Assignment 4

Analysis I (Fall 2022, Semester I)

Deadline: December 07, 2022

November 28, 2022

The underlying scalar field is taken to be $\mathbb{C}$ unless mentioned otherwise

1. Let H be a Hilbert space and $T \in \mathscr{B}(H)$. Then T is hermitian if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.

2. Let H be a Hilbert space and $T \in \mathscr{B}(H)$. Show that there exist unique self-adjoint operators $T_1, T_2 \in \mathscr{B}(H)$ such that $T = T_1 + iT_2$. T_1 and T_2 are called the *real* and *imaginary* parts of T respectively.

3. Let H be a hilbert space and $T \in \mathscr{B}(H)$. T is called *normal* if $TT^* = T^*T$. Show that the following statements are equivalent:

(i) T is normal.

(ii) $\|Tx\| = \|T^*x\|$ for all x.

(iii) The real and imaginary parts of T commute.

4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $k \in L^2(\Omega \times \Omega, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$ be a *kernel.* Given $f \in L^2(\Omega, \mathcal{F}, \mu)$, define the function Kf on Ω as follows:

$$(Kf)(\omega) = \int k(\omega, \omega')f(\omega')d\mu(\omega').$$

(a) Show that K is a bounded linear operator on $L^2(\Omega, \mathcal{F}, \mu)$ with $\|K\| \leq \|k\|_2$.

(b) Let $\{e_i : i \in I\}$ be an ONB for $L^2(\Omega, \mathcal{F}, \mu)$ and

$$\phi_{ij}(\omega, \omega') := e_j(\omega)\,\overline{e_i(\omega')}$$

for $i, j \in I$ and $\omega, \omega' \in \Omega$. Show that $\{\phi_{ij} : i, j \in I\}$ is an ONS in $L^2(\Omega \times \Omega, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$ and that $\langle k, \phi_{ij}\rangle = \langle Ke_i, e_j\rangle$ where the inner products are L^2 in their respective measure spaces.

(c) Show that there are at most a countable number of i and j such that $\langle k, \phi_{ij}\rangle \neq 0$. Let us denote them as $\{\psi_{k\ell} : 1 \leq k, \ell < \infty\}$ and $\psi_{k\ell}(\omega, \omega') = e_k(\omega)\, \overline{e_\ell(\omega')}$. Now define $K_n = KP_n + P_nK - P_nKP_n$ where P_n is the orthogonal projection onto $\operatorname{span}\{e_k : 1 \leq k \leq n\}$. Deduce that K_n is finite-rank.

(d) Show that

$$\|Kf - K_nf\|^2 \leq \sum_{n+1 \leq k, \ell < \infty} |\langle k, \psi_{k\ell}\rangle|^2$$

for any $f \in L^2(\Omega, \mathcal{F}, \mu)$ such that $\|f\| \leq 1$.

(e) Deduce that $\|K - K_n\| \to 0$ and conclude that K is a compact operator.

5. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. For any $\phi \in L^\infty(\Omega, \mathcal{F}, \mu)$, consider the multiplication operator $M_\phi : L^2(\Omega, \mathcal{F}, \mu) \mapsto L^2(\Omega, \mathcal{F}, \mu)$ by $M_\phi f = \phi f$.

(a) Show that M_ϕ is bounded and $\|M_\phi\| = \|\phi\|_\infty$.

(b) No nonzero multiplication operator on $L^2[0, 1]$ is compact.

6. Let H be a Hilbert space and $T \in \mathcal{K}(H)$. Let S be an invariant subspace for H, i.e., $TS \subset S$. Show that $T_{|S}$ is compact.

7. (Hilbert-Schmidt operators) Let H be a separable Hilbert space. An operator $T \in \mathscr{B}(H)$ is called a *Hilbert-Schmidt operator* if there is an ONB $\{e_n\}_{n=1}^\infty$ of H such that $\sum \|Te_n\|^2 < \infty$.

(a) Show that if $\{f_m\}_{m=1}^\infty$ is another ONB of H, then $\sum \|Te_n\|^2 = \sum \|Tf_m\|^2$.

(b) The number $\|T\|_{\text{HS}} = (\sum \|Te_n\|^2)^{1/2}$ is called the *Hilbert-Schmidt norm* of T. Show that $\|T\|_{\text{HS}} \geq \|T\|$.

(c) Show that the operator K considered in Problem 4 is in fact a Hilbert-Schmidt operator. Find its Hilbert-Schmidt norm.

8. Show that every Hilbert-Schmidt operator T on a Hilbert space H is compact. Find a compact operator that is not a Hilbert-Schmidt operator.