REDUCTIONS OF GALOIS REPRESENTATIONS AND THE THETA OPERATOR

EKNATH GHATE AND ARVIND KUMAR

ABSTRACT. Let $p \geq 5$ be a prime, and let f be a cuspidal eigenform of weight at least 2 and level coprime to p of finite slope α. Let $\bar{\rho}_f$ denote the mod p Galois representation associated with f and ω the mod p cyclotomic character. Under an assumption on the weight of f, we prove that there exists a cuspidal eigenform g of weight at least 2 and level coprime to p of slope $\alpha + 1$ such that

$$\bar{\rho}_f \otimes \omega \simeq \bar{\rho}_g,$$

up to semisimplification. The proof uses Hida-Coleman families and the theta operator acting on overconvergent forms. The structure of the reductions of the local Galois representations associated to cusp forms with slopes in the interval $[0, 1)$ were determined by Deligne, Buzzard and Gee and for slopes in $[1, 2)$ by Bhattacharya, Ganguli, Ghate, Rai and Rozensztajn. We show that these reductions, in spite of their somewhat complicated behavior, are compatible with the displayed equation above. Moreover, the displayed equation above allows us to predict the shape of the reductions of a class of Galois representations attached to eigenforms of slope larger than 2. Finally, the methods of this paper allow us to obtain upper bounds on the radii of certain Coleman families.

1. INTRODUCTION

Let p be a prime and let $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous, absolutely irreducible, two-dimensional, odd, mod p Galois representation. Such a representation is said to be of Serre-type. Serre's modularity conjecture in its qualitative form claims that every $\bar{\rho}$ of Serre-type is of the form $\bar{\rho}_f$ for some eigenform f. The refined or quantitative form of the conjecture also specifies a minimal weight $k(\bar{\rho})$, known as the Serre weight, and a level $N(\bar{\rho})$ for f. The level $N(\bar{\rho})$ is taken to be the Artin conductor of $\bar{\rho}$ outside p, whereas the weight $k(\bar{\rho})$ is built out of information on the ramification of $\bar{\rho}$ at p. This conjecture is now a theorem due to the celebrated work of Khare [Kha06], Khare-Wintenberger [KW09] and Kisin [Kis09].

Now suppose f is an eigenform of weight $k \geq 2$ with $\bar{\rho}_f$ absolutely irreducible. Let ω denote the mod p cyclotomic character. Since $\bar{\rho}_f$ is of Serre-type, the twisted representation $\bar{\rho}_f \otimes \omega$ is again of Serre-type, hence by Serre's modularity conjecture, it arises from an eigenform, say g, i.e.,

$$\bar{\rho}_f \otimes \omega \simeq \bar{\rho}_g.$$

Serre's conjecture gives the minimal weight, level and character of the eigenform g from the corresponding data for f. It would be interesting to investigate how the slope of g depends on the slope of f. The conjecture does not give any information about this. In many cases, computational

evidence suggests that

$$\text{slope of } g \;\; = \;\; \text{slope of } f + 1, \tag{1}$$

if f and g are normalized to have first Fourier coefficient 1. But in fact (1) is not always true, see Sec. 6 for some examples. However, we prove the following general result.

Theorem 1.1. *Let $p \geq 5$ be a prime and N be a positive integer such that $(p, N) = 1$. Suppose that $f \in S_k(N, \chi)$ is an eigenform of weight $k \geq 2$, level N, character χ and slope α with a p-stabilization f_k of slope α, and suppose $\bar{\rho}_f$ denotes the mod p Galois representation associated with f. Let M_{f_k} be the non-negative integer in Definition 2.12 and let δ_{f_k} be the Kronecker delta function defined in (8). If $k \equiv 2 - \kappa \mod p^{M_{f_k} + \delta_{f_k}}$ with $\kappa \in \{2, 3, \ldots, p^{M_{f_k} + \delta_{f_k}} + 1\}$, then there is an eigenform $g \in S_l(N, \chi)$ of slope $\alpha + \kappa - 1$ such that*

$$\bar{\rho}_f \otimes \omega^{\kappa-1} \simeq \bar{\rho}_g, \tag{2}$$

up to semisimplification. Moreover, if f is a newform, we may choose g to be a newform.

If the slope α of f is smaller than $\frac{k-1}{2}$, there is always a p-stabilization f_k of f of slope α, so in many cases the hypothesis imposed on the p-stabilization in Theorem 1.1 holds automatically. Also, there is (another) non-negative integer M such that the weight l of g in Theorem 1.1 can be chosen to be any integer satisfying the following conditions:

(i) $l > 2\alpha + 2\kappa$,
(ii) $l = (k - 2 + \kappa)p^M + \kappa + n(p - 1)p^M$, for any $n \in \mathbb{Z}$.

The simplest case of the theorem above is the case $\kappa = 2$, which is of special interest.

Corollary 1.2. *Let $p \geq 5$ be a prime and N be a positive integer such that $(p, N) = 1$. Suppose that $f \in S_k(N, \chi)$ is an eigenform of finite slope α as in Theorem 1.1. If $\alpha > 0$, assume that $k \equiv 0 \mod p^{M_{f_k} + \delta_{f_k}}$. Then there is an eigenform $g \in S_l(N, \chi)$ of slope $\alpha + 1$ such that*

$$\bar{\rho}_f \otimes \omega \simeq \bar{\rho}_g, \tag{3}$$

up to semisimplification.

Let f be as in Corollary 1.2. Then there are three forms satisfying condition (3), which are natural choices for the form g in Corollary 1.2. These are

(a) θf, where the theta operator $\theta = q\frac{d}{dq}$ is defined on q-expansions by

$$\theta \left(\sum_{n=0}^{\infty} a_n q^n \right) = \sum_{n=0}^{\infty} n a_n q^n, \tag{4}$$

(b) a minimal weight form associated to $\bar{\rho}_f \otimes \omega$ by Serre's conjecture if $\bar{\rho}_f$ is irreducible,
(c) $f_\omega := f \otimes \omega$, the twist of f by the (Teichmüller lift of the) character ω.

But in Corollary 1.2, we cannot simply take g to be any of the forms above. The form θf in (a) has the right slope $\alpha + 1$ but it is not a classical eigenform because, e.g., the local Galois representation corresponding to θf is still crystalline but has Hodge-Tate weights $(1, k)$ instead of $(0, l)$, for some integer l. Furthermore, the form obtained from Serre's conjecture in (b) and the twisted form f_ω in (c) are classical but they do not have the right slopes. Indeed, as mentioned above (see Sec. 6), the slope of the form obtained in (b) is not necessarily $\alpha + 1$ whereas the form in (c) has infinite slope. However, in this paper we develop a method to construct a classical form g of finite slope $\alpha + 1$ which is closely related to all three forms above in the sense that $g \equiv \theta f \mod p$. In fact, more generally, given an eigenform f of slope α as in Theorem 1.1, we prove in Sec. 3 that there exists an eigenform g of slope $\alpha + \kappa - 1$ such that

$$g \equiv \theta^{\kappa-1} f \mod p \tag{5}$$

(see Theorem 3.2). Theorem 1.1 and Corollary 1.2 now follow immediately from (5). The proof of (5) uses families of overconvergent eigenforms and congruences between them. We spend some time recalling the necessary background about such families in Sec. 2. Finally, we remark that while the forms f and f_k satisfying the congruence conditions on the weight k in Theorem 1.1 and Corollary 1.2 may not be typical, examples of such forms are not hard to write down under some plausible assumptions on the size of M_{f_k}.

One of the chief motivations of this paper was to develop and use a result like Corollary 1.2 to study the images of the reductions of local Galois representations associated with eigenforms of arbitrary weights and slopes. Assume $p \geq 5$. Given a Galois representation $\bar{\rho}_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ coming from an eigenform f, we can restrict it to the subgroup $G_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ to get a local representation $\bar{\rho}_{f,p} := \bar{\rho}_f|_{G_p}$. This representation has been much studied, but its shape is not known in general. However, if f and g are as in Corollary 1.2, then clearly $\bar{\rho}_{f,p}$ is irreducible (respectively, reducible) if and only if $\bar{\rho}_{g,p}$ is irreducible (respectively, reducible). So we obtain the following.

Corollary 1.3. *The phenomenon of irreducibility (respectively, reducibility) of the reduction of local modular Galois representations tends to propagate as the slope increases by one.*

To say more, we work in the more general setting of crystalline representations of G_p. Let $I_p \subset G_p$ denote the inertia subgroup at p. Let ω_2 denote the mod p fundamental character of level 2 of $G_{p^2} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_{p^2}) \subset G_p$. If t is an integer with $p + 1 \nmid t$, let $\mathrm{ind}(\omega_2^t)$ be the unique irreducible two-dimensional mod p representation of G_p, with determinant ω^t and with restriction to I_p given by $\omega_2^t \oplus \omega_2^{pt}$. Let E be a finite extension of $\mathbb{Q}_p$ and $a_p \in \mathfrak{m}_E$, the maximal ideal in the ring of integers $\mathcal{O}_E$. Let v denote the normalized p-adic valuation so that $v(p) = 1$. For $k \geq 2$, let V_{k,a_p} be the irreducible crystalline representation of G_p defined over E, with Hodge-Tate weights $(0, k-1)$ and slope $v(a_p)$, such that $D_{\mathrm{cris}}(V_{k,a_p}^*) = D_{k,a_p}$, where $D_{k,a_p} = Ee_1 \oplus Ee_2$ is the filtered ϕ-module defined in [Ber11]. Let $\bar{V}_{k,a_p}^{ss}$ denote the semisimplification of the mod p reduction of any G_p-stable

$\mathcal{O}_E$-lattice in V_{k,a_p}; it is independent of the choice of the lattice. For any normalized eigenform $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(N)$, with $k \geq 2$, $(p, N) = 1$, $\chi = 1$ and slope $v(a_p) > 0$, it is known that

$$\bar{\rho}_{f,p} \simeq \bar{V}_{k,a_p}^{ss}.$$

For a fixed k and a_p, there are only finitely many possibilities for $\bar{V}_{k,a_p}^{ss}$, up to unramified characters. But computations show that the behavior of this reduction is quite mysterious. The shape of $\bar{V}_{k,a_p}^{ss}$ is known when k is small ($k \leq 2p + 1$), by the work of Fontaine, Edixhoven and Breuil [Edi92], [Bre03]. On the other hand, Berger, Li and Zhu [BLZ04] computed its shape when the slope is large compared to k, that is, when $v(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$. Deligne [Del74] obtained the shape of $\bar{\rho}_{f,p}$ in the case $v(a_p) = 0$. Recently, Buzzard and Gee [BGe09, BGe13] computed the reduction $\bar{V}_{k,a_p}^{ss}$, when $v(a_p) \in (0, 1)$ and Bhattacharya, Ganguli, Ghate, Rai and Rozensztajn [GG15, BG15, BGR18, GR19] computed the reduction if $v(a_p) \in [1, 2)$, for all weights $k \geq 2$.

