CONNECTIVITY FUNCTIONS FOR THE VACANT SET OF RANDOM INTERLACEMENTS

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Abstract

We consider the percolation of vacant set of random interlacements in dimensions three and higher, and derive lower bounds on the truncated two-point function for all but the corresponding critical parameter u_*. These bounds are sharp to principal exponential order, and match the upper bounds derived in a companion article [12]. In dimension three, our results imply that the two-point function grows at an atypical rate, with a logarithmic correction and a precise pre-factor that converges to 0 as the parameter u approaches the critical point u_* from either side. A particular challenge comes from the combined effects of lack of monotonicity when dealing with *truncation* in the super-critical phase and the precise (*rotationally invariant*) control we seek in dimension three. These rely on rather fine estimates for hitting probabilities of the random walk in arbitrary direction e, which witness this invariance at the discrete level, and preclude straightforward applications of projection arguments.

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1 Introduction

We consider the vacant set of random interlacements $(\mathcal{V}^u)_{u>0}$ on $\mathbb{Z}^d$ in dimensions $d \geq 3$ and its percolative properties (see [20]). As shown in the successive works [20, 19, 24, 18], the random set $\mathcal{V}^u$, which is decreasing in u, undergoes a percolation phase transition across a non-degenerate threshold $u_* = u_*(d) \in (0,\infty)$ for all $d \geq 3$. This transition entails that whenever $u < u_*$, there exists a unique infinite cluster in $\mathcal{V}^u$ with probability one. In stark contrast, for all $u > u_*$ the connected components (clusters) of $\mathcal{V}^u$ are finite almost surely.

We now come to the statement of our main result which concerns the *truncated two-point function* of $(\mathcal{V}^u)_{u>0}$. To this end let

$$\tau_u^{\mathrm{tr}}(x,y) = \mathbb{P}[x \overset{\mathcal{V}^u}{\longleftrightarrow} y, x \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty], \text{ for } x,y \in \mathbb{Z}^d \text{ and } u \in (0,\infty). \tag{1.1}$$

Note that $\tau_u^{\mathrm{tr}}(x,y) = \tau_u^{\mathrm{tr}}(y,x)$ and that $\tau_u^{\mathrm{tr}}(x,y) = \tau_u^{\mathrm{tr}}(0,y-x) \equiv \tau_u^{\mathrm{tr}}(y-x)$ by translation invariance of $\mathcal{V}^u$. When $u > u_*$ the 'truncation' $\{0 \not\longleftrightarrow \infty \text{ in } \mathcal{V}^u\}$ has probability one and can be safely omitted, so that $\tau_u^{\mathrm{tr}}(x) = \mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} x] \equiv \tau_u(x)$, the usual two-point function. When $u < u_*$ however, $\tau_u(x) \geq \mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \infty]^2 > 0$, which does not decay at all. the next result gives sharp asymptotics for τ_u^{tr} at large distances.

In a recent work [8] (see Theorem 1.2, i) and (1.15) there), drawing upon the companion articles [9, 10], it was shown that $\tau_u(x)$ decays stretched-exponentially fast in the sub-critical regime $u > u_*$. In the super-critical regime $u > u_*$, on the other hand, [8, Theorem 1.2] together with the disconnection estimate in [24] yields a similar bound on a related but *strictly smaller* quantity, namely where the disconnection in (1.1) happens at an intensity $v < u$. The main object of this paper is to obtain 'sharp' lower bounds on $\tau_u^{\mathrm{tr}}(x)$. In the sequel, we write $|\cdot|$ for the Euclidean distance on $\mathbb{Z}^d$.

Theorem 1.1. *For all $u \neq u_*$,*

$$\liminf_{|x|\to\infty} \frac{\log|x|}{|x|} \log \tau_u^{\mathrm{tr}}(x) \geq -\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*})^2, \quad d = 3. \tag{1.2}$$

When $d \geq 4$, for all $u \neq u_*$, there exists $C = C(u,d) \in (0,\infty)$ such that for all $x \in \mathbb{Z}^d$,

$$\log \tau_u^{\mathrm{tr}}(x) \geq -C|x|. \tag{1.3}$$

Combined with Theorem 1.4 in the companion article [12], this implies

Corollary 1.2. *For all $u \neq u_*$,*

$$\lim_{|x|\to\infty} \frac{\log|x|}{|x|} \log \tau_u^{\mathrm{tr}}(x) = -\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*})^2, \quad d = 3. \tag{1.4}$$

When $d \geq 4$, for all $u \neq u_*$, there exist $C = C(u,d)$ and $c = c(u,d) \in (0,\infty)$ such that for all $x \in \mathbb{Z}^d$,

$$-c\,|x| \geq \log \tau_u^{\mathrm{tr}}(x) \geq -C\,|x|. \tag{1.5}$$

We now make a few comments and highlight some aspects of the proof focussing *entirely* on dimension 3 which is our prime interest in this paper. One of the main obstacles is the non-monotonic nature (as functions of u) of the function τ_u^{tr}. Theorem 1.1 will follow from a more general result, Theorem 3.1, which yields a lower bound on the probability of a class of events $A^u(x,\delta)$, that, roughly speaking, allow to connect the $|x|^\delta$-neighborhoods of 0 and $x \in \mathbb{Z}^d$ by a sufficiently "straight" cluster (i.e. efficiently in diameter) in the vacant set $\mathcal{V}^u$, which is isolated from infinity (eventually, $\delta \downarrow 0$). The event $A^u(x,\delta)$ in fact more involved, it also entails the flexibility to perform local surgeries near 0 and x (borrowing a technique recently introduced in [9]), which normally would follow by application of the FKG-inequality, which however fails to apply when the event in question is not monotone.

Theorem 3.1 has a lot of mileage, in allowing us to deduce Theorem 1.1, along with other interesting results, among which, bounds on the so-called truncated one-arm events, see Remark 3.2. Another case in point is the following 'local uniqueness' event LU and variants thereof, which play a prominent role in the proof: for $u, v, \varepsilon > 0$, with B_r denoting the ball of radius $r > 0$ around 0, let

$$\mathrm{LU}_{r,\varepsilon}^{u,v} \overset{\text{def.}}{=} \left\{ \begin{array}{r} (B_{(1+\varepsilon)r} \setminus B_{r/2}) \cap \mathcal{V}^u \text{ has at least one crossing cluster,} \\ (B_r \setminus B_{r/2}) \cap \mathcal{V}^v \text{ has at most one crossing cluster} \end{array} \right\}. \tag{1.6}$$

Notice that the event $\mathrm{LU}_{r,\varepsilon}^{u,v}$ becomes *unfavorable* compared to the 'unsprinkled' version $\mathrm{LU}_{r,\varepsilon}^{u,u}$ when $v < u$. Events of this type arise naturally from monotone couplings involving non-monotone events like $\mathrm{LU}_{r,\varepsilon}^{u,u}$ etc. and as such are quite instrumental in understanding the properties of $\mathcal{V}^u$ in the super-critical phase (see, e.g. [15, 3, 21, 23, 8, 9, 10, 17]).

The study of $\mathrm{LU}_{\varepsilon}^{u,v}$ for $v < u$ (and in fact the more general event considered in Theorem 3.1) proceeds through a change of probability method, that involves a carefully designed tilt $\widetilde{\mathbb{P}}_f$ of the canonical law $\mathbb{P}$ of the interlacement point process. As within classical large deviation theory, the gyst is for the change of measure to simultaneously make the event of interest likely (i.e. with probability of order unity rather than, say, the right-hand side of (1.2)), all the while retaining good control on the Radon-Nikodym derivative it induces, which eventually leads to lower bounds as in (1.2). A similar but simpler tilting technique than the one we employ here was used by Li and Sznitman in [14] to study the (monotone) disconnection event from a box in the regime $u < u_*$, see also [22] for other contexts. In the present case, the function $f : \mathbb{Z}^d \to \mathbb{R}$ induces a tilt on *labelled* trajectories entering the interlacement set which acts as a Doob-transform for the trajectories, in a manner that depends on the label (here working with labeled trajectories is intimately linked to the fact that the event (1.6) requires working simultaneously at different levels), in order to induce a propitious spatially modulated level $u(x)$, for which we now give some heuristics.

Roughly speaking, the effect of the function f we choose is to create a 'corridor' T^1 and concentric interface T^2, in which the new vacant sets $\widetilde{\mathcal{V}}^u$, resp. $\widetilde{\mathcal{V}}^v$ declared under $\widetilde{\mathbb{P}}_f$ look slightly super-, resp. sub-critical. That is, f is designed so as to roughly ensure that for any $0 < v \leq u$, and small $\varepsilon > 0$

$$\widetilde{\mathcal{V}}^u|_{T^1} \overset{\text{law}}{\approx} \mathcal{V}^{u_*-\varepsilon}, \quad \widetilde{\mathcal{V}}^v|_{T^2} \overset{\text{law}}{\approx} \mathcal{V}^{u_*+\varepsilon}. \tag{1.7}$$

The rationale behind (1.7) is that the opposite requirements appearing in (1.6) become typical at the effective new levels $u_* \pm \varepsilon$; indeed connection in $\mathcal{V}^u$ is likely below u_*, whereas disconnection is above u_*. The way (1.7) is made precise is by developing couplings allowing to compare tilted and untilted interlacements; see Proposition 6.1. These couplings boil down to rather precise comparison estimates between tilted and untilted harmonic quantities for the random walk (see for instance Proposition 7.3) below, which make the "effective levels" $u_* \pm \varepsilon$ appear, and are ultimately responsible for our choice of tilting function f. One technical point is that the tilting function f is typically not the discrete blow-up of a continuous analogue, which makes proving these comparison estimates somewhat involved.

The remaining bit is to control the relative entropy $H(\widetilde{\mathbb{P}}_f|\mathbb{P})$ between $\widetilde{\mathbb{P}}_f$ and $\mathbb{P}$, and with it the derivative between the two measures. This eventually follows from (killed) capacity estimates for the tube regions T^1 and T^2, somewhat similar in spirit to those that arose in [11] in the context of the Gaussian free field. However, unlike the setup of [11], which dealt with one-arm events (in the present context this would mean choosing T^i, $i = 1, 2$ to be axis-aligned), the efficient connection between 0 and x underlying (1.2) follows the *Euclidean* distance, hence the sets T^1 and T^2 are oblique and directed towards $e = \frac{x}{|x|}$. This setup is paramount in order to yield (1.2) and it precludes the use of certain projection arguments (onto subsets of coordinates) used in [11]. Rather than following a route suggested in [11, Remark 5.17,2)], which would resort to dyadic coupling to Brownian motion, we develop robust techniques, which could be adapted, e.g. to the setup of [5] and involve certain (non-standard) martingale arguments, see also Fig. 1 and the proof of Proposition 2.1. These are of independent interest.

We now describe how this article is organized. Section 2 isolates an estimate on the hitting probability for an obliquely aligned cylinder pertaining to the problem discussed above. In Section 3 we state our general lower bound in Theorem 3.1 and deduce from it our main result, namely Theorem 1.1 above. The rest of the paper is devoted to the proof of Theorem 3.1. Section 4 introduces the tilting functions f of interest, along with the tilted interlacement measure $\widetilde{\mathbb{P}}_f$ they induce. Theorem 3.1 is then reduced to two results, Propositions 4.1 and 4.2, from which the proof of Theorem 3.1 is readily concluded. Proposition 4.1 is the control on the relative entropy $H(\widetilde{\mathbb{P}}_f|\mathbb{P})$. It is proved in Section 5 which draws upon, among others, the aforementioned hitting probability estimate for 'oblique corridors' given by Proposition 2.1 in Section 2. Proposition 4.2 asserts that the event of interest in Theorem 3.1 becomes typical under the tilted measure. The proof of Proposition 4.2 spans Sections 6 and 7. The key ingredient is a coupling between tilted and untilted interlacements at suitable levels that makes precise the above heuristics.

Our convention regarding constants is as follows. Throughout the article $c, c', C, C', \dots$ denote generic constants with values in $(0, \infty)$ which are allowed to change from place to place. All constants may implicitly depend on the dimension $d \geq 3$. Their dependence on other parameters will be made explicit. Numbered constants are fixed when first appearing within the text.

2 Hitting probability for oblique sets

We present an upper bound on the hitting probability of a cylinder aligned along any *arbitrary* direction in $\mathbb{R}^3$. This bound becomes effective when the distance between the underlying starting point and the cylinder is of *lower order* than its *length* which renders the 'standard' estimate obtained using bounds on capacity and Green's function (see, e.g. (2.3) below) *trivial* (i.e. ≥ 1). A similar estimate was obtained in [11, (2.25)] for *axis-aligned* cylinders. The main feature of our bound is that it holds *uniformly* in all directions which is consistent with the rotational invariance apparent in (1.2) and (1.4). The argument in [11] relied heavily on projection arguments, whose straightforward application is precluded when the cylinders are oblique. We proceed using a (different) martingale argument which inherits the required (rotational) symmetry from the large distance behavior of the Green's function (see (2.10) below) *rather than* the Brownian motion, the corresponding scaling limit, as suggested in [11, Remark 5.17,2)]. Consequently, our method could be adapted to very general class of graphs; see, e.g. [5].

Below and in the remainder of the article, we write P_x for the canonical law of the continuous-time random walk on $\mathbb{Z}^d$, $d \geq 3$, with starting point $x \in \mathbb{Z}^d$ and mean one exponential holding times. We write $X = (X_t)_{t\geq 0}$ for the canonical process under P_x. For $K \subset \mathbb{Z}^d$, we introduce the stopping time $H_K = \inf\{t \geq 0 : X_t \in K\}$, $T_K = H_{\mathbb{Z}^d \setminus K}$ and $\widetilde{H}_K = \inf\{t \geq 0 : X_t \in K \text{ and } \exists s \in [0,t] \text{ s.t. } X_s \neq X_0\}$. For $U \subset \mathbb{Z}^d$, we write g_U for the Green's function of the walk killed when exiting U, i.e.

$$g_U(x,y) = E_x\Big[\int_0^{T_U} 1\{X_t = y\}\mathrm{d}t\Big],\ x,y \in \mathbb{Z}^d \tag{2.1}$$

which is symmetric and finite, and for $K \subset U$ we denote by $e_{K,U}$ for the equilibrium measure of K relative to U,

$$e_{K,U}(y) = P_y[\widetilde{H}_K > T_U]1_{\{y\in K\}}. \tag{2.2}$$

Its total mass is denoted by $\mathrm{cap}_U(K)$, the capacity of K (relative to U). We omit U from the notation in (2.1) and (2.2) whenever $U = \mathbb{Z}^d$ (with $T_{\mathbb{Z}^d} = \infty$ by convention). An application of the strong Markov property yields the formula

$$P_y[H_K < T_U] = \sum_{z\in K} g_U(y,z)e_{K,U}(z), \tag{2.3}$$

valid for all $y \in \mathbb{Z}^d$ and K finite set.

