Analysis I (Fall 2022, Semester I)

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Lecture III

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3 Extension of measures

Before addressing the question whether can we extend a given measure from a field to the corresponding σ-field, let us consider the question whether such a measure, if it exists, is unique. In other words, this is equivalent to asking whether the measure on the field *determines* the measure on the corresponding σ-field whenever it can be extended. The general answer to this question is no and you can find a counter-example in Section 1.3.12 in the book by Ash and Dade. But fortunately, this holds for a large enough class of measures containing most of the interesting examples we are going to consider in this course.

Definition 3.1 (σ-finite measure)**.** A measure on a field $\mathcal{F}$ of subsets of Ω is called σ-finite if Ω can be expressed as $\bigcup_n A_n$ where $A_n \in \mathcal{F}$ have finite μ-measures.

In particular, a σ-finite measure can be written as a countable sum of finite measures (Exercise). The following fundamental theorem is the central object of today's lecture.

Theorem 3.2 (Carathéodory Extension Theorem)**.** *Let μ be a σ-finite measure a field $\mathcal{F}_0$ of subsets of Ω. Then μ has a unique extension to a measure to $\mathcal{F} := \sigma(\mathcal{F}_0)$.*

As a first application and in view of our description of Borel σ-field as generated by the field of all finite, disjoint unions of rectangles , we can define the Lebesgue measure on $\mathcal{B}(\mathbb{R}^k)$ as the *unique* extension of the volume of rectangles.

We now proceed to prove the extension theorem starting with the uniqueness part which is shorter.

Proof of uniqueness. Suppose that μ can be extended to a measure, also denoted as μ, to a measure on $\mathcal{F}$. Let λ be another such measure so that $\lambda_{|\mathcal{F}_0} = \mu_{|\mathcal{F}_0} = \mu$. We would like to prove that $\lambda = \mu$. A natural approach — based on what we have seen so far — would be to use the so-called *good sets principle*. To this end we can define our class of good sets to be

$$\mathscr{S} = \{A \in \mathcal{F} : \lambda(A) = \mu(A)\}.$$

It is clear that $\mathcal{F}_0 \subset \mathscr{S}$. However, although it is quite easy to see that $\mathscr{S}$ is closed under countable, *disjoint* union owing to the additivity of λ and μ, it is not obvious how to use similar arguments to deduce the closure under general countable union. When faced with such situations, there are usually two different, albeit related, principles that come to our rescue. The first one is the *Dynkin* $\pi - \lambda$ *theorem* while the second one is the *monotone class theorem*. We will state and use the latter in this note although you should look up and learn the former.

Theorem 3.3 (Monotone class theorem). *Let $\mathcal{F}_0$ be a field of subsets of Ω, and $\mathscr{C}$ a class of subsets of Ω that is monotone, i.e., $\mathscr{C}$ is closed under monotone limit. If $\mathcal{F}_0 \subset \mathscr{C}$, then $\sigma(\mathcal{F}_0) \subset \mathscr{C}$.*

Let us finish the proof of uniqueness assuming the monotone class theorem. However, the class of sets $\mathscr{S}$ is not monotone unless μ is finite (why?). It is at this point we will make use of the σ-finiteness of μ *on* $\mathcal{F}_0$. To this end, let $\Omega = \bigcup_n A_n$ where $A_n \in \mathcal{F}_0$'s are disjoint with $\mu(A_n) < \infty$ for each n. Now define the measures μ_n and λ_n on $\mathcal{F}$ as $\mu_n(A) = \mu(A \cap A_n)$ and $\lambda_n(A) = \lambda(A \cap A_n)$ for $A \in \mathcal{F}$. Since $A_n \in \mathcal{F}_0$ and $\mu_{|\mathcal{F}_0} = \lambda_{|\mathcal{F}_0}$, it follows that both λ_n and μ_n are extensions of the finite measure μ_n on $\mathcal{F}_0$. Now, using continuity of finite measures, we can conclude that the class of sets $\mathcal{S}_n$ comprising all sets in $\mathcal{F}$ where the measures λ_n and μ_n coincide is a monotone class containing the field $\mathcal{F}_0$. By Theorem 3.3, we can then conclude $\mu(A \cap A_n) = \lambda(A \cap A_n)$ for all $A \in \mathcal{F}$ and n. Summing over n, we deduce the uniqueness.