In the remainder of this section, we assume f and g to be eigenforms as in Corollary 1.2 of slope α and $\alpha + 1$, respectively, so $\bar{\rho}_f \otimes \omega \simeq \bar{\rho}_g$. We also assume $\chi = 1$ for simplicity. Let $\alpha \in [0, 1)$. Since the structure of $\bar{\rho}_{f,p}$ is known due to Deligne, Buzzard and Gee, we immediately get the structure of $\bar{\rho}_{g,p}$, by Corollary 1.2. On the other hand, the structure of $\bar{\rho}_{g,p}$ has been independently and directly computed by the authors above. Hence, we can compare the two results regarding the structure of $\bar{\rho}_{g,p}$, one derived from the structure of the smaller slope eigenform f and Corollary 1.2, and the other directly from results in the literature. We do this in Sec. 4. In all cases (see Tables 1-3 in Sec. 4.1), the two methods to compute the structure of $\bar{\rho}_{g,p}$ are compatible (as they should be!). We can use Corollary 1.2 to partially predict the structure of $\bar{\rho}_{g,p}$ for forms g of slope in $[2, \infty)$, a range of slopes for which the shape of the reductions have yet to be determined completely. We illustrate this in Sec. 4.2 for slope in $[2, 3)$. Recently, the first author made a general conjecture, known as the zig-zag conjecture, describing the reductions of local Galois representations associated with cusp forms of positive half-integral slopes $\leq \frac{p-1}{2}$ and exceptional weights, for which computing the reduction is the trickiest (see [Gha21]). We show that Corollary 1.2 is compatible with the zig-zag conjecture in the cases where the reduction is known (cf. Secs. 4.1.1 and 4.1.3). In [Gha21], it was shown that Corollary 1.2 is compatible with the zig-zag conjecture in general (cf. Sec. 4.3).

Coleman families are not defined on all of weight space, whence the notion of the Coleman radius p^{-r} of a Coleman family of finite slope α, where r is a rational number in the range $(-1 + \frac{1}{p-1}, \infty)$. Gouvêa-Mazur [GM92] gave a conjectural upper bound $\lceil \alpha \rceil$ for a quantity closely related to r, which was subsequently shown to be not valid in all cases by Buzzard-Calegari [BC04]. Wan [Wan98] showed that this quantity is bounded above by a quadratic polynomial in α. This should provide lower bounds for the largest Coleman radius p^{-r} of a Coleman family passing through a form of finite slope α. Since the radius of the Coleman family passing through the cusp form g in Corollary 1.2 is involved in defining its weight, as an application of the above-mentioned compatibility, in Sec. 5 we obtain an *upper bound* for the radius of the Coleman family passing through g, when g has slope in $[1, 2)$.

Let f be an eigenform of finite slope α such that $\bar{\rho}_f$ is irreducible. We end this paper in Sec. 6 by giving examples of cases where the minimal weight eigenform h associated to $\bar{\rho}_f \otimes \omega$ by Serre's conjecture does not have slope $\alpha + 1$. This shows that the constructions made to prove Theorem 1.1 and Corollary 1.2 are of some importance.

2. Overconvergent modular forms

In this section, we recall some basic results on families of overconvergent p-adic modular forms that will be needed in our proof. The definition of overconvergent modular forms was first given by Katz (see, e.g., [Wan98, Sec. 2]). Katz's overconvergent modular forms are defined only for integral weights. In [Col97], Coleman defined overconvergent modular forms of tame level N of weights $\kappa \in \mathcal{W} := \text{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$, the p-adic weight space, which is a p-adic rigid analytic space. These forms incorporate the forms of Katz, since we have an embedding $\mathbb{Z} \hookrightarrow \mathcal{W}$ sending $k \in \mathbb{Z}$ to the character $a \mapsto a^k$, for $a \in \mathbb{Z}_p^*$. Coleman's definition of overconvergent modular forms of integral weights is geometric, while forms of general weight in $\mathcal{W}$ are defined using powers of a weight 1 Eisenstein series. We recall the definition of overconvergent forms following Coleman [Col97]. The reader is referred to [Col96, Col97] for a more systematic treatment, and [BGR84] for background on rigid analysis.

Let $N \geq 1$ be the tame level and $p \geq 5$ a prime that is relatively prime to N. Let $I_m := \{v \in \mathbb{Q} : 0 \leq v < \frac{p^{2-m}}{p+1}\}$ and $I_m^* = I_m \setminus \{0\}$, for an integer $m \geq 1$. Let A be a lift of the Hasse invariant to the modular curve $X_1(N)$ (since $p \geq 5$, we can take $A = E_{p-1}$, the Eisenstein series of level 1 and weight $p - 1$). Then, for $v \in I_1$,

$$X_1(N)(v) := \{x \in X_1(N) : v(A(x)) \leq v\}$$

is an affinoid subdomain of $X_1(N)$. Using the canonical subgroup, we may regard $X_1(N)(v)$ as an affinoid subdomain of $X_1(N; p) = X(\Gamma_1(N) \cap \Gamma_0(p))$, by [Col96, Sec. 6]. Let $X_1(Np)(v)$ be the affinoid subdomain of $X_1(Np)$ which is the inverse image of $X_1(N)(v)$ under the natural forgetful map from $X_1(Np)$ to $X_1(N; p)$. Let ω denote the invertible sheaf on $X_1(Np)$ defined as in [Col96, Sec. 2, 8]. On the non-cuspidal locus, ω is the push-forward of the sheaf of relative invariant differentials of the universal elliptic curve. Then, for $k \in \mathbb{Z}$,

$$M_k^\dagger(v) := \omega^k(X_1(Np)(v))$$

is the space of v-overconvergent modular forms of weight k on $\Gamma_1(Np)$ which converge on $X_1(Np)(v)$ [Col97, p. 449]. If $0 < v' < v$, then there is an injection $M_k^\dagger(v) \hookrightarrow M_k^\dagger(v')$ and their direct limit

$$M_k^\dagger(N) = \varinjlim_{v > 0} M_k^\dagger(v)$$

is the space of overconvergent modular forms of weight k and level Np. Let $S_k^\dagger(N)$ denote the subspace of cusp forms in $M_k^\dagger(N)$ (the subspace of functions vanishing at the cusps in $X_1(Np)(0)$). Every overconvergent modular form has a q-expansion and the v-adic valuation of the p-th Fourier coefficient of a (normalized) form is called the *slope* of the form.

Note that $M_k^\dagger(N)$ is an infinite-dimensional p-adic Banach space and contains the classical modular forms of weight k and level Np. Using the Hodge theory of modular curves, Coleman [Col96, Theorem 8.1] proved the following 'control theorem'.

Theorem 2.1. *An overconvergent modular form of weight k and tame level N is a classical modular form if its slope is strictly less than $k - 1$.*

Let $\theta = q\frac{d}{dq}$ be the theta operator whose action on q-expansions is given by (4). We recall that a suitable power of θ preserves overconvergent forms (see [Col96, Proposition 4.3] and [Col97a]).

Theorem 2.2. *Let $\kappa \geq 2$ be an integer and f be an overconvergent modular form of weight $2 - \kappa$ and tame level N and some character. Then $\theta^{\kappa-1}f$ is also an overconvergent form of weight κ, tame level N and the same character. Moreover, $\theta^{\kappa-1}f$ is an eigenform if f is an eigenform, and is N-new if f is N-new.*

Notation. Let $|\cdot|$ be the norm on $\mathbb{C}_p$, defined by $|x| = p^{-v(x)}$. Let π be a $(p-1)$-st root of $-p$, so $|\pi/p| = p^{\frac{p-2}{p-1}} > 1$, and let $\mathfrak{B}^* = B_{\mathbb{Q}_p}(0, |\pi/p|)$ be the extended disc. Let $D = (\mathbb{Z}/p\mathbb{Z})^*$ and $\hat{D} = \mathrm{Hom}(D, \mathbb{C}_p^*) = \mathbb{Z}/(p-1)\mathbb{Z}$. The rigid analytic space $\mathcal{W}^* = \hat{D} \times \mathfrak{B}^*$ sits inside $\mathcal{W}$, where we identify a point $(i, s) \in \mathcal{W}^*$ with the character $a \mapsto \omega^i(a)\langle\langle a \rangle\rangle^s$, for $a \in \mathbb{Z}_p^*$. Here, ω is the Teichmüller character on $\mathbb{Z}_p^*$ and $\langle\langle \cdot \rangle\rangle : \mathbb{Z}_p^* \longrightarrow \mathbb{Z}_p^*$ is the character defined by $\langle\langle a \rangle\rangle = \frac{a}{\omega(a)}$, so $\langle\langle a \rangle\rangle \equiv 1 \mod p$, for all $a \in \mathbb{Z}_p^*$.

Construction of the characteristic power series of the U_p operator. For $v \in I_1^*$, the U_p operator (sometimes we write $U_{(k)}$) is a completely continuous endomorphism of $M_k^\dagger(v)$. The characteristic power series of the U_p operator plays a major role in the theory of overconvergent modular forms. We give some details.

For simplicity, we write $X(v)$ for $X_1(Np)(v)$, for $v \in I_1$. Let E be the weight one modular form on $\Gamma_1(p)$ with character ω^{-1} as defined in [Col97, p. 447, (1)]. One has $E \equiv 1 \mod p$. Let e be the analytic function on $\bigcup_{v \in I_2} X(v)$ with q-expansion $E(q)/E(q^p)$, so $|e - 1|_{X(0)} \leq |p|$. The following fact is standard, see [Was97, Proposition 5.8]: for $s \in \mathbb{C}_p$, the power series $\sum_{n=0}^\infty \binom{s}{n}T^n$ converges for $|T| < |p|r$ if $|s| \leq |\pi/p|(1/r)$, for $r \leq |\pi/p|$. Taking $r = 1$, we obtain

$$e^s := \sum_{n=0}^\infty \binom{s}{n}(e-1)^n$$

is a well-defined function on $X(0)$, for $s \in \mathbb{C}_p$ with $|s| \leq |\pi/p|$. We need a variant of this. As $|e - 1|_{X(0)} = \lim_{v \to 0^+} |e - 1|_{X(v)}$, we have the following lemma.