We write $|\cdot|$ for the Euclidean norm and denote by $d(\cdot,\cdot)$ the Euclidean distance between sets. Let $x \in \mathbb{Z}^d \setminus \{0\}$ and $e = \frac{x}{|x|}$. For a point $z \in \mathbb{R}^d$, let $[z] \in \mathbb{Z}^d$ be a point achieving $d(z, \mathbb{Z}^d)$. We now introduce certain discretized cylindrical sets, namely

$$\begin{aligned} T(x) &= T(x,0) = \{[je] : 0 \leq j \leq \lceil |x| \rceil\}, \\ T(x,r) &= \{y \in \mathbb{Z}^d :\ d(y, T(x)) \leq r\}, \quad r \geq 0. \end{aligned} \tag{2.4}$$

In the sequel, $\partial K \stackrel{\text{def.}}{=} \{x \in K : d(x, K^c) = 1\}$ denotes the inner vertex boundary of $K \subset \mathbb{Z}^d$ whereas $\partial^{\text{out}} K \stackrel{\text{def.}}{=} \partial(\mathbb{Z}^d \setminus K)$ its outer boundary.

Proposition 2.1. *For all $\delta, \varepsilon \in (0,1)$, $|x| \geq C(\delta,\varepsilon)$, if $d = 3$,*

$$\inf_{y \notin T(x,|x|^\delta)} P_y[H_{T(x,|x|^{(1-\varepsilon)\delta})} = \infty] \geq c\delta\varepsilon. \tag{2.5}$$

Proof. We abbreviate $T = T(x)$, $T^2 = T(x, |x|^{\delta(1-\varepsilon)})$ and $T^3 = T(x, |x|^\delta)$ throughout this proof. Let us start with a few reduction steps. By a straightforward application of the strong Markov property at time H_{T^3} it is enough to show (2.5) for $y \in \partial^{\text{out}} T^3$. For $|x| \geq C(\delta)$, we can write $\partial^{\text{out}} T^3 \subset \mathbb{S} \cup \mathbb{L}$, where, writing $R = |x|^\delta$ and $\ell = \{te : t \in \mathbb{R}\}$ with $e = x/|x|$ and tacitly embedding $\mathbb{Z}^d \subset \mathbb{R}^d$ in writing expressions such as $d(z,\ell)$ below, we set

$$\begin{aligned} \mathbb{L} &\stackrel{\text{def.}}{=} \big\{z \in T(x,4R) \setminus T^3 : d(z,\ell) \geq \tfrac{R}{4}\big\}, \\ \mathbb{S} &\stackrel{\text{def.}}{=} \partial^{\text{out}} T^3 \setminus \mathbb{L}. \end{aligned} \tag{2.6}$$

One can reduce the case $y \in \mathbb{S}$ to the case $y \in \mathbb{L}$, as explained at the end of the proof. We now focus on $y \in \mathbb{L}$. For such y, we will in fact show a stronger statement, with T^2 in (2.5) replaced by a certain enlargement $\widetilde{T}^2 \supset T^2$, which we now introduce. This enlargement will later have a 'uniformizing' effect on the martingale we consider, cf. (2.8) and (2.12) below.

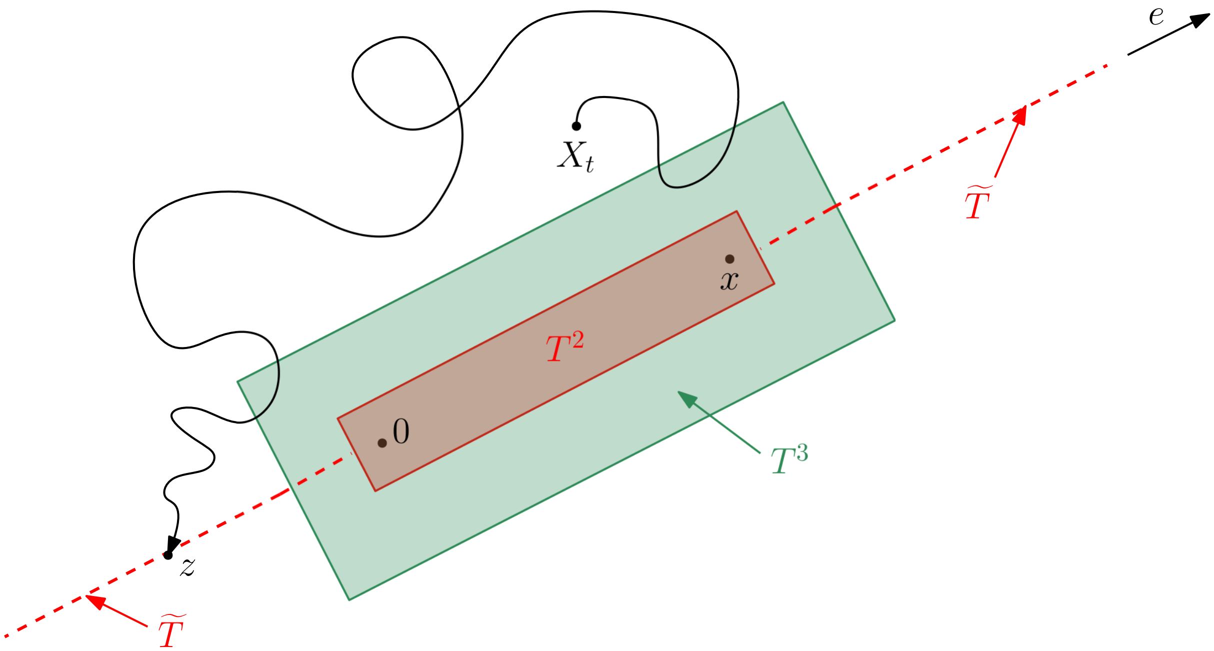

Figure 1 – **A configuration of cylinders in a generic direction $e = x/|x|$.** We illustrate a sample path contributing to the martingale M_t introduced in (2.8). The point z varies over all of $\widetilde{T}$, which includes the two 'stablilizing' rods (dashed). These are used to achieve the desired uniformity for the lower bound in (2.12). In reality, all lines and rectangles drawn are replaced by lattice approximations, cf. (2.4)-(3.1).

Recalling the notation $[z]$ from around (2.4) and for $K \subset \mathbb{Z}^d$, let

$$\widetilde{K} \stackrel{\text{def.}}{=} K \cup \big\{[je] : j \in \mathbb{Z} \cap \big([-|x|, 0] \cup [|x|, 2|x|]\big)\big\}. \tag{2.7}$$

In words, $\widetilde{K}$ is obtained from K by adding two 'rods' of length roughly $|x|$ each, extending from near 0 and x in opposite directions parallel to e. Abbreviating $T(x) \equiv T$ for the set from (2.4) in the sequel, let

$$M_t = \sum_{z \in \widetilde{T}} g(X_t, z), \tag{2.8}$$

and define the stopping time $\tau = H_{\widetilde{T}^2} \wedge T_U = T_{U \setminus \widetilde{T}^2}$ with U given by (4.1).

Since $g(\cdot, z)$ is harmonic outside z and using the fact that $\widetilde{T} \subset \widetilde{T}^2$ it readily follows that $(M_{t \wedge \tau})_{t \geq 0}$ is a martingale under P_y for all $y \in \mathbb{L}$. We will need to separately consider the extremities $\mathrm{Ext}(\widetilde{T}) \subset \widetilde{T}$, defined as follows. Let $x_\pm \in \widetilde{T}$ where $x_+ = [2|x|e] = [2x]$ and $x_- = [-|x|e] = [-x]$ and

$$\mathrm{Ext}(\widetilde{T}) \stackrel{\text{def.}}{=} \{z \in \widetilde{T} : d(z, x_+) \wedge d(z, x_-) \leq |x|/2\}. \tag{2.9}$$

With these definitions, the following hold.

Claim 2.2 $(d=3)$**.** *Let*

$$c_1 = \lim_{|x|\to\infty} |x|^{-1} g(0,x) (= \tfrac{3}{2\pi}) \tag{2.10}$$

(see, e.g. [13, Theorem 1.5.4]). For $|x| \geq C(\delta,\varepsilon)$*, one has:*

$$\text{for all } z \in \mathbb{L}\text{: } E_z[M_0] \leq 2c_1(1-(1-0.1\varepsilon)\delta)\log|x|, \tag{2.11}$$

$$\text{for all } z \in \widetilde{T}^2 \setminus \mathrm{Ext}(\widetilde{T})\text{: } E_z[M_0] \geq 2c_1(1-(1-0.9\varepsilon)\delta)\log|x|. \tag{2.12}$$

The proof of both (2.11) and (2.12) requires slight care owing to discrete effects. To avoid disrupting the flow of reading the proof is postponed to the end of this section. Suppose now that Claim 2.2 holds. We proceed to show (2.5) for $y \in \mathbb{L}$. Since $|M_{t\wedge\tau}| \leq C$ P_y-a.s., the optional stopping theorem applies and yields, upon neglecting the (positive) contributions stemming from both the event $\{H_{\widetilde{T}^2} > T_U\}$ and otherwise from the case where $X_\tau = X_{H_{\widetilde{T}^2}}$ belongs to $\mathrm{Ext}(\widetilde{T})$, that for all $|x| \geq C(\delta,\varepsilon)$ and $y \in \mathbb{L}$,

$$\begin{aligned} 2c_1(1-(1-0.1\varepsilon)\delta)\log|x| &\overset{(2.11)}{\geq} E_y[M_0] \\ &= E_y[M_\tau] \overset{(2.12)}{\geq} P_y\big[H_{\widetilde{T}^2} \leq T_U, X_\tau \notin \mathrm{Ext}(\widetilde{T})\big] \cdot 2c_1(1-(1-0.9\varepsilon)\delta)\log|x|. \end{aligned} \tag{2.13}$$

By definition of $\widetilde{T}$ and $\mathrm{Ext}(\widetilde{T})$, see (2.7) and (2.9), see also (3.1) one has that $d(T(x,4R),\mathrm{Ext}(\widetilde{T})) \geq \frac{|x|}{4}$ when $|x| \geq C(\delta,\varepsilon)$. By [11, Lemma 2.2 and Remark 2.3] (or using (5.5) below), one knows that $\mathrm{cap}\big(\mathrm{Ext}(\widetilde{T})\big) \leq C\frac{|x|}{\log|x|}$. By (2.3), it thus follows that

$$\sup_{y\in T(x,4R)} P_y\big[H_{\mathrm{Ext}(\widetilde{T})} < \infty\big] \leq C(\log|x|)^{-1} \to 0 \text{ as } |x| \to \infty. \tag{2.14}$$

Using (2.14) one deduces in particular that $P_y\big[H_{\widetilde{T}^2} \leq T_U, X_\tau \in \mathrm{Ext}(\widetilde{T})\big] \leq 0.2c_1\delta\varepsilon$ for $y \in \mathbb{L}$ whenever $|x| \geq C(\delta,\varepsilon)$. Using this together with (2.13) and the fact that $T_2 \subset \widetilde{T}_2$ by (2.8), it follows that

$$P_y[H_{T^2} \leq T_U] \leq P_y[H_{\widetilde{T}^2} \leq T_U] \leq 1 - 2c_1\Big(\frac{0.8\delta\varepsilon}{1-(1-0.9\varepsilon)\delta} - 0.1\delta\varepsilon\Big), \tag{2.15}$$

for all $y \in \mathbb{L}$ and $|x| \geq C(\delta,\varepsilon)$. With a similar calculation as in (2.14), one finds that $\sup_{z\in\mathbb{Z}^d} P_z[T_U < H_{T^2} < \infty] \to 0$ as $|x| \to \infty$. Combining with (2.15), one deduces the bound in (2.5) uniformly in $y \in \mathbb{L}$.

To complete the proof of (2.5) it thus remains to treat the case $y \in \mathbb{S}$ in (2.5). To deal with this, we now argue that

$$\inf_{y\in\mathbb{S}} P_y[H_{\mathbb{L}} < H_{T^2}] \geq c. \tag{2.16}$$

Once (2.16) is shown, (2.5) follows by applying the strong Markov property at time $H_{\mathbb{L}}$ and combining (2.16) with the lower bound on $P_y[H_{T^2} = \infty]$ for $y \in \mathbb{L}$ already derived. We show (2.16) using a chaining argument. Let $y \in \mathbb{S}$. By definition, see (2.6) and (3.1) for any such y we can find a finite number of boxes $B_i = B(x_i, \frac{R}{100})$, $1 \leq i \leq n$ (with n uniform in y) such that $B_1 \ni y$, B_{i+1} and B_i have a non-empty overlap containing a translate of $B(0,\frac{R}{300})$ for all $1 \leq i < n$, $B_n \subset \mathbb{L}$ and lastly $B(x_i,\frac{R}{10}) \cap T_2 = \emptyset$ for all $1 \leq i \leq n$. Then, using [17, (A.4)] with the choice $\delta = 1$, one readily infers that $\inf_{z\in B_i} P_z[H_{B_{i+1}} < H_{T^2}] \geq c$ for all $1 \leq i < n$. Applying the strong Markov property repeatedly, (2.16) follows as $B_n \subset \mathbb{L}$. □

We now give the proof of the estimates (2.11) and (2.12), for which the stabilizing effect of the extension $\widetilde{T}$ comes into effect.

