# Analysis I (Fall 2022, Semester I)

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Lecture II

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## 2 Measures

In the previous lecture, we saw that a desirable property of the set function $\mu(A) \coloneqq \int_\Omega \mathbf{I}_A$ is *finite additivity*, i.e., for any collection of disjoint, measurable sets $A_1, \ldots, A_n$, $\mu$ should satisfy

$$\mu\left(\bigcup_{1 \le k \le n} A_n\right) = \sum_{1 \le k \le n} \mu(A_k).$$

**Definition 2.1** (Finitely additive measure)**.** A *finitely additive measure* on a field $\mathcal{F}$ is a finitely additive set function $\mu : \mathcal{F} \mapsto \overline{\mathbb{R}}_{\ge 0} \coloneqq [0, \infty]$ satisfying $\mu(\emptyset) = 0$.

However, since we would like to ensure the continuity of $\mu(A) = \int_\Omega \mathbf{I}_A$ at least under *increasing limits* so that we can obtain 'well-behaved-ness of integrals under taking limits', we should require a more stringent property. For any *increasing* family of sets $A_1, A_2, \ldots \in \mathcal{F}$ *converging* to $A \in \mathcal{F}$, i.e., $A_1 \subset A_2 \subset \ldots$ and $A \coloneqq \lim_n A_n = \bigcup_n A_n \in \mathcal{F}$, the following should be true

$$\lim_n \mu(A_n) = \mu(A).$$

Henceforth, we will refer to this property as the *continuity* of $\mu$ under *increasing limit*. Similarly we can define the continuity under *decreasing* and *monotonic* limit. The following result says that the continuity under increasing limit is equivalent to an extension of finite additivity, namely *countable additivity* which is the property that

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$$

for any finite or countably infinte collection of disjoint sets $A_1, A_2, \ldots \in \mathcal{F}$ such that $\bigcup_n A_n \in \mathcal{F}$. Our next result states that countable additivity and increasing limit are equivalent.

**Proposition 2.2.** *Let $\mu$ be a finitely additive measure on the field $\mathcal{F}$. Then $\mu$ is countably additive if and only if $\mu$ is continuous under increasing limit.*

*Proof. Countable additivity $\implies$ continuity under increasing limit.*
Let $A_1 \subset A_2 \subset \ldots \in \mathcal{F}$ be such that $A := \bigcup_n A_n \in \mathcal{F}$. Clearly, the sets $A_n \setminus A_{n-1} \in \mathcal{F}$ $(n \geq 1)$ are disjoint (here we let $A_0 = \emptyset$) and satisfy,

$$A = \bigcup_n A_{n+1} \setminus A_n.$$

By the countable additivity of $\mu$, we then have

$$\mu(A) = \sum_n \mu(A_n \setminus A_{n-1}) = \lim_n \sum_{1 \leq k \leq n} \mu(A_k \setminus A_{k-1}) = \lim_n \mu(A_n)$$

which yields the implication.
*Continuity under increasing limit. $\implies$ countable additivity* (Exercise). □

**Definition 2.3** (Measure)**.** We call a finitely additive measure a *measure* if it is countably additive. A *measure space* is an ordered triple $(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F})$ is a measurable space and $\mu$ is a measure on $\mathcal{F}$.

**Note.** Although we required $\mu$ to be nonnegative in this and the previous definitions, there is abolutely no problem in allowing $\mu$ to take negative or even complex values. We will call such set functions as *signed* and *complex* measures respectively and use the term measure only when $\mu$ is nonnegative. One reason for this is that the signed and complex measures can be understood using nonnegative measures as we will see later in the course.

**Exercise 2.4** (Monotonicity of measures)**.** Let $\mathcal{F}$ be a field of subsets of $\Omega$ and $\mu$ be a finitely additive measure on $\mathcal{F}$. Show that $\mu$ is monotonic with respect to set inclusion, i.e., $\mu(A) \geq \mu(B)$ (respectively, $\mu(A) \leq \mu(B)$) whenever $B \subset A \in \mathcal{F}$ (respectively, $A \subset B \in \mathcal{F}$).

The following result gives a partial analogue of Proposition 2.2 for decreasing limits.

**Proposition 2.5** (Continuity of measures from above)**.** *Let $\mathcal{F}$ be a field of subsets of $\Omega$ and $\mu$ be a measure on $\mathcal{F}$. Also let $A_1, A_2, \ldots \in \mathcal{F}$ be such that $A_n \downarrow A \in \mathcal{F}$ (i.e., $A_1 \supset A_2 \supset \ldots$ and $A = \bigcap_n A_n = \lim_n A_n$) with $\mu(A_1) < \infty$. Then*

$$\lim_n \mu(A_n) = \mu(A).$$

*Proof.* Since $A_n \downarrow A$, it follows that $A_1 \setminus A_n \uparrow A_1 \setminus A$ and hence by the continuity of $\mu$ under increasing limit we get,

$$\lim_n \mu(A_1 \setminus A_n) = \mu(A_1 \setminus A).$$

Also because $A_n \subset A_1$, $A \subset A_1$ and $\mu(A_1) < \infty$, we have $\mu(A_1 \setminus B) = \mu(A_1) - \mu(B)$ for $B = A_n$ or $A$ and consequently $\lim_n \mu(A_n) = \mu(A)$. □

Our first theorem in this section is one of the most important concerning limits of measures of sets.

