Assignment 3

Analysis I (Fall 2024, Semester I)

Deadline: November 04, 2024

October 29, 2024

1. Let $\{\mu_n\}$ and $\{\lambda_n\}$ be two sequences of finite measures. Put

$$\overline{\mu}_n = \sum_{i=1}^{n} \mu_i,\ \overline{\lambda}_n = \sum_{i=1}^{n} \lambda_i,\ \mu = \sum_{i=1}^{\infty} \mu_i,\ \lambda = \sum_{i=1}^{\infty} \lambda_i.$$

Assume that μ, λ are finite measures, $\overline{\lambda}_n \ll \overline{\mu}_n$, for all $n = 1, 2, \ldots$. Show that $\lambda \ll \mu$ and

$$\frac{d\overline{\lambda}_n}{d\overline{\mu}_n} \to \frac{d\lambda}{d\mu} \text{ a.e. } [\mu].$$

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $A \in \mathcal{F}$ be such that $\mu(A) < \infty$. Also let S be a closed subspace of $\mathbb{C}$. Now suppose that $f \in L^1(\Omega, \mathcal{F}, \mu)$ is such that

$$\frac{1}{\mu(E)} \int_E f d\mu \in S$$

for every (measurable) $E \subset X$ with $\mu(E) > 0$. Show that $f(\omega) \in S$ for almost every $\omega \in A$.

3. Let μ be a complex measure on $\mathcal{F}$ — a σ-algebra of subsets of Ω. Then there is a *choice* h for the Radom-Nikodym derivative $\frac{d\mu}{d|\mu|}$ (where $|\mu|$ is the total variation of μ defined in Problem 3 in Assignment 2) such that $|h(\omega)| = 1$ for all ω.

4. Let $f : [a, b] \mapsto \mathbb{R}$ be a function such that $f(x) - f(a) = \int_{[a,x]} g(t)\, dt$ for all $x \in [a, b]$ where $g : [a, b] \mapsto \mathbb{R}$ is some integrable function. Show that $f'(x)$ exists and equals $g(x)$ for λ almost every x where λ is the standard Lebesgue measure.

5. Let $f : [a, b] \mapsto \mathbb{R}$ be Lipschitz continuous, i.e., $|f(x) - f(y)| \leq L|x - y|$ for some fixed positive number L and all $x, y \in [a, b]$. Show that f is differentiable for λ almost every x on $[a, b]$ where λ is the standard Lebesgue measure.

6. Let $(X_n, \tau_n)_{n\geq 1}$ be a countable family of second countable topological spaces and $(\prod_{n\geq 1} X_n, \prod_{n\geq 1} \tau_n)$ denote their product. Show that $\mathscr{B}(\prod_{n\geq 1} X_n) = \prod_{n\geq 1} \mathscr{B}(X_n)$ where the former is the Borel σ-field for the product space $(\prod_{n\geq 1} X_n, \prod_{n\geq 1} \tau_n)$ and the latter is the product σ-field of $\mathscr{B}(X_n); n \geq 1$.

7. (**Not for submission**) Does the equality in the previous problem remain true for any uncountable family of topological spaces $(X_\alpha, \tau_\alpha)_\alpha$?