Proof of Claim 2.2. Recall M_t from (2.8), which involves summation along the oblique line $\widetilde{T}$, see (2.4) regarding $T = T(x)$ and (2.7) regarding its extension $\widetilde{T} \supset T$. As before, with $|\cdot|$ denoting the Euclidean norm, let $e = \frac{x}{|x|}$ for $x \neq 0$, and viewing $\mathbb{Z}^d \subset \mathbb{R}^d$, consider the line $\ell = \{te : t \in \mathbb{R}\}$. Given $z \notin \ell$, define $y_z \in \ell$ as the point minimizing the distance to z, so that, with $\langle \cdot, \cdot \rangle$ denoting the standard inner product,

$$\langle z - y_z, e \rangle = 0 \tag{2.17}$$

and let $j_z \in \mathbb{Z}$ be such that $|je - y_z|$ is minimized when $j = j_z$. In particular, it follows that

$$|j_z e - y_z| \leq C. \tag{2.18}$$

From this, one obtains that for all $y \in \widetilde{T}$, i.e. for all y of the form $y = [je]$ for some $j \in \mathbb{Z} \cap [-|x|, 2|x|]$, and $z \in \mathbb{Z}^d$, abbreviating $d_z = d(z, \ell)$ (recall that $d(\cdot, \cdot)$ refers to the Euclidean distance between sets), whence $d_z = |z - y_z|$, that

$$\begin{aligned} |y - z| = |[je] - z| &\geq |je - z| - C \\ &\overset{(*)}{=} \sqrt{|je - y_z|^2 + d_z^2} - C \geq d_z \sqrt{1 + \frac{((|j - j_z| - C_1) \vee 0)^2}{d_z^2}} - C, \end{aligned} \tag{2.19}$$

where we used in deriving $(*)$ that $y_z - e$ is collinear with e, whence $\langle je - y_z, z - y_z \rangle = 0$, and in the last step that $|je - y_z| \geq |j - j_z| - |j_z e - y_z| \geq |j - j_z| - C$ using (2.18). Now if $z \in \mathbb{L}$ as in (2.11), then in view of (2.6) one has that $j_z \in [-|x|/2, 3|x|/2]]$ whenever $|x| \geq C(\delta, \varepsilon)$. For such z, let $I_k = \{j \in \mathbb{Z} : |j - j_z| - C_1 \in [kd_z, (k+1)d_z)\}$ and $K = 2|x|/d_z$, so that

$$\{j \in \mathbb{Z} : [je] \in \widetilde{T}\} \setminus [j_z - 2C_2, j_z + 2C_2] \subset \bigcup_{k=1}^{K} I_k \tag{2.20}$$

It follows that for $z \in \mathbb{L}$ and $|x| \geq C'(\delta, \varepsilon)$, recalling M_t from (2.8), one has

$$\begin{aligned} (1 + \tfrac{\varepsilon}{10^3})^{-1} E_z[M_0] &\overset{(2.10)}{\leq} c_1 \sum_{y \in \widetilde{T}} \frac{1}{|z - y|} \\ &\overset{(2.20)}{\leq} C + c_1 \sum_{k=1}^{K} \sum_{y \in \widetilde{T}} \frac{1\{y = [je] \text{ for a } j \in I_k\}}{|z - y|} \overset{(2.19)}{\leq} C + c_1 d_z^{-1} \sum_{k=1}^{K} |I_k| \frac{1}{\sqrt{1 + k^2} - \frac{C}{d_z}}. \end{aligned} \tag{2.21}$$

Now, using the fact that $c|x|^\delta \leq d_z = d(z, \ell) \leq C|x|^\delta$ for all x and $z \in \mathbb{L}$ one infers that the last fraction in (2.21) is at most $(1 + \frac{\varepsilon}{10^3})k^{-1}$ for $|x| \geq C(\delta)$, that $K \leq C|x|^{1-\delta}$ and that $|I_k| = 2d_z$ for any such k. Substituting these bounds into (2.21), the bound (2.11) readily follows upon performing the harmonic sum and using that $\sum_{1 \leq k \leq K} \frac{1}{k} \leq 1 + \log K$.

To obtain (2.12), one proceeds similarly with a few modifications, which we now highlight. It is here that the extension $\widetilde{T}$ bears its fruits over T and allows a not too wasteful estimate (in particular, one correctly producing the pre-factor 2 appearing on the right-hand side of (2.12)). One readily derives a companion bound to (2.19), yielding that for all $z \in \mathbb{Z}^d$ and $j \in \mathbb{Z}$,

$$|[je] - z| \le d_z \sqrt{1 + \frac{(|j - j_z| + C_2)^2}{d_z^2}} + C. \tag{2.22}$$

The fact that j_z as defined above (2.18) continues to range in $[-|x|/2, 3|x|/2]]$ when $z \in \widetilde{T}^2 \setminus \mathrm{Ext}(\widetilde{T})$ as in (2.12) follows readily using the definition of $\mathrm{Ext}(\widetilde{T})$ in (2.9). Thus, in view of the definition of $\widetilde{T}$ in (2.7), if one sets $K' = c|x|/d_z$ for sufficiently small c and defines $I'_k = \{j \in \mathbb{Z} : |j - j_z| \in [kd_z, (k+1)d_z)\}$, then one obtains the inclusion

$$\{j \in \mathbb{Z} : [je] \in \widetilde{T}\} \supset \bigcup_{k=1}^{K'} I'_k. \tag{2.23}$$

Equipped with (2.22) and (2.23), now playing the role of (2.19) and (2.20), respectively, one readily performs a computation akin to (2.21), using among others the complementary bound stemming from (2.10) as well as the fact that the sets I'_k are disjoint, to conclude (2.12). □

3 Generalized lower bound

We now state a general lower bound on a certain class of events involving simultaneous connection and disconnection events in the vacant set. The precise statement is given in Theorem 3.1 below. The event of interest in Theorem 3.1 (cf. (3.5)) has three distinguishing features, which together allow for various interesting consequences. First, it is quantitative in the sense that connection and disconnection are implemented in specific (tubular) regions introduced below. Second, the event allows a small amount of unfavorable sprinkling between the levels involved in the connection and disconnection events (parametrized by $\eta > 0$ below). Third, it leaves the flexibility for a good event $G^u(x)$, see (3.4), which in practice allows for local surgery around 0 and x: indeed, the event in Theorem 3.1 only connects their r_1-neighborhhood, where r_1 is chosen below, see (3.2).

Theorem 1.1, as well as various other interesting lower bounds (see for instance Remark 3.2), are then derived in the remainder of this section, using Theorem 3.1 as crucial ingredient. Theorem 1.1 follows rather straightforwardly once one implements a suitable surgery to connect 0 and x (without spoiling disconnection).

We will work with some special cylindrical sets (recall (2.4)) which will play a role in the sequel. To this end, let $x \in \mathbb{Z}^d \setminus \{0\}$ (eventually this will correspond to the point x appearing in the statement of Theorem 1.1) and $\delta \in (0, \frac{1}{6})$. Our construction involves the nested sets

$$\begin{aligned} &T^1 \subset \cdots \subset T^6, \text{ where} \\ &T^i = T(x, (|x| \vee 100)^{i\delta}), \ 1 \le i \le 5, \quad T^6 = T(x, 4(|x| \vee 100)^{5\delta}), \end{aligned} \tag{3.1}$$

which implicitly depend on x and δ. Let

$$r_0 = r_0(x, \delta) \overset{\text{def.}}{=} 100 \vee \frac{|x|^\delta}{100}, \quad r_1 = r_0^{1/4}. \tag{3.2}$$

In the sequel $(\ell^u_y)_{y \in \mathbb{Z}^d, u > 0}$ denotes the occupation time field of the interlacement point process; for its explicit definition cf. (6.1) below. In what follows we consider, for $u > 0$, $x \in \mathbb{Z}^d$,

$\delta \in (0,1)$ and $\eta \in [0,1)$ (often kept implicit in our notation), a generic event G^u of the form $G^u = \bigcap_{i\in I} G^u_{B_i}$, where $(B_i : i \in I)$ is a finite sequence of boxes of radius $10r_1$ with centers in $T(x)$. It may well be $B_i = B_j$ for some $i \neq j$. We further assume that $G_B = G^u_B$ can be expressed measurably in terms of (at most) two local occupation fields $(\ell^v_y)_{y\in B}$ and $(\ell^w_y)_{y\in B}$, where $v, w \in \{u(1-\frac{k}{4}\eta\sqrt{\delta}) : k = 0,1,2,3,4\}$ with $v > w$, and that G_B is increasing in $(\ell^v_y)_{y\in B}$ and decreasing in $(\ell^w_y)_{y\in B}$.

Theorem 3.1. *Let G^u be of the above form and define, for $u \neq u_*$, $\eta \in [0,1)$, $x \in \mathbb{Z}^d$ and $\delta \in (0,1)$, the event*

$$A^u(x,\delta) = \big\{G^u,\ B(0,r_1) \overset{\mathcal{V}^u\cap T^1}{\longleftrightarrow} B(x,r_1),\ T^4 \overset{\mathcal{V}^{u(1-\eta\sqrt{\delta})}}{\not\longleftrightarrow} \partial T^5\big\}. \tag{3.3}$$

For $u > u_*$, $A^u(x,\delta)$ is declared as in (3.3) but omitting the disconnection event. If

$$\lim_{|x|\to\infty} |I| \cdot \sup_i \mathbb{P}[(G^u_{B_i})^c] = 0, \text{ for all } u \in [2^{-1}u_*, 2u_*], \tag{3.4}$$

then for all $u \neq u_*$, $0 \le \eta \le c_2$, one has when $d = 3$ that

$$\liminf_{\delta\downarrow 0} \liminf_{|x|\to\infty} \frac{\log|x|}{|x|} \log \mathbb{P}\big[A^u(x,\delta)\big] \ge -\frac{\pi}{3}\big(\sqrt{u_*} - \sqrt{u}(1-C\eta)\big)^2. \tag{3.5}$$

As a first consequence of Theorem 3.1, we derive the asserted lower bound on the truncated two-point function τ^{tr}_u from (1.1). The event G^u will thereby allow for local surgeries (around 0 and x) at affordable *multiplicative* cost. This will notably involve a sprinkled finite energy technique recently developed in [9, Section 3].

Proof of Theorem 1.1. We first prove (1.2), and start with the case $u > u_*$. For $x \in \mathbb{Z}^d$ and $\delta > 0$, abbreviating $B_x = B(x,r_1)$ with r_1 as in (3.2) and applying the FKG-inequality for $\mathcal{V}^u$ yields that

$$\log \mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} x] \ge \log \mathbb{P}[B_0 \overset{\mathcal{V}^u}{\longleftrightarrow} B_x,\ (B_0 \cup B_x) \subset \mathcal{V}^u] \ge \log \mathbb{P}[B_0 \overset{\mathcal{V}^u}{\longleftrightarrow} B_x] - u\,\mathrm{cap}(B_0 \cup B_x). \tag{3.6}$$

By subadditivity, $\mathrm{cap}(B_0 \cup B_x) \le C|x|^{\delta d/4}$. Thus for all $\delta < \frac{4}{d}$, the second term on the right hand side vanishes when multiplied with $\frac{\log|x|}{|x|}$ in the limit when $|x| \to \infty$. The event $\{B_0 \overset{\mathcal{V}^u}{\longleftrightarrow} B_x\}$ is clearly implied by the event in (3.3) (in absence of the disconnection event since $u > u_*$) for the trivial choice $G^u(x) = \Omega$, the space on which $\mathbb{P}$ is defined, and (1.2)-(1.3) thus follow for $u > u_*$ using (3.5) upon letting first $|x| \to \infty$ and then $\delta \downarrow 0$.

For $u < u_*$, the absence of monotonicity of the event appearing in (1.1) precludes the application of the FKG-inequality. To address this issue we make use of the event G^u. For $y \in \mathbb{Z}^d$, let $A_y \subset \widetilde{A}_y$ denote the annuli defined as $A_y = B(x,6r_1) \setminus B(x,4r_1)$ and $\widetilde{A}_y = B(y,7r_1) \setminus B(x,3r_1)$. By choice of r_1 in (3.2) and in view of (3.1), we have that $(\widetilde{A}_0 \cup \widetilde{A}_x) \subset T^1$, as will be needed for the event defined momentarily in (3.7) to satisfy the assumptions of Theorem 3.1. Let $\eta \in (0,c_2)$ be small enough so that $\frac{u}{1-\eta} < u_*$. We will eventually apply Theorem 3.1 with $v = \frac{u}{1-\eta\sqrt{\delta}}$ $(< u_*)$ in place of u. To this effect, let

$$G^v(x) \overset{\text{def.}}{=} \bigcap_{z\in\{0,x\}} G^1_z \cap G^2_z \cap G^3_z \tag{3.7}$$

where, setting $v_2 = v(1 - \frac{\eta\sqrt{\delta}}{2})$ so that $u < v_2 < v (< u_*)$, we define

$$G_z^1 \stackrel{\text{def.}}{=} \bigcap_{y,y' \in \mathcal{I}^{v_2} \cap A_z} \{ y \xleftrightarrow{\mathcal{I}^v \cap \widetilde{A}_z} y' \},$$
$$G_z^2 \stackrel{\text{def.}}{=} \big\{ (\mathcal{I}^{v_2} \setminus \mathcal{I}^u) \cap B(z, 2r_1) \neq \emptyset \big\}, \text{ and } G_z^3 \stackrel{\text{def.}}{=} \bigcap_{y \in B(z, 9r_1)} \{ \ell_y^v \leq r_1 \}, \tag{3.8}$$

and $\ell_\cdot^v$ denote the (discrete) occupation time field of random interlacements at level v. As we now briefly elaborate, one readily checks that the event $G^v = G^v(x)$ is of the form required by Theorem 3.1, for a choice of boxes B_i, $i \in I$, with $|I| = 6$, each centered at 0 or x. For instance, the event G_0^1 in (3.8) can be represented as G_B^v with $B = B(0, 10r_1) (\subset T^1)$ and expressed as a function increasing in $(\ell_x^v)_{x \in B}$ and decreasing in $(\ell_x^{v_2})_{x \in B}$. The other events appearing in (3.8) are dealt with similarly.

As $r_1 = r_1(x) \to \infty$ as $|x| \to \infty$, the fact that (3.4) holds with v in place of u (for any $v > 0$) follows by standard arguments, see in particular [7, Theorem 5.1] to deal with G_z^1. Thus, all in all, (3.5) yields for $d = 3$ that

$$\liminf_{\delta \downarrow 0} \liminf_{|x| \to \infty} \frac{\log |x|}{|x|} \log \mathbb{P}\big[G^v(x),\, B(0, r_1) \xleftrightarrow{\mathcal{V}^v \cap T^1} B(x, r_1),\, T^4 \overset{\mathcal{V}^u}{\not\leftrightarrow} \partial T^5 \big] \geq -\frac{\pi}{3} \big(\sqrt{u_*} - \sqrt{u}(1 - C\eta) \big)^2, \tag{3.9}$$

for all sufficiently small $\eta > 0$, with G^v as in (3.7) and obvious amendments when $d \geq 4$.

