B_\alpha^{i_0+\sum\limits_{j=1}^{f-1}i_jp^j}\right)$$

$$\cdot A_\alpha^{p^f\left(\sum\limits_{j=0}^{f-1}(p^{j+1}-p^j-k_jp^j)\right)}B_\alpha^{p^f\left(\sum\limits_{j=0}^{f-1}k_jp^j\right)}$$

$$= A_\alpha^{r-i-2}B_\alpha^i\left(A_\alpha^{q+q^2-iq}B_\alpha^{iq}-A_\alpha^{1-i+q}B_\alpha^i\right)A_\alpha^{q^2-q-kq}B_\alpha^{kq} = A_\alpha^{r+q^2-1-i-(i+k)q}B_\alpha^{i+(i+k)q}-A_\alpha^{r+q^2-1-(2i+kq)}B_\alpha^{2i+kq},$$

where in the last but one equality $k=\sum_{j=0}^{f-1}k_jp^j$. Thus $\psi_{P\otimes Q}$ is a linear combination of functions of the form f_i above and hence is $T(\mathbb{F}_q)$-linear.

$G(\mathbb{F}_q)$**-linearity:** Let $g=\left(\begin{smallmatrix} u & v\\ w & z\end{smallmatrix}\right)\in G(\mathbb{F}_q)$. Note that,

$$g\cdot(P\otimes Q) \;=\; \bigotimes_{j=0}^{f-1}P_j(U_j,V_j)\otimes\bigotimes_{j=0}^{f-1}Q_j(U_j',V_j') =: \bigotimes_{j=0}^{f-1}P_j'\otimes\bigotimes_{j=0}^{f-1}Q_j',$$

where $U_j=u^{p^j}X_j+w^{p^j}Y_j$, $V_j=v^{p^j}X_j+z^{p^j}Y_j$, $U_j'=u^{p^j}S_j+w^{p^j}T_j$, $V_j'=v^{p^j}S_j+z^{p^j}T_j$. Now,

$$\psi\,(g\cdot P\otimes Q)\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)$$
$$=\left(A_\alpha\frac{\partial}{\partial X_0}+B_\alpha\frac{\partial}{\partial Y_0}\right)^{r_0-2}(P_0')\prod_{j=1}^{f-1}\left(A_\alpha^{p^j}\frac{\partial}{\partial X_j}+B_\alpha^{p^j}\frac{\partial}{\partial Y_j}\right)^{r_j}(P_j')\Big|_{(A_\alpha,B_\alpha,\dots,A_\alpha^{p^{f-1}},B_\alpha^{p^{f-1}})}^{(A_\alpha^{p^f},B_\alpha^{p^f},\dots,A_\alpha^{p^{2f-1}},B_\alpha^{p^{2f-1}})}$$
$$\cdot\prod_{j=0}^{f-1}Q_j'(A_\alpha^{p^{f+j}},B_\alpha^{p^{f+j}}). \tag{4.22}$$

Applying Lemma 2.10 (twice), for $0\le j\le f-1$ and $k\ge 0$ we have

$$\left(A_\alpha^{p^j}\frac{\partial}{\partial X_j}+B_\alpha^{p^j}\frac{\partial}{\partial Y_j}\right)^k(P_j')\Big|_{(A_\alpha^{p^j},B_\alpha^{p^j})}^{(A_\alpha^{p^{f+j}},B_\alpha^{p^{f+j}})}$$
$$=\left((u^{p^j}A_\alpha^{p^j}+w^{p^j}B_\alpha^{p^j})\frac{\partial}{\partial X_j}+(v^{p^j}A_\alpha^{p^j}+z^{p^j}B_\alpha^{p^j})\frac{\partial}{\partial Y_j}\right)^k(P_j)\Big|_{(u^{p^j}A_\alpha^{p^j}+w^{p^j}B_\alpha^{p^j},\,v^{p^j}A_\alpha^{p^j}+z^{p^j}B_\alpha^{p^j})}^{(u^{p^j}A_\alpha^{p^{f+j}}+w^{p^j}B_\alpha^{p^{f+j}},\,v^{p^j}A_\alpha^{p^{f+j}}+z^{p^j}B_\alpha^{p^{f+j}})}$$
$$=\left(A_\alpha'^{p^j}\frac{\partial}{\partial X_j}+B_\alpha'^{p^j}\frac{\partial}{\partial Y_j}\right)^k(P_j)\Big|_{(A_\alpha'^{p^j},B_\alpha'^{p^j})}^{(A_\alpha'^{p^{f+j}},B_\alpha'^{p^{f+j}})},$$

where $A_\alpha'=uA_\alpha+wB_\alpha$ and $B_\alpha'=vA_\alpha+zB_\alpha$. Observe that, in the last equality in writing the top limits in terms of A_α' and B_α', we use the fact that u,v,w and z are in $\mathbb{F}_q$.

Now taking, $k=r_0-2$ for $j=0$ and $k=r_j$ for $1\le j\le f-1$, the expression in (4.22) equals

$$\left(A_\alpha'\frac{\partial}{\partial X_0}+B_\alpha'\frac{\partial}{\partial Y_0}\right)^{r_0-2}(P_0)\prod_{j=1}^{f-1}\left(A_\alpha'^{p^j}\frac{\partial}{\partial X_j}+B_\alpha'^{p^j}\frac{\partial}{\partial Y_j}\right)^{r_j}(P_j)\Big|_{(A_\alpha',B_\alpha',\dots,A_\alpha'^{p^{f-1}},B_\alpha'^{p^{f-1}})}^{(A_\alpha'^{p^f},B_\alpha'^{p^f},\dots,A_\alpha'^{p^{2f-1}},B_\alpha'^{p^{2f-1}})}$$
$$\cdot\prod_{j=0}^{f-1}Q_j(A_\alpha'^{p^{f+j}},B_\alpha'^{p^{f+j}})$$
$$=\psi(P\otimes Q)\left(\begin{pmatrix} ua+wb & va+zb\\ uc+wd & vc+zd\end{pmatrix}\right) = g\cdot(\psi(P\otimes Q))\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right). \qquad \square$$

The following lemmas will be useful in the proof of the main theorem.

Lemma 4.7. *Let*

$$P = \sum_{i_0=0}^{r_0+p-1} \sum_{i_1=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1} b_{i_0,\ldots,i_{f-1}} X_0^{r_0+p-1-i_0} Y_0^{i_0} \left(\prod_{l=1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \in V_{r_0+p-1} \otimes \bigotimes_{l=1}^{f-1} V_{p-1}^{\mathrm{Fr}^l}.$$

Then,

(1) *we have $P \in \mathrm{Im}\ D_0$ if and only if*

$$i_0 b_{i_0,0,\ldots,0} = (r_0 - i_0) b_{i_0+p-1,p-1,\ldots,p-1}$$

for $1 \leq i_0 \leq r_0 - 1$, and

$$b_{i_0,\ldots,i_{f-1}} = 0$$

for $r_0 \leq i_0 \leq p-1$ and all $0 \leq i_j \leq p-1$ for $1 \leq j \leq f-1$.