It remains to give the:

Proof of Theorem 3.3. This is an instance of a very powerful problem solving technique known as *bootstrapping*. Let $\mathscr{M}$ be the smallest monotone class containing $\mathcal{F}_0$, i.e., the intersection of all monotone class containing $\mathcal{F}_0$ (why is this a monotone class?). We want to show that $\mathcal{F} \subset \mathscr{M}$ which in turn implies $\mathcal{F} = \mathscr{M}$ (why?).

Fix $A \in \mathscr{M}$ and let $\mathscr{M}_A := \{B \in \mathscr{M} : A \cap B, A \cap B^c \text{ and } A^c \cap B \in \mathscr{M}\}$. Then $\mathscr{M}_A$ is a monotone class which implies $\mathscr{M}_A = \mathscr{M}$ due to the minimality of $\mathscr{M}$ and the fact that $\mathcal{F}_0 \subset \mathscr{M}_A$ by the properties of fields. But this shows that for any $B \in \mathscr{M}$, we have $A \cap B, A \cap B^c, A^c \cap B \in \mathscr{M}$ for any $A \in \mathcal{F}_0$ yielding $\mathcal{F}_0 \subset \mathscr{M}_B$. Again by minimality of $\mathscr{M}$, $\mathscr{M}_B = \mathscr{M}$.

We thus get that $\mathscr{M}$ is a field (because if $A, B \in \mathscr{M} = \mathscr{M}_A$, then $A \cap B, A \cap B^c, A^c \cap B \in \mathscr{M}$) and a monotone class that is also a field is a σ-field. Hence $\mathscr{M}$ is a σ-field containing $\mathcal{F}_0$. □

Now we proceed to prove the existence.

We will extend the measure μ in multiple steps and in the process will introduce several notions of fundamental importance.

Step I. *Extending by taking increasing limit.*
Consider the σ*-closure* $\mathcal{G}$ of $\mathcal{F}_0$ defined as follows:

$$\mathcal{G} = \{A \subset \Omega : \exists A_n \in \mathcal{F}_0 \text{ such that } A_n \uparrow A\}.$$

Exercise 3.4. $\mathcal{G}$ can also be defined as the collection of all countable unions of sets in $\mathcal{F}_0$. Is $\mathcal{G}$ a σ-field?

Lemma 3.5. *$\mathcal{F}_0 \subset \mathcal{G}$ and $\mathcal{G}$ is closed under finite union, intersection, countable union and increasing limit.*

Proof. It follows by using the characterization of $\mathcal{G}$ given in Exercise 3.4, the fact that $\mathcal{F}_0$ is a field and the distributive properties of the algebra of sets. □

The natural choice for $\mu(A)$ when $A \in \mathcal{G}$ is as follows (we denote the extended set function as μ^σ):

$$\mu^\sigma(A) = \lim_n \mu(A_n). \tag{3.1}$$

where $A_n \in \mathcal{F}_0$ are such that $A_n \uparrow A$. However, it is not immediately clear whether this is well-defined since there can be multiple sequences $(A_n)_{n\geq 1}$ in $\mathcal{F}_0$ increasing to A.

Lemma 3.6. *Let $A_n \in \mathcal{F}_0 \uparrow A$ and $A'_n \in \mathcal{F}_0 \uparrow A'$ such that $A \subset A'$, then*

$$\lim_m \mu(A_m) \leq \lim_n \mu(A'_n).$$

In particular, μ^σ in (3.1) is well-defined.

Proof. Since $A'_n \uparrow A' \supset A$ and $A_m \subset A$, it follows that $A_m \cap A'_n \in \mathcal{F}_0 \uparrow_{n\to\infty} A_m \cap A' = A_m \in \mathcal{F}_0$ for each m. Hence

$$\lim_n \mu(A'_n) \geq \lim_n \mu(A_m \cap A'_n) = \mu(A_m),$$

where we used the monotonicity and continuity of the measure μ under increasing limit in the first and second steps respectively. Now the result follows by sending $n \to \infty$. □

The following result says that μ^σ is well-behaved on $\mathcal{G}$.