Lemma 2.3. (cf. [Col97, Lemma B3.1]) *For any $\epsilon \in \mathbb{R}$ with $|p| < \epsilon$, there exists a $v \in I_2^*$ such that e is defined on $X(v)$ and $|e - 1|_{X(v)} < \epsilon$.*

We obtain the following.

Lemma 2.4. *If $t \in [1, |\pi/p|)$, then there exists a $v \in I_2^*$ such that $|e - 1|_{X(v)} < |\pi|/t$. The function e^s is defined on $X(v)$, for all s such that $|s| \leq t$.*

Proof. By Lemma 2.3 with $\epsilon = |\pi|/t$, there exists a $v \in I_2^*$ such that $|e - 1|_{X(v)} < |\pi|/t$. From the standard fact above applied with $r = |\pi/p|(1/t) \leq |\pi/p|$, we obtain that e^s is defined on $X(v)$, for all s with $|s| \leq t$. $\qquad\square$

For all $s \in \mathfrak{B}^*$ with $|s| \geq 1$ and $v \in I_2^*$ with $|e - 1|_{X(v)} < |\pi/s|$, we see that e^s is defined on $X(v)$, so the operator $u_s := U_{(0)} \circ m_{e^s}$ on $M_0^\dagger(v)$ is well defined, where m_{e^s} is 'multiplication by the function e^s'. Clearly u_s is a completely continuous operator because $U_{(0)}$ is. Let $k \in \mathbb{Z}$ and $v \in I_2^*$ be such that e^k is defined on $X(v)$. Then Coleman observed [Col97, p. 451, (1)] that the Fredholm theory of the operator $U_{(k)}$ on $M_k^\dagger(v)$ is equivalent to that of u_k on $M_0^\dagger(v)$, i.e., we have an equality of Fredholm determinants

$$\det(1 - TU_{(k)}|M_k^\dagger(v)) = \det(1 - Tu_k|M_0^\dagger(v)). \tag{6}$$

Coleman interpolates these Fredholm determinants by constructing a power series $P(s, T)$, for $s \in \mathfrak{B}^*$, which we describe now. Let

$$\mathscr{T}^* := \{(t, v) \in (|\mathbb{C}_p| \cap [1, |\pi/p|)) \times I_2^* : |e - 1|_{X(v)} < |\pi|/t\}.$$

The set $\mathscr{T}^*$ is non-empty, by Lemma 2.4. Let K be a finite extension of $\mathbb{Q}_p$. Put $\mathcal{V}^* := \bigcup_{(t,v) \in \mathscr{T}^*} Z_t(v)$, which is a rigid analytic subspace of $\mathbb{A}^1_{/K} \times X_1(Np)_{/K}$ admissibly covered by the affinoids

$$Z_t(v) = B_K[0, t] \times_K X(v)_{/K}.$$

Write $A(X)$ for the algebra of rigid analytic functions on a rigid analytic space X. Let

$$M(t, v) := A(Z_t(v)) = A(B_K[0, t]) \,\hat{\otimes}_K\, M_0^\dagger(v)$$

be the algebra of rigid analytic functions on $Z_t(v)$.

Let $(t, v) \in \mathscr{T}^*$ and $s \in B_K[0, t]$. The projection

$$Z_t(v) \to B_K[0, t]$$

makes $M(t, v)$ into an $A(B_K[0, t])$-module. We may view u_s as a completely continuous operator $u_s : A(Z_t(v)_s) \to A(Z_t(v)_s)$, where $Z_t(v)_s = \{s\} \times X(v)$ is the fiber above s. Coleman shows there is a completely continuous operator $U_{(t,v)}$ on $M(t, v)$ over $A(B_K[0, t])$ whose restriction to the fiber $Z_t(v)_s$ above s is u_s, i.e., the following diagram commutes:

$$\begin{array}{ccc}
A(Z_t(v)) & \longrightarrow & A(Z_t(v)_s) \\
\downarrow{\scriptstyle U_{(t,v)}} & & \downarrow{\scriptstyle u_s} \\
A(Z_t(v)) & \longrightarrow & A(Z_t(v)_s).
\end{array} \tag{7}$$

Since $A(B_K[0, t])$ is a PID, using [Col97, Lemma A5.1] we see that $M(t, v)$ is orthonormizable over $A(B_K[0, t])$. Thus, for any $(t, v) \in \mathscr{T}^*$, we may define the characteristic power series

$$P_{(t,v)}(s, T) := \det(1 - TU_{(t,v)}|M(t, v)) \in A(B_K[0, t])[[T]].$$

The series $P_{(t,v)}$ is analytic on $B_K[0,t] \times \mathbb{C}_p$, and is independent of (t,v), in the sense that if (t,v) and (t',v') lie in $\mathscr{T}^*$ with $t \leq t'$, then the restriction of $P_{(t',v')}$ from $B_K[0,t'] \times \mathbb{C}_p$ to $B_K[0,t] \times \mathbb{C}_p$ is $P_{(t,v)}$. Using (6) and (7), we have the following result.

Theorem 2.5. ([Col97, Theorem B3.2]) *There is a unique rigid analytic function $P(s,T)$ on $\mathfrak{B}^* \times \mathbb{C}_p$ defined over $\mathbb{Q}_p$, such that for $k \in \mathbb{Z}$ and $v \in I_1^*$,*

$$P(k,T) = \det(1 - TU_{(k)}|M_k^\dagger(v)).$$

Note that D acts via the diamond operators on all the spaces $M_k^\dagger(v)$ and $M(t,v)$. From the decomposition of the space $M(t,v) = \oplus_{\varepsilon \in \widehat{D}} M(t,v,\varepsilon)$, we get

$$P(s,T) = \prod_{\varepsilon \in \widehat{D}} P_\varepsilon(s,T).$$

Similarly, the decomposition $S(t,v) = \oplus_{\varepsilon \in \widehat{D}} S(t,v,\varepsilon)$ for cusp forms induces

$$P^0(s,T) = \prod_{\varepsilon \in \widehat{D}} P_\varepsilon^0(s,T).$$

If i denotes an integer such that $0 \leq i < p-1$, we set $P_i(s,T) = P_{\omega^i}(s,T)$ and $P_i^0(s,T) = P_{\omega^i}^0(s,T)$. We obtain the following.

Theorem 2.6. ([Col97, Theorem B3.3]) *For each $0 \leq i < p-1$, there exists a rigid analytic function $P_i(s,T) \in \mathbb{Q}_p[[s,T]]$ which converges on the region $\mathfrak{B}^* \times \mathbb{C}_p$ such that for an integer k, $P_i(k,T)$ is the characteristic series of the U_p operator acting on overconvergent forms of weight k and character ω^{i-k}.*

Similarly, for cusp forms, we have

$$P_i^0(k,T) = \det(1 - TU_{(k)}|S_k^\dagger(N,\omega^{i-k})).$$

Let $d(k,\varepsilon,\alpha)$ (respectively, $d^0(k,\varepsilon,\alpha)$) be the dimension of the subspace of classical modular forms (respectively, classical cusp forms) of weight k and character $\varepsilon\omega^{-k}$ consisting of forms of finite slope α, for $\varepsilon \in \widehat{D}$. Then from Theorem 2.6, the series $P_\varepsilon(s,T)$ is continuous in the s-variable, so for integers k, k' which are sufficiently close p-adically, the series $P_\varepsilon(k,T)$ and $P_\varepsilon(k',T)$ are p-adically close (see [GM93, Theorem 1]), so their Newton polygons are close and therefore equal, so the number of zeros of $P_\varepsilon(k,T)$ and $P_\varepsilon(k',T)$ of valuation $-\alpha$ in the T-variable are the same. If k, $k' > \alpha+1$, the corresponding forms are classical by the control theorem (Theorem 2.1). A similar argument works for $P_\varepsilon^0(s,T)$. We obtain the following.

Theorem 2.7. ([Col97, Theorem B3.4]) *If $\varepsilon \in \widehat{D}$, $\alpha \in \mathbb{Q}$ and k, $k' > \alpha+1$ are integers which are sufficiently close p-adically, then*

$$
\begin{aligned}
d(k,\varepsilon,\alpha) &= d(k',\varepsilon,\alpha), \\
d^0(k,\varepsilon,\alpha) &= d^0(k',\varepsilon,\alpha).
\end{aligned}
$$

This hints at the existence of p-adic families of overconvergent modular forms to which we turn now.

Coleman families. Coleman defined a Hecke algebra which acts on the space of families of overconvergent modular forms and used it, together with Riesz theory over affinoids, to prove that any overconvergent modular form lives in a family. We describe this result in detail. For a similar result in the more general Hilbert modular setting, see [AIP16, Theorems 3.16 and 3.23].

We first describe the notion of a Coleman family. For each $0 \leq i < p - 1$, let

$$M^\dagger(N, i) = \varprojlim_{t \leq |\pi/p|} \varinjlim_{(t,v) \in \mathscr{T}^*} M(t, v, \omega^i)$$

and

$$S^\dagger(N, i) = \varprojlim_{t \leq |\pi/p|} \varinjlim_{(t,v) \in \mathscr{T}^*} S(t, v, \omega^i),$$

and let $M^\dagger(N) = \oplus_i M^\dagger(N, i)$, $S^\dagger(N) = \oplus_i S^\dagger(N, i)$. These are $A(\mathfrak{B}^*)$-modules.