(2) *for $1 \leq j \leq f-1$, we have $P \in \mathrm{Im}\ D_j$ if and only if*

$$b_{i_0,0,\ldots,0,i_j,\ldots,i_{f-1}} = -b_{i_0+p,p-1,\ldots,p-1,i_j-1,i_{j+1},\ldots,i_{f-1}}$$

where $0 \leq i_0 \leq r_0 - 1$, $1 \leq i_j \leq p-1$ and $0 \leq i_{j+1}, \ldots, i_{f-1} \leq p-1$, and in the remaining cases

$$b_{i_0,\ldots,i_{f-1}} = 0.$$

Proof. The conditions are clearly necessary, and can be checked to be sufficient. □

Lemma 4.8. *Let $1 \leq r_0 \leq p-1$ and let*

$$P = \sum_{i_0=0}^{r_0+p-1} \sum_{i_1=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1} b_{i_0,\ldots,i_{f-1}} X_0^{r_0+p-1-i_0} Y_0^{i_0} \left(\prod_{l=1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \in V_{r_0+p-1} \otimes \bigotimes_{l=1}^{f-1} V_{p-1}^{\mathrm{Fr}^l}.$$

Then we have $P \in \langle D_0, \ldots, D_{f-1} \rangle$ if and only if the following hold:

(1) *for $1 \leq i_0 \leq r_0 - 1$, we have*

$$i_0 b_{i_0,0,\ldots,0} = (r_0 - i_0) b_{i_0+p-1,p-1,\ldots,p-1}$$

(2) *for $r_0 \leq i_0 \leq p-1$ and all $0 \leq i_j \leq p-1$ for $1 \leq j \leq f-1$, we have*

$$b_{i_0,\ldots,i_{f-1}} = 0,$$

(3) *for $0 \leq i_0 \leq r_0 - 1$ and for all $1 \leq t \leq f-1$, $1 \leq i_t \leq p-1$ and $0 \leq i_{t+1}, \ldots, i_{f-1} \leq p-1$, we have*

$$b_{i_0,0,\ldots,0,i_t,\ldots,i_{f-1}} = -b_{i_0+p,p-1,\ldots,p-1,i_t-1,i_{t+1},\ldots,i_{f-1}}.$$

Proof. Say $P \in \langle D_0, \ldots, D_{f-1} \rangle$. We show conditions (1), (2), (3) are satisfied. Write $P = \sum_{j=0}^{f-1} P_j$ where

$$P_j = \sum_{i_0=0}^{r_0+p-1} \cdots \sum_{i_{f-1}=0}^{p-1} b^{(j)}_{i_0,\ldots,i_{f-1}} X_0^{r_0+p-1-i_0} Y_0^{i_0} \prod_{l=1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \in \mathrm{Im}\ D_j$$

for all $0 \leq j \leq f-1$. Then

$$b_{i_0,\ldots,i_{f-1}} = \sum_{j=0}^{f-1} b^{(j)}_{i_0,\ldots,i_{f-1}}$$

with each $b^{(j)}_{i_0,\ldots,i_{f-1}}$ satisfying the conditions of Lemma 4.7. Now (2) is clear since if $r_0 \leq i_0 \leq p-1$, each term on the right vanishes by (both parts of) Lemma 4.7. Condition (1) is also clear, since if $1 \leq i_0 \leq r_0 - 1$ and the other i_j are all 0 or all $p-1$, then all the terms on the right vanish for $j \geq 1$, by the second part of Lemma 4.7, and the $j = 0$ term on the right satisfies the desired identity by the first part. Similarly (3) holds, since if $0 \leq i_0 \leq r_0 - 1$ and the other i_j are not all 0 or not all $p-1$, then the $j = 0$ term on the right vanishes by the first part of the lemma and the remaining terms satisfy the desired identity by the second part, whence so does their sum.

For the converse, note that P can be written as

$$\sum_{t=1}^{f-1}\left(\sum_{i_0=0}^{r_0-1}\sum_{i_1=0}^{0}\cdots\sum_{i_{t-1}=0}^{0}\sum_{i_t=1}^{p-1}\sum_{i_{t+1}=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1} b_{i_0,0,\dots,0,i_t,\dots,i_{f-1}} X_0^{r_0+p-1-i_0}Y_0^{i_0}\prod_{l=1}^{t-1}X_l^{p-1}\prod_{l=t}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}\right)$$
$$+\sum_{i_0=1}^{r_0-1} b_{i_0,0,\dots,0}X_0^{r_0+p-1-i_0}Y_0^{i_0}\prod_{l=1}^{f-1}X_l^{p-1}+b_{0,\dots,0}X_0^{r_0+p-1}\prod_{l=1}^{f-1}X_l^{p-1}$$
$$+\sum_{i_0=r_0}^{p-1}\sum_{i_1=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1} b_{i_0,\dots,i_{f-1}}X_0^{r_0+p-1-i_0}Y_0^{i_0}\prod_{l=1}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}$$
$$+\sum_{t=1}^{f-1}\left(\sum_{i_0=p}^{r_0+p-1}\sum_{i_1=p-1}^{p-1}\cdots\sum_{i_{t-1}=p-1}^{p-1}\sum_{i_t=0}^{p-2}\sum_{i_{t+1}=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1} b_{i_0,p-1,\dots,p-1,i_t,\dots,i_{f-1}}X_0^{r_0+p-1-i_0}Y_0^{i_0}\prod_{l=1}^{t-1}Y_l^{p-1}\prod_{l=t}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}\right)$$
$$+\sum_{i_0=p}^{r_0+p-2} b_{i_0,p-1,\dots,p-1}X_0^{r_0+p-1-i_0}Y_0^{i_0}\prod_{l=1}^{f-1}Y_l^{p-1}+b_{r_0+p-1,p-1,\dots,p-1}Y_0^{r_0+p-1}\prod_{l=1}^{f-1}Y_l^{p-1},$$

which, by the transformations $i_0 \mapsto i_0+p$ and $i_t \mapsto i_t-1$ in the fourth sum above (and dropping the summations for $i_1,\dots,i_{t-1}$ in the first and fourth sum) and by the transformation $i_0 \mapsto i_0+p-1$ in the last sum, can be rewritten as

