Analysis I (Fall 2022, Semester I)

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Lecture VI

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6 The Riesz-Markov Representation Theorem

In the ensuing few lectures, we will discuss a very important result in analysis which is going mark our transition from measure theory to functional analysis. This result characterizes the linear functionals on a locally compact Hausdorff space X as integrals with respect to regular complex Borel measures. Recall that we motivated the definition of Lebesgue integrals as "continuous" linear functionals on the linear space of measurable functions. The Riesz-Markov theorem makes that heuristic precise.

Let us start with some warm up drills on function spaces, in particular two special subspaces of $C(K)$ — the space of all continuous functions on a topological space K. Based on whether the codomain is $\mathbb{C}$ or $\mathbb{R}$, the corresonding spaces will be denoted as $C_{\cdot}(K, \mathbb{R})$ or $C_{\cdot}(K, \mathbb{C})$ in the sequel. Unless specifically mentioned, the statements involving $C_{\cdot}(K)$ are supposed to hold for both $C_{\cdot}(K, \mathbb{R})$ and $C_{\cdot}(K, \mathbb{C})$.

Definition 6.1 (The spaces $C_c(K)$ and $C_0(K)$)**.** For any topological space K, we denote by $C_c(K)$ the set of complex-valued continuous functions on K with compact support, i.e.,

$$C_c(K) \coloneqq \left\{ f \in C(K) : \overline{\{x \in K : |f(x)| > 0\}} \text{ is compact} \right\}.$$

We further define,

$$C_0(K) \coloneqq \{ f \in C(K) : \text{for all } \varepsilon > 0, \{x \in K : |f(x)| \geq \varepsilon\} \text{ is compact} \}.$$

Recall from a previous lecture that a normed linear space $(X, \|\cdot\|)$ over a scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is simply a $\mathbb{K}$-vector space equipped with a norm $\|\cdot\|$. As before, all the statements about normed linear spaces in the sequel will be assumed to hold for both $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ unless mentioned otherwise.

Exercise 6.2. Show that both $C_c(K)$ and $C_0(K)$ are normed linear spaces under the sup norm $\|\cdot\|_\infty$ defined as $\|f\|_\infty := \sup\{|f(x)| : x \in K\}$. Furthermore, $C_c(K)$ is a subspace of $C_0(K)$.

Henceforth, we will assume $C_c(K)$ and $C_0(K)$ to be equipped with the sup norm. A *subspace* of $(X, \|\cdot\|_X)$ is a normed linear space $(Y, \|\cdot\|_Y)$ where Y is a linear subspace of X and $\|\cdot\|_Y$ is $\|\cdot\|$ restricted to Y. The following is an important notion in the study of normed linear spaces.

Definition 6.3 (Linear isometry)**.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces. A linear map $T : X \mapsto Y$ is called a *(linear) isometry* if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$. In plain words, an isometry is an injective, linear map that preserves norms. If $T : (X, \|\cdot\|_X) \mapsto (Y, \|\cdot\|_Y)$ is an isometry, we say that $(X, \|\cdot\|_X)$ is *isometrically embedded* inside $(Y, \|\cdot\|_Y)$.

In this note, we will be exclusively invested in the case when K is *locally compact Hausdorff*. As illustrated by the simple result below, there are good enough reasons for that!

Lemma 6.4. *Let K be locally compact Hausdorff space that is not compact and denote by $\overline{K} := K \cup \{\infty\}$ the* one-point compactification *of K. For any $f \in C(K)$, define Tf to be the function on $\overline{K}$ defined as $Tf(x) = f(x)$ when $x \in K$ and $Tf(\infty) = 0$. Then a function $f \in C(K)$ lies in $C_0(K)$ iff $Tf \in C(\overline{K})$. This justifies the sobriquet "the space of functions vanishing at ∞" for $C_0(K)$. Also, the map $T : C_0(K) \mapsto C(\overline{K})$ is a linear isometry.*

Proof. Exercise. □

Remember that a normed linear space is called a *Banach space* if the underlying metric space is complete.