As now explain, the event G^v allows to obtain the bound

$$\big(\tau_u^{\text{tr}}(x) \geq \big)\ \mathbb{P}\big[0 \xleftrightarrow{\mathcal{V}^u \cap T^1} x,\, T^4 \overset{\mathcal{V}^u}{\not\leftrightarrow} \partial T^5 \big] \geq e^{-Cr_1^{2d}} \mathbb{P}\big[G^v(x),\, B(0, r_1) \xleftrightarrow{\mathcal{V}^v \cap T^1} B(x, r_1),\, T^4 \overset{\mathcal{V}^u}{\not\leftrightarrow} \partial T^5 \big], \tag{3.10}$$

valid for all $d \geq 3$, $\delta \in (0, c)$ and $\eta \in (0, c(u))$. Once (3.10) is shown, taking logarithms, using (3.10) and (3.2), the desired lower bound in (1.2) for $u < u_*$ follows upon taking successively the limits $|x| \to \infty$ (for fixed $\delta < \frac{1}{2d}$), then $\delta \downarrow 0$ and finally $\eta \downarrow 0$. To obtain (1.3) one proceeds similarly but simply fixes any admissible value of $\eta > 0$, for instance $\eta = c_2 \wedge (1 - \frac{2u}{u + u_*})$.

It remains to argue that (3.10) holds. The events G_z^i defined in (3.8) correspond to those appearing in [9, (3.8)] for the choices $B = B(z, r_1)$ and $u_1 = u_2 = u_3 = v$, $2\delta_1 = \delta_2 = \eta\sqrt{\delta} v$. In particular, the intersection $G_z^1 \cap G_z^2 \cap G_z^3$ implies the event referred to as $\widetilde{F}_B$ therein (see for instance [9, (3.9)] or [9, Remark 3.2]), on which the conclusions of [9, Proposition 3.1] hold for $r = r_1$ and with (v, u) above corresponding to $(u, u - \delta)$ therein. In particular, this entails the following. Let $\widehat{B} = B(z, 8r_1)$ and $\omega_{\widehat{B}}^-$ denote the point measure obtained from $\omega = \sum_i \delta_{(w_i^*, u_i)}$, the canonical interlacement point measure, by removing from each trajectory w_i^* intersecting $\widehat{B}$ all excursions between $\widehat{B}$ and $\partial_{\text{out}} \widehat{B}$. Then abbreviating by $\mathcal{F} = \sigma(\omega_{\widehat{B}}^-, \mathcal{I}^v \cap \widehat{B})$, one has by [9, (3.2)] that $\widetilde{F}_B \in \mathcal{F}$ and moreover that

$$\mathbb{P}\big[B \subset \mathcal{V}^u \,\big|\, \mathcal{F} \big] 1_{\widetilde{F}_B} \geq e^{-Cr_1^{2d}}, \quad B = B(z, r_1),\, z \in \{0, x\}. \tag{3.11}$$

Thus, returning to the probability on the right-hand side of (3.10), using for fixed z (say $z = 0$) the inclusion $G_z^1 \cap G_z^2 \cap G_z^3 \subset \widetilde{F}_B$, observing that each of the events G_x^1, G_x^2, G_x^3, $\widetilde{F}_B$,

$\{B(0,r_1) \xleftrightarrow{\mathcal{V}^v \cap T^1} B(x,r_1)\}$ and $\{T^4 \xleftrightarrow{\mathcal{V}^u} \partial T^5\}$ is $\mathcal{F}$-measurable (with regards to G_x^i, using that $100r_1 \leq |x|$), applying (3.11) allows to enforce the event $\{B(0,r_1) \subset \mathcal{V}^u\}$ at the cost of a multiplicative factor $e^{C{r_1}^{2d}}$. Repeating the procedure for $z = x$ (which now also includes the event $\{B(0,r_1) \subset \mathcal{V}^u\} \in \mathcal{F}$, where the latter follows using again that $100r_1 \leq |x|$) yields the desired connection between 0 and x in $\mathcal{V}^u$, and (3.10) follows. This completes the verification of (1.2).

We now prove (1.3), which is far simpler. Let $\ell \subset \mathbb{Z}^d$ denote a connected set of minimal cardinality intersecting both 0 and x. Thus $|x| \leq |\ell| \leq C|x|$, where $|\ell|$ denotes the cardinality of ℓ. Defining $\Sigma = \partial B(\ell, 1)$ we note that any unbounded path intersecting ℓ must also intersect Σ. Let $\mathcal{N}^u$ denote the number of trajectories of the interlacement at level u intersecting Σ, a Poisson variable with mean $u \cdot \mathrm{cap}(\Sigma)$. By subadditivity of the capacity, one has that $\mathrm{cap}(\Sigma) \leq |\Sigma| \leq C|x|$, whence

$$\tau_u^{\mathrm{tr}}(x) \geq \mathbb{P}[\ell \subset \mathcal{V}^u,\, \Sigma \subset \mathcal{I}^u] \geq cue^{-Cu|x|}\mathbb{P}\big[\ell \subset \mathcal{V}^u,\, \Sigma \subset \mathcal{I}^u \,\big|\, \mathcal{N}^u = 1\big], \tag{3.12}$$

where the first inequality is an inclusion by construction of ℓ and on account of the above observation on Σ. Moreover, in bounding $\mathbb{P}[\mathcal{N}^u = 1]$ from below to obtain the second inequality, we also used that $\mathrm{cap}(\Sigma) \geq \mathrm{cap}(\{0\}) \geq c$. Now, the conditional probability on the right-hand side of (3.12) is bounded from below by

$$\inf_{z \in \Sigma} P_z[\mathrm{range}(X) \supset \Sigma,\, \mathrm{range}(X) \cap \ell = \emptyset],$$

where X is the simple random walk starting from z under P_z. We claim that the latter probability is bounded from below by $e^{-C|x|}$, which, if true and when substituted in (3.12), completes the proof. The former can be seen as follows. Fix a deterministic path γ contained in Σ starting in $z \in \Sigma$ whose range covers Σ (which is a connected set). One can choose γ in such a way that its length is bounded by $C|x|$, uniformly in z. Forcing X to follow this path in its first steps ensures that $\mathrm{range}(X) \supset \Sigma$, at a cost bounded from below by $(2d)^{-C|x|}$. Upon completing this requirement, one forces X at a similar cost to move away from Σ without intersecting ℓ following another deterministic path until reaching distance $|x|$ from Σ. For z' at distance $|x|$ from Σ, denoting by H_Σ the entrance time in Σ one readily infers that, as $|x| \to \infty$,

$$P_{z'}[H_\Sigma < \infty] \leq C|x|^{2-d}\mathrm{cap}(\Sigma) \leq C'|x|^{3-d} \to 0.$$

The assertion now follows by applying the Markov property for X and combining the various ingredients. □

Remark 3.2 (one-arm estimates). 1) By picking $x = Ne_1$, one immediately infers from Theorem 1.1 similar bounds for the (truncated) one-arm probability, by which for all $u \neq u_*$,

$$\text{for } d = 3: \quad \liminf_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \xleftrightarrow{\mathcal{V}^u} \partial B_N, 0 \stackrel{\mathcal{V}^u}{\not\longleftrightarrow} \infty] \geq -\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*})^2, \tag{3.13}$$

$$\text{for } d \geq 4: \quad \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}[0 \xleftrightarrow{\mathcal{V}^u} \partial B_N, 0 \stackrel{\mathcal{V}^u}{\not\longleftrightarrow} \infty] \geq -C(u). \tag{3.14}$$

2) The following lower bound, which both i) localizes the disconnection from ∞, and ii) allows for a small unfavorable sprinkling can be useful in applications and is of independent

interest (see below (1.6)). Let $\delta \in (0,1)$ and $T_\delta = [-N^\delta, N+N^\delta] \times [-N^\delta, N^\delta]^{d-1}$. Then for all $u < u_*$ and $0 \le \eta < c_2 \wedge (1-\frac{u}{u_*})$, when $d=3$, one has that

$$\liminf_{\delta \downarrow 0} \liminf_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}\Big[0 \xleftrightarrow{\mathcal{V}^u \cap T_\delta} Ne_1, T_{2\delta} \overset{\mathcal{V}^{u(1-\eta\sqrt{\delta})}}{\not\longleftrightarrow} \partial T_{3\delta}\Big] \\ \ge -\frac{\pi}{3}\big(\sqrt{u_*} - \sqrt{u}(1-C\eta)\big)^2. \tag{3.15}$$

The estimate (3.15) follows by a slight modification of the above proof of Theorem 1.1, which we now explain. Inspecting that proof, one chooses $x = Ne_1$ and replaces the occurrences of v_2 and u when defining the events G^i_z in (3.8) by $v_2 := v(1-\frac{\eta\sqrt{\delta}}{4})$ and $v_3 := v(1-\frac{\eta\sqrt{\delta}}{2})$, respectively. The resulting events are still of the form required above Theorem 3.1, and (3.4) continues to hold. Following (3.10) and the subsequent arguments, these choices effectively allow to replace u by v_3 in (3.11). Together with the connection event in $\mathcal{V}^v \cap T^1$ appearing on the right of (3.10), this leads to a connection between 0 and Ne_1 at level $v_3 > u$, where u is the level of disconnection in (3.10). The lower bound (3.15) then readily follows after a straightforward reparametrization.

When $d \ge 4$, (3.15) remains true upon making the following amendments. The factor $\log N$ appearing on the left-hand side is removed and the sets $T_\delta, T_{2\delta}$ and $T_{3\delta}$, are replaced by $[0,N]\times\{0\}^{d-1}$, $[0,N]\times\{0\}^{d-1}$ and $\partial([0,N]\times[-1,1]^{d-1})$, respectively. The bound on the right-hand side is then replaced by $-C(\eta,u)$; this bound follows directly by inspection of the proof of (1.3), for the choice $x = Ne_1$ (see the argument around (3.12)).

4 The tilted measure $\widetilde{\mathbb{P}}_f$

The remaining sections are devoted to the proof of Theorem 3.1. In the present section we introduce a certain tilt of the measure which will effectively render the event appearing in Theorem 3.1 typical. The tilted interlacement measure $\widetilde{\mathbb{P}}_f$ is introduced in (4.9) below. It depends on a profile function f which corresponds to a carefully chosen spatial modulation of the intensity. The key features of the measure $\widetilde{\mathbb{P}}_f$ are given by Propositions 4.1 and 4.2 below, from which Theorem 3.1 is deduced at the end of the present section.

We start by introducing a minimal amount of notation that will be needed from here onwards. Recall the tubes $T^i = T^i(x)$ from (3.1), which depend on a choice of point $x \in \mathbb{Z}^d \setminus \{0\}$ and $\delta \in (0,1)$. In view of (3.1), one has the inclusions

$$(T^1 \subset \cdots \subset) T^6 \subset U, \text{ where } U = B(0, 10^3|x|). \tag{4.1}$$

We now introduce

$$\mathcal{F} = \{f_0, f_1\} \tag{4.2}$$

where $f_0, f_1 : \mathbb{Z}^d \to \mathbb{R}_+$ are defined as follows. For $\varepsilon \in (0, \frac{u_*}{10} \wedge 1)$ and $u > u_*$, let

$$f_0(y) \equiv f_0^{u;\varepsilon}(y) = 1 - \Big(1 - \sqrt{\frac{u_* - \varepsilon}{u}}\Big) h_0(y), \tag{4.3}$$

where $h_0(y) = P_y[H_{T^1} < T_U]$ (see above (2.1) for notation) and similarly for $0 < v \leq u < u_*$ let

$$f_1(y) \equiv f_1^{v,u;\varepsilon}(y) = 1 + \Big(\sqrt{\frac{u_* + \varepsilon}{v}} - 1\Big) h_2(y) - \Big(\sqrt{\frac{u_* + \varepsilon}{v}} - \sqrt{\frac{u_* - \varepsilon}{u}}\Big) h_1(y), \tag{4.4}$$

where $h_1(y) = P_y[H_{T^2} < T_{T^3}]$, $h_2(y) = P_y[H_{T^6} < T_U]$. For later purposes, we record that

$$f_0(y) = \begin{cases} \sqrt{\frac{u_* - \varepsilon}{u}} & x \in T^1 \\ 1 & x \notin U \end{cases} \tag{4.5}$$

and that

$$f_1(y) = \begin{cases} \sqrt{\frac{u_* - \varepsilon}{u}}, & x \in T^2 \\ \sqrt{\frac{u_* + \varepsilon}{v}}, & x \in T^6 \setminus T^3 \\ 1 & x \notin U \end{cases} \tag{4.6}$$

In particular, any function $f \in \mathcal{F}$ is finite, strictly positive and identically equal to 1 outside of a finite set.

In light of this, we can now define a tilted interlacement measure $\widetilde{\mathbb{P}}_f$ for $f \in \mathcal{F}$, which we introduce next. The following construction is essentially the same as in [14, Section 2], to which we frequently refer, except that we retain information on the labels u for reasons related to the function f_1 and the non-monotonicity of the event of interest in the supercritical phase. Let W_+, resp. W denote the space of infinite, resp. bi-infinite continuous-time transient $\mathbb{Z}^d$-valued trajectories, with finitely many jumps in bounded intervals of times. We write X_t, $t \in \mathbb{R}$ (resp. $t \geq 0$) for the canonical coordinates on W (resp. W_+). Identifying trajectories $w, w' \in W$ using the equivalence relation $w \sim w'$ if $w(\cdot) = w'(\cdot + t)$ for some $t \in \mathbb{R}$ yields the space $W^* = W \backslash \sim$ of trajectories modulo time-shift and the canonical projection $\pi : W \to W^*$. We write $W_U^* \subset W^*$ for the set of trajectories entering U. If $w^* \in W_U^*$, we write $s_U^+(w^*)$ for the trajectory in W_+ obtained by considering any $w \in W$ such that $\pi^*(w) = w^*$ and restricting w to the trajectory in W_+ obtained after w first enters U.