$$\sum_{t=1}^{f-1}\left(\sum_{i_0=0}^{r_0-1}\sum_{i_t=1}^{p-1}\sum_{i_{t+1}=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1} b_{i_0,0,\dots,0,i_t,\dots,i_{f-1}}X_0^{r_0+p-1-i_0}Y_0^{i_0}\left(\prod_{l=1}^{t-1}X_l^{p-1}\right)X_t^{p-1-i_t}Y_t^{i_t}\prod_{l=t+1}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}\right)$$
$$+\sum_{t=1}^{f-1}\left(\sum_{i_0=0}^{r_0-1}\sum_{i_t=1}^{p-1}\sum_{i_{t+1}=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1} b_{i_0+p,p-1,\dots,p-1,i_t-1,\dots,i_{f-1}}X_0^{r_0-i_0-1}Y_0^{i_0+p}\prod_{l=1}^{t-1}Y_l^{p-1}X_t^{p-i_t}Y_t^{i_t-1}\prod_{l=t+1}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}\right)$$
$$+\sum_{i_0=1}^{r_0-1} b_{i_0,0,\dots,0}X_0^{r_0+p-1-i_0}Y_0^{i_0}\prod_{l=1}^{f-1}X_l^{p-1}+\sum_{i_0=1}^{r_0-1} b_{i_0+p-1,p-1,\dots,p-1}X_0^{r_0-i_0}Y_0^{i_0+p-1}\prod_{l=1}^{f-1}Y_l^{p-1}$$
$$+\sum_{i_0=r_0}^{p-1}\sum_{i_1=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1} b_{i_0,\dots,i_{f-1}}X_0^{r_0+p-1-i_0}Y_0^{i_0}\prod_{l=1}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}$$
$$+b_{0,\dots,0}X_0^{r_0+p-1}\prod_{l=1}^{f-1}X_l^{p-1}+b_{r_0+p-1,p-1,\dots,p-1}Y_0^{r_0+p-1}\prod_{l=1}^{f-1}Y_l^{p-1}. \tag{4.23}$$

Now suppose the conditions (1), (2), (3) hold. Then by (4.23), we can write the polynomial P as

$$\sum_{t=1}^{f-1}\left(-\sum_{i_0=0}^{r_0-1}\sum_{i_t=1}^{p-1}\sum_{i_{t+1}=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1}\frac{b_{i_0,0,\dots,0,i_t,\dots,i_{f-1}}}{i_t}D_t\left(X_0^{r_0-1-i_0}Y_0^{i_0}X_t^{p-i_t}Y_t^{i_t}\prod_{l=t+1}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}\right)\right)$$
$$+\sum_{i_0=1}^{r_0-1}\frac{b_{i_0,0,\dots,0}}{(r_0-i_0)}D_0\left(X_0^{r_0-i_0}Y_0^{i_0}\right)+\frac{1}{r_0}D_0\left(b_{0,\dots,0}X_0^{r_0}+b_{r_0+p-1,p-1,\dots,p-1}Y_0^{r_0}\right).$$

This implies that $P\in\langle D_0,\dots,D_{f-1}\rangle$. □

Remark 4. In Lemma 4.8, there are no conditions on the coefficients $b_{0,\dots,0}$ and $b_{r_0+p-1,p-1,\dots,p-1}$ of P. So if $1\le r_0\le p-1$, then the dimension of $\langle D_0,\dots,D_{f-1}\rangle$ over $\mathbb{F}_q$ is

$$2+(r_0-1)+\sum_{t=1}^{f-1}r_0(p-1)p^{f-1-t}=r_0+1+r_0(p^{f-1}-1)=r_0p^{f-1}+1.$$

Theorem 4.9. *Let $r=r_0+r_1p+\cdots+r_{f-1}p^{f-1}$, where $2\le r_0\le p-1$ and $r_j=0$ for all $1\le j\le f-1$. Recall that*

$$D_0=X_0^pX_1^{p-1}\cdots X_{f-1}^{p-1}\frac{\partial}{\partial X_0}+Y_0^pY_1^{p-1}\cdots Y_{f-1}^{p-1}\frac{\partial}{\partial Y_0}$$

and

$$D_j=X_0^pX_1^{p-1}\cdots X_{j-1}^{p-1}\frac{\partial}{\partial X_j}+Y_0^pY_1^{p-1}\cdots Y_{j-1}^{p-1}\frac{\partial}{\partial Y_j},$$

for all $1 \leq j \leq f-1$. *Then over* $\mathbb{F}_{q^2}$ *we have*

$$\frac{\bigotimes_{j=0}^{f-1} V_{r_j+p-1}^{\mathrm{Fr}^j}}{\langle D_0, \ldots, D_{f-1} \rangle} \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j} \simeq \operatorname{ind}_{T(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \omega_{2f}^r.$$

Proof. Define $\psi : \bigotimes_{j=0}^{f-1} V_{r_j+p-1}^{\mathrm{Fr}^j} \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j} \to \operatorname{ind}_{T(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \omega_{2f}^r$ as in Lemma 4.6. Recall that for $P \otimes Q := \bigotimes_{j=0}^{f-1} P_j \otimes \bigotimes_{j=0}^{f-1} Q_j$, we have $\psi(P \otimes Q) = \psi_{P \otimes Q}$, where now noting that $r_j = 0$ for $1 \leq j \leq f-1$, we have $\psi_{P \otimes Q} : G(\mathbb{F}_q) \to \mathbb{F}_{q^2}$ is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \nabla_0^{r_0-2}(P_0) P_1 \cdots P_{f-1} \Big|_{(A_\alpha, B_\alpha, \ldots, A_\alpha^{p^{f-1}}, B_\alpha^{p^{f-1}})}^{(A_\alpha^{p^f}, B_\alpha^{p^f}, \ldots, A_\alpha^{p^{2f-1}}, B_\alpha^{p^{2f-1}})} \cdot \prod_{j=0}^{f-1} Q_j(A_\alpha^{p^{f+j}}, B_\alpha^{p^{f+j}}),$$

where $\nabla_0 = A_\alpha \frac{\partial}{\partial X_0} + B_\alpha \frac{\partial}{\partial Y_0}$. By Lemma 4.6 (1), the map $\psi_{P \otimes Q}$ is $T(\mathbb{F}_q)$-linear and hence ψ is well defined. It is also $G(\mathbb{F}_q)$-linear by Lemma 4.6 (2).

We show that $\ker \psi = \langle D_0, \ldots, D_{f-1} \rangle \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j}$. The proof occupies the rest of this paper. First we show $\langle D_0, \ldots, D_{f-1} \rangle \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j} \subset \ker \psi$.

