Assignment 4

Analysis I (Fall 2024, Semester I)

Deadline: December 04, 2024

November 20, 2024

1. Let G be a locally compact Hausdorff topological group. A *left Haar measure* (respectively *right Haar measure*) on G is a nonzero locally finite regular Borel measure μ on G such that $\mu(gA) = \mu(A)$ (respectively $\mu(Ag) = \mu(A)$) for all $g \in G$ and all measurable subsets A of G. In the remainder of this problem we will assume that μ is a left Haar measure on G.

(a) Show that $\mu(U) > 0$ for all open $U \subset G$ and also that $\int_G f d\mu > 0$ for any nonnegative $f \in C_c(G)$ that is not identically 0.

(b) Show that there always exists a nonnegative $\varphi \in C_c(G)$ that is not identically 0.

(c) Show that $C_c(G) \subset L^1(G)$.

(d) Show that $\int_G f(hg) d\mu(g) = \int_G f(g) d\mu(g)$ for all $f \in L^1(G, \mathscr{B}(G), \mu)$ and $h \in G$.

(e) Consider a function $\varphi \in C_c(G)$ that is not identically 0 and a left Haar measure ν (not necessarily equal to μ). Now define, for any $f \in C_c(G)$,

$$F_f(g,h) = \frac{f(g)\,\varphi(hg)}{\int_G \varphi(kg) d\nu(k)}\,.$$

Show that $F_f \in L^1(G \times G, \mathscr{B}(G)^{\otimes 2}, \mu \otimes \nu)$.

(f) Show that $\int_G f d\mu = \int_{G\times G} F_f(g,h)\, d\mu(g) d\nu(h) = \int_{G\times G} F_f(h^{-1}, gh)\, d\mu(g) d\nu(h)$ and deduce that $\frac{\int_G f d\mu}{\int_G \varphi d\mu}$ is independent of the *choice* of μ.

(g) (Uniqueness of Haar measure upto scaling) Conclude that any left Haar measure ν satisfies $\nu = a\mu$ for some $a > 0$.

HINT: Use the Riesz representation theorem.

2. Let Y be a closed subspace of a Banach space X.

(a) Show that if X is separable, then Y and X/Y are separable.

(b) Show that if Y and X/Y are separable, then X is separable.

3. Let $(X, \|\cdot\|_X)$ be an infinite-dimensional separable Banach space and $\{e_\gamma\}$ be an *algebraic* basis for X. Define a new norm $\|\cdot\|$ on X by $\|x\| = \sum |x_\gamma|$ for $x = \sum x_\gamma e_\gamma$. Show that $\|\cdot\|$ is indeed a norm on X and prove that it is not equivalent to $\|\cdot\|_X$.

4. Let X be a Banach space and f be a linear functional on X.

(a) Show that $f \in X^*$ if and only if $f^{-1}(0)$ is closed.

(b) Show that if f is not continuous, then $f^{-1}(0)$ is dense in X.

5. Show that if X is an infinite-dimensional Banach space, then X admits a discontinuous linear functional. Conclude that a Banach space X is infinite-dimensional if and only if it has a subspace that is not closed.

6. Let H be a Hilbert space. Show that there exists an abstract set Γ such that H is isometric to $\ell_2(\Gamma)$

7. Let $C^1[0,1]$ be the normed space of all real-valued functions on $[0,1]$ with a continuous derivative, endowed with the supremum norm. Define a linear map T from $C^1[0,1]$ into $C[0,1]$ by $T(f) = f'$. Show that T is closed. Prove that T is not bounded. Explain why the closed graph theorem cannot be used here.

For the remaining problems, assume the underlying scalar field to be $\mathbb{C}$ unless mentioned otherwise.

8. Let H be a Hilbert space and $T \in \mathscr{B}(H)$. Then T is hermitian if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.

9. Let H be a Hilbert space and $T \in \mathscr{B}(H)$. Show that there exist unique self-adjoint operators $T_1, T_2 \in \mathscr{B}(H)$ such that $T = T_1 + iT_2$. T_1 and T_2 are called the *real* and *imaginary* parts of T respectively.