Suppose α is a positive rational and $k_0 \in \mathfrak{B}^*(K)$ is an integer. Coleman showed that there is an integer $d \geq 0$ such that there exists an affinoid disk $B = B_K[k_0, p^{-r}]$, with $0 < p^{-r} < |\pi/p|$ (so $r \in (-1 + \frac{1}{p-1}, \infty)$), such that the slope $-\alpha$ affinoid in the zero locus of $P_i^0(s, T)$, for $0 \leq i < p - 1$, is finite of degree d over B. The integer d equals the dimension of $S_{k_0}^\dagger(N, \omega^{i-k_0})_\alpha$, the subspace of $S_{k_0}^\dagger(N, \omega^{i-k_0})$ consisting of forms of slope α. If Q is the corresponding factor of $P_i^0(s, T)$ over B, then by the Riesz decomposition theorem [Col97, Theorem A4.3] for the module $S^\dagger(N, i)_B := S^\dagger(N, i) \otimes_{A(\mathfrak{B}^*)} A(B)$, we get a free closed submodule $H := N_{U_B}(Q)$ of rank d over $A(B)$ such that $Q^*(U_B)H = 0$, where U_B is the restriction of U_p on $S^\dagger(N, i)_B$ and for a polynomial $Q(T)$ of degree d, we define $Q^*(T) = T^d Q(T^{-1})$. Let $\mathbb{T}$ denote the Hecke algebra generated over $A(\mathfrak{B}^*)$ by the Hecke operators $T(n)$. Let R be the image of $\mathbb{T} \otimes_{A(\mathfrak{B}^*)} A(B)$ in $\mathrm{End}_{A(B)}(H)$, so R is free of finite rank d over $A(B)$. Moreover, R is the ring of rigid analytic functions on an affinoid $X(R)$ with a finite morphism to B of degree d. We obtain the following theorem (this is [Col97, Theorem B5.7], for weights larger than $\alpha + 1$ in B).

Theorem 2.8. *Suppose $L \subset \mathbb{C}_p$ is a finite extension of K. For $x \in X(R)(L)$, let $\eta_x : R \to L$ be the corresponding homomorphism and set*

$$f_x(q) = \sum_{n=1}^\infty \eta_x(T(n)) q^n.$$

Now suppose k is an integer such that $k \in B$. Then the mapping from $X(R)_k(L)$ to $L[[q]]$, $x \in X(R)_k(L) \mapsto f_x(q)$, is a bijection onto the set of q-expansions of normalized overconvergent cuspidal eigenforms on $X_1(Np)$ over L of weight k, character ω^{i-k} and slope α. Moreover, if $k > \alpha + 1$, the bijection is onto the corresponding space of classical forms.

We need a slight refinement of the above result. We have a surjection

$$R \otimes_{A(B)} \mathrm{QF}(A(B)) \twoheadrightarrow \prod_\lambda K_\lambda,$$

where K_λ are finite extensions of the quotient field $\mathrm{QF}(A(B))$ of $A(B)$ and the $\lambda : R \to K_\lambda$ vary over the (finitely many) $\overline{\mathrm{QF}(A(B))}$-valued points of R. The above surjection is an isomorphism if R is semi-simple, but the semi-simplicity of the U_p-operator does not seem to be known. Let A_λ be the integral closure of $A(B)$ in K_λ. It is also an affinoid, being a finitely generated $A(B)$-module. We let $\mathcal{U}_\lambda = X(A_\lambda)$ be the corresponding rigid analytic space. There is a morphism $\mathcal{U}_\lambda \to B$ which is of finite degree $d_\lambda \geq 0$, and $\sum_\lambda d_\lambda = d$ if R is semi-simple.

Definition 2.9. *The q-expansion $\mathcal{F}_\lambda = \sum_{n=1}^\infty \lambda(T(n))q^n \in A_\lambda[[q]]$ is called a Coleman family.*

Note that $a_n := \lambda(T(n)) \in A_\lambda$, so the a_n may be thought of as rigid analytic functions on $\mathcal{U}_\lambda$ and so may be evaluated at points in $\mathcal{U}_\lambda$ lying over integer points in B. We say that an overconvergent cuspidal eigenform f of integral weight k_0 and character ω^{i-k_0} lives in the Coleman family F_λ if there is a homomorphism $\eta_y : A_\lambda \to L$, corresponding to $y \in \mathcal{U}_\lambda$, for L the field of definition of f, such that $\eta_y(F_\lambda) = \sum_{n=1}^\infty \eta_y(a_n)q^n = \sum_{n=1}^\infty a_n(y)q^n \in L[[q]]$ is the q-expansion of f.

Theorem 2.10. *Every normalized overconvergent cuspidal eigenform f of integral weight k_0 and character ω^{i-k_0} lives in a Coleman family (of type i).*

Proof. Let L be the field of definition of f. By the previous theorem, there is an affinoid ball $B = B_K[k_0, p^{-r}]$, for some r as above, and a morphism $\eta_x : R \to L$ such that $f_x = f$. Let $\mathfrak{m}_x = \ker \eta_x$, a maximal ideal of R. Take a minimal prime $\mathfrak{p}$ of R such that $\mathfrak{p} \subset \mathfrak{m}_x$. Then $R/\mathfrak{p}$ is a domain and its quotient field is a finite integral extension of $\mathrm{QF}(A(B))$. Consider the homomorphism $\lambda : R \to R/\mathfrak{p} \subset A_\lambda$. We claim that f lies in the Coleman family $\mathcal{F}_\lambda$. But this is obvious in view of the fact that η_x is the composition of the maps λ and (any extension η_y to A_λ of) the canonical projection $R/\mathfrak{p} \to R/\mathfrak{m}_x = L$. $\qquad\square$

Everything we have said above holds for the subspaces of N-new forms, as is easy to see. Coleman calls these forms p'-new on [Col97, Definition, p. 467]. Summarizing the above discussion, and including the case of newforms, and the case $\alpha = 0$ which was known earlier by work of Hida [Hid86], [Hid86b], and introducing an auxiliary character χ of level N, we obtain the following theorem:

Theorem 2.11. *Let $p \geq 5$, $p \nmid N \geq 1$, χ a character of level N and $k_0 \in \mathbb{Z}$. Suppose that f is a normalized overconvergent cuspidal eigenform of weight k_0, level Np, character $\chi\omega^{i-k_0}$ and slope $\alpha \geq 0$, defined over a finite extension L of K. Then there exists an $r \in \mathbb{Q} \cap (-1 + \frac{1}{p-1}, \infty)$ and a rigid analytic space $\mathcal{U} \to B_K[k_0, p^{-r}]$ and a family $\mathcal{F} = \sum_{n=1}^\infty a_n q^n$, where a_n's are rigid analytic functions on $\mathcal{U}$ such that the following statements hold:*

(1) *For every integer $k \in B_K[k_0, p^{-r}]$, and every point $\eta_y \in \mathcal{U}$ lying over k, the series $f_k := \sum_{n=1}^\infty \eta_y(a_n)q^n$ coincides with the q-expansion of a normalized overconvergent cuspidal eigenform of weight k, level Np, character $\chi\omega^{i-k}$ and slope α. Moreover, if f is N-new, then all the f_k are N-new.*

(2) *There is an $\eta_{y_0} \in \mathcal{U}$ lying over k_0 such that the series $\sum_{n=1}^{\infty} \eta_{y_0}(a_n)q^n$ coincides with the q-expansion of f.*

Definition 2.12. *We call the radius p^{-r} appearing in Theorem 2.11 the Coleman radius of the family $\mathcal{F}$. Let r_f be such that p^{-r_f} is the supremum of the numbers p^{-r} for which Theorem 2.11 holds. Let $M_f \in \mathbb{Z}_{\geq 0}$ be the smallest non-negative integer such that p^{-M_f} is a Coleman radius.*

We remark that if $r_f \in \mathbb{Z}$, then $M_f = r_f$ if and only if p^{-r_f} is a Coleman radius. If the slope $\alpha = 0$, we know that[1] $M_f = 0$ by the work of Hida [Hid86], [Hid86b].

Congruences. We now show that the forms f_k in a Coleman family are all congruent modulo p, at least if one takes k in the interior of the ball $B_K[k_0, p^{-M_f}]$ in the case when $M_f = r_f$. To this end, let $\delta_f = \delta_{M_f, r_f}$ denote the Kronecker delta function defined by
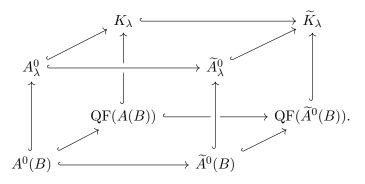
$$\delta_f = \begin{cases} 1, & \text{if } M_f = r_f, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

If the slope $\alpha = 0$, we have $\delta_f = 0$.

Proposition 2.13. *Let the notation be as in Theorem 2.11 and M_f be as in Definition 2.12. For every integer $k \in B_K[k_0, p^{-M_f}]$, with $k \equiv k_0 \mod p^{M_f + \delta_f}$, we have*

$$f_k \equiv f \mod p. \tag{9}$$

Proof. Choose r such that $r_f < r < M_f + \delta_f$, and let $B = B_K[k_0, p^{-r}]$. The Hecke operators in R have norm bounded by 1, so the a_n also have norm bounded by 1. In particular, the a_n lie in the subring of power bounded elements A_λ^0 of A_λ. Let $\mathcal{O}_K$ be the ring of integers of K. Note that A_λ^0 contains $A^0(B) = \mathcal{O}_K\langle (s - k_0)/p^r \rangle$, the power bounded elements in $A(B)$, and that A_λ^0 is the integral closure of $A^0(B)$ in K_λ. Let $\widetilde{A}(B)$ be the completion of $A(B)$ in the s'-adic topology with $s' = (s - k_0)/p^r$ and $\widetilde{A}^0(B) = \mathcal{O}_K[[(s - k_0)/p^r]]$ be the subring of $\widetilde{A}(B)$ consisting of power bounded elements. Let $\widetilde{K}_\lambda$ be the smallest field extension of $\mathrm{QF}(\widetilde{A}^0(B))$ containing K_λ, and $\widetilde{A}_\lambda^0$ the integral closure of $\widetilde{A}^0(B)$ in $\widetilde{K}_\lambda$. Since $\widetilde{A}^0(B)$ is a complete local ring, we see that $\widetilde{A}_\lambda^0$ is a complete local ring. Clearly, $A_\lambda^0 \subset \widetilde{A}_\lambda^0$. The above rings sit in the following diagram:

---

[1]There is an isomorphism $B(0, |\pi/p|) \xrightarrow{\sim} B(0, 1)$ induced by $s \mapsto (1 + p)^s - 1$. Coleman works on the left hand side of this isomorphism whereas Hida works on the right, so that when $\alpha = 0$, we have $r_f = -1 + \frac{1}{p-1}$ and $M_f = 0$.