Lemma 6.5. *$C_c(K)$ is a dense subspace of $C_0(K)$ when K is Hausdorff. Consequently, $C_c(K)$ is not a Banach space in general.*

Proof. Let $f \in C_0(K)$ and $F^\varepsilon := \{x \in K : |f(x)| \geq \varepsilon\} \subset\subset K$ ("$\subset\subset$" is the notation for "compactly contained", i.e., the left-hand side is a compact set) where $\varepsilon > 0$. Observe that

$$F^\varepsilon \subset \text{int}(F^{\varepsilon/2}) \subset F^{\varepsilon/2} \subset\subset K.$$

$F^{\varepsilon/2}$ is a compact Hausdorff space and hence is normal. Also F^ε is a closed subspace of $F^{\varepsilon/2}$ which is disjoint from $F^{\varepsilon/2} \setminus \text{int}(F^{\varepsilon/2})$. Hence by Urysohn's lemma, there exists a continuous $\phi : F^{\varepsilon/2} \mapsto [0,1]$ such that $\phi_{|F^\varepsilon} \equiv 1$ and $\phi_{|F^{\varepsilon/2} \setminus \text{int}(F^{\varepsilon/2})} \equiv 0$. By pasting lemma, ϕ can be extended to a continuous function (also denoted by ϕ) on K such that $\phi_{|K \setminus F^{\varepsilon/2}} \equiv 0$. The lemma now follows upon noticing that $g := f\phi \in C_c(K)$ and $\|f - g\|_\infty \leq \varepsilon$. □

Definition 6.6 (Linear functionals and the dual space X^*)**.** Let $(X, \|\cdot\|)$ be a normed linear space. A *linear functional* on X is just a linear map $f : X \mapsto \mathbb{K}$. We denote by X^* the linear space of all *continuous* linear functionals on X.

However, the space $C_0(K)$ is Banach when K is also locally compact.

Lemma 6.7. *$C_0(K)$ is a Banach space when K is locally compact Hausdorff.*

Proof. When K is compact, $C_0(K) = C(K)$ which is clearly complete w.r.t. the sup metric due to properties of uniform convergence. If K is not compact, we use the isometry T from Lemma 6.4 and notice that $T(C_0(K))$ is a *closed* subspace of $C(\overline{K})$ and hence is complete. □

When X is a subspace of $C(K)$, there is a natural notion of *positivity* of linear functionals on X.

Definition 6.8 (Positive linear functionals)**.** Let X be a subspace of $C(K)$. A linear functional Λ on X is *positive* if $\Lambda f \geq 0$ whenever $f \geq 0$.

We are now ready to state the first version of Riesz-Markov theorem, also called the Riesz–Markov–Kakutani representation theorem.

Theorem 6.9 (Riesz–Markov–Kakutani representation theorem)**.** *Let K be a locally compact Hausdorff space and Λ be a positive linear functional on $C_c(K)$. Then there is a unique Radon measure μ on K such that*

$$\Lambda f = \int f d\mu \tag{6.1}$$

for each $f \in C_c(K)$.

Proof. The proof of this theorem will occupy a big chunk of this note. Let us clarify a simple fact first which says that it suffices to prove the result when $\mathbb{K} = \mathbb{C}$.

Claim. Any positive linear functional Λ on $C_c(K, \mathbb{C})$ also lies in $C_c(K, \mathbb{R})$, i.e., it maps real-valued (continuous) functions on K to real numbers. Consequently, it suffices to prove (6.1) over $\mathbb{R}$.

Proof of Claim. If $f \in C_c(K, \mathbb{R})$, then both f^+ and f^- lie in $C_c(K)$. Consequently, $\Lambda f = \Lambda f^+ - \Lambda f^- \in \mathbb{R}$.

Step I. *Uniqueness of μ.*

As is often the case in this business, the proof of uniqueness is simpler. We will use the following basic lemma.

Lemma 6.10. *Let K be a locally compact Hausdorff space and $S \subset\subset U \subset K$ where U is open. Then there exists a function $f \in C_c(K)$ satisfying $\mathbf{I}_S \leq f \leq \mathbf{I}_U$ and $\operatorname{supp}(f) := \overline{\{x \in K : |f(x)| > 0\}} \subset U$.*

Proof. Since K is locally compact Hausdorff, there is an open set V with compact closure satisfying $S \subset V \subset \overline{V} \subset U$ (why?). The lemma now follows from a similar argument as used in the proof of Lemma 6.7. $\square$

Using Lemma 6.10, we can prove

Lemma 6.11. *Let K be a locally compact Hausdorff space and let μ be a Radon measure on K. If $U \subset K$ is open, then*

$$\mu(U) = \sup\left\{\int f d\mu : f \in C_c(K), \operatorname{supp}(f) \subset U \text{ and } 0 \leq f \leq \mathbf{I}_U\right\}.$$

Proof. Let us denote the supremum on the right-hand side by s. By monotonicity of integrals, we have

$$\mu(U) \geq s.$$

On the other hand, for any $S \subset\subset U$, let $f \in C_c(K)$ be given by Lemma 6.10 for S and U. Consequently,

$$s \geq \int f d\mu \geq \int \mathbf{I}_S \, d\mu = \mu(S).$$

By regularity of μ, we then get

$$s \geq \sup\{\mu(S) : S \subset\subset U\} = \mu(U). \qquad \square$$

Now assume that μ and ν are two Radon measures satisfying (6.1). By Lemma 6.11, $\mu(U) = \nu(U)$ for all $U \subset K$ open. By regularity of μ and ν, it then follows that $\mu = \nu$.

