

Assignment 1

Analysis I (Fall 2026, Semester I)

August 31, 2026

1. Consider a *countably infinite* family $\{A_1, A_2, \dots\}$ of subsets of Ω . We are interested in the description of the σ -algebra $\mathcal{F} = \sigma(\{A_1, A_2, \dots\})$ generated by this family.

To this end, for each $\varepsilon \in \{0, 2\}^{\mathbb{N}^*}$, let $\mathbf{A}^\varepsilon$ denote the set $\bigcap_{i \in \mathbb{N}^*} A_i^{\varepsilon_i}$ where $\mathbb{N}^* = \{1, 2, \dots\}$ is the set of all positive integers and $A_i^\varepsilon = A_i$ if $\varepsilon = 0$ and A_i^c if $\varepsilon = 2$. Note that any $\varepsilon \in \{0, 2\}^{\mathbb{N}^*}$ corresponds to a number in the real interval $[0, 1]$, namely $\sum_{i \in \mathbb{N}^*} \frac{\varepsilon_i}{3^i} \in [0, 1]$. Furthermore this correspondence is *injective*. Identifying ε with this number, we can therefore define, for any $B \subset [0, 1]$, the set $\mathbf{A}^B = \bigcup_{\varepsilon \in B} \mathbf{A}^\varepsilon$.

- (a) Show that $A_i = \mathbf{A}^{T_i}$ where T_i is the set of all $\varepsilon \in \{0, 2\}^{\mathbb{N}^*}$ satisfying $\varepsilon_i = 0$.
 - (b) A (closed) *triadic* subinterval of $[0, 1]$ is an interval of the form $[\frac{k}{3^n}, \frac{k+1}{3^n}]$ for some $k \in \mathbb{N}$ satisfying $k+1 \leq 3^n$. Show that $\mathbf{A}^T \in \mathcal{F}$ for any triadic subinterval T of $[0, 1]$.
 - (c) Conclude that $\mathcal{F} = \{\mathbf{A}^B : B \in \mathcal{B}(\mathcal{C})\}$ where $\mathcal{C}$ is the Cantor (ternary) set.
2. Consider n measurable spaces $(\Omega_k, \mathcal{F}_k)$; $1 \leq k \leq n$ and let $\Omega := \prod_{1 \leq k \leq n} \Omega_k$ be the cartesian product of Ω_k 's. A *measurable rectangle* $R \subset \Omega$ is a set of the form $\prod_{1 \leq k \leq n} A_k$ where $A_k \in \mathcal{F}_k$ for each $1 \leq k \leq n$.
 - (a) Show that the class of all measurable rectangles in Ω is a semifield and hence the class of all finite, disjoint unions of measurable rectangles is a field $\mathcal{A}_0$ of subsets of Ω . The σ -field generated by $\mathcal{A}_0$, denoted as $\prod_{1 \leq k \leq n} \mathcal{F}_k$, is called the *product σ -field* and the measurable space $(\Omega, \prod_{1 \leq k \leq n} \mathcal{F}_k)$ is called the *product space*.
 - (b) Let $j < n$ and fix $\omega_{\leq j} \in \prod_{1 \leq k \leq j} \Omega_k$. For any $A \subset \Omega$, define the *section* of A at $\omega_{\leq j}$ as follows:

$$A(\omega_{\leq j}) = \left\{ \omega_{>j} \in \prod_{j < k \leq n} \Omega_k : (\omega_{\leq j}, \omega_{>j}) \in A \right\} \subset \prod_{j < k \leq n} \Omega_k.$$

Show that if $A \in \prod_{1 < k \leq n} \mathcal{F}_k$, then $A(\omega_{\leq j}) \in \prod_{j < k \leq n} \mathcal{F}_k$.

- (c) Define product σ -field for an arbitrary collection of measurable spaces and prove (b).