Let η_y, η_{y_0} be two homomorphisms $A_\lambda^0 \to \mathcal{O}_L$ lying above k and k_0 corresponding to f_k and f_{k_0}, with kernels the prime ideals $\mathfrak{p}_y$, $\mathfrak{p}_{y_0}$, respectively. Because of our choice of k and r, the point k lies in the interior of the ball B. So the restrictions of η_y, η_{y_0} to $A^0(B) = \mathcal{O}_K\langle(s - k_0)/p^r\rangle$ extend to homomorphisms

$$\widetilde{A}^0(B) = \mathcal{O}_K[[(s - k_0)/p^r]] \to \mathcal{O}_L.$$

Since $\widetilde{A}_\lambda^0$ is a finite integral extension of $\widetilde{A}^0(B)$, we can extend the above homomorphisms to homomorphisms $\widetilde{\eta}_y$, $\widetilde{\eta}_{y_0} : \widetilde{A}_\lambda^0 \to \mathcal{O}_{L'}$, for some finite extension L' of L. Let $\widetilde{\mathfrak{m}}_\lambda$ be the maximal ideal of the local ring $\widetilde{A}_\lambda^0$. Let $\widetilde{\mathfrak{p}}_y$ and $\widetilde{\mathfrak{p}}_{y_0}$ denote the prime ideals corresponding to the kernels of $\widetilde{\eta}_y$ and $\widetilde{\eta}_{y_0}$. Clearly the two projections $\widetilde{A}_\lambda^0 \to \widetilde{A}_\lambda^0/\widetilde{\mathfrak{p}}_y \subset \mathcal{O}_{L'}$ and $\widetilde{A}_\lambda^0 \to \widetilde{A}_\lambda^0/\widetilde{\mathfrak{p}}_{y_0} \subset \mathcal{O}_{L'}$ both further project to give the *same* morphism $\widetilde{A}_\lambda^0 \to \widetilde{A}_\lambda^0/\widetilde{\mathfrak{m}}_\lambda \subset \mathcal{O}_{L'}/\mathfrak{m}_{L'}$. It follows that the projections $A_\lambda^0 \to A_\lambda^0/\mathfrak{p}_y \subset \mathcal{O}_L$ and $A_\lambda^0 \to A_\lambda^0/\mathfrak{p}_{y_0} \subset \mathcal{O}_L$ both further project to give the same morphism $A_\lambda^0 \to \mathcal{O}_L/\mathfrak{m}_L$. $\qquad\square$

We make some remarks. Firstly, the proposition implies that $\bar{\rho}_{f_k} \simeq \bar{\rho}_f$ up to semisimplification. Secondly, one might expect that the congruences (9) hold for all weights $k \in B_K[k_0, p^{-M_f}]$, as is the case when $\alpha = 0$ by Hida theory, but we have not been able to show this (but see [CM98, Theorem D]). However, when the map $\mathcal{U} \to B$ has degree 1, so that $\mathcal{U} = B$ is a ball, one can indeed show this under an additional assumption. In fact, one can prove the following stronger Kummer-like congruences for the members of the Coleman family, though they will not be used later.

The open ball $\mathfrak{B}^* = B_K(0, |\pi/p|)$ is isomorphic to the open unit ball $B^0 = B_K(0, 1)$. The map $s \mapsto (1 + p)^s - 1$ induces a morphism of rigid analytic functions $\mathcal{O}_K[[T]] \subset A(B^0) \to A(\mathfrak{B}^*)$ given by $f(T) \mapsto f((1 + p)^s - 1)$, for $f(T) \in \mathcal{O}_K[[T]]$. Recall that, with notation as above, we have $B = B_K[k_0, p^{-r}] \subset \mathfrak{B}^*$, for some r, so there is a map $A(\mathfrak{B}^*) \to A(B)$ obtained by restricting functions. Clearly, the pullback of the specialization map $\eta_k : A(B) \to \bar{\mathbb{Q}}_p$, $s \mapsto k$ under the composition of the two maps above

$$\mathcal{O}_K[[T]] \to A(\mathfrak{B}^*) \to A(B)$$

is the specialization map $\mathcal{O}_K[[T]] \to \bar{\mathbb{Q}}_p$, $T \mapsto (1 + p)^k - 1$.

Now let $\mathcal{F} = \sum_{n=1}^\infty a_n q^n$ be a Coleman family specializing to f at weight k_0, with a_n rigid analytic functions on $\mathcal{U}$. If $\mathcal{U} = B$, then $a_n \in A(B)$, for all $n \geq 1$. For each $a_n \in A(B)$, assume there is a power series $f_n(T) \in \mathcal{O}_K[[T]]$ such that $a_n(s) = f_n((1 + p)^s - 1)$, for all $s \in B$. Write $f_n(T) = \sum_{n=0}^\infty c_\nu T^\nu$, where $c_\nu \in \mathcal{O}_K$. Then for all integers k of the form $k = k_0 + p^{\lceil r \rceil + t}a$, for some

integers a and $t \geq 0$ (so $k \in \mathbb{Z} \cap B$), we have, for each $n \geq 1$,

$$
\begin{aligned}
|a_n(k) - a_n(k_0)| &= |f_n((1+p)^k - 1) - f_n((1+p)^{k_0} - 1)| \\
&= \left| \sum_{\nu=0}^{\infty} c_\nu [((1+p)^k - 1)^\nu - ((1+p)^{k_0} - 1)^\nu] \right| \\
&\leq \sup_{\nu \geq 1} |c_\nu| |(1+p)^k - (1+p)^{k_0}| \\
&\leq |p(k - k_0)| \\
&\leq |p^{t+1}|.
\end{aligned}
$$

Taking the infimum over all r, and recalling that M_f is the non-negative integer defined in Definition 2.12, we obtain a system of Kummer congruences:

$$
k \equiv k_0 \mod p^{M_f + t} \implies f_k \equiv f \mod p^{t+1},
$$

for all $t \geq 0$.

3. Main result

We now prove Theorem 1.1. We start with a useful definition.

Definition 3.1. *Let χ be a character of level N. If f is an overconvergent form of integer weight k, level Np and character $\chi\omega^j$, for some integer j, then we say that f has weight-character $\chi\omega^j \langle\langle \cdot \rangle\rangle^k$.*

Theorem 3.2. *Suppose $p \geq 5$ is a prime and N is a positive integer such that $(N, p) = 1$. Let $f \in S_k(N, \chi)$ be a classical eigenform of finite slope α. Assume that a p-stabilization f_k of f has slope α. Let M_{f_k} be as in Definition 2.12 and δ_{f_k} as in (8). Let $\kappa \in \{2, 3, \ldots, p^{M_{f_k} + \delta_{f_k}} + 1\}$ be the unique integer such that $k \equiv 2 - \kappa \mod p^{M_{f_k} + \delta_{f_k}}$. Then there exists a classical eigenform $g \in S_l(N, \chi)$ of slope $\alpha + \kappa - 1$ such that*

$$
g \equiv \theta^{\kappa - 1} f \mod p.
$$

If f is a newform, so is g. Moreover, there is a non-negative integer M such that the weight l of g can be chosen to be any integer satisfying the following two conditions:

(i) $l > 2\alpha + 2\kappa$,
(ii) $l = (k - 2 + \kappa)p^M + \kappa + n(p-1)p^M$, *for any $n \in \mathbb{Z}$.*

Proof. It is well known that the p-stabilization f_k satisfies $f_k \equiv f \mod p$, since the prime Fourier coefficient of both sides away from p are equal, and the p-th Fourier coefficients are either both units with the same reduction or of positive slope with reduction equal to zero. Let $\mathcal{F}$ be an overconvergent family of slope α as in Theorem 2.11 of radius $p^{-M_{f_k}}$ passing through f_k. Since $k \equiv 2 - \kappa \mod p^{M_{f_k}}$, by Theorem 2.11, we see that there exists an overconvergent eigenform $f_{2-\kappa}$ of tame level N, weight-character $\chi\omega^{k-2+\kappa} \langle\langle \cdot \rangle\rangle^{2-\kappa}$ and slope α in the family $\mathcal{F}$. Since $k \equiv 2 - \kappa \mod p^{M_{f_k} + \delta_{f_k}}$, where δ_{f_k} is as in (8), we have

$$
f \equiv f_k \equiv f_{2-\kappa} \mod p, \tag{10}
$$

by (9). Now, consider the form

$$g_\kappa = \theta^{\kappa-1} f_{2-\kappa}.$$

By Theorem 2.2, g_κ is an overconvergent eigenform of tame level N and weight-character $\chi \omega^{k-2+\kappa} \langle\langle \cdot \rangle\rangle^\kappa$. Clearly g_κ has slope $\alpha + \kappa - 1$ because $f_{2-\kappa}$ has slope α. Applying $\theta^{\kappa-1}$ to (10), we obtain

$$g_\kappa \equiv \theta^{\kappa-1} f \mod p. \tag{11}$$

Let $\mathcal{G}$ be a family consisting of overconvergent eigenforms of slope $\alpha + \kappa - 1$ and tame level N such that g_κ is a weight κ specialization of the family $\mathcal{G}$ (cf. Theorem 2.11). Let M_{g_κ} and δ_{g_κ} be as in Definition 2.12 and (8) for the form g_κ, and set

$$M := M_{g_\kappa} + \delta_{g_\kappa}. \tag{12}$$

Choose a weight $l \equiv \kappa \mod p^M$ and assume that

$$\begin{aligned} &(i) \quad l > 2\alpha + 2\kappa, \\ &(ii) \quad l \equiv k - 2 + 2\kappa \mod (p-1), \end{aligned} \tag{13}$$

and let g_l be an eigenform of weight l in the family $\mathcal{G}$. Note that the weight-character of g_l is $\chi \omega^{k-2+2\kappa-l} \langle\langle \cdot \rangle\rangle^l$ which is equal to $\chi \langle\langle \cdot \rangle\rangle^l$ because of the second assumption above. Furthermore, in view of the first assumption, by the control theorem (Theorem 2.1), we see that g_l is a classical eigenform of weight l, tame level N and character χ. By (9) and (11), we get

$$g_l \equiv g_\kappa \equiv \theta^{\kappa-1} f \mod p. \tag{14}$$

By the Chinese remainder theorem, any simultaneous solution l of the congruence (ii) in (13) and the congruence $l \equiv \kappa \mod p^M$ will be of the form given in the statement of the theorem.

