Analysis I (Fall 2022, Semester I)

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Lecture I

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1 Fields, σ-fields and measurable spaces

In the next few lectures, we will discuss the theory of Lebesgue integration which generalizes the notion of Riemann(-Stieltjes) integrals both in terms of functions and the underlying domain. In particular, this enables a unified treatment of (weighted) sums and integrals which makes it very useful to work with. A remarkable property of Lebesgue integrals — as we shall see — is that they commute with the operation of taking limits under fairly moderate conditions (as opposed to Riemann integrals). One very desirable consequence of this feature is that we can now exchange of derivative and integral of a function in many situations.

One of the main reasons why Lebesgue integrals are so well-behaved with respect to taking limits is that they are defined as limits of integrals of functions that take only **finitely** many values. Such functions are called *simple functions*. Let us study these functions in more detail.

Consider a (non-empty) set Ω and a simple function $f : \Omega \mapsto \mathbb{R}$. By definition, there exist (distinct) points $x_1, \ldots, x_n$ such that $f^{-1}(\{x_1, \ldots, x_n\}) = \Omega$. Consequently,

$$\Omega = \bigcup_{1 \le i \le n} A_i,\ A_i \coloneqq f^{-1}(\{x_i\})$$

where A_i's are clearly disjoint.

Definition 1.1 (Indicator functions)**.** For any $A \subset \Omega$, the *indicator* of A is the function $\mathbf{I}_A : \Omega \mapsto \mathbb{R}$ defined as

$$\mathbf{I}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{otherwise.} \end{cases}$$

In some books, $\mathbf{I}_A$ is also called the *characteristic function* of A, denoted as χ_A. It is clear that

$$f = \sum_{1 \le i \le n} x_i \, \mathbf{I}_{A_i} \, . \tag{1.1}$$

Henceforth, we will take this decomposition as the definition of a simple function.

Note. The sets A_i's in the decomposition (1.1) are disjoint but do not need to form a partition of Ω, i.e., $\cup_{1 \le i \le n} A_i \subset \Omega$. This is because we can omit $x_i \mathbf{I}_{A_i}$ if $x_i = 0$. Similarly we can drop the requirement that x_i's are distinct. However, notice that with this new convention, the decomposition (1.1) is *non-unique*.

Let us denote the 'hypothesised' Lebesgue integral of f on Ω as $\int_\Omega f$. Any reasonable definition of an integral should be linear in the *integrand* f. In more formal terms, an integral should be a linear functional on the space of *integrable functions* which is always a *vector space*. Therefore, we can write

$$\int_\Omega f = \sum_{1 \le i \le n} \int_\Omega x_i \, \mathbf{I}_{A_i} = \sum_{1 \le i \le n} x_i \int_\Omega \mathbf{I}_{A_i} \, .$$

One has to supply the values of the integrals $\int_\Omega \mathbf{I}_{A_i}$. In fact, $\int_\Omega \mathbf{I}_A$ does not need to be defined for all subsets of Ω which, as we shall see, is inevitable in many situations. Let $\mathcal{F}$ denote the class of all subsets A of Ω *for which* $\int_\Omega \mathbf{I}_A$ is defined. Then $\int_\Omega \mathbf{I}_{\cdot} : \mathcal{F} \mapsto \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ defines a *set function* which we will denote as μ in the sequel and accordingly $\int_\Omega f$ as $\int_\Omega f d\mu$ (often denoted as $\int_\Omega f \mu(d\omega)$) in order to incorporate the dependence of the latter on the choice of μ. We should keep here in mind that the notation $d\mu$ (or $\mu(d\omega)$) is merely to have resemblance to the classical notations for Riemann integrals and not because of any differentiable structure on Ω.

One important thing that came out for the discussion above is that $\mathcal{F}$ already puts a restriction on the type of simple functions that can be integrated by requiring that $f^{-1}(\{x_i\}) \in \mathcal{F}$ for each i. We will come back to this notion when we discuss *measurable functions*. For the time being, let us 'loosely' use the term measurable function for any map f for which it might *make sense* to talk of an integral.

We might now wonder what conditions must $\mathcal{F}$ satisfy in order that our notion of integral stays reasonable and effective. Let us start with some obvious ones first. Below we ignore the issues of adding or subtracting ∞ for now.