3. Consider the field $\mathcal{F}_0$ of subsets of $\mathbb{R}^n$ consisting of all finite, disjoint unions of right-semiclosed rectangles, i.e., sets of the form $\prod_{1 \leq j \leq n} (a_j, b_j]$ where $-\infty \leq a_j < b_j < \infty$ for all $1 \leq j \leq n$. Now let λ denote the standard volume function on the class of rectangles, i.e.,

$$\lambda \left(\prod_{1 \leq j \leq k} (a_j, b_j] \right) = \prod_{1 \leq j \leq k} (b_j - a_j).$$

Show that λ extends to a measure on $\mathcal{F}_0$. Consequently, λ extends to a unique (σ -finite) measure on $\mathcal{B}(\mathbb{R}^n)$ called the *Lebesgue measure*. Show that the Lebesgue measure is *translation invariant*, i.e., $\lambda(A + x) = \lambda(A)$ for all $x \in \mathbb{R}^n$ and $A \in \mathcal{B}(\mathbb{R}^n)$.

4. Keeping with the notation in the previous problem, let λ_n denote the n -dimensional Lebesgue measure on $\mathcal{B}(\mathbb{R}^n)$. Now suppose that μ is a translation invariant measure on $\mathcal{B}(\mathbb{R}^n)$ satisfying $\mu(I^n) = 1$ where $I^n := (0, 1]^n$. Through the following steps, we would prove that $\mu = \lambda_n$. All the rectangles below are assumed to be right-semiclosed.

- (a) Show that $\mu(R) = \lambda_n(R)$ for any rectangle R with integer side-lengths.
- (b) Show that $\mu(R) = \lambda_n(R)$ for any rectangle R with rational lengths.
- (c) Show that $\mu(R) = \lambda_n(R)$ for all rectangles R and conclude that $\mu = \lambda_n$.

5. In this exercise, we will see a way to attach functions to measures on $\mathcal{B}(\mathbb{R}^n)$ and study their properties.

We need some notations. For $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$, we write $b \geq a$ if $b_i \geq a_i$ for all $1 \leq i \leq n$. For $a, b \in \mathbb{R}^n$ such that $a \leq b$, we define the interval $(a, b]$ as $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : a_i < x_i \leq b_i\}$. The interval $(-\infty, b]$ is defined similarly.

Now let μ be a finite measure on $\mathcal{B}(\mathbb{R}^n)$ and define $F : \mathbb{R}^n \rightarrow \mathbb{R}$ as $F(x) = \mu(-\infty, x]$.

- (a) Prove that F is *right-continuous*, i.e., for any sequence $x^1 \geq x^2 \geq \dots \rightarrow x \in \mathbb{R}^n$, we have $F(x^k) \rightarrow F(x)$.
- (b) For $a \leq b \in \mathbb{R}^n$, prove that

$$\mu(a, b] = F_0 - F_1 + F_2 - \dots + (-1)^n F_n$$

where F_k is the sum of all $\binom{n}{k}$ terms of the form $F(c_1, \dots, c_n)$ with $c_i = a_i$ for exactly k integers in $(1, 2, \dots, n)$, and $c_i = b_i$ for the remaining $n - k$ integers. In the sequel, we will denote the expression on the right-hand side above as $\Delta_{b,a} F$. Note that when $n = 1$, this is simply $F(b) - F(a)$.

6. We will now go in the reverse direction. So let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be increasing, i.e., $\Delta_{b,a}F \geq 0$ for all $a \leq b \in \mathbb{R}^n$ and right-continuous (revisit the definitions from the previous exercise). We call such a function as a *distribution* function. Now set $\mu(a, b] = \Delta_{b,a}F$ for all $a \leq b$. Show that μ has a unique extension to a σ -finite measure on $\mathbb{R}^n$.

7. Recall the definition of an *outer measure* λ on Ω . Also recall that a set $E \subset \Omega$ is called λ -*measurable* if

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \quad \text{for all } A \subset \Omega.$$

Let $\mathcal{M}$ denote the class of all λ -measurable sets.

- (a) Show that $\mathcal{M}$ is a field.
(b) Show that for any finite sequence of disjoint sets $E_1, \dots, E_n \in \mathcal{M}$ and $A \subset \Omega$, one has

$$\lambda\left(A \cap \bigcup_{1 \leq k \leq n} E_k\right) = \sum_{1 \leq k \leq n} \lambda(A \cap E_k).$$

- (c) Extend (b) to countable, disjoint unions.
(d) Conclude that $\mathcal{M}$ is a σ -field and that λ is a measure on $\mathcal{M}$.
(e) Let μ be a measure on a field $\mathcal{F}_0$ of subsets of Ω . If $A \subset \Omega$, define

$$\mu^*(A) = \inf \left\{ \sum_n \mu(E_n) : A \subset \bigcup_n E_n, E_n \in \mathcal{F}_0 \right\}.$$

Show that μ^* is an outer measure on Ω and that $\mu^* = \mu$.

