Final Exam.

Analysis I (Fall 2022, Semester I)

November 30, 2022, **Time: 3 hours**

Note: This paper carries 110 points. Answer as many questions as you can. Your final score will be the minimum of 100 and your total points. State each result clearly that you are using.

1. Let $\{\mu_n\}$ and $\{\lambda_n\}$ be two sequences of finite measures. Put

$$\overline{\mu}_n = \sum_{i=1}^{n} \mu_i,\ \overline{\lambda}_n = \sum_{i=1}^{n} \lambda_i,\ \mu = \sum_{i=1}^{\infty} \mu_i,\ \lambda = \sum_{i=1}^{\infty} \lambda_i.$$

Assume that μ, λ are finite measures, $\overline{\lambda}_n \ll \overline{\mu}_n$, for all $n = 1, 2, \ldots$. Show that $\lambda \ll \mu$ and

$$\frac{d\overline{\lambda}_n}{d\overline{\mu}_n} \to \frac{d\lambda}{d\mu} \text{ a.e. } [\mu].$$

[15]

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space where μ is σ-finite. Also let $f \geq 0$ and $g \geq 0$ be measurable maps on $(\Omega, \mathcal{F})$ and $p > 1$.

(a) Show that

$$\int_0^\infty u^{-p} \Big(\int_\Omega \min(f, u^p)\, d\mu \Big) du = \frac{p}{p-1} \int_\Omega f^{1/p}\, d\mu$$

(b) Show that if

$$\mu\{\omega : f(\omega) \geq t\} \leq t^{-1} \int_{\{\omega: f(\omega) \geq t\}} g\, d\mu, \quad \forall\, t \in (0, \infty),$$

then

$$\|f\|_p \leq \frac{p}{p-1} \|g\|_p.$$

[5 + 10 = 15]

3. Let $(\Omega, \mathcal{F})$ be a measurable space, μ be a measure on $(\Omega, \mathcal{F})$ and λ be a signed measure on $(\Omega, \mathcal{F})$. Show that if λ is finite, then $\lambda \ll \mu$ iff $\lim_{\mu(A)\to 0} \lambda(A) = 0$. [10]

4. Let X be a separable Banach space and μ be a locally finite and translation invariant measure on X. Show that either X is finite-dimensional or $\mu \equiv 0$. [10]

5. A closed subspace Y of a Banach space X is said to *complemented in* X if there exists a continuous projection P from X onto Y.

(a) Show that any finite-dimensional subspace of a Banach space X is complemented in X.

(b) Let X be a Banach space containing a subspace Y isometric to the Banach space ℓ_∞. Show that Y is complemented in X by a norm 1 projection.
[Hint: Hahn-Banach Theorem] [10 + 20 = 30]

6. Let H be a separable Hilbert space. A linear map $T : H \mapsto H$ is called *diagonal or diagonalizable* if there is an ONB $\{e_n\}$ of H and a sequence $\{\alpha_n\}$ of scalars such that $Te_n = \alpha_n e_n$ for all n. Show that a diagonal operator T is bounded iff $\{\alpha_n\} \in \ell^\infty$. And in that case, T is normal with norm $\|(\alpha_n)\|_\infty$. [15]

7. Let H be a separable Hilbert space. An operator $T \in \mathscr{B}(H)$ is called a *Hilbert-Schmidt operator* if there is an ONB $\{e_n\}_{n=1}^\infty$ of H such that $\sum \|Te_n\|^2 < \infty$.

(a) Show that if $\{f_m\}_{m=1}^\infty$ is another ONB of H, then $\sum \|Te_n\|^2 = \sum \|Tf_m\|^2$.

(b) The number $\|T\|_{\text{HS}} = (\sum \|Te_n\|^2)^{1/2}$ is called the *Hilbert-Schmidt norm* of T. Show that $\|T\|_{\text{HS}} \geq \|T\|$.

(c) Show that every Hilbert-Schmidt operator T on a Hilbert space H is compact. [5+5+5 = 15]