We claim that g_l is the p-stabilization of a form g of slope $\alpha + \kappa - 1$. First note that the eigenform g_l is p-old. Indeed, if g_l is a p-new, by [Miy89, Theorem 4.6.17(ii)], we get

$$c_p^2 = \chi(p) p^{l-2},$$

where c_p denotes the p-th Fourier coefficient of g_l. It follows that the slope of g_l is $(l-2)/2$, i.e.,

$$l = 2\alpha + 2\kappa,$$

which contradicts the first assumption on l in (13). Therefore, the form g_l is p-old and so is a p-stabilization of an eigenform $g(z) = \sum_{n=1}^\infty b_n q^n \in S_l(N, \chi)$. We now show that g has slope $\alpha + \kappa - 1$. We know that g_l is either g_{α_p} or g_{β_p}, where

$$g_{\alpha_p}(z) := g(z) - \beta_p g(pz) \quad \text{and} \quad g_{\beta_p}(z) := g(z) - \alpha_p g(pz)$$

and α_p, β_p are their U_p eigenvalues given by the roots of $X^2 - b_p X + \chi(p) p^{l-1}$. Clearly,

$$\alpha_p + \beta_p = b_p, \tag{15}$$

$$\alpha_p \beta_p = \chi(p) p^{l-1} \implies v(\alpha_p) + v(\beta_p) = l - 1. \tag{16}$$

Since c_p is either α_p or β_p and $v(c_p) = \alpha + \kappa - 1$, by using (16) we conclude that

$$v(\alpha_p) \neq v(\beta_p),$$

otherwise we would get $(l-1)/2 = \alpha + \kappa - 1$, which is not possible because of condition (i) in (13). Therefore, there is no loss of generality in assuming that $v(\alpha_p) < v(\beta_p)$ and hence, by (16), we get

$$v(\alpha_p) < \frac{l-1}{2} \quad \text{and} \quad v(\beta_p) > \frac{l-1}{2}.$$

We now claim that $g_l = g_{\alpha_p}$. If not, $g_l = g_{\beta_p}$, hence their U_p eigenvalues satisfy $c_p = \beta_p$, which gives

$$\alpha + \kappa - 1 = v(\beta_p) > \frac{l-1}{2},$$

which again contradicts condition (i) in (13). We thus get $g_l(z) = g_{\alpha_p}(z) = g(z) - \beta_p g(pz)$ which gives $c_p = \alpha_p$. Since $v(\alpha_p) < v(\beta_p)$, by using (15), we conclude that $v(b_p) = v(\alpha_p) = \alpha + \kappa - 1$, proving that g has slope $\alpha + \kappa - 1$. Since $\beta_p \equiv 0 \mod p$, we have $g \equiv g_l \mod p$. If follows from this and (14), that $g \equiv \theta^{\kappa-1} f \mod p$, which completes the proof when f is an eigenform. If f is N-new, then the forms f_k, $f_{2-\kappa}$, g_κ, g_l and g above are all N-new, completing the proof of the theorem. $\square$

As remarked in the Introduction, the theorem above implies Theorem 1.1, and so also Corollary 1.2.

4. Compatibility with Reductions of Galois representations

Fix a prime $p \geq 5$ and a positive integer N such that $(p, N) = 1$. Let $f = \sum_{n=1}^\infty a_n q^n \in S_k(N)$ be a classical normalized eigenform (with character $\chi = 1$) of slope α, having a p-stabilization f_k of slope α. Assume also that $k \equiv 0 \mod p^{M_{f_k}+\delta_{f_k}}$ if $\alpha > 0$. Let $g = \sum_{n=1}^\infty b_n q^n \in S_l(N)$ be a normalized eigenform (with character $\chi = 1$) of slope $\alpha + 1$ produced by Corollary 1.2, so

$$\bar{\rho}_g \simeq \bar{\rho}_f \otimes \omega. \tag{17}$$

For small slopes $\alpha + 1$, the shape of the local Galois representation $\bar{\rho}_g|_{G_p}$ can be obtained in two ways: one by using (17), thereby reducing the problem to determining the shape of $\bar{\rho}_f|_{G_p}$ which is a form of smaller slope α, and the other directly. Since the reductions $\bar{\rho}_f|_{G_p}$ are known for all slopes smaller than 2, we can compare these two methods to compute $\bar{\rho}_g|_{G_p}$, when $\alpha + 1 \in [1, 2)$. We will see that the computation of $\bar{\rho}_g|_{G_p}$ using these two methods is compatible in all cases (Sec. 4.1). When $\alpha + 1 \geq 2$, we can also use (17) to produce new examples of reductions $\bar{\rho}_g|_{G_p}$ which have not, as far as we know, been shown to exist. We do this in Sec. 4.2 when $\alpha + 1 \in [2, 3)$. Finally, in Sec. 4.3, we recall that (17) is compatible with the zig-zag conjecture of [Gha21].

4.1. **Compatibility for $\alpha \in [0, 1)$.** We divide our discussion into three cases. Sec. 4.1.1 treats the case $\alpha = 0$, Sec. 4.1.2 the case $\alpha \in (0, 1)$, excluding weights $k \not\equiv 3 \mod (p-1)$ if $\alpha = \frac{1}{2}$, and finally Sec. 4.1.3 the exceptional case $\alpha = \frac{1}{2}$ and $k \equiv 3 \mod (p-1)$. Each section contains a table, whose columns we describe now. In the first column, we write down the structure of the reductions of the local Galois representations attached to f on the inertia subgroup I_p using an appropriate reference. Using Corollary 1.2, or more precisely (17), we immediately obtain the shape of the local

Galois representation attached to g on I_p. It is stated in the second column. Clearly, the slope $\alpha + 1$ of g lies in the interval $[1, 2)$. Corollary 1.2 shows that the weight l of g is of the form

$$l = kp^M + 2 + n(p-1)p^M,$$

for any $n \in \mathbb{Z}$. This information is enough to compute the reduction $\bar{\rho}_g|_{I_p}$ directly, using the recent work of the first author and his collaborators. It is listed in the third column. In spite of the rather complicated behavior of the representations involved in the tables, the representations listed in columns 2 and 3 match in all cases.

4.1.1. *Compatibility for $\alpha = 0$.* In an unpublished letter to Serre [Del74], Deligne obtained the shape of $\bar{\rho}_f|_{I_p}$ when $v(a_p) = 0$. It is stated in the first column of Table 1. By (17), we obtain the structure of $\bar{\rho}_g|_{I_p}$. This is listed in the second column. In the third column, we use [BGR18] to directly compute the shape of $\bar{\rho}_g|_{I_p}$. To do this we need some notation. Set $r = l - 2 = kp^M + np^M(p-1)$ and suppose that $r \equiv b \mod (p-1)$ for the set of representatives $2 \le b \le p$ modulo $(p-1)$. Also, if $b = 2$, set

$$t' = v(l-4) \quad \text{and} \quad \tau' = v\left(\frac{b_p^2 - \binom{l-2}{2}p^2}{pb_p}\right) \ge 0. \tag{18}$$

Using [BGR18], we obtain column 3 of Table 1.

Deligne [Del74]	Corollary 1.2	Bh-Gh-Ro [BGR18]
$f \in S_k(N)$ with $v(a_p) = 0$	$g \in S_l(N)$ with $v(b_p) = 1$	
$\bar{\rho}_f\|_{I_p} \simeq \omega^{k-1} \oplus 1.$	$\bar{\rho}_g\|_{I_p} \simeq \omega^k \oplus \omega.$	**Case (i)**: $b = 2$. $M \ge 1 \implies \tau' = 0 = t'.$ $\implies \bar{\rho}_g\|_{I_p} \simeq \omega^b \oplus \omega.$ **Case (ii)**: $b = 3, \ldots, p-1$. $M \ge 1 \implies p \nmid r - b.$ $\implies \bar{\rho}_g\|_{I_p} \simeq \omega^b \oplus \omega.$ **Case (iii)**: $b = p$. $M \ge 1 \implies p \mid r - b.$ $\implies \bar{\rho}_g\|_{I_p} \simeq \omega \oplus \omega.$

Table 1. $(\alpha = 0)$

Since $b \equiv k \mod (p-1)$, the reductions $\bar{\rho}_g|_{I_p}$ listed in columns 2 and 3 of Table 1 match.

4.1.2. *Compatibility for $\alpha \in (0, 1)$ with $k \not\equiv 3 \mod (p-1)$ if $\alpha = \frac{1}{2}$.* In Table 2 we compare the reductions obtained in [BGe09] for slopes in $(0, 1)$ and [BG15] for slopes in $(1, 2)$. These papers do not treat completely the difficult cases of exceptional weights $k \equiv 3 \mod (p-1)$ if $\alpha = \frac{1}{2}$ and $l \equiv 5 \mod (p-1)$ if $\alpha = \frac{3}{2}$, respectively, but we shall deal with them in the following section. To use [BGe09], let $t - 1$ be the residue class of $k - 2$ modulo $(p-1)$, for $1 \le t \le p-1$. To use [BG15], we let $r = l - 2$ and b be as in Table 1. We obtain Table 2.

Buzzard-Gee [BGe09]	Corollary 1.2	Bh-Gh [BG15]
$f \in S_k(N)$ with $v(a_p) \in (0,1)$, $v(a_p) \neq \frac{1}{2}$ if $k \equiv 3 \mod (p-1)$	$g \in S_l(N)$ with $v(b_p) \in (1,2)$, $v(b_p) \neq \frac{3}{2}$ if $l \equiv 5 \mod (p-1)$	
$\bar{\rho}_f\|_{G_p} \simeq \mathrm{ind}\left(\omega_2^t\right)$.	$\bar{\rho}_g\|_{G_p} \simeq \mathrm{ind}\left(\omega_2^{t+p+1}\right)$.	**Case (i)**: $b = 2$. $M \geq 1 \implies p \mid r(r-1)$. $\implies \bar{\rho}_g\|_{G_p} \simeq \mathrm{ind}\left(\omega_2^{b+p}\right)$. **Case (ii)**: $b = 3, \ldots, p-1$. $M \geq 1 \implies p \nmid r - b$. $\implies \bar{\rho}_g\|_{G_p} \simeq \mathrm{ind}\left(\omega_2^{b+p}\right)$. **Case (iii)**: $b = p$. $M \geq 2 \implies p^2 \nmid r - b$. $\implies \bar{\rho}_g\|_{G_p} \simeq \mathrm{ind}\left(\omega_2^{2p}\right)$.

Table 2. ($\alpha \in (0,1)$ with $k \not\equiv 3 \mod (p-1)$ if $\alpha = \frac{1}{2}$)

Note $l \equiv k + 2 \mod (p-1)$ implies $b = t + 1$, so the shape of $\bar{\rho}_g\|_{G_p}$ in columns 2 and 3 of Table 2 match.

