# Assignment 1

## Analysis I (Fall 2022, Semester I)

**Deadline: September 16, 2022**

September 15, 2022

**1**. Consider the field $\mathcal{F}_0$ of subsets of $\mathbb{R}^n$ consisting of all finite, disjoint unions of right-semiclosed rectangles, i.e, sets of the form $\prod_{1\le j\le n}(a_j, b_j]$ where $-\infty \le a_j < b_j < \infty$ for all $1 \le j \le n$. Now let $\lambda$ denote the standard volume function on the class of rectangles, i.e.,

$$\lambda\left(\prod_{1\le j\le k}(a_j, b_j]\right) = \prod_{1\le j\le k}(b_j - a_j).$$

Show that $\lambda$ extends to a measure on $\mathcal{F}_0$. Consequently, $\lambda$ extends to a unique ($\sigma$-finite) measure on $\mathscr{B}(\mathbb{R}^n)$ called the *Lebesgue measure*. Show that the Lebesgue measure is *translation invariant*, i.e., $\lambda(A + x) = \lambda(A)$ for all $x \in \mathbb{R}^n$ and $A \in \mathscr{B}(\mathbb{R}^n)$.

**2**. Consider $n$ measurable spaces $(\Omega_k, \mathcal{F}_k)$; $1 \le k \le n$ and let $\Omega := \prod_{1\le k\le n}\Omega_k$ be the cartesian product of $\Omega_k$'s. A *measurable rectangle* $R \subset \Omega$ is a set of the form $\prod_{1\le k\le n} A_k$ where $A_k \in \mathcal{F}_k$ for each $1 \le k \le n$.

(a) Show that the class of all measurable rectangles in $\Omega$ is a semifield and hence the class of all finite, disjoint unions of measurable rectangles is a field $\mathcal{A}_0$ of subsets of $\Omega$. The $\sigma$-field generated by $\mathcal{A}_0$, denoted as $\prod_{1\le k\le n}\mathcal{F}_k$, is called the *product $\sigma$-field* and the measurable space $(\Omega, \prod_{1\le k\le n}\mathcal{F}_k)$ is called the *product space*.

(b) Let $j < n$ and fix $\omega_{\le j} \in \prod_{1\le k\le j}\Omega_k$. For any $A \subset \Omega$, define the *section* of $A$ at $\omega_{\le j}$ as follows:

$$A(\omega_{\le j}) = \left\{\omega_{>j} \in \prod_{j<k\le n}\Omega_k : (\omega_{\le j}, \omega_{>j}) \in A\right\} \subset \prod_{j<k\le n}\Omega_k.$$

Show that if $A \in \prod_{1<k\le n}\mathcal{F}_k$, then $A(\omega_{\le j}) \in \prod_{j<k\le n}\mathcal{F}_k$.

(c) (**Not for submission**) Define product $\sigma$-field for an arbitrary collection of measurable spaces and prove (b).

**3**. Keeping with the notation in the previous problem, let $\lambda_n$ denote the $n$-dimensional Lebesgue measure on $\mathscr{B}(\mathbb{R}^n)$. Now suppose that $\mu$ is a translation invariant measure on $\mathscr{B}(\mathbb{R}^n)$ satisfying $\mu(I^n) = 1$ where $I^n := (0, 1]^n$. Through the following steps, we would prove that $\mu = \lambda_n$. All the rectangles below are assumed to be right-semiclosed.

(a) Show that $\mu(R) = \lambda_n(R)$ for any rectangle $R$ with integer side-lengths.

(b) Show that $\mu(R) = \lambda_n(R)$ for any rectangle $R$ with rational lengths.

(c) Show that $\mu(R) = \lambda_n(R)$ for all rectangles $R$ and conclude that $\mu = \lambda_n$.

**4**. Recall the definition of an *outer measure* $\lambda$ on $\Omega$. Also recall that a set $E \subset \Omega$ is called $\lambda$-*measurable* if

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \quad \text{for all} \quad A \subset \Omega.$$

Let $\mathscr{M}$ denote the class of all $\lambda$-measurable sets.

(a) Show that $\mathscr{M}$ is a field.

(b) Show that for any finite sequence of disjoint sets $E_1, \ldots, E_n \in \mathscr{M}$ and $A \subset \Omega$, one has

$$\lambda\left(A \cap \bigcup_{1\le k\le n} E_k\right) = \sum_{1\le k\le n} \lambda(A \cap E_k).$$

(c) Extend (b) to countable, disjoint unions.

(d) Conclude that $\mathscr{M}$ is a $\sigma$-field and that $\lambda$ is a measure on $\mathscr{M}$.