With these notations, we then introduce, for $f \in \mathcal{F}$, the function $F_f : W^+ \to \mathbb{R}$ defined as

$$F_f(w) = \int_0^\infty V_f(w(s))\mathrm{d}s, \quad V_f = -\frac{\Delta f}{f} \tag{4.7}$$

where $\Delta f(x) = \frac{1}{2d}\sum_{|e|=1} f(x+e) - f(x)$ for $f : \mathbb{Z}^d \to \mathbb{R}$ denotes the discrete Laplacian. The integral (4.7) is well-defined for any $f \in \mathcal{F}$ because $V_f = 0$ outside U, a finite set and the trajectory w is transient. In turn, the function F_f in (4.7) induces a function F_f^* on $W^* \times (0, \infty)$ given by

$$F_f^*(w^*, v) = \begin{cases} F_f(s_U^+(w^*)) & \text{whenever } (w^*, v) \in W_U^* \times [0, u]) \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

with U as in (3.1). The function F_f^* defines the exponential tilt of the interlacement measure $\mathbb{P}$, as follows. The measure $\mathbb{P}$ is formally defined on the space $\Omega = \{\omega = \sum_i \delta_{(w_i^*, u_i)} : w_i^* \in W^*,\ u_i \in (0, \infty)$ for all $i \geq 0$, and $\omega(W_K^* \times [0, u]) < \infty$ for all $K \subset\subset \mathbb{Z}^d$ and $u > 0\}$. For suitable $G : W^* \times (0, \infty)$ we write $\langle \omega, G \rangle$ for the canonical pairing, i.e. the integral of G with respect to the point measure $\omega \in \Omega$. In particular, one readily checks from (4.8) that $\langle \omega, F_f^* \rangle$ leads to a finite sum and hence is well-defined. We then define the measure $\widetilde{\mathbb{P}}_f$ via the Radon-Nikodym derivative

$$\frac{d\widetilde{\mathbb{P}}_f}{d\mathbb{P}} = e^{\langle \omega, F_f^* \rangle}. \tag{4.9}$$

We now gather a few essential features of $\widetilde{\mathbb{P}}_f$, which will be used in the sequel. By a slight adaptation of [14, Proposition 2.1], one has for each $f \in \mathcal{F}$ that $\widetilde{\mathbb{P}}_f$ defined by (4.9) is a probability measure. Moreover, denoting by $\nu \otimes \lambda$ the intensity measure on $W^* \times (0, \infty)$ of the interlacement process ω under $\mathbb{P}$, where λ denotes Lebesgue measure, one has that the canonical point measure ω under $\widetilde{\mathbb{P}}_f$ is a Poisson point process with intensity measure $e^{F_f^*}(\nu \otimes \lambda)$.

The measure $\widetilde{\mathbb{P}}_f$ retains an interlacement-like character. In particular, we will use the following fact in the sequel. For $\omega = \sum_i \delta_{(w_i^*, u_i)} \in \Omega$ and $K \subset\subset \mathbb{Z}^d$, let

$$\mu_K(\omega) = \sum_{i \geq 0} \delta_{(s_K^+(w_i^*), u_i)} 1\{w_i^* \in W_K^*\},$$

a locally finite (by definition of Ω) point measure on $W_+ \times \mathbb{R}_+$. In words, μ_K retains all labeled trajectories (w_i^*, u_i) in the support of ω which enter K, and replaces w_i^* for such points by the forward trajectory $s_K^+(w_i^*) \in W_+$ obtained after w_i^* first enters K. Then (see [14, (2.9)] for a similar result),

$$\text{under } \widetilde{\mathbb{P}}_f,\ \mu_K(\omega) \text{ is a PPP on } W^+ \times \mathbb{R}_+ \text{ with intensity measure } \widetilde{P}^f_{\tilde{e}_K} \otimes \lambda \tag{4.10}$$

where $\widetilde{P}^f_{\tilde{e}_K} = \sum_x \tilde{e}_K(x) \widetilde{P}^f_x$, which we proceed to define. The measure $\widetilde{P}^f_x$ is given by

$$\frac{d\widetilde{P}^f_x}{dP_x} = \frac{1}{f(x)} e^{\int_0^\infty V_f(X_s) ds} \tag{4.11}$$

with V_f as in (4.7). On account of [14, Lemma 1.2 and Corollary 1.3], $\widetilde{P}^f_x$ is a probability measure for all $f \in \mathcal{F}$, and the canonical process $(X_t)_{t \geq 0}$ under $\widetilde{P}^f_x$ is a Markov chain on $\mathbb{Z}^d$ with reversible measure $\tilde{\lambda}(x) = f^2(x)$, $x \in \mathbb{Z}^d$, whose semi-group on $L^2(\tilde{\lambda})$ is given by $(e^{t\tilde{\Delta}})_{t \geq 0}$ where $\tilde{\Delta} h(x) = \frac{1}{2d} \sum_{|e|=1} \frac{f(x+e)}{f(x)} (h(x+e) - h(x))$, for $h \in L^2(\tilde{\lambda})$. The equilibirum measure $\tilde{e}_K$ appearing in (4.10) is defined as

$$\tilde{e}_K(x) = \tilde{e}_K(x) = \widetilde{P}^f_x[\tilde{H}_K = \infty] 1_K(x) f(x) \Big(\frac{1}{2d} \sum_{|e|=1} f(x+e) \Big), \text{ for } x \in \mathbb{Z}^d. \tag{4.12}$$

For later reference, we record that, upon introducing the tilted Green's function

$$\tilde{g}(x, y) = \frac{1}{f(y)^2} \widetilde{E}^f_x \Big[\int_0^\infty 1\{X_s = y\} \mathrm{d}s \Big], \tag{4.13}$$

one has in analogy with (2.3) that

$$1 = \sum_{y \in K} \tilde{g}(x, y) \tilde{e}_K(y), \text{ for all } x \in K. \tag{4.14}$$

Following are the two key properties of the measure $\widetilde{\mathbb{P}}_f$ defined by (4.9). The first is a bound on the cost of tilting by f in terms of the relative entropy $H(\widetilde{\mathbb{P}}_f | \mathbb{P}) \stackrel{\text{def.}}{=} \widetilde{\mathbb{E}}_f \big[\frac{\mathrm{d}\widetilde{\mathbb{P}}_f}{\mathrm{d}\mathbb{P}} \big]$.

Proposition 4.1. *There exists $C_3 \in (0,\infty)$ such that for all $u < u_*$, if $d = 3$, the bound*

$$\limsup_{\delta\downarrow 0}\limsup_{\varepsilon\downarrow 0}\limsup_{|x|\to\infty}\frac{\log|x|}{|x|}H(\widetilde{\mathbb{P}}_f|\mathbb{P}) \leq C_3(\sqrt{u}-\sqrt{u_*})^2(1+C\eta) \tag{4.15}$$

holds for $f = f_1^{u(1-\eta\sqrt{\delta}),u;\varepsilon}$ and all $\eta \in [0,c)$. For $u > u_*$, (4.15) remains valid for $f = f_0^{u;\varepsilon}$ with $\eta = 0$ on the right-hand side. Moreover, one can choose $C_3 = \frac{\pi}{3}$.

The second characteristic feature of the measure $\widetilde{\mathbb{P}}_f$ is the mass it assigns to the event $A^u(x,\delta)$ from (3.3), which involves the event G^u. In the next proposition all assumptions on G^u made in Theorem 3.1 are tacitly assumed to hold; in particular, this includes (3.4). Recall the dependence of $A^u(x,\delta)$ on the sprinkling parameter $\eta \in [0,1)$ when $u < u_*$, which is implicit in our notation.

Proposition 4.2. *For all $\delta \in (0,1)$, $u \neq u_*$, $\eta \in [0,c)$ and $\varepsilon \in (0, u_*(1 \wedge \frac{\eta\sqrt{\delta}}{10}))$,*

$$\lim_{|x|\to\infty}\widetilde{\mathbb{P}}_f[A^u(x,\delta)] = 1 \quad \textit{holds for} \begin{cases} u > u_* \textit{ with } f = f_0^{u;\varepsilon} \\ u < u_* \textit{ with } f = f_1^{u(1-\eta\sqrt{\delta}),u;\varepsilon}. \end{cases} \tag{4.16}$$

The proofs of Propositions 4.1 and 4.2 are postponed to later sections. We now conclude the proof of Theorem 3.1 using these two results.

Proof of Theorem 3.1. Let $u > u_*$ and $f = f_0^{u;\varepsilon}$ for $\varepsilon > 0$. Since $\widetilde{\mathbb{P}}_f$ is absolutely continuous with respect to $\mathbb{P}$ in view of (4.9), one classically obtains by Jensen's inequality (see for instance the discussion following (2.7) in [2] for a proof) that for all $x \in \mathbb{Z}^d$ and $\delta \in (0,1)$,

$$\log\mathbb{P}[A^u(x,\delta)] \geq \log(\widetilde{\mathbb{P}}_f[A^u(x,\delta)]) - \frac{H(\widetilde{\mathbb{P}}_f|\mathbb{P}) + e^{-1}}{\widetilde{\mathbb{P}}_f[A^u(x,\delta)]}. \tag{4.17}$$

Multiplying by $\frac{\log|x|}{|x|}$ on both sides of (4.17) and subsequently letting first $|x| \to \infty$, $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, the claim (3.5) (with $\eta = 0$) follows upon inserting the bounds (4.15) and (4.16) on the right-hand side of (4.17). For $u < u_*$, the proof is analogous but choosing $f = f_1^{u(1-\eta\sqrt{\delta}),u;\varepsilon}$ instead, and the conclusions hold for all $0 \leq \eta < c$. □

5 Relative entropy estimate

In the previous section, the proof of Theorem 3.1 was completed, subject to the validity of two results, stated as Propositions 4.1 and 4.2. In the present section we prove Proposition 4.1. As will turn out, the estimate on the relative entropy will involve bounding potential theoretic quantities related to the tubes T^i, which recall are oriented towards $\frac{x}{|x|}$. Importantly, the bounds derived need to be sufficiently sharp to make the correct (rotationally invariant!) functional $\frac{\log|x|}{|x|}$ in (4.15) appear in the large-scale limit. This is a somewhat delicate matter when x is in generic position and one cannot exploit (lattice) symmetries.

We start with some preparation. For $f : \mathbb{Z}^d \to \mathbb{R}$, we consider the Dirichlet form $\mathcal{E}(f,f)$ associated to the random walk, defined as

$$\mathcal{E}(f,f) = \frac{1}{2}\sum_{|x-y|=1}\frac{1}{2d}(f(x)-f(y))^2, \tag{5.1}$$

where the sum ranges over all points $x, y \in \mathbb{Z}^d$ satisfying the constraint. Let $H^2 = \{f : \mathbb{Z}^d \to \mathbb{R} : \mathcal{E}(f,f) < \infty\}$. For $f, g \in H^2$, the energy $\mathcal{E}(f,g)$ is declared by polarization. Recall the functions $f_0 = f_0^{u;\varepsilon}$ and $f_1 = f_1^{v,u;\varepsilon}$ from (4.3)-(4.4), which depend implicitly on both $x \in \mathbb{Z}^d$ and $\delta \in (0,1)$ via the choice of sets T^i and U, cf. (3.1) and (4.1), entering their definition. Recall the notation for the (relative) capacity from below (2.2).

Lemma 5.1. *For all $u, \varepsilon > 0$ and $0 < v \le u$, one has $f_0, f_1 \in H^2$. Moreover, for $|x| \ge C(\delta)$,*

$$\mathcal{E}(f_0, f_0) = u^{-1}\big(\sqrt{u} - \sqrt{u_* - \varepsilon}\big)^2 \mathrm{cap}_U(T^1), \tag{5.2}$$

$$\mathcal{E}(f_1, f_1) = v^{-1}\big[\big(\sqrt{u_* + \varepsilon} - \sqrt{v}\big)^2 \mathrm{cap}_U(T^6) + \big(\sqrt{u_* + \varepsilon} - \sqrt{v/u}\sqrt{u_* - \varepsilon}\big)^2 \mathrm{cap}_{T^3}(T^2)\big]. \tag{5.3}$$

Proof. On account of (4.5), (4.6) and (4.1), the functions f_0, f_1 are constant outside the finite set U hence the sum in (5.1) is effectively finite, whence $f_0, f_1 \in H^2$. We now show (5.3). The proof of (5.2) is similar, but simpler.

Introducing the shorthands $\alpha = (\frac{u_*+\varepsilon}{v})^{1/2} - 1$ and $\beta = (\frac{u_*+\varepsilon}{v})^{1/2} - (\frac{u_*-\varepsilon}{u})^{1/2}$, the formula (4.4) defining f_1 reads $f_1 - 1 = \alpha h_2 - \beta h_1$. The functions $h_1(x) = P_x[H_{T^2} < T_{T^3}]$ and $h_2(x) = P_x[H_{T^6} < T_U]$ have the property that for any neighboring pair of points $x, y \in \mathbb{Z}^d$, $h_1(x) = h_1(y)$ or $h_2(x) = h_2(y)$ whenever $|x| \ge C(\delta)$; indeed for such x we may assume in view of (3.1) that the 1-neighborhood of T^3 is contained in T^6. It follows that $h_2(x) = h_2(y) = 1$ whenever at least one of the neighbors x, y lies in T_3, whereas $h_1(x) = h_1(y) = 0$ whenever $x, y \notin T_3$. All in all, it follows that $\mathcal{E}(h_1, h_2) = 0$. Hence,

$$\mathcal{E}(f_1, f_1) = \mathcal{E}(f_1 - 1, f_1 - 1) = \alpha^2 \mathcal{E}(h_2, h_2) + \beta^2 \mathcal{E}(h_1, h_1). \tag{5.4}$$

The claim (5.3) now follows from (5.4) using the classical fact that $\mathrm{cap}_U(K) = \mathcal{E}(V, V)$ with $V(x) = P_x[H_K < T_U]$ for all $U \subset\subset \mathbb{Z}^d$, $K \subset U$, $K \neq \emptyset$. □

Next we collect bounds for the relative capacities involved in Lemma 5.1. The proof of (5.7) below crucially involves Proposition 2.1.