4.1.3. *Compatibility for $\alpha = \frac{1}{2}$ and $k \equiv 3 \mod (p-1)$.* In Table 3, we compare the results of [BGe13] which treats the exceptional weights $k \equiv 3 \mod (p-1)$ if the slope is $\frac{1}{2}$ and the recent work [GR19] which treats the exceptional weights $l \equiv 5 \mod (p-1)$ when the slope is $\frac{3}{2}$. In order to use these works we introduce the following notation. Set

$$t = v(k-3) \quad \text{and} \quad \tau = v\left(\frac{a_p^2 - (k-2)p}{pa_p}\right) \geq -\frac{1}{2}, \tag{19}$$

$$t' = v(l-5) \quad \text{and} \quad \tau' = v\left(\frac{b_p^2 - (l-4)\binom{l-3}{2}p^3}{pb_p}\right) \geq \frac{1}{2}. \tag{20}$$

As above, we obtain Table 3.

Buzzard-Gee [BGe13]	Corollary 1.2	Gh-Ra [GR19]
$f \in S_k(N)$ with $v(a_p) = \frac{1}{2}$ and $k \equiv 3 \mod (p-1)$	$g \in S_l(N)$ with $v(b_p) = \frac{3}{2}$ and $l \equiv 5 \mod (p-1)$	
$\bar{\rho}_f\|_{I_p} \simeq$ $\begin{cases} \omega_2^2 \oplus \omega_2^{2p}, & \tau < t, \\ \omega \oplus \omega, & \tau \geq t. \end{cases}$	$\bar{\rho}_g\|_{I_p} \simeq$ $\begin{cases} \omega_2^{p+3} \oplus \omega_2^{3p+1}, & \tau < t, \\ \omega^2 \oplus \omega^2, & \tau \geq t. \end{cases}$	$M \geq 1 \implies t' = 0.$ $\implies \bar{\rho}_g\|_{I_p} \simeq$ $\begin{cases} \omega_2^{p+3} \oplus \omega_2^{3p+1}, & \frac{1}{2} \leq \tau' < 1, \\ \omega^2 \oplus \omega^2, & \tau' \geq 1. \end{cases}$

Table 3. ($\alpha = \frac{1}{2}$ and $k \equiv 3 \mod (p-1)$)

Again, we see that the shapes of $\bar{\rho}_g|_{I_p}$ in columns 2 and 3 of Table 3 are compatible. In fact, Table 3 shows that if $\tau < t$ (respectively, $\tau \geq t$), then we must have $\tau' < 1$ (respectively, $\tau' \geq 1$).

4.2. **Extrapolation of the shape of $\bar{\rho}_g|_{G_p}$.** Let f be an eigenform of slope $\alpha \in [1, 2)$ as in Corollary 1.2. Using the results of [BGR18], [BG15] and [GR19] and (17), we see that there is an eigenform $g = \sum_{n=1}^{\infty} b_n q^n$ of weight l, level coprime to p, and slope $v(b_p) = \alpha + 1 \in [2, 3)$ such that $\bar{\rho}_g|_{I_p}$ has one of the following structures, although of course there may be other structures (e.g., those not coming from the theta operator).

Case (i) $v(b_p) = 2$.
$$\bar{\rho}_g|_{I_p} \simeq \begin{cases} \mathrm{ind}\left(\omega_2^{b+p+2}\right), & b = 2, 3, \ldots, p-1, \\ \omega^{b+1} \oplus \omega^2, & b = 2, 3, \ldots, p, \\ \mathrm{ind}\left(\omega_2^{b+2p+1}\right), & b = 2, p, \end{cases}$$
where $2 \leq b \leq p$ is an integer such that $b \equiv l - 4 \mod (p-1)$.

Case (ii) $v(b_p) \in (2, 3)$ and $l \not\equiv 7 \mod (p-1)$ if $v(b_p) = \frac{5}{2}$.
$$\bar{\rho}_g|_{I_p} \simeq \begin{cases} \mathrm{ind}\left(\omega_2^{b+p+2}\right), & b = 2, 3, \ldots, p-1, \\ \mathrm{ind}\left(\omega_2^{b+2p+1}\right), & b = 2, 3 \ldots, p, \\ \omega^2 \oplus \omega^2, & b = p, \end{cases}$$
where $2 \leq b \leq p$ is an integer such that $b \equiv l - 4 \mod (p-1)$.

Case (iii) $v(b_p) = \frac{5}{2}$ if $l \equiv 7 \mod (p-1)$.
$$\bar{\rho}_g|_{I_p} \simeq \begin{cases} \mathrm{ind}\left(\omega_2^{p+5}\right), \\ \omega^4 \oplus \omega^2, \\ \mathrm{ind}\left(\omega_2^{2p+4}\right), \\ \omega^3 \oplus \omega^3. \end{cases}$$

4.3. **Zig-zag conjecture.** Let $f \in S_k(N)$ be a normalized eigenform of half-integral slope α such that $0 < \alpha \leq \frac{p-1}{2}$. Then f is said to have exceptional weight k for slope α if $k \equiv 2\alpha + 2 \mod (p-1)$. If $b = 2\alpha$, then the first author conjectured that in the general exceptional case, there are $b + 1$ possibilities for the reduction $\bar{\rho}_f|_{I_p}$ with various irreducible and reducible cases occurring alternately (in fact, the conjecture is for general crystalline representations). The precise version is outlined in a conjecture called the zig-zag conjecture [Gha21, Conjecture 1.1]. It is known that the zig-zag conjecture holds for exceptional weights corresponding to slopes $\frac{1}{2}$, 1 and $\frac{3}{2}$ (by [BGe13, BGR18, GR19], respectively). Tables 1 and 3 show that these known cases of the zig-zag conjecture are compatible with the theta operator (more precisely, Corollary 1.2). In [Gha21, Sec. 4.2], the first author showed that the general zig-zag conjecture is compatible with the theta operator. See [Gha21] for further details.

5. Upper bounds for the radii of Coleman families

Let $f \in S_k(N)$ and $g \in S_l(N)$ be two normalized eigenforms as in Corollary 1.2 (with trivial character $\chi = 1$) and slopes α and $\alpha + 1$, respectively. Recall that $p^{-M_{g_2}}$ is a radius for a Coleman family passing through $g_2 = \theta f_0$ (see the proof of Theorem 3.2), and that the integer $M = M_{g_2} + \delta_{g_2}$ defined in (12) appears in the formula for the weight l of g, namely $l = kp^M + 2 + n(p-1)p^M$, for any $n \in \mathbb{Z}$. Assume that $M_{g_2} \neq r_{g_2}$, i.e., $\delta_{g_2} = 0$, so that

$$M = M_{g_2}.$$

In this section, we obtain lower bounds for M_{g_2} (so upper bounds for the radii of the Coleman family passing through g_2) when $\alpha \in [0, 1)$ using the compatibility results in Secs. 4.1.1, 4.1.2 and 4.1.3, regarding the reductions of Galois representations of slopes α and $\alpha + 1$.

Case (1): $\alpha = 0$. In this case $g = \sum_{n=1}^{\infty} b_n q^n$ has slope 1. Let τ' and t' be the parameters defined in (18) when $l \equiv 4 \mod (p-1)$.

Proposition 5.1. *Assume g has slope 1. If $l \not\equiv 4 \mod (p-1)$, then $M \geq 1$. If $l \equiv 4 \mod (p-1)$, then $\tau' = t'$.*

Proof. Recall $r = l - 2 = kp^M + np^M(p-1)$, and $b \equiv r \equiv k \mod (p-1)$ where $2 \leq b \leq p$ is the representative of r modulo $(p-1)$. First assume $b = 3, \ldots p-1$, which is **Case (ii)** of Table 1. Suppose $M = 0$. Choosing $n \equiv k - b \mod p$, we see that $r - b \equiv 0 \mod p$, so the main theorem of [BGR18] shows that $\bar{\rho}_g|_{G_p} \simeq \mathrm{ind}(\omega_2^{b+1})$ is irreducible, whereas the middle column of Table 1 (which comes from Corollary 1.2) yields that $\bar{\rho}_g|_{I_p} \simeq \omega^k \oplus \omega$. This is a contradiction. Hence $M \geq 1$. Now assume $b = p$, which is **Case (iii)** of Table 1. If $M = 0$, choosing $n \not\equiv k \mod p$, we have $r - b \equiv k - n \not\equiv 0 \mod p$, so by [BGR18], $\bar{\rho}_g|_{G_p} \simeq \mathrm{ind}(\omega_2^{b+p})$ is irreducible, another contradiction. Finally, assume $b = 2$. If $\tau' \neq t'$, [BGR18] shows that $\bar{\rho}_g|_{G_p} \simeq \mathrm{ind}(\omega_2^{b+1})$ or $\mathrm{ind}(\omega_2^{b+p})$, both of which are irreducible, giving a contradiction. $\square$

Case (2): $\alpha \in (0, 1)$ and $\alpha \neq \frac{1}{2}$ if $k \equiv 3 \mod (p-1)$.

Proposition 5.2. *If g has slope in $(1, 2)$ and $l \not\equiv 5 \mod (p-1)$ if the slope is $\frac{3}{2}$, then*

$$M \geq \begin{cases} 2, & \text{if } l \equiv 3 \mod (p-1), \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Let notation be as in the proof of the previous proposition. First assume $l \equiv 3 \mod (p-1)$, so $b = p$. This is **Case (iii)** of Table 2. If $M = 1$, then choosing $n \equiv k - 1 \mod p$, gives $r - b \equiv 0 \mod p^2$. Applying the main theorem of [BG15], we see that $\bar{\rho}_g|_{I_p} \simeq \omega \oplus \omega$ is reducible whereas from the middle column of Table 2, it is $\mathrm{ind}(\omega_2^{b+p})$, so irreducible (on G_p), giving a contradiction. If $M = 0$, choosing $n \equiv k - p + kp \mod p^2$ leads to the same contradiction. Therefore, $M \geq 2$. If $l \not\equiv 3 \mod (p-1)$, then we are in **Case (i)** and **(ii)** of Table 2. If $M = 0$, then choosing $n \equiv k - b \mod p$, we see that $p \nmid r(r-1)$ if $b = 2$ and $p \mid r - b$ if $b = 3, \ldots, p-1$. Either way, Bhattacharya and

Ghate [BG15] show that $\bar{\rho}_g|_{G_p} \simeq \mathrm{ind}(\omega_2^{b+1})$ is a *different* irreducible representation, a contradiction, so $M \geq 1$. Note that since we have excluded the exceptional weights $l \equiv 5 \mod (p-1)$ when the slope of g is $\frac{3}{2}$, the main result of [BG15] does indeed apply. $\qquad\square$

<u>Case (3)</u>: $\alpha = \frac{1}{2}$ and $k \equiv 3 \mod (p-1)$. In this case $g = \sum_{n=1}^{\infty} b_n q^n$ has slope $\frac{3}{2}$. Let τ' and t' be as in (20). In this case we are only able to prove the following result.

