Assignment 1

Analysis I (Fall 2024, Semester I)

Deadline: September 04, 2024

September 2, 2024

1. Consider a *countably infinite* family $\{A_1, A_2, \ldots\}$ of subsets of Ω. We are interested in the description of the σ-algebra $\mathcal{F} = \sigma(\{A_1, A_2, \ldots\})$ generated by this family.

To this end, for each $\boldsymbol{\varepsilon} \in \{0,2\}^{\mathbb{N}^*}$, let $\boldsymbol{A}^{\boldsymbol{\varepsilon}}$ denote the set $\bigcap_{i \in \mathbb{N}^*} A_i^{\varepsilon_i}$ where $\mathbb{N}^* = \{1, 2, \ldots\}$ is the set of all positive integers and $A_i^{\varepsilon} = A_i$ if $\varepsilon = 0$ and A_i^c if $\varepsilon = 2$. Note that any $\boldsymbol{\varepsilon} \in \{0,2\}^{\mathbb{N}^*}$ corresponds to a number in the real interval $[0,1]$, namely $\sum_{i \in \mathbb{N}^*} \frac{\varepsilon_i}{3^i} \in [0,1]$. Furthermore this correpondence is *injective*. Identifying $\boldsymbol{\varepsilon}$ with this number, we can therefore define, for any $B \subset [0,1]$, the set $\boldsymbol{A}^B = \bigcup_{\boldsymbol{\varepsilon} \in B} \boldsymbol{A}^{\boldsymbol{\varepsilon}}$.

(a) Show that $A_i = \boldsymbol{A}^{T_i}$ where T_i is the set of all $\boldsymbol{\varepsilon} \in \{0,2\}^{\mathbb{N}^*}$ satisfying $\varepsilon_i = 0$.

(b) A (closed) *triadic* subinterval of $[0,1]$ is an interval of the form $[\frac{k}{3^n}, \frac{k+1}{3^n}]$ for some $k \in \mathbb{N}$ satisfying $k + 1 \leq 3^n$. Show that $\boldsymbol{A}^T \in \mathcal{F}$ for any triadic subinterval T of $[0,1]$.

(c) Conclude that $\mathcal{F} = \{\boldsymbol{A}^B : B \in \mathscr{B}(\mathcal{C})\}$ where $\mathcal{C}$ is the Cantor (ternary) set.

2. Consider n measurable spaces $(\Omega_k, \mathcal{F}_k)$; $1 \leq k \leq n$ and let $\Omega \coloneqq \prod_{1 \leq k \leq n} \Omega_k$ be the cartesian product of Ω_k's. A *measurable rectangle* $R \subset \Omega$ is a set of the form $\prod_{1 \leq k \leq n} A_k$ where $A_k \in \mathcal{F}_k$ for each $1 \leq k \leq n$.

(a) Show that the class of all measurable rectangles in Ω is a semifield and hence the class of all finite, disjoint unions of measurable rectangles is a field $\mathcal{A}_0$ of subsets of Ω. The σ-field generated by $\mathcal{A}_0$, denoted as $\prod_{1 \leq k \leq n} \mathcal{F}_k$, is called the *product σ-field* and the measurable space $(\Omega, \prod_{1 \leq k \leq n} \mathcal{F}_k)$ is called the *product space*.

(b) Let $j < n$ and fix $\omega_{\leq j} \in \prod_{1 \leq k \leq j} \Omega_k$. For any $A \subset \Omega$, define the *section* of A at $\omega_{\leq j}$ as follows:

$$A(\omega_{\leq j}) = \left\{ \omega_{>j} \in \prod_{j < k \leq n} \Omega_k : (\omega_{\leq j}, \omega_{>j}) \in A \right\} \subset \prod_{j < k \leq n} \Omega_k.$$

Show that if $A \in \prod_{1<k\leq n} \mathcal{F}_k$, then $A(\omega_{\leq j}) \in \prod_{j<k\leq n} \mathcal{F}_k$.

(c) (**Not for submission**) Define product σ-field for an arbitrary collection of measurable spaces and prove (b).

3. Consider the field $\mathcal{F}_0$ of subsets of $\mathbb{R}^n$ consisting of all finite, disjoint unions of right-semiclosed rectangles, i.e, sets of the form $\prod_{1\leq j\leq n}(a_j, b_j]$ where $-\infty \leq a_j < b_j < \infty$ for all $1 \leq j \leq n$. Now let λ denote the standard volume function on the class of rectangles, i.e.,

$$\lambda\left(\prod_{1\leq j\leq k}(a_j, b_j]\right) = \prod_{1\leq j\leq k}(b_j - a_j).$$

Show that λ extends to a measure on $\mathcal{F}_0$. Consequently, λ extends to a unique (σ-finite) measure on $\mathscr{B}(\mathbb{R}^n)$ called the *Lebesgue measure*. Show that the Lebesgue measure is *translation invariant*, i.e., $\lambda(A + x) = \lambda(A)$ for all $x \in \mathbb{R}^n$ and $A \in \mathscr{B}(\mathbb{R}^n)$.

4. Keeping with the notation in the previous problem, let λ_n denote the n-dimensional Lebesgue measure on $\mathscr{B}(\mathbb{R}^n)$. Now suppose that μ is a translation invariant measure on $\mathscr{B}(\mathbb{R}^n)$ satisfying $\mu(I^n) = 1$ where $I^n := (0, 1]^n$. Through the following steps, we would prove that $\mu = \lambda_n$. All the rectangles below are assumed to be right-semiclosed.

(a) Show that $\mu(R) = \lambda_n(R)$ for any rectangle R with integer side-lengths.