Lemma 5.2. *For all $\delta \in (0, \frac{1}{2})$, $|x| \ge C(\delta)$ and $i = 1, \dots, 6$, when $d = 3$,*

$$\mathrm{cap}(T^i) \le (1 + C\delta)\frac{\pi |x|}{3 \log |x|}, \tag{5.5}$$

$$\mathrm{cap}_U(T^i) \le \Big(1 + \frac{C}{\log |x|}\Big)\mathrm{cap}(T^i), \tag{5.6}$$

$$\mathrm{cap}_{T^3}(T^2) \le C\delta^{-1}\mathrm{cap}(T^2). \tag{5.7}$$

Proof. The bound (5.5) can be obtained by combining [16, (2.24)] and [11, Lemma 2.2] (see, e.g., [6, below (5.9)] for a similar argument). Next, we aim to show (5.6). To do this, consider $y \notin U$ and notice that $\min_{z \in \partial T_i} |y - z| \ge 9|x|$ on account of (4.1) whence $g(y,z) \le C|x|^{-1}$ for $z \in T^i$. Using this, a last-exit decomposition, and (5.5), it follows that $P_y[H_{T^i} < \infty] \le C|x|^{-1}\mathrm{cap}(T^i) \le C'(\log |x|)^{-1}$. For $z \in T^i$, decomposing the (bare) equilibrium measure of T^i over the exit location of U and feeding this bound yields that

$$P_x[\widetilde{H}_{T^i} = \infty] = \sum_{y \in \partial U} P_x[\widetilde{H}_{T^i} > T_U, X_{T_U} = y] P_y[H_{T^i} = \infty] \ge \Big(1 - \frac{C}{\log |x|}\Big) P_x[\widetilde{H}_{T^i} > T_U]$$

Summing both sides over $x \in T^i$ and rearranging yields (5.6). It remains to prove (5.7). Using the inequality

$$e_{T^2}(z) \geq e_{T^2,T^3}(z) \inf_{y \notin T^3} P_y[H_{T^2} = \infty] \tag{5.8}$$

valid for all $z \in \mathrm{supp}(e_{T^2})$, which follows readily from (2.2) upon applying the strong Markov property at time T_{T^3}, one obtains (5.7) by summing (5.8) over z and using Proposition 2.1. □

Equipped with Lemmas 5.1 and 5.2, we are ready to proceed with the:

Proof of Proposition 4.1. Let $f \in \mathcal{F}$ as in (4.2). The relative entropy $H(\widetilde{\mathbb{P}}_f|\mathbb{P})$ can be recast as

$$H(\widetilde{\mathbb{P}}_f|\mathbb{P}) = \widetilde{\mathbb{E}}_f\Big[\frac{\mathrm{d}\widetilde{\mathbb{P}}_f}{\mathrm{d}\mathbb{P}}\Big] \overset{(4.9)}{=} \widetilde{\mathbb{E}}_f\big[\langle \omega, F_f^*\rangle\big] \overset{(4.10)}{=} u\widetilde{E}^f_{\tilde{e}_U}[F_f(X)], \tag{5.9}$$

where in applying (4.10) one notes that the Poisson variable $\langle \omega, F_f^*\rangle$ depends on ω 'through' μ_U in view of (4.7), (4.8) and since V_f vanishes outside U. By definition of F_f in (4.7) one obtains, noting that all sums below are effectively finite, that

$$\begin{aligned}\widetilde{E}^f_{\tilde{e}_U}[F_f(X)] = \int_0^\infty \widetilde{E}^f_{\tilde{e}_U}[V_f(X_s)]\mathrm{d}s &\overset{(4.13)}{=} \sum_{y,z} \tilde{e}_U(y)\tilde{g}(y,z)f^2(z)V_f(z) \\ &\overset{(4.14)}{=} \sum_z f^2(z)V_f(z) \overset{(4.7)}{=} \sum_z f(z)(-\Delta f)(z) \overset{(5.1)}{=} \mathcal{E}(f,f),\end{aligned} \tag{5.10}$$

where the last equality follows by a discrete version of the Gauss-Green theorem, see for instance [1, Theorem 1.24]. We now focus on the case $u < u_*$, whence $f = f_1^{u(1-\eta\sqrt{\delta}),u;\varepsilon}$. The case $u > u_*$ follows by a similar resoning, simply using (5.2) instead of (5.3) in what follows. Combining (5.9), (5.10) and (5.3) with the choice $v = u(1-\eta\sqrt{\delta})$ for $\eta \in [0,1]$, and subsequently feeding the bounds (5.6)-(5.7) in combination with (5.5), one finds that

$$\begin{aligned}H(\widetilde{\mathbb{P}}_f|\mathbb{P}) \leq \frac{1+C\delta}{1-\eta\sqrt{\delta}}\frac{C_3|x|}{\log|x|}\Big[\Big(\sqrt{u_*+\varepsilon} - \sqrt{u(1-\eta\sqrt{\delta})}\Big)^2\Big(1+\frac{C}{\log|x|}\Big) \\ + C\delta^{-1}\Big(\sqrt{u_*+\varepsilon} - \sqrt{(1-\eta\sqrt{\delta})(u_*-\varepsilon)}\Big)^2\Big],\end{aligned}$$

for all $u, \varepsilon > 0$, $\eta \in [0,c)$ and $|x| \geq C(\delta)$. Multiplying by $\frac{\log|x|}{|x|}$ on both sides, the desired bound (4.15) follows upon taking the successive limits $|x| \to \infty$, $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, using for the term in the second line that $|1-\sqrt{1-x}| \leq Cx$ for $0 < x < c$ with $x = \eta\sqrt{\delta}$, which upon squaring precludes the explosion of the factor δ^{-1} in the limit $\delta \downarrow 0$. □

6 Coupling tilted interlacements

The remainder of this article deals with the proof of Proposition 4.2, which concerns the effect of the tilted measure $\widetilde{\mathbb{P}}_f$ introduced in Section 4. Towards this goal, we first show that the tilted interlacements can be controlled locally in terms of regular interlacements but with a modified (and spatially inhomogenous) intensity close to u_*. Herein enter the specifics in our choice of tilt f from Section 4.

The control is stated in terms of a coupling between the corresponding occupation time fields, which appears in Proposition 6.1. This is the main result of this section. From Proposition 6.1 we first deduce Proposition 4.2. The proof of Proposition 6.1 then occupies the remainder of Section 6, and relies on two key intermediate results, Lemmas 6.2 and 6.3. The former compares certain entrance laws for tilted vs. untilted walk, while the latter exhibits concentration of certain associated excursion counts. Both of these results hinge on fine properties of the tilted random walk measure $\widetilde{P}_x^f$. Their proofs are given separately in Section 7.

Recall that, for a realization $\omega = \sum_i \delta_{(w_i^*, u_i)} \in \Omega$ of the Poisson process (declared under either $\mathbb{P}$ or $\widetilde{\mathbb{P}}_f$, cf. (4.9)), one introduces the occupation time at level $u > 0$ and $x \in \mathbb{Z}^d$ as

$$\ell_x^u = \ell_x^u(\omega) = \sum_{i: u_i \le u} \int_{-\infty}^{\infty} 1\{w_i^*(t) = x\}\, dt. \tag{6.1}$$

To distinguish between the two possible reference measures, we will henceforth write $\tilde{\ell}^u_\cdot$ to refer to a random field having the same law as $\ell^u_\cdot$ under $\widetilde{\mathbb{P}}_f$ (the choice of f will always be clear from the context). We consider boxes

$$B_i^z = B\big(z, r_1^{(i+2)/8}\big), \text{ for } z \in \mathbb{Z}^d \text{ and } i \in \{1, \dots, 4\}, \tag{6.2}$$

where r_0 is given by (3.2). We will almost exclusively consider centers $z \in \Gamma = \Gamma_{\text{int}} \cup \Gamma_{\text{ext}}$, where $\Gamma_{\text{int}} = T(x)$ and $\Gamma_{\text{ext}} = \partial T^4$, with $T(x)$ and T^4 as in (2.4) and (3.1). Note that, with the choices of radii in (6.2), whenever $|x| \ge C(\delta)$ we have that $B_i^z \subset B_4^z \subset T^1$ for all $z \in \Gamma_{\text{int}}$, and $B_i^z \subset (T^5 \setminus T^3)$ for all $z \in \Gamma_{\text{ext}}$. The following proposition essentially asserts that the effect of tilting is to make the law of the (tilted) occupation times at level u look slightly supercritical in the region T^1, i.e. roughly like untilted interlacement occupation times at level $u_* - O(\varepsilon)$, and slightly subcritical in the region $T^3 \setminus T^2$.

Proposition 6.1. *For all $\delta \in (0,1)$, $\varepsilon \in (0, \frac{u_*}{10} \wedge 1)$ and $\eta \in [0, c_3)$, the following holds.*

i) If $u < u_*$ and $z \in \Gamma_{\text{ext}}$, then with $f = f_1^{v,u;\varepsilon}$, $v = u(1 - \eta\sqrt{\delta})$ and abbreviating $B = B_1^z$, there exists a coupling Q_B of $(\ell_x^{u_* + \frac{1}{2}\varepsilon})_{x \in B}$, $(\ell_x^{u_* + \frac{3}{2}\varepsilon})_{x \in B}$ (each under $\mathbb{P}$) and $(\tilde{\ell}_x^v)_{x \in B}$ (having the same law as $(\ell_x^v)_{x \in B}$ under $\widetilde{\mathbb{P}}_f$ by above convention), such that for $\tilde{c} = \tilde{c}(u, \delta, \varepsilon, \eta)$,

$$Q_B\big[\ell_x^{u_* + \frac{1}{2}\varepsilon} \le \tilde{\ell}_x^v \le \ell_x^{u_* + \frac{3}{2}\varepsilon},\, x \in B\big] \ge 1 - e^{-\tilde{c} r_0^{\tilde{c}}}. \tag{6.3}$$

ii) If $u < u_*$ and $z \in \Gamma_{int}$, with $s_i(t) = t(1 - \gamma_i)^{\sigma_i}$ for $\gamma_i \in [0,1)$, $\sigma_i \in \{\pm 1\}$, $i = 1, 2$, there exists a coupling Q_B of $(\ell_x^{s_i(u_* - \frac{2k-1}{2}\varepsilon)})_{x \in B,\, k, i \in \{1,2\}}$ and $(\tilde{\ell}_x^{s_i(u)})_{x \in B, i=1,2}$, such that

$$Q_B\big[\ell_x^{s_i(u_* - \frac{3}{2}\varepsilon)} \le \tilde{\ell}_x^{s_i(u)} \le \ell_x^{s_i(u_* - \frac{1}{2}\varepsilon)},\, x \in B,\, i = 1, 2\big] \ge 1 - e^{-\tilde{c} r_0^{\tilde{c}}}, \tag{6.4}$$

for $\tilde{c} = \tilde{c}(u, \delta, \varepsilon, \eta, \gamma_i, \sigma_i)$. If $u > u_*$, then for all $z \in \Gamma_{int}$ there exists a coupling Q_B with the same properties, but with $f = f_0^{u;\varepsilon}$ now underlying the marginal law of $(\tilde{\ell}_x^{s_i(u)})_{x \in B, i=1,2}$.

With the aid of Proposition 6.1, we first give the proof of Proposition 4.2.

Proof of Proposition 4.2. We will freely (and tacitly) assume that various statements hold for $|x| \ge C(\delta)$, which is no loss of generality in view of (4.16) since the latter concerns the limit

$|x| \to \infty$ only. Let $u < u_*$ and $f = f_1^{v,u;\varepsilon}$ with $v = u(1-\eta\sqrt{\delta})$ and $\eta \in [0, c_4)$ as for the conclusions of Proposition 6.1 to hold. The following conclusions will always hold uniformly in η as above, $\delta \in (0,1)$ and $\varepsilon \in (0, u_*(1 \wedge \frac{\eta\sqrt{\delta}}{10}))$, as postulated in the statement of Proposition 4.2. Using Proposition 6.1 we argue separately that

$$\lim_{|x|\to\infty} \widetilde{\mathbb{P}}_f[T^4 \stackrel{\mathcal{V}^v}{\longleftrightarrow} \partial T^5] = 0, \tag{6.5}$$

$$\lim_{|x|\to\infty} \widetilde{\mathbb{P}}_f[G^u, B(0,r_1) \stackrel{\mathcal{V}^u \cap T^1}{\longleftrightarrow} B(x,r_1)] = 1. \tag{6.6}$$

Recalling $A^u(x,\delta)$ from (3.3), the claim (4.16) for $u > u_*$ immediately follows from (6.5) and (6.6). To obtain (6.5) applying the relevant coupling from Proposition 6.1,*i)*, one finds that for all $z \in \Gamma_{\text{int}}$, with $B = B_1^z$,

$$\widetilde{\mathbb{P}}_f[z \stackrel{\mathcal{V}^v}{\longleftrightarrow} \partial B] = Q_B[z \stackrel{\{\tilde{\ell}^v_\cdot = 0\}}{\longleftrightarrow} \partial B] \stackrel{(6.3)}{\leq} Q_B[z \stackrel{\{\ell^{u_*+\frac{1}{2}\varepsilon}_\cdot = 0\}}{\longleftrightarrow} \partial B] + e^{-c|x|^{c'(\delta)}}$$
$$= \mathbb{P}\big[z \stackrel{\mathcal{V}^{u_*+\frac{1}{2}\varepsilon}}{\longleftrightarrow} \partial B\big] + e^{-cr_0^{\tilde{c}}} \leq Ce^{-c|x|^{c'(\delta)}},$$

where the last step follows using [8, Theorem 1.2,i)] and recalling that $r_0 \geq c|x|^\delta$. Applying a union bound over $z \in \partial T^4 (= \Gamma_{\text{ext}})$ and using the previous bound on $\widetilde{\mathbb{P}}_f[z \stackrel{\mathcal{V}^v}{\longleftrightarrow} \partial B]$ yields (6.5). To deal with (6.6), we use part *ii)* of Proposition 6.1. We consider the two events appearing in (6.6) separately, and start with the connection event appearing there. Throughout the rest of the proof, constants may implicitly depend on all of $u, \delta, \varepsilon, \eta$.