Lemma 3.7. *$\mu^\sigma = \mu$ on $\mathcal{F}_0$ and satisfies $0 \leq \mu^\sigma(A) \leq \mu(\Omega)$ for all $A \in \mathcal{G}$. It further satisfies the following properties:*

(a) If $G_1, G_2 \in \mathcal{G}$, then $\mu^\sigma(G_1 \cup G_2) + \mu^\sigma(G_1 \cap G_2) = \mu^\sigma(G_1) + \mu^\sigma(G_2)$.

(b) (Monotonicity) If $G_1, G_2 \in \mathcal{G}$ such that $G_1 \subset G_2$, then $\mu^\sigma(G_1) \leq \mu^\sigma(G_2)$.

(c) (Continuity under increasing limit) If $G_n \in \mathcal{G}$ such that $G_n \uparrow G$, then $\mu^\sigma(G_n) \to \mu^\sigma(G)$.

Proof. We will only give the proof of (c), since the other properties follow in a straightforward manner from Lemmas 3.5—3.6 as well as the fact that μ is a measure on $\mathcal{F}_0$ (Exercise: check this).

From (b), we get that $\lim_n \mu^\sigma(G_n) \geq \mu^\sigma(G)$ and hence we only need to prove the reverse inequality. To this end we will make use of a *diagonal argument*. The objective is to construct a sequence of increasing sets $C_n \subset G_n$ in $\mathcal{F}_0$ so that $C_n \uparrow G$. For then we would be able to immediately conclude $\mu(G) = \lim_n \mu^\sigma(C_n) \leq \lim_n \mu^\sigma(G_n)$.

It remains to construct C_n's. Let $A_{mn} \in \mathcal{F}_0 \uparrow_{m\to\infty} G_n$ for each n and define $C_n = \bigcup_{1\leq i\leq n} A_{in}$. Since A_{ij}'s are increasing in i, it follows that C_n's are increasing and satisfy

$$C_n \subset \bigcup_{1\leq i\leq n} G_i = G_n \tag{3.2}$$

for each n. On the other hand, $A_{mn} \subset \bigcup_j C_j$ for all m, n and hence $G_n \subset \bigcup_j C_j$ for all n which implies — together with the (3.2) — that $C_n \uparrow \bigcup_j C_j = \bigcup_j G_j = G$. □

Step II. *Extending to an outer measure on Ω.*

We will now extend μ^σ from $\mathcal{G}$ to $\mathcal{P}(\Omega)$ by taking limits on sets *from above* through sets in $\mathcal{G}$. More precisely, we define for each $A \subset \Omega$,

$$\mu^*(A) \coloneqq \inf\{\mu^\sigma(G) : G \in \mathcal{G}, G \supset A\}.$$

It is clear from this definition and the monotonicity of μ^σ that $\mu^* = \mu$ on $\mathcal{G}$ and that $0 \leq \mu^*(A) \leq 1$ for all $A \subset \Omega$. There is a class of 'measure-like' set functions defined on $\mathcal{P}(\Omega)$ which arise quite often in practice — especially when Ω is a metric space — and are particularly useful for constructing measures on suitable σ-fields. These are known as *outer measures*.

Definition 3.8 (Outer measure)**.** A *outer measure* on Ω is a nonnegative, $\overline{\mathbb{R}}_{\geq 0}$-valued set function λ on $\mathcal{P}(\Omega)$, satisfying

(a) $\lambda(\emptyset) = 0$.

(b) (Monotonicity) $A \subset B$ implies $\lambda(A) \leq \lambda(B)$.

(c) (Countable subadditivity) $\lambda\left(\bigcup_n A_n\right) \leq \sum_n \lambda(A_n)$.

Proposition 3.9. *μ^* is an outer measure on Ω.*

In order to prove this proposition, we need the following lemma.

Lemma 3.10. *μ^* is finitely subadditive, monotonic and continuous under increasing limit.*

Let us first finish the proof of Proposition 3.9 assuming Lemma 3.10.