10. Let H be a hilbert space and $T \in \mathscr{B}(H)$. T is called *normal* if $TT^* = T^*T$. Show that the following statements are equivalent:

(i) T is normal.

(ii) $\|Tx\| = \|T^*x\|$ for all x.

(iii) The real and imaginary parts of T commute.

11. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $k \in L^2(\Omega \times \Omega, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$ be a *kernel*. Given $f \in L^2(\Omega, \mathcal{F}, \mu)$, define the function Kf on Ω as follows:

$$(Kf)(\omega) = \int k(\omega, \omega') f(\omega') d\mu(\omega').$$

(a) Show that K is a bounded linear operator on $L^2(\Omega, \mathcal{F}, \mu)$ with $\|K\| \leq \|k\|_2$.

(b) Let $\{e_i : i \in I\}$ be an ONB for $L^2(\Omega, \mathcal{F}, \mu)$ and

$$\phi_{ij}(\omega, \omega') := e_j(\omega)\, \overline{e_i(\omega')}$$

for $i, j \in I$ and $\omega, \omega' \in \Omega$. Show that $\{\phi_{ij} : i, j \in I\}$ is an ONS in $L^2(\Omega \times \Omega, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$ and that $\langle k, \phi_{ij} \rangle = \langle Ke_i, e_j \rangle$ where the inner products are L^2 in their respective measure spaces.

(c) Show that there are at most a countable number of i and j such that $\langle k, \phi_{ij} \rangle \neq 0$. Let us denote them as $\{\psi_{k\ell} : 1 \leq k, \ell < \infty\}$ and $\psi_{k\ell}(\omega, \omega') = e_k(\omega)\, \overline{e_\ell(\omega')}$. Now define $K_n = KP_n + P_nK - P_nKP_n$ where P_n is the orthogonal projection onto $\text{span}\{e_k : 1 \leq k \leq n\}$. Deduce that K_n is finite-rank.

(d) Show that

$$\|Kf - K_nf\|^2 \leq \sum_{n+1 \leq k, \ell < \infty} |\langle k, \psi_{k\ell} \rangle|^2$$

for any $f \in L^2(\Omega, \mathcal{F}, \mu)$ such that $\|f\| \leq 1$.

(e) Deduce that $\|K - K_n\| \to 0$ and conclude that K is a compact operator.

12. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. For any $\phi \in L^\infty(\Omega, \mathcal{F}, \mu)$, consider the multiplication operator $M_\phi : L^2(\Omega, \mathcal{F}, \mu) \mapsto L^2(\Omega, \mathcal{F}, \mu)$ by $M_\phi f = \phi f$.

(a) Show that M_ϕ is bounded and $\|M_\phi\| = \|\phi\|_\infty$.

(b) No nonzero multiplication operator on $L^2[0, 1]$ is compact.

13. Let H be a Hilbert space and $T \in \mathcal{K}(H)$. Let S be an invariant subspace for H, i.e., $TS \subset S$. Show that $T_{|S}$ is compact.

14. (Hilbert-Schmidt operators) Let H be a separable Hilbert space. An operator $T \in \mathscr{B}(H)$ is called a *Hilbert-Schmidt operator* if there is an ONB $\{e_n\}_{n=1}^\infty$ of H such that $\sum \|Te_n\|^2 < \infty$.

(a) Show that if $\{f_m\}_{m=1}^\infty$ is another ONB of H, then $\sum \|Te_n\|^2 = \sum \|Tf_m\|^2$.

(b) The number $\|T\|_{\text{HS}} = (\sum \|Te_n\|^2)^{1/2}$ is called the *Hilbert-Schmidt norm* of T. Show that $\|T\|_{\text{HS}} \geq \|T\|$.

(c) Show that the operator K considered in Problem 4 is in fact a Hilbert-Schmidt operator. Find its Hilbert-Schmidt norm.

15. Show that every Hilbert-Schmidt operator T on a Hilbert space H is compact. Find a compact operator that is not a Hilbert-Schmidt operator.