Case 1: Suppose $j \neq 0$. We show that $\operatorname{Im} D_j \subset \ker \psi$. For $0 \leq i_0 \leq r_0 - 1$, we have

$$\begin{aligned} D_j &\left(X_0^{r_0-1-i_0} Y_0^{i_0} X_j^{p-i_j} Y_j^{i_j} \prod_{l=j+1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \\ &= X_0^{r_0+p-1-i_0} Y_0^{i_0} \left(\prod_{l=1}^{j-1} X_l^{p-1} \right) (-i_j) X_j^{p-1-i_j} Y_j^{i_j} \left(\prod_{l=j+1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \\ &\quad + X_0^{r_0-1-i_0} Y_0^{i_0+p} \left(\prod_{l=1}^{j-1} Y_l^{p-1} \right) i_j X_j^{p-i_j} Y_j^{i_j-1} \left(\prod_{l=j+1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right). \end{aligned}$$

We claim

$$\begin{aligned} &- \nabla_0^{r_0-2} \left(X_0^{r_0+p-1-i_0} Y_0^{i_0} \right) \left(\prod_{l=1}^{j-1} X_l^{p-1} \right) X_j^{p-1-i_j} Y_j^{i_j} \left(\prod_{l=j+1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \Big|_{(A_\alpha, B_\alpha, \ldots, A_\alpha^{p^{f-1}}, B_\alpha^{p^{f-1}})}^{(A_\alpha^{p^f}, B_\alpha^{p^f}, \ldots, A_\alpha^{p^{2f-1}}, B_\alpha^{p^{2f-1}})} \\ &+ \nabla_0^{r_0-2} \left(X_0^{r_0-1-i_0} Y_0^{i_0+p} \right) \left(\prod_{l=1}^{j-1} Y_l^{p-1} \right) X_j^{p-i_j} Y_j^{i_j-1} \left(\prod_{l=j+1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \Big|_{(A_\alpha, B_\alpha, \ldots, A_\alpha^{p^{f-1}}, B_\alpha^{p^{f-1}})}^{(A_\alpha^{p^f}, B_\alpha^{p^f}, \ldots, A_\alpha^{p^{2f-1}}, B_\alpha^{p^{2f-1}})} = 0. \end{aligned} \tag{4.24}$$

Indeed, we have

$$\begin{aligned} \nabla_0^{r_0-2} \left(X_0^{r_0+p-1-i_0} Y_0^{i_0} \right) &= \left(A_\alpha \frac{\partial}{\partial X_0} + B_\alpha \frac{\partial}{\partial Y_0} \right)^{r_0-2} \left(X_0^{r_0+p-1-i_0} Y_0^{i_0} \right) \\ &= \sum_{k_0=0}^{r_0-2} \binom{r_0-2}{k_0} A_\alpha^{r_0-2-k_0} B_\alpha^{k_0} [r_0+p-1-i_0]_{r_0-2-k_0} [i_0]_{k_0} X_0^{p+1-(i_0-k_0)} Y_0^{i_0-k_0} \\ &= \binom{r_0-2}{i_0-1} A_\alpha^{r_0-1-i_0} B_\alpha^{i_0-1} (r_0-1-i_0)! i_0! X_0^p Y_0 \\ &\quad + \binom{r_0-2}{i_0} A_\alpha^{r_0-2-i_0} B_\alpha^{i_0} (r_0-1-i_0)! i_0! X_0^{p+1}, \end{aligned} \tag{4.25}$$

and similarly

$$\begin{aligned} \nabla_0^{r_0-2} \left(X_0^{r_0-1-i_0} Y_0^{i_0+p} \right) &= \binom{r_0-2}{i_0-1} A_\alpha^{r_0-1-i_0} B_\alpha^{i_0-1} (r_0-1-i_0)! i_0! Y_0^{p+1} \\ &\quad + \binom{r_0-2}{i_0} A_\alpha^{r_0-2-i_0} B_\alpha^{i_0} (r_0-1-i_0)! i_0! X_0 Y_0^p. \end{aligned} \tag{4.26}$$

Ignoring the factor $(r_0-1-i_0)!i_0!$ and using (4.25), the first summand on the left side of (4.24) becomes

$$-\binom{r_0-2}{i_0-1}A_\alpha^{r_0-1-i_0}B_\alpha^{i_0-1}X_0^pY_0\left(\prod_{l=1}^{j-1}X_l^{p-1}\right)X_j^{p-1-i_j}Y_j^{i_j}\left(\prod_{l=j+1}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}\right)\Big|^{(A_\alpha^{p^f},B_\alpha^{p^f},\ldots,A_\alpha^{p^{2f-1}},B_\alpha^{p^{2f-1}})}_{(A_\alpha,B_\alpha,\ldots,A_\alpha^{p^{f-1}},B_\alpha^{p^{f-1}})}$$
$$-\binom{r_0-2}{i_0}A_\alpha^{r_0-2-i_0}B_\alpha^{i_0}X_0^{p+1}\left(\prod_{l=1}^{j-1}X_l^{p-1}\right)X_j^{p-1-i_j}Y_j^{i_j}\left(\prod_{l=j+1}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}\right)\Big|^{(A_\alpha^{p^f},B_\alpha^{p^f},\ldots,A_\alpha^{p^{2f-1}},B_\alpha^{p^{2f-1}})}_{(A_\alpha,B_\alpha,\ldots,A_\alpha^{p^{f-1}},B_\alpha^{p^{f-1}})},$$

which, by noting

$$\sum_{l=1}^{j-1}(p-1)p^{f+l}+(p-1-i_j)p^{f+j}+\sum_{l=j+1}^{f-1}(p-1-i_l)p^{f+l}=p^{2f}-p^{f+1}-\sum_{l=j}^{f-1}i_lp^{f+l}$$

and, dividing by p^f,

$$\sum_{l=1}^{j-1}(p-1)p^{l}+(p-1-i_j)p^{j}+\sum_{l=j+1}^{f-1}(p-1-i_l)p^{l}=p^{f}-p-\sum_{l=j}^{f-1}i_lp^{l},$$