Proof of Proposition 3.9. (a) holds since $\mu^* = \mu$ on $\mathcal{F}_0$ whereas (b) is precisely the monotonocity of μ^*. As to (c), we can write

$$\mu^*\left(\bigcup_n A_n\right) = \lim_n \mu^*\left(\bigcup_{1\le k\le n} A_k\right) \le \lim_n \sum_{1\le k\le n} \mu^*(A_k) = \sum_n \mu^*(A_n)$$

where, in the first step, we used the continuity of μ^* under increasing limit and in the second step we used the finite subadditivity of μ^*. □

Now we can return to

Proof of Lemma 3.10. We can in fact prove a stronger statement than mere finite subadditivity:

$$\mu^*(A \cup B) + \mu^*(A \cap B) \le \mu^*(A) + \mu^*(B). \tag{3.3}$$

To see this, for $\varepsilon > 0$, choose $G_1, G_2 \in \mathcal{G}$, $G_1 \supset A$, $G_2 \supset B$, such that $\mu^\sigma(G_1) \le \mu^*(A)+\varepsilon/2$ and $\mu^\sigma(G_2) \le \mu^*(B) + \varepsilon/2$. By Lemma 3.7–(a),

$$\begin{aligned}\mu^*(A) + \mu^*(B) + \varepsilon &\ge \mu^\sigma(G_1) + \mu^\sigma(G_2) = \mu^\sigma(G_1 \cup G_2) + \mu^\sigma(G_1 \cap G_2)\\ &\ge \mu^*(A \cup B) + \mu^*(A \cap B).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (3.3) follows.

Monotonicity of μ^* follows from the definition.

By monotonicity, $\mu^*(A) \ge \lim_n \mu^*(A_n)$. In order to prove the reverse inequality, choose for each n, $G_n \in \mathcal{G}$, $A_n \subset G_n$ such that

$$\mu^*(G_n) \le \mu^*(A_n) + \varepsilon 2^{-n}$$

where $\varepsilon > 0$. We assume that $\mu^*(A_n) < \infty$ for each n as otherwise the inequality is trivial. Now since $A = \bigcup_n A_n \subset \bigcup_n G_n \in \mathcal{G}$, we have

$$\mu^*(A) \overset{\text{monoton.}}{\le} \mu^*\left(\bigcup_n G_n\right) \overset{\mu^*_{|\mathcal{G}}=\mu^\sigma_{|\mathcal{G}}}{=} \mu^\sigma\left(\bigcup_n G_n\right) \overset{\text{Lem. 3.7-(c)}}{=} \lim_n \mu^\sigma\left(\bigcup_{1\le k\le n} G_k\right)$$

Thus we will be done if we can show

$$\mu^\sigma\left(\bigcup_{1\le k\le n} G_k\right) \le \mu^*(A_n) + \varepsilon \sum_{1\le k\le n} 2^{-k} \tag{3.4}$$

for all n. This is true for $n = 1$ by the choice of G_1. Suppose that it holds for some $n \ge 1$. Now applying Lemma 3.7–(b) to the sets $\bigcup_{1\le k\le n}$ and G_{n+1}, we get

$$\begin{aligned}\mu^\sigma\left(\bigcup_{1\le k\le n+1} G_k\right) &= \mu^\sigma\left(\bigcup_{1\le k\le n} G_k\right) + \mu^\sigma(G_{n+1}) - \mu^\sigma\left(\left(\bigcup_{1\le k\le n} G_k\right) \cap G_{n+1}\right)\\ &\le \mu^*(A_n) + \varepsilon \sum_{1\le k\le n} 2^{-k} + \mu^*(A_{n+1}) + \varepsilon 2^{-(n+1)} - \mu^*(A_n)\\ &= \mu^*(A_{n+1}) + \varepsilon \sum_{1\le k\le n+1} 2^{-k},\end{aligned}$$

where in the second step, we used the observation that

$$\left(\bigcup_{1\leq k\leq n} G_k\right)\cap G_{n+1}\supset G_n\cap G_{n+1}\supset A_n\cap A_{n+1}=A_n$$

along with the monotonicity of μ^*. (3.4) now follows by induction. □

Step III. *Obtaining a measure by restricting μ^* .*
The proof of the following important result will be an assignment problem.