Proposition 5.3. *If g has slope $\frac{3}{2}$ and $l \equiv 5 \mod (p-1)$, then $\tau' > t'$.*

Proof. The proof is similar to the proof of the last case of Propositions 5.1. If $\tau' \leq t'$, then [GR19] shows that $\bar{\rho}_g|_{I_p} \simeq \omega_2^4 \oplus \omega_2^{4p}$ or $\omega^3 \oplus \omega$, neither of which occur in the middle column of Table 3, giving a contradiction. $\qquad\square$

Corollary 5.4. *In the cases of slopes $\alpha \in [0,1)$ treated above, the least M we can take as one varies over all g of slope $\alpha + 1$ coming from Corollary 1.2 is at least $\lceil \alpha + 1 \rceil$.*

In [Ber19, Theorem 7.4, Remark 6.8], Bergdall obtains general lower bounds for M via a similar analysis involving the reductions of crystalline representations, but using [BLZ04] instead of [BGR18], [BG15], [GR19]. However, we remark that the bound he obtains is trivial, i.e., $M \geq 0$, when the slope is small (e.g., smaller than 2).

6. Slope of the form obtained from Serre's conjecture

Let p be an odd prime in this section. Let $f \in S_k(N, \chi)$ be an eigenform of weight $k \geq 2$, level N coprime to p, character χ and finite slope α such that $\bar{\rho}_f$ is irreducible. Consider the twisted representation $\bar{\rho}_f \otimes \omega$. It is known that $\rho_f|_{I_p} \simeq V_{k, a_p \sqrt{\chi(p)}}$, by [Bre03, Theorem 6.2.1]. By Serre's conjecture (proved in [Kha06], [KW09], [Kis09]), there is a normalized eigenform h of minimal weight (Serre weight) $k(\bar{\rho}_h)$ satisfying

$$\bar{\rho}_h \simeq \bar{\rho}_f \otimes \omega.$$

The aim of this section is to show that the slope of h is not necessarily $\alpha + 1$. In order to show this we give two examples.

Example (1): Assume the form f has slope $\alpha = 0$ so that $\bar{\rho}_f|_{I_p} \simeq \left(\begin{smallmatrix} \omega^{k-1} & * \\ 0 & 1 \end{smallmatrix}\right)$. Suppose that $k \equiv 0 \mod (p-1)$. Then

$$\bar{\rho}_h|_{I_p} = \bar{\rho}_f \otimes \omega|_{I_p} \simeq \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right). \tag{21}$$

By using Serre's recipe for the Serre weight (see [Edi92]), we see that

$$k(\bar{\rho}_h) = \begin{cases} 2 & \text{if } * = 0, \\ 2p & \text{if } * \neq 0. \end{cases}$$

In the latter case ($* \neq 0$), if the slope of h is 1, then by [Bre03], $\bar{\rho}_h|_{I_p} \simeq \left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right)$, which is compatible with (21). However, in the former case ($* = 0$), if the slope of h is 1, then by [Edi92], we would have $\bar{\rho}_h|_{I_p} \simeq \left(\begin{smallmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{smallmatrix}\right)$ which contradicts (21). So we see that if $* = 0$, the form h cannot have slope 1. Forms f of slope 0 with $* = 0$ are rare (the vanishing is related to the existence of companion forms

[Gro90]), but as an example of a form f satisfying this vanishing condition and all the hypotheses of this section, one may take $p = 3$ and the form f corresponding to the elliptic curve of conductor $N = 89$.

Example (2): Assume the form f has slope $\frac{1}{2}$. Suppose $3 < k \equiv 3 \mod (p-1)$ and $\tau \geq t$, where τ and t are defined in (19). Then by [BGe13], we have $\bar{\rho}_f|_{I_p} \simeq \omega \oplus \omega$ up to semisimplification, and so

$$\bar{\rho}_h|_{I_p} \simeq \omega^2 \oplus \omega^2, \tag{22}$$

up to semisimplification. A Serre weight computation gives

$$k(\bar{\rho}_h) = 2p + 3,$$

independently of whether $\bar{\rho}_h|_{I_p}$ is semi-simple or not. Now assume that h has slope $\frac{3}{2}$. The shape of $\bar{\rho}_h|_{G_p}$ has been recently worked out in [GR19]. With notation as in Sec. 4, we have $r = 2p + 1$, so $r \equiv 3 \mod (p-1)$, so $b = 3$. Then $\tau' = \frac{1}{2}$ and $t' = 0$, where τ' and t' are defined by (20) with b_p the p-th Fourier coefficient of h. By the main theorem of [GR19], we have $\bar{\rho}_h|_{I_p} \simeq \omega_2^{p+3} \oplus \omega_2^{3p+1}$ which contradicts (22). Thus h cannot have slope $\frac{3}{2}$. It should be possible to produce a numerical example of a form f satisfying all the hypotheses of this example.

References

[AIP16] F. Andreatta, A. Iovita, V. Pilloni, *On overconvergent Hilbert modular cusp forms*, Astérisque, no. 382 (2016), 163–193.

[Ber19] J. Bergdall, *Upper bounds for constant slope p-adic families of modular forms*, Selecta Math. (N.S.) **25** (2019), no. 4, Paper No. 59, 24 pp.

[Ber11] L. Berger, *La correspondance de Langlands locale p-adique pour* $\mathrm{GL}_2(\mathbf{Q}_p)$, Astérisque (2011), 157–180.

[BLZ04] L. Berger, H. Li, H. Zhu, *Construction of some families of* 2-*dimensional crystalline representations*, Math. Ann. **329** (2004), no. 2, 365–377.

[BG15] S. Bhattacharya, E. Ghate, *Reductions of Galois representations for slopes in* $(1, 2)$, Doc. Math. **20** (2015), 943–987.

[BGR18] S. Bhattacharya, E. Ghate, S. Rozensztajn, *Reductions of Galois representations of slope* 1, J. Algebra **508** (2018), 98–156.

[BGR84] S. Bosch, U. Güntzer, R. Remmert, Non-Archimedean analysis. A systematic approach to rigid analytic geometry. Grundlehren der Mathematischen Wissenschaften **261**, Springer-Verlag, Berlin, 1984.

[Bre03] C. Breuil, *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *II*, J. Inst. Math. Jussieu **2** (2003), no. 3, 23–58.

[BC04] K. Buzzard, F. Calegari, *A counterexample to the Gouvêa-Mazur conjecture*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 10, 751–753.

[BGe09] K. Buzzard, T. Gee, *Explicit reduction modulo p of certain two-dimensional crystalline representations*, Int. Math. Res. Not. **12** (2009), 2303–2317.

[BGe13] K. Buzzard, T. Gee, *Explicit reduction modulo p of certain 2-dimensional crystalline representations, II*, Bull. Lond. Math. Soc. **45** (2013), no. 4, 779–788.

[Col96] R. F. Coleman, *Classical and overconvergent modular forms*, Invent. Math. **124** (1996), no. 1-3, 215–241.

[Col97a] R. F. Coleman, *Classical and overconvergent modular forms of higher level*, J. Théor. Nombres Bordx. **9** (1997), 395–403.

[Col97] R. F. Coleman, *p-adic Banach spaces and families of modular forms*, Invent. Math. **127** (1997), no. 3, 417–479.

[CM98]   R. F. Coleman, B. Mazur, *The eigencurve*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser. **254** (1998), 1–113.

[Del74]   P. Deligne, Letter to J-P. Serre, 1974.

[Edi92]   B. Edixhoven, *The weight in Serre's conjectures on modular forms*, Invent. Math. **109** (1992), no. 3, 563–594.

[GG15]   A. Ganguli, E. Ghate, *Reductions of Galois representations via the mod p Local Langlands Correspondence*, J. Number Theory **147** (2015), 250–286.

[Gha21]   E. Ghate, *A zig-zag conjecture and local constancy for Galois representations*, RIMS Kôkyûroku Bessatsu **B86** (2021), 249–268.

[GR19]   E. Ghate, V. Rai, *Reductions of Galois representations of slope $\frac{3}{2}$*, arXiv:1901.01728 (2019), submitted.

[GM92]   F. Gouvêa, B. Mazur, *Families of modular eigenforms*, Math. Comp. **58** (1992), no. 198, 793–805.

[GM93]   F. Gouvêa, B. Mazur, *On the characteristic power series of the U operator*, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 2, 301–312.

[Gro90]   B. H. Gross, *A tameness criterion for Galois representations associated to modular forms (mod p)*, Duke Math. J. **61** (1990), no. 2, 445–517.

[Hid86]   H. Hida, *Galois representations into $\mathrm{GL}_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms*, Invent. Math. **85** (1986), 545–613.

[Hid86b] H. Hida, *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup. (4) **19** (1986), 231–273.

[Kha06]   C. Khare, *Serre's modularity conjecture: the level one case*, Duke Math. J. **134** (2006), 557–589.

[KW09]   C. Khare, J.-P. Wintenberger. *Serre's modularity conjecture. I and II.*, Invent. Math. **178** (2009), no. 3, 485–586.

[Kis09]   M. Kisin, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180.

[Miy89]   T. Miyake, Modular forms, *Springer-Verlag*, Berlin, 1989.

[Wan98]   D. Wan, *Dimension variation of classical and p-adic modular forms*, Invent. Math. **133** (1998), no. 2, 449–463.

[Was97]   L. C. Washington, Introduction to Cyclotomic Fields, 2nd ed., Graduate Texts in Mathematics, vol. 83, *Springer-Verlag*, New York, Heidelberg, Berlin, 1997.

(Eknath Ghate) School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India.

   E-mail address: `eghate@math.tifr.res.in`

(Arvind Kumar) Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmund Safra Campus, Jerusalem 91904, Israel.

Current Address: Department of Mathematics, Indian Institute of Technology Jammu, Jagti NH-44, PO Nagrota, Jammu 181221, India.

   E-mail address: `arvind.kumar@iitjammu.ac.in`