**Theorem 2.6** (Fatou's lemma and the continuity of finite measures)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $A_1, A_2, \ldots \in \mathcal{F}$. Then we have,*

$$\mu\left(\liminf_n A_n\right) \leq \liminf_n \mu(A_n).$$

*Furthermore, if $\mu(\Omega) < \infty$, i.e., $\mu$ is finite, we also have*

$$\mu(\liminf_n A_n) \leq \liminf_n \mu(A_n) \leq \limsup_n \mu(A_n) \leq \mu(\limsup_n A_n).$$

*In particular, if $\lim_n A_n$ exists, we have $\mu(\lim_n A_n) = \lim_n \mu(A_n)$ for all finite measures.*

*Proof.* Recall that $\liminf_n A_n = \bigcup_n B_n$ where $B_n := \bigcap_{k \geq n} A_k$, i.e., $B_n \uparrow \liminf_n A_n$. Thus, by the continuity of $\mu$ under increasing limit as well as its monotonicity, we get

$$\mu\left(\liminf_n A_n\right) = \lim_n \mu(B_n) \leq \liminf_n \mu(A_n).$$

The other inequality follows by applying this result to $A_n^c$'s and noting that $\mu$ is finite. □

**Examples.**

1) *Dirac measures and weighted sums.* Possibly the simplest non-trivial measure (i.e., a measure which is not identically zero) one can define on a field $\mathcal{F}$ of subsets of $\Omega$ is the *Dirac measure.* Let $\omega \in \Omega$ and define the measure $\delta_\omega$ on $\mathcal{F}$ as follows:

$$\delta_\omega(A) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{otherwise.} \end{cases}$$

   It is straightforward to check that $\delta_\omega$ is a measure on $\mathcal{F}$ called the Dirac measure at $\omega$. We can use positive linear combinations of Dirac measures to define a large class of measures.

**Exercise 2.7.** Let $\{\mu_i : i \in I\}$ be a family of measures on a field $\mathcal{F}$ of subsets of $\Omega$. Given any family of nonnegative numbers $\{p_i : i \in I\}$, consider the set function $\mu = \sum_{i\in I} \mu_i$ defined as

$$\mu(A) = \sum_{i\in I} p_i\mu_i(A) := \sup_{J\subset\subset I} \sum_{j\in J} p_j\mu_j(A), \quad A \in \mathcal{F}$$

($J \subset\subset \cdot$ means that $J$ is finite). Show that $\mu$ is a measure.

In view of Exercise 2.7, given any subset $\Omega' \subset \Omega$ and nonnegative numbers $\{p_\omega : \omega \in \Omega'\}$, the set function $\sum_{\omega\in\Omega'} p_\omega\delta_\omega$ is a measure on $\mathcal{P}(\Omega)$ and hence on any field $\mathcal{F}$ of subsets of $\Omega$. We can rightly think of such measures as *weighted sums* especially when $\Omega'$ is countable. In the special case when $\Omega' = \Omega$ and $p_\omega$'s are all 1, $\mu(A)$ is simply the cardinality of the set $A$ and the measure is called the *counting measure* on $\Omega$. Taking this cue, we will henceforth think of measures as ways of assigning 'size values' to sets.

2) *Length, area and volume.* Recall from the previous lecture that the intervals (open, closed or semiclosed) generate the Borel $\sigma$-algebra $\mathscr{B}(\mathbb{R})$. Similarly, the *rectangles*, i.e., $k$-fold product of real intervals generate $\mathcal{B}(\mathbb{R}^k)$ for any $k \geq 1$. The natural way to assign a size value to an interval $(a, b]$ (or $(a, b)$, $[a, b]$, or $(a, b]$) is to use its length $b - a$. In case of $\mathbb{R}^k$, we can similarly use its volume. But unfortunately, the measure on a $\sigma$-field is not always determined by its value on a generating set.

**Exercise 2.8.** Give an example of two different measures that agree on a generating set for the underlying $\sigma$-field. One can even produce an example where $\Omega$ is a compact metric space, $\mathcal{F} = \mathscr{B}(\Omega)$ and $\mathscr{C} = \{\text{closed balls in } \Omega\}$ (why is this a generating set for $\mathscr{B}(\Omega)$?). But this is more difficult.

However, in a previous exercise, we saw that the family of right-semiclosed intervals of $\mathbb{R}$ forms a semifield and the collection of all finite, disjoint unions of such intervals is a field (say $\mathcal{B}(\mathbb{R})$). The length function admits of a natural extension to $\mathcal{B}(\mathbb{R})$. It is less immediate, however, that this function is in fact a measure which will be an assignment problem. Whether it extends to a measure on all Borel sets is a question that requires more work.

The take-home message is that it is often relatively easier to define a measure on a field and hence they are good starting points for the construction of measures. But can we always extend such measures to the corresponding $\sigma$-field and are such extensions unique whenever they exist? We will address these two questions in the next lecture.

**Suggested reading.** Section 1.2 in *Probability and Measure Theory* by Robert B. Ash and Catherine A. Doléans-Dade.