- Both $\emptyset$ and Ω must lie in $\mathcal{F}$ since we should be always able to integrate the constant function $\mathbf{I}_\Omega \equiv 1$ and $\mu(\emptyset)$ should be 0.
- Since $\mathbf{I}_A + \mathbf{I}_B = \mathbf{I}_{A \cup B}$ whenever A and B are disjoint and $\int_\Omega \mathbf{I}_A \, d\mu + \int_\Omega \mathbf{I}_B \, d\mu$ is well-defined provided each of the summands is, $\mathcal{F}$ should be closed under *finite, disjoint union*.

- If $A \subset B$, then $\mathbf{I}_B - \mathbf{I}_A = \mathbf{I}_{B \setminus A}$ and by similar reasoning as above, we get that $\mathcal{F}$ should be closed under taking *proper difference*. In particular, this implies that $\mathcal{F}$ should be closed under *complementation* as $\Omega \in \mathcal{F}$ due to the first item.

However, there is one more property we would like $\mathcal{F}$ to have. So far we have only used the intuition that the space of measurable functions should be a vector space. However, for many applications in Analysis and Probability theory, we would like to consider products of functions as well, i.e., the space of measurable functions should be an *algebra*. Applied to the indicator functions and noting that $\mathbf{I}_A \mathbf{I}_B = \mathbf{I}_{A \cap B}$, this suggests that $\mathcal{F}$ should also be closed under *finite intersection*. But this leads to the following.

Exercise 1.2. Let $\mathcal{F}$ be a (non-empty) collection of subsets of Ω that is closed under complementation. Show that $\mathcal{F}$ is closed under finite (respectively countable) union if and only if it is closed under finite (respectively countable) intersection.

We are now ready to give our first, formal definition.

Definition 1.3 (Field)**.** Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a *field* (alternatively called an *algebra*) if and only if the following conditins are satisfied:

(a) $\Omega \in \mathcal{F}$.

(b) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

(c) If $A_1, A_2, \ldots, A_n \in \mathcal{F}$, then $\bigcup_{1 \leq i \leq n} A_i \in \mathcal{F}$.

By Exercise 1.2, it follows that $\mathcal{F}$ is closed under finite intersection.

We are yet to touch upon the aspect of 'well-behaved-ness of integrals under taking limits' that we promised ourselves in the beginning. For this we need to equip the space of measurable functions with a closure under sequential limits (superiors and inferiors). To this end, consider a sequence of sets $A_1, A_2, \ldots$ in $\mathcal{F}$. Now define:

$$\limsup_n A_n = \bigcap_n \bigcup_{k \geq n} A_k, \text{ and } \liminf_n A_n = \bigcup_n \bigcap_{k \geq n} A_k. \tag{1.2}$$

When $\limsup_n A_n = \liminf_n A_n$, we call the common value as the limit of A_n and denote it as $\lim_n A_n$.

Exercise 1.4. Let $A_1, A_2, \ldots$ be a sequence of subsets of Ω. Show that

$$\limsup_n \mathbf{I}_{A_n} = \mathbf{I}_{\limsup_n A_n}, \text{ and } \liminf_n \mathbf{I}_{A_n} = \mathbf{I}_{\liminf_n A_n},$$

and consequently that $\lim_n A_n$ exists if and only if $\lim_n \mathbf{I}_{A_n}$ exists pointwise. How can one describe the sets $\limsup_n A_n$ and $\liminf_n A_n$ in terms of the elements of A_n's.

In view of (1.2) and Exercise 1.2, it follows that the closure of $\mathcal{F}$ under (sequential) limits (superior and inferior) is equivalent to the closure under *countable union*. This is called a σ-field.

Definition 1.5 (σ-Field)**.** A field $\mathcal{F}$ is called a *σ-field* (alternatively called a σ-algebra) if it is closed under countable union. The (ordered) pair $(\Omega, \mathcal{F})$ comprising a set Ω and a σ-field of subsets of Ω is often called a *measurable space* and the sets in $\mathcal{F}$ are called *measurable.*

Examples.

1) *Largest and smallest σ-field.* The largest or the *finest* σ-field of subsets of Ω is simply $\mathcal{P}(\Omega)$ — the power set of Ω, i.e., the collection of all subsets of Ω. The smallest or the *coarsest* σ-field (also called the *trivial* σ-field), on the other hand, is $\{\emptyset, \Omega\}$.