Similarly as in (1.6), let $\text{LU}_z(\mathcal{V}, \mathcal{V}')$ denote the event that $B(z, r_1/2) \setminus B(z, r_1/2)$ has at least one crossing cluster in $\mathcal{V}$ and that $B(z, r_1) \setminus B(z, r_1/2)$ has at most one crossing cluster in $\mathcal{V}'$. For any $v \geq u$, the joint occurrence of $\text{LU}_z(\mathcal{V}^v, \mathcal{V}^u)$ as z varies over $\Gamma_{\text{int}} = T(x)$ is seen to imply E. Let $v = \frac{u}{1-\gamma}$, with γ chosen small enough so that $v \vee w < u_*$, where $w = \frac{u_* - \frac{1}{2}\varepsilon}{1-\gamma}$. Applying Proposition 6.1,*ii)* with $\gamma_1 = 0$ and $\gamma_2 = \gamma$, $\sigma_2 = -1$, one finds that for any $z \in \Gamma_{\text{int}}$,

$$\widetilde{\mathbb{P}}_f[\text{LU}_z(\mathcal{V}^v, \mathcal{V}^u)] \geq \mathbb{P}[\text{LU}_z(\mathcal{V}^w, \mathcal{V}^{u_* - \frac{3}{2}\varepsilon})] - e^{-\tilde{c}r_0^{\tilde{c}}}. \tag{6.7}$$

But as a straightforward consequence of [8, Theorem 1.2,ii)], which applies because $w < u_*$, one obtains that the right-hand side of (6.7) exceeds $1 - Ce^{-\tilde{c}r_0^{\tilde{c}}}$. Together with a union bound, this implies (6.6) in absence of the event G^u, i.e. $\widetilde{\mathbb{P}}_f[E] \to 1$ as $|x| \to \infty$, where $E = \{B(0,r_1) \leftrightarrow B(x,r_1) \text{ in } \mathcal{V}^u \cap T^1\}$. To accommodate the presence of G^u in (6.6), recalling its form specified above Theorem 3.1 and applying a union bound, one sees that it is enough to argue that

$$\lim_{|x|\to\infty} |I| \cdot \sup_i \widetilde{\mathbb{P}}_f[(G_{B_i})^c] = 0. \tag{6.8}$$

This is obtained by combining (3.4) and Proposition 6.1,*ii)*. Indeed, recall to this effect (cf. above Theorem 3.1) that B_i is a box of radius $10r_1$ with center $z \in \Gamma_{\text{int}}$. By choice of r_1 in (3.2) and in view of (6.2), this means that $B_i \subset B_1^z = B$. In particular, the event G_{B_i} is thus measurable relative to $(\tilde{\ell}_x^{s_i(u)})_{x\in B, i=1,2}$, where $s_i(u) = u(1-\gamma_i)$ for suitable choice of $\gamma_i \in \{\frac{k}{4}\eta\sqrt{\delta} : k = 0,1,2,3,4\}$. Assume for concreteness that $s_1(u) > s_2(u)$, so that G_{B_i} is increasing in $(\tilde{\ell}_x^{s_1(u)})_{x\in B}$ and decreasing in $(\tilde{\ell}_x^{s_2(u)})_{x\in B}$. With $(v,w) = (s_1(u), s_2(u))$, $(v',w') = (s_1(u_* - \frac{3}{2}\varepsilon), s_2(u_* - \frac{1}{2}\varepsilon))$, it then follows by application of (6.4), using the (separate) monotonicity of $G_{B_i} = G_{B_i}^{v,w}$, that

$$\widetilde{\mathbb{P}}_f[G_{B_i}^{v,w}] \geq \mathbb{P}[G_{B_i}^{v',w'}] - e^{-\tilde{c}r_0^{\tilde{c}}}.$$

The claim (6.8) now follows using (3.4), noting in particular that the assumption $\varepsilon < u_* \frac{\eta\sqrt{\delta}}{10}$ guarantees that $v' > w'$.

To deduce (4.16) for $u < u_*$, one shows (6.6) now with $f = f_0^{u;\varepsilon}$ in exactly the same manner as above but resorting to the second part of item *ii)* in Proposition 6.1 instead. □

It thus remains to prove Proposition 6.1. We start with some preparation. For $z \in \Gamma$, with $B = B_1^z$ and $U = B_2^z$ (recall (6.2)) we will consider both for tilted and untilted walks successive excursions between B and U^c. In light of this, we introduce a reference Poisson point process $\eta_B = \sum_{n\geq 0} \delta_{(\zeta_n,u_n)}$ governed by the probability measure Q_B on the state space $\Xi \times \mathbb{R}_+$, where $\Xi = \Xi_{B,U}$ denotes the space of relevant excursions, i.e. of finite length nearest neighbour trajectories starting on ∂B, with range contained in U up to their terminal point, a vertex in U^c. The intensity measure of η is $\nu_B \otimes \lambda$, where λ denotes the Lebesgue measure and

$$\nu_B(\cdot) \stackrel{\text{def.}}{=} \sum_{x\in\partial B} P_x\big[(X_t)_{0\leq t\leq T_U} \in \cdot\big]. \tag{6.9}$$

Importantly for what is to follow, if $f = f_1^{v,u;\varepsilon}$ for any $0 < v \leq u < u_*$ and $\varepsilon \in (0, \frac{u_*}{10} \wedge 1)$, then regardless of the choice of $z \in \Gamma$, by (4.6) the function function f is constant in the one-neighborhood of U, which in turn implies that $V_f(x) = 0$ for all $x \in U$. In view of (4.11), this means that P_x in (6.9) can be freely replaced by $\widetilde{P}_x^f$, i.e. excursions between B and U^c do not witness the tilt. The same conclusions hold for $f = f_0^{u;\varepsilon}$, $u > u_*$ and $\varepsilon \in (0, 1)$ if $z \in \Gamma_{\text{int}}$ in view of (4.5).

We now proceed to define two Markov chains $Z = (Z_n)_{n\geq 1}$ and $\widetilde{Z} = (\widetilde{Z}_n)_{n\geq 1}$ on Ξ, as follows. To this effect, we introduce the entrance distribution and potential of B, for $x, y \in \mathbb{Z}^d$, as

$$h_B(x,y) = P_x\big[H_B < \infty, X_{H_B} = y\big], \quad h_B(x) = \sum_y h_B(x,y). \tag{6.10}$$

The corresponding tilted quantities $\tilde{h}_B(x,y)$ and $\tilde{h}_B(x)$ are obtained by replacing P_x by the tilted measure $\widetilde{P}_x \equiv \widetilde{P}_x^f$. Both Z and $\widetilde{Z}$ are specified in terms of their transition densities π and $\tilde{\pi}$ relative to ν_B in (6.9), that is,

$$Z_1 \stackrel{\text{law}}{=} \pi_0(\zeta)\nu_B(d\zeta), \quad P[Z_{k+1} \in d\zeta | Z_1, \dots, Z_k] = \pi(Z_k, \zeta)\nu_B(d\zeta), \text{ for } k \geq 1 \tag{6.11}$$

and similarly for $\widetilde{Z}$ with $\tilde{\pi}$ in place of π, where, for $\zeta = (\zeta_0, \dots, \zeta_n) \in \Xi$ and with $\bar{e}_B = e_B/\text{cap}(B)$ the normalized equilibrium measure,

$$\pi_0(\zeta) \stackrel{\text{def.}}{=} \bar{e}_B(\zeta_0), \quad \pi(\zeta, \zeta') \stackrel{\text{def.}}{=} h_B(\zeta_n, \zeta_0') + \bar{e}_B(\zeta_0')(1 - h_B)(\zeta_n). \tag{6.12}$$

The corresponding tilted transition densities are declared by replacing $\bar{e}_B$ and h_B by their tilted analogues. One readily sees from (6.11) and (6.12) in combination with the observation made below (6.9) that Z, resp. $\widetilde{Z}$ has the same law as the excursions from B to U^c induced by the interlacement process ω under $\mathbb{P}$, resp. $\widetilde{\mathbb{P}}_f$, when ordering them according to increasing label u and in order of appearance within a given random walk trajectory in the support of the point measure ω. The key behind the coupling(s) postulated in Proposition 6.1 is encapsulated in the following comparison of transition densities.

To state it concisely, it will be convenient to introduce the notation Γ_f for $f \in \mathcal{F}$, as follows. Recall $\Gamma = \Gamma_{\text{int}} \cup \Gamma_{\text{ext}}$ from below (6.2), the set of centers z under consideration, along with the boxes B_i^z, $1 \le i \le 4$ defined there. We set $\Gamma_{f_0} = \Gamma_{\text{int}}$ and $\Gamma_{f_1} = \Gamma$ in the sequel. With this definition, in light of the statement of Proposition 6.1, when working with $f \in \mathcal{F}$ we only ever have to deal with boxes $B = B_1^z$ having centers $z \in \Gamma_f$. Note that, albeit implicit in our notation, the set of centers $\Gamma_f = \Gamma_f(x)$ depends on $x \in \mathbb{Z}^d$ via the oblique cylinders introduced in (3.1). These cylinders, as well as the boxes B_i^z, depend on one further parameter $\delta \in (0, \frac{1}{6})$.

Lemma 6.2. *For all $f \in \mathcal{F}$, $x \in \mathbb{Z}^d$, $z \in \Gamma_f(x)$, with $\Xi = \Xi_{B_1^z, B_2^z}$,*

$$\sup_{\zeta, \zeta' \in \Xi} \left\{ \left| \frac{\tilde{\pi}_0(\zeta)}{\pi_0(\zeta)} - 1 \right|, \left| \frac{\tilde{\pi}(\zeta', \zeta)}{\pi(\zeta', \zeta)} - 1 \right| \right\} \le C r_0^{-c}. \tag{6.13}$$

The proof of Lemma 6.2 requires some work and is postponed to the next section. The main issue is that between successive excursions, the walk under $\widetilde{P}_x$, $x \in B$, may in principle travel far (with polynomial probability in r_0), thereby exploring regions in which the effect of the tilt is severely felt. An additional source of difficulty stems from the fact that (6.13) requires a (stronger) control of ratios rather than of mere differences $|\tilde{\pi} - \pi|$.

Assuming Lemma 6.2 to hold, the method of soft local time [15], see also [4] which will be sufficient for our purposes, allows to define under Q_B an event U_n^v for each integer $n \ge 1$ and $v \in (0, c)$ such that, whenever $|x| \ge C(v)$ (so that the bound in (6.13) is smaller than $\frac{v}{3}$), one has

$$\begin{aligned} &Q_B[U_n^v] \ge 1 - C\exp(-cvn), \text{ and} \\ &\text{on } U_n^v, \text{ for all } m \ge n\text{: } \begin{array}{l} \{Z_1, \dots, Z_{(1-v)m}\} \subset \{\widetilde{Z}_1, \dots, \widetilde{Z}_{(1+4v)m}\} \\ \{\widetilde{Z}_1, \dots, \widetilde{Z}_{(1-v)m}\} \subset \{Z_1, \dots, Z_{(1+4v)m}\}. \end{array} \end{aligned} \tag{6.14}$$

To obtain (6.14) one retraces the steps of [4, Lemma 2.1], and the key assumption [4, (2.5)] appearing in that context is replaced by (6.13).

Next, we attach to each of Z and $\widetilde{Z}$ a sequence $\sigma = (\sigma_k)_{k \ge 1}$ and $\tilde{\sigma}$ of labels in $\{0, 1\}$, as follows. We $\sigma_1 = \tilde{\sigma}_1 = 1$ and for each $k \ge 2$, the label σ_k has conditional law given Z_{k-1}, Z_k given by

$$P[\sigma_k = 0 \,|\, Z_{k-1}, Z_k] = 1 - P[\sigma_k = 1 \,|\, Z_{k-1}, Z_k] = \frac{h_B(Z_{k-1}^e, Z_k^i)}{\pi(Z_{k-1}, Z_k)}, \tag{6.15}$$

where $Z_\cdot^{i/e}$ refer to the initial/end-point of $Z_\cdot$. The prescription for $\tilde{\sigma}_k$ is identical to (6.15) but using $\tilde{h}_B$ and $\tilde{\pi}$ instead. The label σ recovers information about the trajectories underlying the excursions forming Z: the label $\sigma_k = 1$ signals excursions $Z_k, \dots$ stemming from a new random walk trajectory. We henceforth assume that Q_B is suitably enlarged as to carry the sequences σ and $\tilde{\sigma}$ with the correct law, independently of each other conditionally on $Z, \widetilde{Z}$.

As a last ingredient, we assume Q_B to carry additionally two independent Poisson counting processes n_B and $\tilde{n}_B$ on $[0, \infty)$ with intensity $\text{cap}(B)$ and $\widetilde{\text{cap}}(B)$, respectively, and write $n_B(t) = n_B([0, t])$ for $t \ge 0$, a Poisson variable with mean $\text{cap}(B)t$. We consider the random variables

$$\mathcal{N}^u = \sup\left\{ n \ge 1 : \textstyle\sum_{1 \le k \le n} \sigma_k \le n_B(u) \right\}, \quad u > 0, \tag{6.16}$$

(with the convention $\sup \emptyset = 0$), and similarly $\widetilde{\mathcal{N}}^u$, using $\tilde{\sigma}_k$ and $\tilde{n}_B$ instead. These random variables admit the following comparison estimates, which depend on the function f underlying the law of $\widetilde{Z}$ determining $\widetilde{\mathcal{N}}^u$.

[v can be anything]

Lemma 6.3. *For all $\delta \in (0,1)$, $\varepsilon \in (0, \frac{u_*}{10} \wedge 1)$ and $\eta \in [0, c_4)$, the following hold.*

i) If $u < u_*$ and $z \in \Gamma_{\text{ext}}$, then with $f = f_1^{v,u;\varepsilon}$, $v = u(1-\eta\sqrt{\delta})$,

$$Q_B\big[(1+c\varepsilon)\mathcal{N}^{u_*+\frac{1}{2}\varepsilon} \leq \widetilde{\mathcal{N}}^v \leq (1+c\varepsilon)^{-1}\mathcal{N}^{u_*+\frac{3}{2}\varepsilon}\big] \geq 1 - e^{-\tilde{c}r_0^{\tilde{c}}}. \tag{6.17}$$

ii) If $u \neq u_*$ and $z \in \Gamma_{int}$, then with $s_i(t) = t(1-\gamma_i)^{\sigma_i}$ for $\gamma_i \in [0,1)$, $\sigma_i \in \{\pm 1\}$, $i = 1,2$, one has for f as above when $u < u_*$ and $f = f_0^{u;\varepsilon}$ when $u > u_*$ that

$$Q_B\big[(1+\varepsilon)\mathcal{N}^{s_i(u_*-\frac{3}{2}\varepsilon)} \leq \widetilde{\mathcal{N}}^{s_i(u)} \leq (1+\varepsilon)^{-1}\mathcal{N}^{s_i(u_*-\frac{1}{2}\varepsilon)}\big] \geq 1 - e^{-\tilde{c}r_0^{\tilde{c}}},\ i = 1,2, \tag{6.18}$$

for suitable $\tilde{c}$ depending on $u, \delta, \varepsilon, \eta, \gamma_i, \sigma_i$.

