Assignment 2

Analysis I (Fall 2026, Semester I)

September 22, 2026

1. Throughout this problem, $(f_n)_{n\geq 1}$ and f stand for extended-real valued measurable functions on a measure space $(\Omega, \mathcal{F}, \mu)$.

(a) We say $f_n \to f$ *in measure* if $\mu(|f_n - f| > \varepsilon) \to 0$ for every $\varepsilon > 0$. Prove that $f_n \to f$ in measure if and only if every subsequence has a further subsequence converging to f a.e.

(b) Prove or disprove: if $f_n \to f$ a.e., then $f_n \to f$ in measure.

(c) Prove or disprove: If $f_n \to f$ in measure, then $f_n \to f$ a.e. $[\mu]$.

(d) Do the answers to the previous two items change if μ is given to be a *finite* measure?

(e) We say that $f_n \to f$ almost uniformly if for every $\varepsilon > 0$, there exists $A \in \mathcal{F}$ with $\mu(A) < \varepsilon$ such that $f_n \to f$ uniformly on A^c. Now suppose that μ is a finite measure. Prove that if $f_n \to f$ a.e., then $f_n \to f$ almost uniformly.

2. (Uniform integrability) A family $\mathcal{S}$ of integrable functions defined on a *finite* measure space $(\Omega, \mathcal{F}, \mu)$ is called *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{f\in\mathcal{S}} \int_{\{|f|>M\}} |f| d\mu = 0.$$

Suppose that $f_n \to f$ in measure and (f_n) is uniformly integrable. Prove that f is integrable and $\int_\Omega |f_n - f| d\mu \to 0$.

3. A *complex measure* λ on a σ-field $\mathcal{F}$ of subsets of Ω is a complex-valued, countably additive set function of the form $\lambda_1 + i\lambda_2$ where λ_1 and λ_2 are finite signed measures. Define the *total variation* $|\lambda|$ of λ as in the previous problem.

(a) Show that $|\lambda|$ is a measure on $\mathcal{F}$.

(b) Define $\lambda \ll \mu$ in the usual manner for any measure μ. Show that $\lambda \ll \mu$ iff $|\lambda| \ll \mu$.

4. (**Scheffé's lemma**) Let $(f_n)_{n\geq 1}$ be a sequence of integrable functions on a measure space $(\Omega, \mathcal{F}, \mu)$ that converges almost everywhere to an integrable function f. Show that $\int_\Omega |f_n - f| d\mu \to 0$ if and only if $\int_\Omega |f_n| d\mu \to \int_\Omega |f| d\mu$.

5. Let f be an integrable function on a measure space $(\Omega, \mathcal{F}, \mu)$. Show that

$$\lim_{\mu(A)\to 0} \int_A f d\mu = 0$$

in the sense that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\int_A f d\mu| < \varepsilon$ whenever $\mu(A) < \delta$.

6. (**Chain rule**) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $g : (\Omega, \mathcal{F}) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be non-negative. Define a measure λ on $\mathcal{F}$ by

$$\lambda(A) = \int_A g\, d\mu\,, \quad A \in \mathcal{F}.$$

Show that if $f : (\Omega, \mathcal{F}) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$,

$$\int_\Omega f\, d\lambda = \int_\Omega fg\, d\mu$$

in the sense that if one of the integrals exists, so does the other, and the two integrals are equal. In particular, prove that if $\lambda \ll \nu$ and $\nu \ll \mu$ (so that $\lambda \ll \mu$) where λ is a signed measure and ν, μ are measures on $\mathcal{F}$ and both $\frac{d\lambda}{d\nu}$ and $\frac{d\nu}{d\mu}$ exist, then $\frac{d\lambda}{d\mu}$ exists and equals $\frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}$ a.e. $[\mu]$.

7. Let λ and μ be σ-finite and mutually absolutely continuous. Prove that

$$\frac{d\mu}{d\lambda} = \left(\frac{d\mu}{d\lambda}\right)^{-1} \; a.e. \, [\lambda].$$

Show also that $\frac{d\lambda}{d\mu} > 0$ a.e. $[\mu]$.