- (f) Show that the σ -field of μ^* -measurable sets contains $\mathcal{F}_0$.

8. Let (X, d) be a metric space. An outer measure λ on X is called a *metric outer measure* if

$$\lambda(A \cup B) = \lambda(A) + \lambda(B)$$

for all pairs of *positively separated* subsets A and B of X , i.e., $d(A, B) := \inf_{a \in A, b \in B} d(a, b) > 0$. Through the following steps, we would prove that all Borel subsets of X are λ -measurable.

- (a) Let $F \subset X$ be closed and $A \subset X$. Define, for each $n \geq 1$, $A_n = \{x \in A : d(x, F) \geq 1/n\}$. Show that

$$\lambda(A) \geq \lambda(A \cap F) + \lambda(A_n).$$

- (b) Show that $A \setminus F = \bigcup_n A_n$.

(c) Show that

$$\lambda(A) \geq \lambda(A \cap F) + \lambda(A \cap F^c).$$

(d) Conclude that all sets in $\mathcal{B}(X)$ are λ -measurable.

9. Let (X, d) be a metric space and $A \subset X$. Define, for any $\alpha \geq 0$ and $\delta > 0$,

$$\mathcal{H}_\delta^\alpha(A) = \inf \left\{ \sum_n (\text{diam } U_n)^\alpha : A \subset \bigcup_n U_n, U_n \subset X, \text{diam } U_n < \delta \right\}$$

where $\text{diam } U := \sup\{d(x, y) : x, y \in U\}$ is the d -diameter of U . **By convention**, we set $0^0 = 1$ and $\text{diam}(\emptyset) = 0$. $\mathcal{H}_\infty^\alpha(A)$ is called the *Hausdorff content* of A . Clearly, $\mathcal{H}_\delta^\alpha(A)$ is decreasing in δ . Now define the α -dimensional Hausdorff measure of A , denoted as $\mathcal{H}^\alpha(A)$, as follows:

$$\mathcal{H}^\alpha(A) = \sup_{\delta > 0} \mathcal{H}_\delta^\alpha(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^\alpha(A)$$

- (a) Prove that $\mathcal{H}^\alpha$ is an outer measure on X and hence, by the previous problem, all Borel subsets of X is $\mathcal{H}^\alpha$ -measurable.
- (b) What if we replaced $(\text{diam}(C))^\alpha$ with any $\overline{\mathbb{R}}_{\geq 0}$ -valued set function τ satisfying $\tau(\emptyset) = 0$.
- (c) Define the *Hausdorff dimension* of $S \subset X$, denoted by $\dim_H(S)$, as follows:

$$\dim_H(S) = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(S) = 0\}.$$

Show that $\mathcal{H}^d(S) = \infty$ for all $d < \dim_H(S)$ so that $\dim_H(S)$ can be alternatively defined as

$$\dim_H(S) = \sup\{\alpha \geq 0 : \mathcal{H}^\alpha(S) = \infty\}.$$

10. Let (X, d) be a metric space and μ be a Borel measure on X . μ is called *regular* if

$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ compact}\} = \inf\{\mu(U) : U \supset A \text{ open}\}$$

for all $A \in \mathcal{B}(X)$. The first condition is called *inner regularity* whereas the second condition is called *outer regularity*. μ is called a *Radon measure* if it is regular and *locally finite*, i.e., every point of X has a neighborhood with finite measure. Through the following steps, we would prove that any *finite*, Borel measure on X is a Radon measure if X is a *Polish space* — a separable, completely metrizable topological space.

- (a) Show that for any $\varepsilon > 0$ and $A \in \mathcal{B}(X)$, there exists a closed set $F \subset A$ and an open set $G \supset A$ such that $\mu(G \setminus F) < \varepsilon$.
- (b) (**Tightness**) Show that for any $\varepsilon > 0$, there exists $K \subset X$ compact such that $\mu(X \setminus K) < \varepsilon$.
- (c) Conclude that μ is regular.