equals

$$-\binom{r_0-2}{i_0-1}A_\alpha^{r_0-1-i_0}B_\alpha^{i_0-1}\left(A_\alpha^{p^{f+1}+p^{2f}-p^{f+1}-\sum_{l=j}^{f-1}i_lp^{f+l}}B_\alpha^{p^f+\sum_{l=j}^{f-1}i_lp^{f+l}}-A_\alpha^{p+p^f-p-\sum_{l=j}^{f-1}i_lp^l}B_\alpha^{1+\sum_{l=j}^{f-1}i_lp^l}\right)$$
$$-\binom{r_0-2}{i_0}A_\alpha^{r_0-2-i_0}B_\alpha^{i_0}\left(A_\alpha^{p^{f+1}+p^f+p^{2f}-p^{f+1}-\sum_{l=j}^{f-1}i_lp^{f+l}}B_\alpha^{\sum_{l=j}^{f-1}i_lp^{f+l}}-A_\alpha^{p+1+p^f-p-\sum_{l=j}^{f-1}i_lp^l}B_\alpha^{\sum_{l=j}^{f-1}i_lp^l}\right)$$
$$=-\binom{r_0-2}{i_0-1}\left(A_\alpha^{r_0-i_0-\sum_{l=j}^{f-1}i_lp^{f+l}}B_\alpha^{i_0-1+p^f+\sum_{l=j}^{f-1}i_lp^{f+l}}-A_\alpha^{r_0-1-i_0+p^f-\sum_{l=j}^{f-1}i_lp^l}B_\alpha^{i_0+\sum_{l=j}^{f-1}i_lp^l}\right)$$
$$-\binom{r_0-2}{i_0}\left(A_\alpha^{r_0-1-i_0+p^f-\sum_{l=j}^{f-1}i_lp^{f+l}}B_\alpha^{i_0+\sum_{l=j}^{f-1}i_lp^{f+l}}-A_\alpha^{r_0-1-i_0+p^f-\sum_{l=j}^{f-1}i_lp^l}B_\alpha^{i_0+\sum_{l=j}^{f-1}i_lp^l}\right). \tag{4.27}$$

Again, ignoring the common factor $(r_0-1-i_0)!i_0!$ and using (4.26), a similar computation shows that the second summand on the left hand side of (4.24) is

$$\binom{r_0-2}{i_0-1}\left(A_\alpha^{r_0-i_0-\sum_{l=j}^{f-1}i_lp^{f+l}}B_\alpha^{i_0-1+p^f+\sum_{l=j}^{f-1}i_lp^{f+l}}-A_\alpha^{r_0-1-i_0+p^f-\sum_{l=j}^{f-1}i_lp^l}B_\alpha^{i_0+\sum_{l=j}^{f-1}i_lp^l}\right)$$
$$+\binom{r_0-2}{i_0}\left(A_\alpha^{r_0-1-i_0+p^f-\sum_{l=j}^{f-1}i_lp^{f+l}}B_\alpha^{i_0+\sum_{l=j}^{f-1}i_lp^{f+l}}-A_\alpha^{r_0-1-i_0+p^f-\sum_{l=j}^{f-1}i_lp^l}B_\alpha^{i_0+\sum_{l=j}^{f-1}i_lp^l}\right), \tag{4.28}$$

which is the negative of (4.27). This proves the claim (4.24). Hence $\mathrm{Im}\,D_j\subset\ker\psi$ for all $1\le j\le f-1$.

Case 2: The proof that $\mathrm{Im}\,D_0\subset\ker\psi$ is similar and is omitted.

Combining Cases 1 and 2, we see that $\langle D_0,\ldots,D_{f-1}\rangle\otimes\bigotimes_{j=0}^{f-1}V_{p-1}^{\mathrm{Fr}^j}\subset\ker\psi$.

Next we show that $\ker\psi\subset\langle D_0,\ldots,D_{f-1}\rangle\otimes\bigotimes_{j=0}^{f-1}V_{p-1}^{\mathrm{Fr}^j}$. Let

$$P\otimes Q=\sum_{\vec{i}=\vec{0}}^{r+\vec{q}-1}\sum_{\vec{j}=\vec{0}}^{q-\vec{1}}b_{\vec{i},\vec{j}}X_0^{r_0+p-1-i_0}Y_0^{i_0}\left(\prod_{l=1}^{f-1}X_l^{p-1-i_l}Y_l^{i_l}\right)\left(\prod_{l=0}^{f-1}S_l^{p-1-j_l}T_l^{j_l}\right)\in\ker\psi,$$

where $\vec{i} = (i_0, \ldots, i_{f-1}), \vec{j} = (j_0, \ldots, j_{f-1})$ and $\sum_{\vec{i}=\vec{0}}^{r+\vec{q}-1} := \sum_{i_0=0}^{r_0+p-1} \sum_{i_1=0}^{p-1} \cdots \sum_{i_{f-1}=0}^{p-1}$, $\sum_{\vec{j}=\vec{0}}^{\vec{q}-1} := \sum_{j_0=0}^{p-1} \cdots \sum_{j_{f-1}=0}^{p-1}$. By definition of ψ, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{F}_q)$, we have

$$\sum_{\vec{i}=\vec{0}}^{r+\vec{q}-1} \sum_{\vec{j}=\vec{0}}^{\vec{q}-1} b_{\vec{i},\vec{j}} \nabla_0^{r_0-2} \left(X_0^{r_0+p-1-i_0} Y_0^{i_0} \right) \left(\prod_{l=1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \Big|_{(A_\alpha, B_\alpha, \ldots, A_\alpha^{p^{f-1}}, B_\alpha^{p^{f-1}})}^{(A_\alpha^{p^f}, B_\alpha^{p^f}, \ldots, A_\alpha^{p^{2f-1}}, B_\alpha^{p^{2f-1}})}$$
$$\cdot \prod_{l=0}^{f-1} \left(S_l^{p-1-j_l} T_l^{j_l} \Big|_{(0,0)}^{(A_\alpha^{p^{f+l}}, B_\alpha^{p^{f+l}})} \right) = 0. \tag{4.29}$$

Note, for each $\vec{j}$,

$$\prod_{l=0}^{f-1} \left(S_l^{p-1-j_l} T_l^{j_l} \Big|_{(0,0)}^{(A_\alpha^{p^{f+l}}, B_\alpha^{p^{f+l}})} \right) = A_\alpha^{\sum_{l=0}^{f-1} (p-1-j_l) p^{f+l}} B_\alpha^{\sum_{l=0}^{f-1} j_l p^{f+l}} = A_\alpha^{1-q-\sum_{l=0}^{f-1} j_l p^{f+l}} B_\alpha^{\sum_{l=0}^{f-1} j_l p^{f+l}}. \tag{4.30}$$

