

Assignment 3

Analysis I (Fall 2026, Semester I)

October 12, 2026

Note: Unless specifically mentioned, assume all the functions and function spaces to be complex valued.

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $A \in \mathcal{F}$ be such that $\mu(A) < \infty$. Also let S be a closed subspace of $\mathbb{C}$. Now suppose that $f \in L^1(\Omega, \mathcal{F}, \mu)$ is such that

$$\frac{1}{\mu(E)} \int_E f d\mu \in S$$

for every (measurable) $E \subset X$ with $\mu(E) > 0$. Show that $f(\omega) \in S$ for almost every $\omega \in A$.

2. Let μ be a complex measure on $\mathcal{F}$ — a σ -algebra of subsets of Ω . Then there is a choice h for the Radon-Nikodym derivative $\frac{d\mu}{d|\mu|}$ (where $|\mu|$ is the total variation of μ defined in Problem 3 in Assignment 2) such that $|h(\omega)| = 1$ for all ω .
3. Let $f : [a, b] \mapsto \mathbb{R}$ be a function such that $f(x) - f(a) = \int_{[a, x]} g(t) dt$ for all $x \in [a, b]$ where $g : [a, b] \mapsto \mathbb{R}$ is some integrable function. Show that $f'(x)$ exists and equals $g(x)$ for λ almost every x where λ is the standard Lebesgue measure.
4. Let $f : [a, b] \mapsto \mathbb{R}$ be Lipschitz continuous, i.e., $|f(x) - f(y)| \leq L|x - y|$ for some fixed positive number L and all $x, y \in [a, b]$. Show that f is differentiable for λ almost every x on $[a, b]$ where λ is the standard Lebesgue measure.

5. Let $(X_n, \tau_n)_{n \geq 1}$ be a countable family of second countable topological spaces and $(\prod_{n \geq 1} X_n, \prod_{n \geq 1} \tau_n)$ denote their product. Show that $\mathcal{B}(\prod_{n \geq 1} X_n) = \prod_{n \geq 1} \mathcal{B}(X_n)$ where the former is the Borel σ -field for the product space $(\prod_{n \geq 1} X_n, \prod_{n \geq 1} \tau_n)$ and the latter is the product σ -field of $\mathcal{B}(X_n); n \geq 1$.
6. Does the equality in the previous problem remain true for any uncountable family of topological spaces $(X_\alpha, \tau_\alpha)_\alpha$?
7. Let $(\Omega, \mathcal{F}, \mu)$ be a σ -finite measure space such that $\mathcal{F}$ is countably generated, i.e., there is a countable family $\mathcal{F}_0$ of subsets of Ω such that $\mathcal{F} = \sigma(\mathcal{F}_0)$. Prove that $L^p(\Omega, \mathcal{F}, \mu)$ is separable for all $1 \leq p < \infty$.
8. Show that polynomials with rational coefficients are dense in $L^p([0, 1])$ where the unit interval $[0, 1]$ is equipped with the Borel σ -algebra and the Lebesgue measure.
9. Suppose that the measure space $(\Omega, \mathcal{F}, \mu)$ contains pairwise disjoint sets each with positive measure. So that $L^\infty(\Omega, \mathcal{F}, \mu)$ is NOT separable.
10. Let G be a locally compact topological group. A *left Haar measure* (respectively *right Haar measure*) on G is a nonzero regular Borel measure μ on G such that $\mu(gA) = \mu(A)$ (respectively $\mu(Ag) = \mu(A)$) for all $g \in G$ and all measurable subsets A of G . In the remainder of this problem we will assume that μ is a left Haar measure on G .
 - (a) Show that $\mu(U) > 0$ for all open $U \subset G$ and also that $\int_G f d\mu > 0$ for any nonnegative $f \in C_c(G)$ that is not identically 0.
 - (b) Show that there always exists a nonnegative $\varphi \in C_c(G)$ that is not identically 0.
 - (c) Show that $C_c(G) \subset L^1(G)$.
 - (d) Show that $\int_G f(hg) d\mu(g) = \int_G f(g) d\mu(g)$ for all $f \in L^1(G, \mathcal{B}(G), \mu)$ and $h \in G$.
 - (e) Consider a function $\varphi \in C_c(G)$ that is not identically 0 and a left Haar measure ν (not necessarily equal to μ). Now define, for any $f \in C_c(G)$,

$$F_f(g, h) = \frac{f(g) \varphi(hg)}{\int_G \varphi(kg) d\nu(k)}.$$

Show that $F_f \in L^1(G \times G, \mathcal{B}(G)^{\otimes 2}, \mu \otimes \nu)$.

- (f) Show that $\int_G f d\mu = \int_{G \times G} F_f(g, h) d\mu(g) d\nu(h) = \int_{G \times G} F_f(h^{-1}, gh) d\mu(g) d\nu(h)$ and deduce that $\frac{\int_G f d\mu}{\int_G \varphi d\mu}$ is independent of the choice of μ .
- (g) (Uniqueness of Haar measure upto scaling) Conclude that any left Haar measure ν satisfies $\nu = a\mu$ for some $a > 0$.

HINT: Use the Riesz representation theorem.