Step II. *Construction of an outer measure μ^*.*

Define a set function μ^* on the open subsets of K by

$$\mu^*(U) = \sup\{\Lambda f : f \in C_c(K, \mathbb{R}), \operatorname{supp}(f) \subset U \text{ and } 0 \leq f \leq \mathbf{I}_U\}, \tag{6.2}$$

and extend this to $\mathcal{P}(K)$ by

$$\mu^*(A) = \inf\{\mu^*(U) : U \subset K \text{ is open and } A \subset U\}. \tag{6.3}$$

We now proceed to check the conditions of an outer measure. Since $\emptyset$ is itself an open set, we get $\mu(\emptyset) = 0$. The monotonicity is clear from the definition of μ^*.

Finally, we come to the subadditivity. We will need the following result.

Lemma 6.12. *Let $f \in C_c(K)$ and $U_1 \ldots, U_n$ be open subsets of K such that $\operatorname{supp}(f) \subset \bigcup_{i=1}^n U_i$. Then there are functions $f_1, \ldots, f_n \in C_c(K)$ such that $f = f_1 + \ldots + f_n$ and $\operatorname{supp}(f_i) \subset U_i$ for each i. Also, $f_1, \ldots, f_n$ can be chosen to be nonnegative if f is.*

We will return to the proof of Lemma 6.12 shortly and complete the proof of subadditivity assuming it. Suppose for the moment that the subadditivity holds for countable collections of open sets. For $\varepsilon > 0$ and $n \geq 1$, let $U_n \supset A_n$ be an open set such that

$$\mu^*(U_n) \leq \mu^*(A_n) + \varepsilon 2^{-n}.$$

Then by the montonicity and subadditivity for open sets of μ^*,

$$\mu^*\left(\bigcup_n A_n\right) \leq \mu^*\left(\bigcup_n U_n\right) \leq \sum_n \mu^*(U_n) \leq \sum_n \mu^*(A_n) + \varepsilon.$$

Sending $\varepsilon \to 0$, we conclude the subadditivity.

Let us now finish the proof of subadditivity for open sets. To this end consider a sequence $\{U_n\}$ of open subsets of K. Let $f \in C_c(K, \mathbb{R})$ such that $\mathrm{supp}(f) \subset \bigcup_n U_n$ and $0 \leq f \leq \mathbf{I}_{\bigcup_n U_n}$. By the compactness of $\mathrm{supp}(f)$, there is a positive integer N such that

$$\mathrm{supp}(f) \subset \bigcup_{1 \leq n \leq N} U_i.$$

Lemma 6.12 then implies that there exist nonnegative functions $f_1, \ldots, f_N \in C_c(K)$ such that $f = \sum_{1 \leq n \leq N} f_n$ and $\mathrm{supp}(f_n) \subset U_n$ for each $1 \leq n \leq N$. Clearly $f_n \leq \mathbf{I}_{U_n}$ since $f \leq 1$ on U_n. Therefore,

$$\Lambda f = \sum_{1 \leq n \leq N} \Lambda f_n \leq \sum_{1 \leq n \leq N} \mu^*(U_n) \leq \sum_n \mu^*(U_n).$$

Since f is any function in $C_c(K, \mathbb{R})$ satisfying $\mathrm{supp}(f) \subset \bigcup_n U_n$ and $0 \leq f \leq \mathbf{I}_{\bigcup_n U_n}$, we deduce the subadditivity by taking supremum over all such f. We now return to the