Now by fixing $\vec{j}$ and only considering the sum over $\vec{i}$ in (4.29), we have

$$\sum_{\vec{i}=\vec{0}}^{r+\vec{q}-1} b_{\vec{i},\vec{j}} \left(A_\alpha \frac{\partial}{\partial X_0} + B_\alpha \frac{\partial}{\partial Y_0} \right)^{r_0-2} \left(X_0^{r_0+p-1-i_0} Y_0^{i_0} \right) \left(\prod_{l=1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \Big|_{(A_\alpha, B_\alpha, \ldots, A_\alpha^{p^{f-1}}, B_\alpha^{p^{f-1}})}^{(A_\alpha^{p^f}, B_\alpha^{p^f}, \ldots, A_\alpha^{p^{2f-1}}, B_\alpha^{p^{2f-1}})}$$
$$= \sum_{\vec{i}=\vec{0}}^{r+\vec{q}-1} \sum_{k_0=0}^{r_0-2} b_{\vec{i},\vec{j}} \binom{r_0-2}{k_0} [r_0+p-1-i_0]_{r_0-2-k_0} [i_0]_{k_0} A_\alpha^{r_0-2-k_0} B_\alpha^{k_0}$$
$$\cdot X_0^{p+1-(i_0-k_0)} Y_0^{i_0-k_0} \left(\prod_{l=1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \Big|_{(A_\alpha, B_\alpha, \ldots, A_\alpha^{p^{f-1}}, B_\alpha^{p^{f-1}})}^{(A_\alpha^{p^f}, B_\alpha^{p^f}, \ldots, A_\alpha^{p^{2f-1}}, B_\alpha^{p^{2f-1}})},$$

which by observing $p^{f+1} + p^f - (i_0 - k_0)p^f + \sum_{l=1}^{f-1} (p-1-i_l) p^{f+l} = p^{2f} + p^f(1+k_0) - \sum_{l=0}^{f-1} i_l p^{f+l}$, and writing

$$C_{i_0,k_0} = \binom{r_0-2}{k_0} [r_0+p-1-i_0]_{r_0-2-k_0} [i_0]_{k_0} = (r_0-2)! \binom{r_0+p-1-i_0}{r_0-2-k_0} \binom{i_0}{k_0}, \tag{4.31}$$

equals

$$\sum_{\vec{i}=\vec{0}}^{r+\vec{q}-1} \sum_{k_0=0}^{r_0-2} b_{\vec{i},\vec{j}} C_{i_0,k_0} A_\alpha^{r_0-2-k_0} B_\alpha^{k_0}$$
$$\cdot \left(A_\alpha^{p^{2f}+p^f(1+k_0)-\sum_{l=0}^{f-1} i_l p^{f+l}} B_\alpha^{(i_0-k_0)p^f+\sum_{l=1}^{f-1} i_l p^{f+l}} - A_\alpha^{p^f+1+k_0-\sum_{l=0}^{f-1} i_l p^l} B_\alpha^{i_0-k_0+\sum_{l=1}^{f-1} i_l p^l} \right)$$
$$= \sum_{\vec{i}=\vec{0}'}^{r+\vec{q}-1'} b_{\vec{i},\vec{j}} \left(\sum_{k_0=0}^{r_0-2} C_{i_0,k_0} A_\alpha^{r_0+q-1-k_0(1-q)-q\left(\sum_{l=0}^{f-1} i_l p^l\right)} B_\alpha^{k_0(1-q)+q\left(\sum_{l=0}^{f-1} i_l p^l\right)} \right)$$
$$- \sum_{\vec{i}=\vec{0}'}^{r+\vec{q}-1'} b_{\vec{i},\vec{j}} (r_0-1)! A_\alpha^{r_0+q-1-\sum_{l=0}^{f-1} i_l p^l} B_\alpha^{\sum_{l=0}^{f-1} i_l p^l}. \tag{4.32}$$

The last equality holds since by (2.1), we have

$$\sum_{k_0=0}^{r_0-2} C_{i_0,k_0} = \sum_{k_0=0}^{r_0-2} \binom{r_0-2}{k_0} [r_0+p-1-i_0]_{r_0-2-k_0} [i_0]_{k_0} = (r_0-1)! \mod p.$$

Moreover, we have adorned the limits in the last two sums with $'$s to indicate that we drop the terms corresponding to $\vec{i} = \vec{0}$ and $\vec{i} = r + \vec{q} - 1$. Indeed, if $\vec{i} = \vec{0}$, then $i_0 = 0$ and

$$C_{0,k_0} = \begin{cases} (r_0-1)!, & \text{if } k_0 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

So the term for $\vec{i}=\vec{0}$ in (4.32) is

$$b_{\vec{0},\vec{j}}\left((r_0-1)!A_\alpha^{r_0+q-1}-(r_0-1)!A_\alpha^{r_0+q-1}\right)=0.$$

Similarly, one may check that the term for $\vec{i}=r_0+\vec{q}-1$ is zero.

For notational convenience, set

$$(*):=\left(\sum_{k_0=0}^{r_0-2}C_{i_0,k_0}A_\alpha^{r_0+q-1-k_0(1-q)-q\left(\sum\limits_{l=0}^{f-1}i_lp^l\right)}B_\alpha^{k_0(1-q)+q\left(\sum\limits_{l=0}^{f-1}i_lp^l\right)}\right)-(r_0-1)!A_\alpha^{r_0+q-1-\sum\limits_{l=0}^{f-1}i_lp^l}B_\alpha^{\sum\limits_{l=0}^{f-1}i_lp^l}.$$

Then (4.32) decomposes as

$$\begin{aligned}
&\sum_{i_0=1}^{r_0-1}b_{i_0,0,\dots,0,\vec{j}}(*)+\sum_{i_0=p}^{r_0+p-2}b_{i_0,p-1,\dots,p-1,\vec{j}}(*)\\
&+\sum_{i_0=r_0}^{p-1}\sum_{i_1=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1}b_{i_0,\dots,i_{f-1},\vec{j}}(*)\\
&+\sum_{t=1}^{f-1}\left(\sum_{i_0=0}^{r_0-1}\sum_{i_1=0}^{0}\cdots\sum_{i_{t-1}=0}^{0}\sum_{i_t=1}^{p-1}\sum_{i_{t+1}=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1}b_{i_0,0,\dots,0,i_t,\dots,i_{f-1},\vec{j}}(*)\right)\\
&+\sum_{t=1}^{f-1}\left(\sum_{i_0=p}^{r_0+p-1}\sum_{i_1=p-1}^{p-1}\cdots\sum_{i_{t-1}=p-1}^{p-1}\sum_{i_t=0}^{p-2}\sum_{i_{t+1}=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1}b_{i_0,p-1,\dots,p-1,i_t,\dots,i_{f-1},\vec{j}}(*)\right).
\end{aligned}\tag{4.33}$$

By taking $i_1=i_2=\cdots=i_{f-1}=0$, the coefficient $(*)$ of $b_{i_0,0,\dots,0,\vec{j}}$ in the first sum above is

$$\left(\sum_{k_0=0}^{r_0-2}C_{i_0,k_0}A_\alpha^{r_0+q-1-k_0(1-q)-i_0q}B_\alpha^{k_0(1-q)+i_0q}\right)-(r_0-1)!A_\alpha^{r_0+q-1-i_0}B_\alpha^{i_0}.$$