(b) Show that $\mu(R) = \lambda_n(R)$ for any rectangle R with rational lengths.

(c) Show that $\mu(R) = \lambda_n(R)$ for all rectangles R and conclude that $\mu = \lambda_n$.

5. Recall the definition of an *outer measure* λ on Ω. Also recall that a set $E \subset \Omega$ is called λ-*measurable* if

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \quad \text{for all} \quad A \subset \Omega.$$

Let $\mathscr{M}$ denote the class of all λ-measurable sets.

(a) Show that $\mathscr{M}$ is a field.

(b) Show that for any finite sequence of disjoint sets $E_1, \ldots, E_n \in \mathscr{M}$ and $A \subset \Omega$, one has

$$\lambda\left(A \cap \bigcup_{1\leq k\leq n} E_k\right) = \sum_{1\leq k\leq n} \lambda(A \cap E_k).$$

(c) Extend (b) to countable, disjoint unions.

(d) Conclude that $\mathscr{M}$ is a σ-field and that λ is a measure on $\mathscr{M}$.

(e) Let μ be a measure on a field $\mathcal{F}_0$ of subsets of Ω. If $A \subset \Omega$, define

$$\mu^*(A) = \inf\left\{\sum_n \mu(E_n) : A \subset \bigcup_n E_n, E_n \in \mathcal{F}_0\right\}.$$

Show that μ^* is an outer measure on Ω and that $\mu^* = \mu$.

(f) Show that the σ-field of μ^*-measurable sets contains $\mathcal{F}_0$.

6. Let (X, d) be a metric space. An outer measure λ on X is called a *metric outer measure* if

$$\lambda(A \cup B) = \lambda(A) + \lambda(B)$$

for all pairs of *positively separated* subsets A and B of X, i.e., $d(A, B) := \inf_{a\in A, b\in B} d(a, b) > 0$. Through the following steps, we would prove that all Borel subsets of X are λ-measurable.

(a) Let $F \subset X$ be closed and $A \subset X$. Define, for each $n \geq 1$, $A_n = \{x \in A : d(x, F) \geq 1/n\}$. Show that

$$\lambda(A) \geq \lambda(A \cap F) + \lambda(A_n).$$

(b) Show that $A \setminus F = \bigcup_n A_j$.

(c) Show that

$$\lambda(A) \geq \lambda(A \cap F) + \lambda(A \cap F^c).$$

(d) Conclude that all sets in $\mathscr{B}(X)$ are λ-measurable.

7. Let (X, d) be a metric space and $A \subset X$. Define, for any $\alpha \geq 0$ and $\delta > 0$,

$$\mathcal{H}^\alpha_\delta(A) = \inf\left\{\sum_n (\operatorname{diam} U_n)^\alpha : A \subset \bigcup_n U_n,\, U_n \subset X,\, \operatorname{diam} U_n < \delta\right\}$$

where $\operatorname{diam} U := \sup\{d(x, y) : x, y \in U\}$ is the d-diameter of U. **By convention,** we set $0^0 = 1$ and $\operatorname{diam}(\emptyset)^0 = 0$. $\mathcal{H}^\alpha_\infty(A)$ is called the *Hausdorff content* of A. Clearly, $\mathcal{H}^\alpha_\delta(A)$ is decreasing in δ. Now define the *α-dimensional Hausdorff measure* of A, denoted as $\mathcal{H}^\alpha(A)$, as follows:

$$\mathcal{H}^\alpha(A) = \sup_{\delta>0} \mathcal{H}^\alpha_\delta(A) = \lim_{\delta\to 0} \mathcal{H}^\alpha_\delta(A)$$

(a) Prove that $\mathcal{H}^\alpha$ is an outer measure on X and hence, by the previous problem, all Borel subsets of X is $\mathcal{H}^\alpha$-measurable.

(b) What if we replaced $(\operatorname{diam}(C))^\alpha$ with any $\overline{\mathbb{R}}_{\geq 0}$-valued set function τ satisfying $\tau(\emptyset) = 0$.

(c) Define the *Hausdorff dimension* of $S \subset X$, denoted by $\dim_{\mathrm{H}}(S)$, as follows:

$$\dim_{\mathrm{H}}(S) = \inf\{\alpha \geq 0 : \mathcal{H}^{\alpha}(S) = 0\}.$$

Show that $\mathcal{H}^{d}(S) = \infty$ for all $d < \dim_{\mathrm{H}}(S)$ so that $\dim_{\mathrm{H}}(S)$ can be alternatively defined as

$$\dim_{\mathrm{H}}(S) = \sup\{\alpha \geq 0 : \mathcal{H}^{\alpha}(S) = \infty\}.$$

8. Let (X, d) be a metric space and μ be a Borel measure on X. μ is called *regular* if

$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ compact}\} = \inf\{\mu(U) : U \supset A \text{ open}\}$$

for all $A \in \mathscr{B}(X)$. The first condition is called *inner regularity* whereas the second condition is called *outer regularity*. μ is called a *Radon measure* if it is regular and *locally finite*, i.e., every point of X has a neighborhood with finite measure. Through the following steps, we would prove that any *finite*, Borel measure on X is a Radon measure if X is a *Polish space* — a separable, completely metrizable topological space.

(a) Show that for any $\varepsilon > 0$ and $A \in \mathscr{B}(X)$, there exists a closed set $F \subset A$ and an open set $G \supset A$ such that $\mu(G \setminus F) < \varepsilon$.

(b) (**Tightness**) Show that for any $\varepsilon > 0$, there exists $K \subset X$ compact such that $\mu(X \setminus K) < \varepsilon$.

(c) Conclude that μ is regular.