Assignment 2

Analysis I (Fall 2022, Semester I)

Deadline: October 5, 2022

October 10, 2022

1. Consider a topological space X and a function $f : X \mapsto \mathbb{R}$. Show that the set $C_f \subset X$ of continuity points of f lies in $\mathscr{B}(X)$.

2. For any $s \in \mathbb{R}$, consider the integral $I(s) := \int_{\mathbb{R}_{\geq 0}} x^{s-1}e^{-x}dx$ where $\mathbb{R}_{\geq 0} := [0, \infty)$.

(a) Show that $I(s) < \infty$ if and only if $s > 0$ in which case we call it the *Gamma function*, denoted as $\Gamma(s)$.

(b) Show that $\Gamma(\cdot)$ is differentiable and give an integral formula for $\Gamma'(s)$.

(c) Show that $\Gamma'(1) = \lim_n \int_{[0,n]}(1 - t/n)^n \ln t \, dt$.

3. Let λ be a signed measure on a σ-filed $\mathcal{F}$ of subsets of Ω. Show that

$$|\lambda|(A) = \sup\left\{ \sum_{1\leq j\leq n} |\lambda(E_i)| : E_n \subset A, E_n \in \mathcal{F} \text{ are pairwise disjoint} \right\}.$$

4. A *complex measure* λ on a σ-field $\mathcal{F}$ of subsets of Ω is a complex-valued, countably additive set function of the form $\lambda_1 + i\lambda_2$ where λ_1 and λ_2 are finite signed measures. Define the *total variation* $|\lambda|$ of λ as in the previous problem.

(a) Show that $|\lambda|$ is a measure on $\mathcal{F}$.

(b) Define $\lambda \ll \mu$ in the usual manner for any measure μ. Show that $\lambda \ll \mu$ iff $|\lambda| \ll \mu$.

(c) If λ is finite, then $\lambda \ll \mu$ iff $\lim_{\mu(A)\to 0} \lambda(A) = 0$.

5. Consider the space $L^p(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \lambda)$ where λ is the Lebesgue measure on $\mathscr{B}(\mathbb{R}^n)$ and $1 \le p < \infty$. Through the following steps, we would prove that the normed space $L^p(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \lambda)$ is separable (here we identify all the functions that agree almost everywhere $[\lambda]$).

(a) If $f \in L^p$ and $\varepsilon > 0$, there is a finite-valued simple function $|g| \le |f|$ in L^p such that $\|f - g\|_p < \varepsilon$.

(b) Prove that the Lebesgue measure λ is *regular* according to the definition given in Problem 7 in Assignment 1.

(c) Conclude that L^p is separable.

6. Let (X, d) be a complete, separable, locally compact metric space, μ a finite measure on $\mathscr{B}(X)$ and $C_c(X)$ denote the space of all compactly supported continuous functions on X. Prove that $C_c(X)$ is a dense subspace of $L^p(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu)$ for any $1 \le p < \infty$.

HINT: Use a special property of this type of measures and Urysohn's lemma.

7. (**Chain rule**) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $g : (\Omega, \mathcal{F}) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be non-negative. Define a measure λ on $\mathcal{F}$ by

$$\lambda(A) = \int_A g \, d\mu \,, \quad A \in \mathcal{F}.$$

Show that if $f : (\Omega, \mathcal{F}) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$,

$$\int_\Omega f \, d\lambda = \int_\Omega fg \, d\mu$$

in the sense that if one of the integrals exists, so does the other, and the two integrals are equal. In particular, prove that if $\lambda \ll \nu$ and $\nu \ll \mu$ (so that $\lambda \ll \mu$) where λ is a signed measure and ν, μ are measures on $\mathcal{F}$ and both $\frac{d\lambda}{d\nu}$ and $\frac{d\nu}{d\mu}$ exist, then $\frac{d\lambda}{d\mu}$ exists and equals $\frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}$ a.e. $[\mu]$.