Note that, by (4.31), for $0\le i_0\le r_0-1$, we have

$$C_{i_0,k_0}=\begin{cases}(r_0-2)!i_0, & \text{if } k_0=i_0-1,\\ (r_0-2)!(r_0-1-i_0), & \text{if } k_0=i_0,\\ 0, & \text{otherwise.}\end{cases}$$

Thus the coefficient of $b_{i_0,0,\dots,0,\vec{j}}$ becomes

$$i_0(r_0-2)!A_\alpha^{r_0-i_0}B_\alpha^{i_0+q-1}+\left((r_0-1-i_0)(r_0-2)!-(r_0-1)!\right)A_\alpha^{r_0+q-1-i_0}B_\alpha^{i_0},$$

which further equals

$$i_0(r_0-2)!\left(A_\alpha^{r_0-i_0}B_\alpha^{i_0+q-1}-A_\alpha^{r_0+q-1-i_0}B_\alpha^{i_0}\right).\tag{4.34}$$

Now use the transformation $i_0\mapsto i_0+p-1$ in the second sum of (4.33). By taking $i_1=i_2=\cdots=i_{f-1}=p-1$, the coefficient $(*)$ of $b_{i_0+p-1,p-1,\dots,p-1,\vec{j}}$ is given by

$$\left(\sum_{k_0=0}^{r_0-2}C_{i_0+p-1,k_0}A_\alpha^{r_0+q-1-k_0(1-q)-q(i_0+q-1)}B_\alpha^{k_0(1-q)+q(i_0+q-1)}\right)-(r_0-1)!A_\alpha^{r_0-i_0}B_\alpha^{i_0+q-1}.$$

For $0\le i_0\le r_0-1$, we have

$$C_{i_0+p-1,k_0}=\binom{r_0-2}{k_0}[r_0-i_0]_{r_0-2-k_0}[i_0+p-1]_{k_0}=\begin{cases}(r_0-2)!(i_0-1), & \text{if } k_0=i_0-2,\\ (r_0-2)!(r_0-i_0), & \text{if } k_0=i_0-1,\\ 0, & \text{otherwise.}\end{cases}$$

Thus the coefficient of $b_{i_0+p-1,p-1,p-1,\dots,p-1,\vec{j}}$ becomes

$$\left((r_0-2)!(i_0-1)-(r_0-1)!\right)A_\alpha^{r_0-i_0}B_\alpha^{i_0+q-1}+(r_0-2)!(r_0-i_0)A_\alpha^{r_0+q-1-i_0}B_\alpha^{i_0},$$

which equals

$$-(r_0-2)!(r_0-i_0)\left(A_\alpha^{r_0-i_0}B_\alpha^{i_0+q-1}-A_\alpha^{r_0+q-1-i_0}B_\alpha^{i_0}\right).\tag{4.35}$$

Finally, using $i_0 \mapsto i_0 + p$ and $i_t \mapsto i_t - 1$ in the fifth sum of (4.33), and noting that $\sum_{l=0}^{f-1} i_l p^l$ changes to

$$i_0 + p + \sum_{l=1}^{t-1}(p-1)p^l + (i_t - 1)p^t + \sum_{l=t+1}^{f-1} i_l p^l = i_0 + \sum_{l=t}^{f-1} i_l p^l,$$

the coefficient $(*)$ of $b_{i_0+p,p-1,\ldots,p-1,i_t-1,i_{t+1},\ldots,i_{f-1},\vec{j}}$ equals

$$\sum_{k_0=0}^{r_0-2} C_{i_0+p,k_0} A_\alpha^{r_0+q-1-k_0(1-q)-q\left(i_0+\sum_{l=t}^{f-1} i_l p^l\right)} B_\alpha^{k_0(1-q)+q\left(i_0+\sum_{l=t}^{f-1} i_l p^l\right)} - (r_0-1)! A_\alpha^{r_0+q-1-i_0-\sum_{l=t}^{f-1} i_l p^l} B_\alpha^{i_0+\sum_{l=t}^{f-1} i_l p^l} \tag{4.36}$$

which is exactly the coefficient of $b_{i_0,0,\ldots,0,i_t,\ldots,i_{f-1},\vec{j}}$ obtained by taking $i_1 = i_2 = \cdots = i_{t-1} = 0$ in $(*)$, since $C_{i_0+p,k_0} = C_{i_0,k_0} \mod p$ for $0 \le i_0 \le r_0 - 1$. Summarizing, the transformation $i_0 \mapsto i_0 + p - 1$ in the second sum of (4.33) and the transformations $i_0 \mapsto i_0 + p$ and $i_t \mapsto i_t - 1$ in the fifth sum of (4.33), together with (4.34), (4.35) and (4.36), allow us to rewrite (4.33) as

$$\begin{aligned}&\sum_{i_0=1}^{r_0-1}(r_0-2)!\left(i_0 b_{i_0,0,\ldots,0,\vec{j}} - (r_0-i_0) b_{i_0+p-1,p-1,\ldots,p-1,\vec{j}}\right)\left(A_\alpha^{r_0-i_0}B_\alpha^{i_0+q-1} - A_\alpha^{r_0+q-1-i_0}B_\alpha^{i_0}\right)\\ &\quad + \sum_{i_0=r_0}^{p-1}\sum_{i_1=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1} b_{i_0,\ldots,i_{f-1},\vec{j}}(*)\\ &\quad + \sum_{t=1}^{f-1}\left(\sum_{i_0=0}^{r_0-1}\sum_{i_t=1}^{p-1}\sum_{i_{t+1}=0}^{p-1}\cdots\sum_{i_{f-1}=0}^{p-1}\left(b_{i_0,0,\ldots,0,i_t,\ldots,i_{f-1},\vec{j}} + b_{i_0+p,p-1,\ldots,p-1,i_t-1,\ldots,i_{f-1},\vec{j}}\right)(*)\right).\end{aligned} \tag{4.37}$$

Now, for each $\vec{j} = (j_0, \ldots, j_{f-1})$, let $i = \sum_{l=0}^{f-1} i_l p^l$ and $j = \sum_{l=0}^{f-1} j_l p^l$, and set

$$X_{i,j} = \begin{cases} i_0 b_{i_0,0,\ldots,0,\vec{j}} - (r_0-i_0) b_{i_0+p-1,p-1,\ldots,p-1,\vec{j}}, & \text{if } 1 \le i_0 \le r_0-1, i_t = 0 \text{ for } t \ne 0,\\ b_{i_0,\ldots,i_{f-1},\vec{j}}, & \text{if } r_0 \le i_0 \le p-1, 0 \le i_t \le p-1 \text{ for } t \ne 0,\\ b_{i_0,0,\ldots,0,i_t,\ldots,i_{f-1},\vec{j}} + b_{i_0+p,p-1,\ldots,p-1,i_t-1,\ldots,i_{f-1},\vec{j}}, & \text{if } 0 \le i_0 \le r_0-1, t \ge 1 \text{ smallest s.t. } i_t \ne 0.\end{cases}$$

Note that $1 \le i \le q-1$ and $0 \le j \le q-1$, so there are $(q-1)q$ variables $X_{i,j}$, with the first kind running in the range $1 \le i = i_0 \le r_0 - 1$, and the second and third kind running in the range $r_0 \le i \le q-1$.

