# Midterm Exam.

## Analysis I (Fall 2024, Semester I)

**September 30, 2024**, **Time: 1.5 hours**

**Note: This paper carries 55 points. Answer as many questions as you can. Your final score will be the minimum of 50 and your total points. State each result clearly that you are using.**

**1**. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. Define

$$D(f) = \{x \in \mathbb{R} : f'(x) \text{ exists and is finite}\}.$$

Show that $D(f) \in \mathscr{B}(\mathbb{R})$. [10]

**2**. Let $\mu$ be a *finite* measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ *supported* on the interval $[-1, 1]$, i.e., $\mu(\mathbb{R} \setminus [-1, 1]) = 0$. The (real) *Stieltjes transform* $S_\mu$ of $\mu$ is the function of a real variable whose value at any $x \in \mathbb{R}$ is defined as

$$S_\mu(x) = \int_{\mathbb{R}} \frac{1}{x - y} \, \mu(dy)$$

*provided* the integral exists.

(a) Show that $S_\mu$ is well-defined as a real-valued function on $\mathbb{R} \setminus [-1, 1]$.

(b) Show that $S_\mu$ is differentiable on the open set $\mathbb{R} \setminus [-1, 1]$ and express its derivative at any point $x \in \mathbb{R} \setminus [-1, 1]$ as an integral of some function w.r.t. $\mu$.

[5 + 10 = 15]

**3**. Let $(G, \cdot)$ be a *topological group*, i.e., there is a topology $\tau$ on $G$ such that the multiplication map $\cdot : G \times G \mapsto G, (g, h) \mapsto g \cdot h$ and the inversion map $^{-1} : G \mapsto G, g \mapsto g^{-1}$ are continuous. Show that $\mathscr{B}(G)$ is *closed* under left as well as right translation, i.e., if $A \in \mathscr{B}(G)$ and $g \in G$, then both $g \cdot A \coloneqq \{g \cdot h : h \in A\}$ and $A \cdot g \coloneqq \{h \cdot g : h \in A\}$ lie in $\mathscr{B}(G)$. [10]

**4**. Let $\mathcal{F}$ be a $\sigma$-field of subsets of $\Omega$ that is *countably generated*, i.e., there exists a countable family $\{A_1, A_2, \ldots\}$ of subsets of $\Omega$ such that $\mathcal{F} = \sigma(\{A_1, A_2, \ldots\})$.

(a) Show that there exists a *countable field* $\mathcal{F}_0$ of subsets of $\Omega$ generating $\mathcal{F}$, i.e., $\mathcal{F} = \sigma(\mathcal{F}_0)$.

(b) Given any *finite* measure $\mu$ on $(\Omega, \mathcal{F})$, show that the function $d_\mu : \mathcal{F} \times \mathcal{F} \mapsto \mathbb{R}_{\geq 0}$ defined as $d_\mu(A, B) = \mu(A \Delta B)$ is a *pseudometric* on $\mathcal{F}$, i.e., it satisfies all the properties of a metric except that $d_\mu(A, B)$ may be 0 for $A \neq B$.

(c) Show that the pseudometric space $(\mathcal{F}, d_\mu)$ is separable, i.e., it has a countable dense subset.

[10 + 5 + 5 = 20]