2) *Minimal σ-field.* For any $\mathscr{C} \subset \mathcal{P}(\Omega)$, i.e., a class of subsets of Ω, we can easily check that

$$\sigma(\mathscr{C}) := \bigcap_{\mathscr{C} \subset \mathcal{F}} \mathcal{F}$$

is a σ-field called the *minimal σ-field over $\mathscr{C}$* or the *σ-field generated by $\mathscr{C}$*.

Exercise 1.6. Let $A_1, A_2, \ldots, A_n$ be subsets of Ω. Describe explicitly the σ-field generated by the family $\{A_1, A_2, \ldots, A_n\}$.

3) *Borel subsets of $\mathbb{R}$.* Let $\Omega = \mathbb{R}$ and $\mathscr{C} := \{(a, b] : -\infty \leq a < b < \infty\}$ consist of all right-semiclosed intervals of real numbers. Then $\sigma(\mathscr{C})$, denoted as $\mathscr{B}(\mathbb{R})$, is known as the *Borel σ-field* of $\mathbb{R}$ and the members of $\mathscr{B}(\mathbb{R})$ are called *Borel subsets* of $\mathbb{R}$.

   Similarly when $\Omega = \overline{\mathbb{R}}$, we define $\mathscr{B}(\overline{\mathbb{R}})$ as the σ-field generated by $\mathscr{C} := \{(a, b] : -\infty \leq a < b \leq \infty\}$ (notice the difference).

Exercise 1.7. A *semifield* $\mathcal{S}$ of subsets of Ω is a non-empty subset of $\mathcal{P}(\Omega)$ such that (a) $\mathcal{S}$ is closed under finite, non-empty intersection and (b) the complement of any set in $\mathcal{S}$ is a finite, disjoint union of sets in $\mathcal{S}$. Show that the family $\mathcal{F}$ consisting of all finite (including empty) disjoint unions of sets in $\mathcal{S}$ is a field. Show that the family of right-semiclosed intervals in both cases above are semifields. Show that the corresponding fields are not σ-fields.

Exercise 1.8. Show that $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{C})$ where $\mathscr{C}$ is the class of all intervals of any of the following types:

open intervals, closed intervals, left-semiclosed intervals, (a, ∞) where $a \in \mathbb{R}$.

4) *Topology and σ-field.* More generally, if $(\Omega, \mathcal{T})$ is a topological space where $\mathcal{T}$ is the collection of all open subsets of Ω, then $\mathscr{B}(\Omega) = \sigma(\mathcal{T})$ is called the *Borel σ-field* of the topological space Ω and the members of $\mathscr{B}(\Omega)$ are called the *Borel subsets* of Ω.

1.1 Good sets principle

Every so often in measure theory, we encounter statements of the form 'every member of $\sigma(\mathscr{C})$ satisfies property P' for some given class of subsets $\mathscr{C}$ of Ω and property P of sets. Usually, it is immediate that sets in $\mathscr{C}$ satisfy property P. Let $\mathscr{S}$ be the class of all sets in $\sigma(\mathscr{C})$ — called the *good sets* — satisfying property P. If it is the case that the property P is closed under countable union and complementation and is satisfied by Ω, then by Definition 1.3 and 1.5, $\mathscr{S}$ is a σ-field. However, since $\mathscr{C} \subset \mathscr{S} \subset \sigma(\mathscr{C})$, it follows from the definition of minimal σ-field that $\mathscr{S} = \sigma(\mathscr{C})$ and consequently P is satisfied by all sets in $\sigma(\mathscr{C})$. A simple but useful application of this principle is the following statement.

Exercise 1.9. For any class $\mathscr{C}$ of subsets of Ω and $A \subset \Omega$, let $\mathscr{C} \cap A$ denote the class $\{B \cap A : B \in \mathscr{C}\}$. Show that $\sigma_A(\mathscr{C} \cap A) = \sigma(\mathscr{C}) \cap A$ where $\sigma_A(\mathscr{C} \cap A)$ is the minimal σ-field of subsets of A over $\mathscr{C} \cap A$.

Suggested reading. Sections 1.1–1.2.2 in *Probability and Measure Theory* by Robert B. Ash and Catherine A. Doléans-Dade.