By (4.29), and (4.30), and (4.37) but with the variables $X_{i,j}$, we obtain

$$\begin{aligned}&\sum_{i=1}^{r_0-1}\sum_{j=0}^{q-1}(r_0-2)! X_{i,j}\left(A_\alpha^{r_0-i}B_\alpha^{i+q-1} - A_\alpha^{r_0+q-1-i}B_\alpha^{i}\right)\cdot A_\alpha^{1-q-jq}B_\alpha^{jq}\\ &\quad + \sum_{i=r_0}^{q-1}\sum_{j=0}^{q-1} X_{i,j}\left(\left(\sum_{k_0=0}^{r_0-2} C_{i_0,k_0} A_\alpha^{r_0+q-1-k_0(1-q)-iq}B_\alpha^{k_0(1-q)+iq}\right) - (r_0-1)! A_\alpha^{r_0+q-1-i}B_\alpha^{i}\right)\\ &\qquad\qquad \cdot A_\alpha^{1-q-qj}B_\alpha^{qj} = 0,\end{aligned}$$

which by dividing by $(r_0-2)!$ and setting (cf. (4.31))

$$Z_{i_0,k_0} = \frac{C_{i_0,k_0}}{(r_0-2)!} = \binom{r_0+p-1-i_0}{r_0-2-k_0}\binom{i_0}{k_0}$$

yields the system of equations (that was obtained earlier in (4.6) for $q = p$)

$$\begin{aligned}&\sum_{i=1}^{r_0-1}\sum_{j=0}^{q-1} X_{i,j}\left(A_\alpha^{r_0+1-q-i-jq}B_\alpha^{i+jq+q-1} - A_\alpha^{r_0-i-jq}B_\alpha^{i+jq}\right)\\ &\quad + \sum_{i=r_0}^{q-1}\sum_{j=0}^{q-1} X_{i,j}\left(\left(\sum_{k_0=0}^{r_0-2} Z_{i_0,k_0} A_\alpha^{r_0-k_0(1-q)-(i+j)q}B_\alpha^{k_0(1-q)+(i+j)q}\right) - (r_0-1) A_\alpha^{r_0-i-jq}B_\alpha^{i+jq}\right) = 0.\end{aligned}$$

As before, we separate the equations according to the congruence class $1 \le n \le q-1$ of the sum $i+j$ so that we obtain $q-1$ separate systems of equations (each with q distinct variables). Again in order to work in the basis $\mathcal{B}_q$, we convert all flips and flops to elements of $\mathcal{B}_q$ using (4.20) and (4.21). The resulting equations and corresponding coefficient matrices obtained and the computation using row operations to show that these matrices have non-zero determinant are identical to the case of $q = p$ treated earlier. The only real difference is that in the formulas for the determinant (obtained earlier in three cases depending on the relative size of n), one needs to replace p by q everywhere. We conclude that all $X_{i,j} = 0$. That is, for each $\vec{j}$, we have

(1) if $1 \leq i_0 \leq r_0 - 1$ and $i_j = 0$ for $j \neq 0$, then
$$i_0 b_{i_0,0,\ldots,0,\vec{j}} = (r_0 - i_0) b_{i_0+p-1,p-1,\ldots,p-1,\vec{j}},$$
(2) if $r_0 \leq i_0 \leq p-1$ and $0 \leq i_j \leq p-1$ for $1 \leq j \leq f-1$, then
$$b_{i_0,\ldots,i_{f-1},\vec{j}} = 0,$$
(3) if $0 \leq i_0 \leq r_0 - 1$ and there is a (smallest) $1 \leq t \leq f-1$ with $i_t \neq 0$ (so $i_j = 0$ for $1 \leq j \leq t-1$ and $0 \leq i_{t+1}, \ldots, i_{f-1} \leq p-1$), then
$$b_{i_0,0,\ldots,0,i_t,\ldots,i_{f-1},\vec{j}} = -b_{i_0+p,p-1,\ldots,p-1,i_t-1,i_{t+1},\ldots,i_{f-1},\vec{j}}.$$

Then,
$$P \otimes Q = \sum_{\vec{j}=\vec{0}}^{\vec{q}-1} \left(\sum_{\vec{i}=\vec{0}}^{r+\vec{q}-1} b_{\vec{i},\vec{j}} X_0^{r_0+p-1-i_0} Y_0^{i_0} \prod_{l=1}^{f-1} X_l^{p-1-i_l} Y_l^{i_l} \right) \otimes \prod_{l=0}^{f-1} S_l^{p-1-j_l} T_l^{j_l},$$
where each polynomial in the parentheses satisfies (1), (2), (3). So by Lemma 4.8, we conclude $P \otimes Q \in \langle D_0, \ldots, D_{f-1} \rangle \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j}$, showing $\ker \psi \subset \langle D_0, \ldots, D_{f-1} \rangle \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j}$. We finally have
$$\ker \psi = \langle D_0, \ldots, D_{f-1} \rangle \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j}.$$
By Remark 4, the dimension of $\langle D_0, \ldots, D_{f-1} \rangle$ over $\mathbb{F}_q$ equals $r_0 p^{f-1} + 1$, so after tensoring with $\mathbb{F}_{q^2}$,
$$\dim_{\mathbb{F}_{q^2}} \left(\frac{\bigotimes_{j=0}^{f-1} V_{r_j+p-1}^{\mathrm{Fr}^j}}{\langle D_0, \ldots, D_{f-1} \rangle} \right) = (r_0 + p) p^{f-1} - r_0 p^{f-1} - 1 = q - 1.$$
Thus, over $\mathbb{F}_{q^2}$ we have $\dim_{\mathbb{F}_{q^2}} \left(\frac{\bigotimes_{j=0}^{f-1} V_{r_j+p-1}^{\mathrm{Fr}^j}}{\langle D_0, \ldots, D_{f-1} \rangle} \otimes \bigotimes_{j=0}^{f-1} V_{p-1}^{\mathrm{Fr}^j} \right) = q(q-1) = \dim_{\mathbb{F}_{q^2}} \mathrm{ind}_{T(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \omega_{2f}^r$, and so ψ must be an isomorphism. This completes the proof of Theorem 4.9 (which is also Theorem 1.6). $\square$

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