Proof of Lemma 6.12. Since K is locally compact Hausdorff and $\mathrm{supp}(f) =: C$ is compact, each $x \in C$ has a neighborhood V_x such that $\overline{V}_x \subset\subset U_i$ for some i. There are points $x_1, x_2, \ldots, x_m$ such that $C \subset V_{x_1} \cup \ldots \cup V_{x_m}$. For $1 \leq i \leq n$, let C_i be the union of those $\overline{V}_{x_j}$ which lie in U_i. By Lemma 6.10, there are functions g_i satisfying $\mathbf{I}_{C_i} \leq g_i \leq \mathbf{I}_{U_i}$ such that $\mathrm{supp}(g_i) \subset U_i$. Define

$$\begin{aligned}
&\varphi_1 = g_1 \\
&\varphi_2 = (1 - g_1) g_2 \\
&\cdots\cdots\cdots\cdots \\
&\varphi_n = (1 - g_1)(1 - g_2) \cdots g_n
\end{aligned}$$

which are nonnegative functions. Clearly, $\mathrm{supp}(\varphi_i) \subset \mathrm{supp}(g_i) \subset U_i$ for each i. Also,

$$\varphi_1 + \varphi_2 + \ldots + \varphi_n = 1 - (1 - g_1)(1 - g_2) \ldots (1 - g_n)$$

Since each $x \in C$ lies in C_i for some $1 \leq i \leq n$ and $g_i \geq \mathbf{I}_{C_i}$, it follows that

$$\varphi_1 + \varphi_2 + \ldots + \varphi_n = 1.$$

Now let $f_i = f \varphi_i$ for each i. □

Step III. *Every Borel subset of K is μ^*-measurable.*

Since the class of μ^*-measurable sets is a σ-field, it suffices to prove that all open subsets of K are μ^*-measurable, i.e.,

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c) \tag{6.4}$$

holds for each $A \subset K$ and $U \subset K$ open. Now by (6.3) there exists, for any $\varepsilon > 0$, an open set $V \subset K$ such that $\mu^*(V) < \mu^*(A) + \varepsilon$. Hence if we can show,

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon, \tag{6.5}$$

it would follow that

$$\mu^*(A) + \varepsilon \geq \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon,$$

and we would be able to deduce (6.4) as ε is arbitrary.

In order to verify (6.5), let us first consider $f_1 \in C_c(K)$ such that $0 \leq f_1 \leq \mathbf{I}_{V\cap U}$ with $\text{supp}(f_1) \subset V \cap U$ and $\Lambda f_1 > \mu^*(V \cap U) - \varepsilon$. Such a function exists due to (6.2). Similarly letting $S = \text{supp}(f_1)$, we choose a function $f_2 \in C_c(K)$ such that $0 \leq f_2 \leq \mathbf{I}_{V\cap S^c}$ with $\text{supp}(f_2) \subset V \cap K^c$ and $\Lambda f_2 > \mu^*(V \cap S^c) - \varepsilon$. Clearly, $0 \leq f_1 + f_2 \leq \mathbf{I}_V$ with $\text{supp}(f_1 + f_2) \subset V$ and hence

$$\mu^*(V) \geq \Lambda(f_1 + f_2) \geq \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon,$$

thus proving (6.5).

Henceforth we will refer to the restriction of μ^* to $\mathscr{B}(K)$ as μ.

Step IV. *Regularity of μ.*

The following lemma allows us to compare $\mu^*(A)$ and Λf when f either dominates or is dominated by $\mathbf{I}_A$.

Lemma 6.13. *Let $A \subset K$ and $f \in C_c(K)$. If $\mathbf{I}_A \leq f$, then $\mu^*(A) \leq \Lambda f$, while if $0 \leq f \leq \mathbf{I}_A$ and if A is compact, then $\Lambda f \leq \mu^*(A)$.*

Proof. Let $\mathbf{I}_A \leq f$ and $\varepsilon \in (0,1)$, and define U_ε by $U_\varepsilon = \{x \in K : f(x) > 1 - \varepsilon\}$. Then U_ε is open and covers A, and each g in $C_c(K)$ that satisfies $g \leq \mathbf{I}_{U_\varepsilon}$ also satisfies $g \leq \mathbf{I}_{U_\varepsilon}$ as $f > 1 - \varepsilon$. By the positivity and linearity of Λ it follows that

$$\Lambda g \leq \frac{1}{1-\varepsilon}\Lambda f.$$

Because U_ε is open it follows from (6.2) and the inequality above that

$$\mu^*(U_\varepsilon) \leq \sup\{\Lambda g\} \leq \frac{1}{1-\varepsilon}\Lambda f.$$

Now recall that A is covered by U_ε and that μ^* is monotonic. It follows from this as well as that ε can be arbitrarily close to zero that $\mu^*(A) \le \Lambda f$ as required.