(e) Let $\mu$ be a measure on a field $\mathcal{F}_0$ of subsets of $\Omega$. If $A \subset \Omega$, define

$$\mu^*(A) = \inf\left\{\sum_n \mu(E_n) : A \subset \bigcup_n E_n, E_n \in \mathcal{F}_0\right\}.$$

Show that $\mu^*$ is an outer measure on $\Omega$ and that $\mu^* = \mu$.

(f) Show that the $\sigma$-field of $\mu^*$-measurable sets contains $\mathcal{F}_0$.

**5**. Let $(X, d)$ be a metric space. An outer measure $\lambda$ on $X$ is called a *metric outer measure* if

$$\lambda(A \cup B) = \lambda(A) + \lambda(B)$$

for all pairs of *positively separated* subsets $A$ and $B$ of $X$, i.e., $d(A, B) := \inf_{a\in A, b\in B} d(a, b) > 0$. Through the following steps, we would prove that all Borel subsets of $X$ are $\lambda$-measurable.

(a) Let $F \subset X$ be closed and $A \subset X$. Define, for each $n \geq 1$, $A_n = \{x \in A : d(x, F) \geq 1/n\}$. Show that

$$\lambda(A) \geq \lambda(A \cap F) + \lambda(A_n).$$

(b) Show that $A \setminus F = \bigcup_n A_j$.

(c) Show that

$$\lambda(A) \geq \lambda(A \cap F) + \lambda(A \cap F^c).$$

(d) Conclude that all sets in $\mathscr{B}(X)$ are $\lambda$-measurable.

**6**. Let $(X, d)$ be a metric space and $A \subset X$. Define, for any $\alpha \geq 0$ and $\delta > 0$,

$$\mathcal{H}^\alpha_\delta(A) = \inf\left\{\sum_n (\operatorname{diam} U_n)^\alpha : A \subset \bigcup_n U_n,\, U_n \subset X,\, \operatorname{diam} U_n < \delta\right\}$$

where $\operatorname{diam} U := \sup\{d(x, y) : x, y \in U\}$ is the $d$-diameter of $U$. **By convention,** we set $0^0 = 1$ and $\operatorname{diam}(\emptyset)^0 = 0$. $\mathcal{H}^\alpha_\infty(A)$ is called the *Hausdorff content* of $A$. Clearly, $\mathcal{H}^\alpha_\delta(A)$ is decreasing in $\delta$. Now define the *$\alpha$-dimensional Hausdorff measure* of $A$, denoted as $\mathcal{H}^\alpha(A)$, as follows:

$$\mathcal{H}^\alpha(A) = \sup_{\delta>0} \mathcal{H}^\alpha_\delta(A) = \lim_{\delta\to 0} \mathcal{H}^\alpha_\delta(A)$$

(a) Prove that $\mathcal{H}^\alpha$ is an outer measure on $X$ and hence, by the previous problem, all Borel subsets of $X$ is $\mathcal{H}^\alpha$-measurable.

(b) What if we replaced $(\operatorname{diam}(C))^\alpha$ with any $\overline{\mathbb{R}}_{\geq 0}$-valued set function $\tau$ satisfying $\tau(\emptyset) = 0$.

(c) Define the *Hausdorff dimension* of $S \subset X$, denoted by $\dim_{\mathrm{H}}(S)$, as follows:

$$\dim_{\mathrm{H}}(S) = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(S) = 0\}.$$

Show that $\mathcal{H}^d(S) = \infty$ for all $d < \dim_{\mathrm{H}}(S)$ so that $\dim_{\mathrm{H}}(S)$ can be alternatively defined as

$$\dim_{\mathrm{H}}(S) = \sup\{\alpha \geq 0 : \mathcal{H}^\alpha(S) = \infty\}.$$

**7**. Let $(X, d)$ be a metric space and $\mu$ be a Borel measure on $X$. $\mu$ is called *regular* if

$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ compact}\} = \inf\{\mu(U) : U \supset A \text{ open}\}$$

for all $A \in \mathscr{B}(X)$. The first condition is called *inner regularity* whereas the second condition is called *outer regularity*. $\mu$ is called a *Radon measure* if it is regular and *locally finite*, i.e., every point of $X$ has a neighborhood with finite measure. Through the following steps, we would prove that any *finite*, Borel measure on $X$ is a Radon measure if $X$ is a *Polish space* — a separable, completely metrizable topological space.

(a) Show that for any $\varepsilon > 0$ and $A \in \mathscr{B}(X)$, there exists a closed set $F \subset A$ and an open set $G \supset A$ such that $\mu(G \setminus F) < \varepsilon$.

(b) (**Tightness**) Show that for any $\varepsilon > 0$, there exists $K \subset X$ compact such that $\mu(X \setminus K) < \varepsilon$.

(c) Conclude that $\mu$ is regular.