Assignment 3

Analysis I (Fall 2022, Semester I)

Deadline: November 18, 2022

November 1, 2022

1. Let G be a locally compact topological group. A *left Haar measure* (respectively *right Haar measure*) on G is a nonzero regular Borel measure μ on G such that $\mu(gA) = \mu(A)$ (respectively $\mu(Ag) = \mu(A)$) for all $g \in G$ and all measurable subsets A of G. In the remainder of this problem we will assume that μ is a left Haar measure on G.

(a) Show that $\mu(U) > 0$ for all open $U \subset G$ and also that $\int_G f d\mu > 0$ for any nonnegative $f \in C_c(G)$ that is not identically 0.

(b) Show that there always exists a nonnegative $\varphi \in C_c(G)$ that is not identically 0.

(c) Show that $C_c(G) \subset L^1(G)$.

(d) Show that $\int_G f(hg)d\mu(g) = \int_G f(g)d\mu(g)$ for all $f \in L^1(G, \mathscr{B}(G), \mu)$ and $h \in G$.

(e) Consider a function $\varphi \in C_c(G)$ that is not identically 0 and a left Haar measure ν (not necessarily equal to μ). Now define, for any $f \in C_c(G)$,

$$F_f(g, h) = \frac{f(g)\,\varphi(hg)}{\int_G \varphi(kg) d\nu(k)}\,.$$

Show that $F_f \in L^1(G \times G, \mathscr{B}(G)^{\otimes 2}, \mu \otimes \nu)$.

(f) Show that $\int_G f d\mu = \int_{G\times G} F_f(g, h)\, d\mu(g) d\nu(h) = \int_{G\times G} F_f(h^{-1}, gh)\, d\mu(g) d\nu(h)$ and deduce that $\frac{\int_G f d\mu}{\int_G \varphi d\mu}$ is independent of the *choice* of μ.

(g) (Uniqueness of Haar measure upto scaling) Conclude that any left Haar measure ν satisfies $\nu = a\mu$ for some $a > 0$.

HINT: Use the Riesz representation theorem.

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $A \in \mathcal{F}$ be such that $\mu(A) < \infty$. Also let S be a closed subspace of $\mathbb{C}$. Now suppose that $f \in L^1(\Omega, \mathcal{F}, \mu)$ is such that

$$\frac{1}{\mu(E)} \int_E f d\mu \in S$$

for every (measurable) $E \subset X$ with $\mu(E) > 0$. Show that $f(\omega) \in S$ for almost every $\omega \in A$.

3. Let μ be a complex measure on $\mathcal{F}$ — a σ-algebra of subsets of Ω. Then there is a *choice* h for the Radom-Nikodym derivative $\frac{d\mu}{d|\mu|}$ (where $|\mu|$ is the total variation of μ defined in Problem 4 in Assignment 2) such that $|h(\omega)| = 1$ for all ω.

4. Let Y be a closed subspace of a Banach space X.

(a) Show that if X is separable, then Y and X/Y are separable.

(b) Show that if Y and X/Y are separable, then X is separable.

5. Let $(X, \|\cdot\|_X)$ be an infinite-dimensional separable Banach space and $\{e_\gamma\}$ be an *algebraic* basis for X. Define a new norm $\|\cdot\|$ on X by $\|x\| = \sum |x_\gamma|$ for $x = \sum x_\gamma e_\gamma$. Show that $\|\cdot\|$ is indeed a norm on X and prove that it is not equivalent to $\|\cdot\|_X$.

6. Let X be a Banach space and f be a linear functional on X.

(a) Show that $f \in X^*$ if and only if $f^{-1}(0)$ is closed.

(b) Show that if f is not continuous, then $f^{-1}(0)$ is dense in X.

7. Show that if X is an infinite-dimensional Banach space, then X admits a discontinuous linear functional. Conclude that a Banach space X is infinite-dimensional if and only if it has a subspace that is not closed.

8. Let H be a Hilbert space. Show that there exists an abstract set Γ such that H is isometric to $\ell_2(\Gamma)$

9. Let $C^1[0, 1]$ be the normed space of all real-valued functions on $[0, 1]$ with a continuous derivative, endowed with the supremum norm. Define a linear map T from $C^1[0, 1]$ into $C[0, 1]$ by $T(f) = f'$. Show that T is closed. Prove that T is not bounded. Explain why the closed graph theorem cannot be used here.