Now suppose that $0 \le f \le \mathbf{I}_A$ and that A is compact. Let U be an open set that includes A. Then $0 \le f \le \mathbf{I}_U$ and $\operatorname{supp}(f) \subset A \subset U$ and so by (6.2)

$$\Lambda f \le \mu^*(U).$$

Since U is an arbitrary open set that includes A, (6.3) implies that

$$\Lambda f \le \mu^*(A). \qquad \square$$

First we argue the local finiteness of μ which is equivalent to finiteness on compact sets when the underlying space is locally compact. To this end, for any $S \subset\subset K$, let us apply Lemma 6.10 to the sets S and K to get a function $f \in C_c(K)$ satisfying $\mathbf{I}_S \le f$. The first part of Lemma 6.13 then implies $\mu^*(S) = \mu(S) \le \Lambda f < \infty$.

The outer regularity is an immediate consequence of (6.3).

For the inner regularity, observe that

$$\begin{aligned}\mu^*(U) &\overset{(6.2)}{=} \sup\left\{\Lambda f : f \in C_c(K,\mathbb{R}),\ \operatorname{supp}(f) \subset U \text{ and } 0 \le f \le \mathbf{I}_U\right\} \\ &\overset{\substack{\text{Lemma 6.13,}\\ \text{second part}}}{\le} \sup\left\{\mu(\operatorname{supp}(f)) : f \in C_c(K,\mathbb{R}),\ \operatorname{supp}(f) \subset U \text{ and } 0 \le f \le \mathbf{I}_U\right\}.\end{aligned}$$

Since $\operatorname{supp}(f)$ is compact, it then follows from the previous display that

$$\mu(U) \le \sup\left\{\mu(K) : K \subset\subset U\right\}.$$

The reverse inequality, on the other hand, follows from the monotonicity of the outer measure μ^*.

Step IV. $\Lambda f = \int f d\mu$ *for any* $f \in C_c(K)$.

Due to same reasons as discussed in the beginning of the proof of Theorem 6.9, it suffices to prove this equality only for nonnegative functions. The idea is to obtain identical approximations for both Λf and $\int f d\mu$ upto arbitrary precision.

Define, for any given $\varepsilon > 0$ and for each positive integer n, a function $f_n : K \mapsto \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} 0, & \text{if } f(x) \le (n-1)\varepsilon, \\ f(x) - (n-1)\varepsilon, & \text{if } (n-1)\varepsilon < f(x) \le n\varepsilon, \\ \varepsilon, & \text{if } n\varepsilon < f(x). \end{cases}$$

It is easy to check that $f_n \in C_c(K)$ and also $f = \sum_n f_n$. Since f is bounded, there exits $N \in \mathbb{N}$ such that $f_n = 0$ for all $n > N$. Let $K_0 \coloneqq \operatorname{supp}(f) \subset\subset K$ and each $n > 0$, let

$$K_n \coloneqq \{x \in K : f(x) \ge n\varepsilon\}.$$

Clearly $K_n \subset K_{n+1} \subset\subset K$ for each $n \geq 0$. Also we have the inequality $\varepsilon \mathbf{I}_{K_n} \leq f_n \leq \varepsilon \mathbf{I}_{K_{n-1}}$ for each $n \geq 1$. From Lemma 6.13, we get that

$$\varepsilon\mu(K_n) \leq \Lambda f_n \leq \varepsilon\mu(K_{n-1}).$$

Similarly by positivity of integrals we get

$$\varepsilon\mu(K_n) \leq \int f_n d\mu \leq \varepsilon\mu(K_{n-1}).$$

Summing both sides of each of these two displays over $1 \leq n \leq N$, we get

$$\sum_{1\leq n\leq N} \varepsilon\mu(K_n) \leq \Lambda f, \int f d\mu \leq \sum_{0\leq n\leq N-1} \varepsilon\mu(K_n).$$

But the length of the interval defined by two sides above is at most

$$\varepsilon\big(\mu(K_0) - \mu(K_N)\big) \leq \varepsilon\mu(\mathrm{supp}(f))$$

which can be made arbitrarily close to 0 by sending ε to 0 and thus we have accomplished our goal. □

We end this note with the statement of:

Theorem 6.14. *Let K be a locally compact Hausdorff space and Λ be a continuous linear functional on $C_0(K)$. Then there is a unique regular complex Borel measure μ on K such that*

$$\Lambda f = \int f d\mu \tag{6.6}$$

for each $f \in C_0(K)$. Furthermore, the norm of Λ is the total variation of μ, i.e.,

$$\|\Lambda\| = |\mu|(K).$$

Suggested reading. Sections 2 in *Real and Complex Analysis* by Walter Rudin.