Assignment 2

Analysis I (Fall 2024, Semester I)

Deadline: October 16, 2024

October 3, 2024

1. For any $s \in \mathbb{R}$, consider the integral $I(s) := \int_{\mathbb{R}_{\geq 0}} x^{s-1}e^{-x}dx$ where $\mathbb{R}_{\geq 0} := [0, \infty)$.

(a) Show that $I(s) < \infty$ if and only if $s > 0$ in which case we call it the *Gamma function*, denoted as $\Gamma(s)$.

(b) Show that $\Gamma(\cdot)$ is differentiable and give an integral formula for $\Gamma'(s)$.

(c) Show that $\Gamma'(1) = \lim_n \int_{[0,n]} (1 - t/n)^n \ln t \, dt$.

2. Let λ be a signed measure on a σ-filed $\mathcal{F}$ of subsets of Ω. Show that

$$|\lambda|(A) = \sup \left\{ \sum_{1 \leq j \leq n} |\lambda(E_i)| : E_n \subset A, E_n \in \mathcal{F} \text{ are pairwise disjoint} \right\}.$$

3. A *complex measure* λ on a σ-field $\mathcal{F}$ of subsets of Ω is a complex-valued, countably additive set function of the form $\lambda_1 + i\lambda_2$ where λ_1 and λ_2 are finite signed measures. Define the *total variation* $|\lambda|$ of λ as in the previous problem.

(a) Show that $|\lambda|$ is a measure on $\mathcal{F}$.

(b) Define $\lambda \ll \mu$ in the usual manner for any measure μ. Show that $\lambda \ll \mu$ iff $|\lambda| \ll \mu$.

4. (**Scheffé's lemma**) Let $(f_n)_{n \geq 1}$ be a sequence of integrable functions on a measure space $(\Omega, \mathcal{F}, \mu)$ that converges almost everywhere to an integrable function f. Show that $\int_\Omega |f_n - f| d\mu \to 0$ if and only if $\int_\Omega |f_n| d\mu \to \int_\Omega |f| d\mu$.

5. Let f be an integrable function on a measure space $(\Omega, \mathcal{F}, \mu)$. Show that

$$\lim_{\mu(A)\to 0} \int_A f d\mu = 0$$

in the sense that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\int_A f d\mu| < \varepsilon$ whenever $\mu(A) < \delta$.

6. (**Chain rule**) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $g : (\Omega, \mathcal{F}) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be non-negative. Define a measure λ on $\mathcal{F}$ by

$$\lambda(A) = \int_A g\, d\mu\,, \quad A \in \mathcal{F}.$$

Show that if $f : (\Omega, \mathcal{F}) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$,

$$\int_\Omega f\, d\lambda = \int_\Omega fg\, d\mu$$

in the sense that if one of the integrals exists, so does the other, and the two integrals are equal. In particular, prove that if $\lambda \ll \nu$ and $\nu \ll \mu$ (so that $\lambda \ll \mu$) where λ is a signed measure and ν, μ are measures on $\mathcal{F}$ and both $\frac{d\lambda}{d\nu}$ and $\frac{d\nu}{d\mu}$ exist, then $\frac{d\lambda}{d\mu}$ exists and equals $\frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}$ a.e. $[\mu]$.