Proposition 3.11. *Let λ be an outer measure on Ω. A set E is called λ-measurable if*

$$\lambda(A)=\lambda(A\cap E)+\lambda(A\cap E^c)$$

for all $A\subset\Omega$. The class $\mathscr{M}$ of all λ-measurable sets is a σ-field and λ restrcited to $\mathscr{M}$ is a measure.

In view of Proposition 3.9, μ^* restricted to $\sigma(\mathcal{F}_0)$ is a measure provided the sets in $\mathcal{F}_0$ are μ^*-measurable.

Lemma 3.12. *The sets in $\mathcal{F}_0$ are μ^*-measurable.*

Proof. By the subadditivity, we already have $\mu^*(A)\leq\mu^*(A\cap E)+\mu^*(A\cap E^c)$ for all $A,E\subset\Omega$. Hence, we only need to show the reverse inequality. To this end let $A\subset G\in\mathcal{G}$ so that, by Lemma 3.7–(a), we have for any $E\in\mathcal{F}_0$,

$$\mu^\sigma(G)=\mu^\sigma(G\cap E)+\mu^\sigma(G\cap E^c)\geq\mu^*(A\cap E)+\mu^*(A\cap E^c)$$

where in the final step we simply used the definition of μ^*. The lemma now follows by taking infimum over all $G\supset A$. □

Now we have accomplished what we had set out for.

Although it is usually difficult to give explicit descriptions of sets in a σ-field in terms of the sets in an underlying field, the following theorem says we can do so upto a set of very small μ-measure when the measure is σ-finite on the field.

Theorem 3.13 (Approximation theorem)**.** *Let $(\Omega,\mathcal{F},\mu)$ be a measure space and let $\mathcal{F}_0$ be a field of subsets of Ω such that $\sigma(\mathcal{F}_0)=\mathcal{F}$. Assume that μ is σ-finite on $\mathcal{F}_0$, and let $\varepsilon>0$ be given. If $A\in\mathcal{F}$ and $\mu(A)<\infty$, there is a set $B\in\mathcal{F}_0$ such that $\mu(A\Delta B)<\varepsilon$.*

Proof. The proof follows directly from the construction of the measure in the extension part of the proof of the Carathéodory extension theorem when μ is finite (why?). Otherwise consider the decomposition of μ as $\sum_n \mu_n$ where $\mu_n(\cdot) := \mu(\cdot \cap A_n)$ with $A_n \in \mathcal{F}_0$ satisfying $\mu(A_n) < \infty$.

Since μ_n is a finite measure on $\mathcal{F}$, for any $A \in \mathcal{F}$, there exists a set $B_n \in \mathcal{F}_0$ such that $\mu_n(A\Delta B_n) < \varepsilon 2^{-n}$. Since

$$\mu_n(A\Delta B_n) = \mu((A\Delta B_n) \cap A_n) = \mu((A\Delta(B_n \cap A_n)) \cap A_n) = \mu_n(A\Delta(B_n \cap A_n)),$$

and $B_n \cap A_n \in \mathcal{F}_0$, we may assume $B_n \subset A_n$ (here we used the hypothesis that μ is σ-finite on $\mathcal{F}_0$, not just on $\mathcal{F}$). If $C = \bigcup_n B_n$, then $C \cap A_n = B_n$, so that

$$\mu_n(A\Delta C) = \mu((A\Delta C) \cap A_n) = \mu((A\Delta B_n) \cap A_n) = \mu_n(A\Delta B_n),$$

hence

$$\mu(A\Delta C) = \sum_n \mu_n(A\Delta C) < \varepsilon. \text{ But } \bigcup_{1\le k\le N} B_k \setminus A \uparrow C \setminus A \text{ as } N \to \infty,$$

and $A - \bigcup_{1\le k\le n} B_k \downarrow A \setminus C$. If $A \in \mathcal{F}$ and $\mu(A) < \infty$, it follows from the continuity of measures on finite measure sets that $\mu(A\Delta \bigcup_{1\le k\le n} B_k) \to \mu(A\Delta C)$ as $N \to \infty$, hence is less than ε for large enough N. Set $B = \bigcup_{1\le k\le n} B_k \in \mathcal{F}_0$. $\square$

Suggested reading. Sections 1.2.3–1.3.12 in *Probability and Measure Theory* by Robert B. Ash and Catherine A. Doléans-Dade.