The bounds (6.17) and (6.18) are tailored to our purposes. Underlying them are similar controls on the behavior of the tilted walk $\widetilde{P}_x^f$ that are also needed to prove (6.13) (essentially because the relevant quantities $\tilde{\pi}$ and $\tilde{h}_B$ also crucially appear in (6.15) and thus govern the law of the variables $(\tilde{\sigma}_k)_{k\geq 1}$ entering $\widetilde{\mathcal{N}}^u$ in (6.16)). The proof of Lemma 6.3 thus appears jointly with that of Lemma 6.2 in the next section. With both Lemmas 6.3 and 6.2 at our disposal, we are ready to give the short:

Proof of Proposition 6.1. We choose Q_B the coupling constructed above. Recall from (4.10) the induced interlacement process $\mu_B = \mu_B(\omega)$ (declared under either of $\mathbb{P}$ and $\widetilde{P}_f$), collecting the labeled trajectories entering B after their entrance time in B. Observe that under Q_B, the random measures

$$(\xi^u, \tilde{\xi}^v)_{u,v>0} \overset{\text{def.}}{=} \Big(\sum_{1\leq k\leq \mathcal{N}^u} \delta_{Z_k}, \sum_{1\leq k\leq \widetilde{\mathcal{N}}^v} \delta_{\widetilde{Z}_k}\Big)_{u,v>0} \tag{6.19}$$

have the same law as the excursions between B and U^c induced by the trajectories in the support of μ_B with label at most u and v under $\mathbb{P}$ and $\widetilde{\mathbb{P}}_f$, respectively. As can be seen from (6.1), the occupation time field $(\ell_x^u)_{x\in B, u>0}$ is clearly a measurable function of the excursions induced by $\mu_B(\omega)$ under $\mathbb{P}$, and similarly for $(\tilde{\ell}_x^u)_{x\in B,u>0}$ under $\widetilde{\mathbb{P}}_f$ (recall our notational convention from below (6.1)). Hence, $\ell_x^u, \tilde{\ell}_x^v$, $x \in B, u, v > 0$ can be viewed (in law) as functionals of the point measures in (6.19), as

$$\ell_x^u \overset{\text{law}}{=} \ell_x(\xi^u) \overset{\text{def.}}{=} \sum_{1\leq k\leq \mathcal{N}^u} \int_0^{\text{len}(Z_k)} 1\{Z_k(t) = x\}\text{d}t, \quad x \in B,\ u > 0, \tag{6.20}$$

where $\text{len}(Z_k)$ refers to the length (duration) of the excursion Z_k, and similarly for $\tilde{\ell}_x^u$. In particular, Q_B thus furnishes a coupling of the occupation time fields $(\ell_x^u, \tilde{\ell}_x^v : x \in B, u, v > 0)$.

We now argue that (6.3) holds (which implicitly entails that z, f and $B = B_1^z$ have been chosen accordingly). For two point measures ξ, ξ' on the excursion space Ξ, as in (6.19) for instance, we write $\xi \leq \xi'$ if $\text{supp}(\xi) \subset \text{supp}(\xi')$. As follows plainly from (6.20), the occupation time field is clearly monotone with respect to this order, i.e. $\xi \leq \xi'$ implies $\ell(\xi) \leq \ell(\xi')$. Fix $v = c\varepsilon$ with c small enough so that the conclusions of (6.14) hold and $\frac{1+4v}{1-v} \leq 1 + c\varepsilon$. Let $n = \mathcal{N}^{u_*}$. With these choices for v and n, if the event U_n^v appearing in (6.14) and the event on the left-hand side of (6.17) jointly occur (under Q_B), one deduces using $\mathcal{N}^{u_*+\frac{1}{2}\varepsilon} \geq \mathcal{N}^{u_*}$ when applying (6.14) that

$$\{Z_1, \dots, Z_{\mathcal{N}^{u_*+\frac{1}{2}\varepsilon}}\} \overset{(6.14)}{\subset} \{\widetilde{Z}_1, \dots, \widetilde{Z}_{(1+c\varepsilon)\mathcal{N}^{u_*+\frac{1}{2}\varepsilon}}\} \overset{(6.17)}{\subset} \{\widetilde{Z}_1, \dots, \widetilde{Z}_{\widetilde{\mathcal{N}}^v}\}.$$

With a view towards (6.19), this yields that $\xi^{u_*+\frac{1}{2}\varepsilon} \leq \tilde{\xi}^v$ and hence $\ell_x^{u_*+\frac{1}{2}\varepsilon} = \ell_x(\xi^{u_*+\frac{1}{2}\varepsilon}) \leq \ell_x(\tilde{\xi}^v) = \tilde{\ell}_x^v$ for all $x \in B$ by monotonicity. The other inequality $\tilde{\ell}_x^v \leq \ell_x^{u_*+\frac{3}{2}\varepsilon}$ inherent to the event in (6.3) is obtained similarly, now exploiting (6.14) together with the second inequality in (6.17). To conclude the proof of (6.3) it thus suffices to combine the bound on the event in (6.14), together with a suitable estimate on $Q_B[U^{c\varepsilon}_{n=\mathcal{N}^{u_*}}]$. The latter is obtained by combining (6.14), first for $|x| \geq C(\varepsilon)$ but eventually for all x by possibly adapting the constant $\tilde{c}$, and the fact (see, for instance, below (7.26)) that $Q_B[\mathcal{N}^{u_*} \geq \frac{1}{2}u_*\text{cap}(B)] \geq 1 - e^{-c\cdot\text{cap}(B)}$.

The proof of item *ii)* in Proposition 6.1 follows a similar reasoning as that of item *i)*, now combining (6.14) and (6.18) to deduce (6.4). $\square$

7 Tilted harmonic measure

In this section we prove Lemmas 6.2 and 6.3, which concern the tilted random walk measure $\widetilde{P}_x^f$ introduced in (4.11). The functions $f \in \mathcal{F} = \{f_0, f_1\}$ are defined (4.2) and implicitly depend on parameters u, v and ε, as well as on $x \in \mathbb{Z}^d$ and $\delta \in (0, \frac{1}{6})$, which determine the underlying (oblique) regions T^i, cf. (3.1) and (4.3)-(4.4). In order to keep notation reasonable, whenever the parameters $\delta, u, v, \varepsilon$ are not further specified below, it is tacitly assumed that the conclusions hold for all $\delta \in (0, \frac{1}{6})$, $\varepsilon \in (0, \frac{u_*}{10} \wedge 1)$, and all $u > u_*$ when $f = f_0$ or all $0 < v \leq u < u_*$ when $f = f_1$. Recall that, when working with $f \in \mathcal{F}$ we only have to deal with boxes $B = B_1^z$ having centers $z \in \Gamma_f = \Gamma_f(x)$ (see above Lemma 6.2 for notation).

The following result will be key.

Lemma 7.1. *For* $f \in \mathcal{F}$*,*

$$\beta^f(x) \overset{\text{def.}}{=} \sup_{i=1,2}\ \sup_{z\in\Gamma_f(x),\, y\in\partial B^z_{i+1}} \widetilde{P}_y^f[H_{B_i^z} < \infty] \to 0 \ \textit{as}\ |x| \to \infty. \tag{7.1}$$

Proof. For simplicity we assume that $i = 2$ in the sequel. The other case is treated in the same way

Let $f \in \mathcal{F}$. Recall from (4.7) that $V_f = -\Delta f / f$. We will prove with y ranging over $\bigcup_{z\in\Gamma_f} \partial^{\text{out}} B_4^z$ below (noting that this range depends on $x \in \mathbb{Z}^d$ as $\Gamma_f = \Gamma_f(x)$) that

$$\sup_{x,y} P_y[I_f(\alpha)] \overset{\alpha\to\infty}{\longrightarrow} 1, \text{ where } I_f(\alpha) \overset{\text{def.}}{=} \Big\{ \int_0^\infty V_f(X_s)\text{d}s \geq -\alpha \Big\}. \tag{7.2}$$

We start by explaining how (7.2) yields the claim. To this effect, first note that, uniformly in $x \in \mathbb{Z}^d$, $z \in \Gamma_f(x)$ and $y \in \partial^{\text{out}} B_4^z$, abbreviating $B_i = B_i^z$, one has by virtue of (6.2) (cf. also (3.2) regarding r_1) that $P_y[\widetilde{H}_{B_3} = \infty] \geq c_5$. Applying (7.2) we then fix α large enough such that $P_y[I_f(\alpha)] \geq 1 - \frac{c_5}{2}$. Using that $f \leq C(u, v, \varepsilon)$, as can be seen by inspection of (4.3), (4.4), whence $\frac{d\widetilde{P}_y^f}{dP_y} 1_{I_f(\alpha)} \geq c_6(u, v, \varepsilon)$, it thus follows that (uniformly in x, y as above)

$$\widetilde{P}_y^f[H_{B_3} = \infty] \geq \widetilde{P}_y^f[H_{B_3} = \infty,\, I_f(\alpha)] \geq c_6(P_y[H_{B_3} = \infty] - P_y[I_f(\alpha)^c]) \geq \frac{c_5 c_6}{2}. \tag{7.3}$$

Now to (7.1). Let $\beta_f^0(\cdot)$ be defined as $\beta_f(\cdot)$ in (7.1) but with ∞ replaced by $T_{B_4^z}$, the exit time from B_4^z. We first argue that

$$\beta_0^f(x) \to 0 \text{ as } |x| \to \infty. \tag{7.4}$$

To see this, first observe that, prior to exiting $B_4 = B_4^z$, the law of the tilted random walk is identical to that of the simple random walk (indeed $B_4^z \subset T^1$ whenever $z \in \Gamma_{f_0}$ and $B_4 \subset T^1 \cup (T^5 \setminus T^3)$ whenever $z \in \Gamma_{f_1}$, and f_0, f_1 are constant on these respective sets; cf. (4.5), (4.6)). In particular this implies for y as in (7.1) that $\widetilde{P}_y^f[H_{B_2} < T_{B_4}] = P_y[H_{B_2} < T_{B_4}]$ and the latter is bounded by $P_y[H_{B_2} < \infty] \leq r_1^{-c}$ using standard arguments and the choice of boxes in (6.2). All in all, (7.4) thus follows.

Consider now the quantity $\widetilde{P}_y^f[T_{B_4} \leq H_{B_2} < \infty]$. Applying the strong Markov property at times T_{B_4} and $H_{B_3} \circ \theta_{T_{B_4}}$, where θ_t denotes the canonical shift by $t > 0$, it readily follows for y as in (7.1) that $\widetilde{P}_y^f[T_{B_4} \leq H_{B_2} < \infty] \leq \beta_0^f(x) \sup_{y'} \widetilde{P}_{y'}^f[H_{B_3} < \infty]$, with the supremum ranging over $y' \in \partial^{\text{out}} B_4$, whence

$$\widetilde{P}_y^f[H_{B_2} < \infty] = \widetilde{P}_y^f[H_{B_2} < T_{B_4}] + \widetilde{P}_y^f[T_{B_4} \leq H_{B_2} < \infty] \leq \frac{\beta_0^f(x)}{1 - \sup_{y'} \widetilde{P}_{y'}^f[\tilde{H}_{B_3} < \infty]}. \tag{7.5}$$

Applying both (7.4) and using (7.3) to control the denominator in (7.5), (7.1) follows.

It thus remains to prove (7.2). For real-valued V let $V^- = (-V) \vee 0$. In the rest of the proof constants $c, C, \dots$ may freely depend on all of u, v and ε (as entering the definition of $f \in \mathcal{F}$). We will show that for all $x \in \mathbb{Z}^d$ and $y \in \bigcup_{z \in \Gamma_f} \partial^{\text{out}} B_4^z$,

$$E_y\Big[\int_0^\infty V_f^-(X_s)ds\Big] \leq C. \tag{7.6}$$

First we deal with the case of $f = f_0$. Recalling its form from (4.3), see also (4.5) and exploiting the fact that f_0 is harmonic at $z \in \mathbb{Z}^d$ unless $z \in \partial T^1 \cup \partial^{\text{out}} U$, it follows that $V_{f_0}(z)$ vanishes unless z belongs to this set, whence

$$\begin{aligned} E_y\Big[\int_0^\infty V_{f_0}^-(X_s)ds\Big] \\ = E_y\Big[\int_0^\infty V_{f_0}^-(X_s) 1\{X_s \in \partial T^1 \cup \partial^{\text{out}} U\} ds\Big] = \sum_{z \in \partial T^1 \cup \partial^{\text{out}} U} V_{f_0}^-(z) g(y,z). \end{aligned} \tag{7.7}$$

Abbreviating $c_7 = 1 - (\frac{u_* - \varepsilon}{u})^{\frac{1}{2}} \in (0,1)$ (recall that $u > u_*$ when $f = f_0 = f_0^{u,\varepsilon}$), so that $f_0 = 1 - c_7 h_0$, we have that for all $z \in \mathbb{Z}^d$,

$$V_{f_0}(z) = -\frac{\Delta f_0(z)}{f_0(z)} = \frac{c_7 \Delta h_0(z)}{1 - c_7 h_0(z)} = \frac{c_7}{1 - c_7 h_0(z)} \frac{1}{2d} \sum_{z' \sim z} (h_0(z') - h_0(z)). \tag{7.8}$$

Moreover, if $z \in \partial^{\text{out}} U$ then $h_0(z) = P_z[H_{T^1} < T_U] = 0$, which effectively disappears from the right-hand side of (7.8), so that $V_{f_0}^-(z) = 0$, hence the sum over $z \in \partial^{\text{out}} U$ can be ignored in (7.7). Meanwhile, $h_0(z) = 1$ whenever $z \in \partial T^1$, hence (7.7) yields for such z that

$$V_{f_0}^-(z) = -V_{f_0}(z) = \frac{c_7}{1 - c_7} \frac{1}{2d} \sum_{z' \sim z} P_{z'}[H_{T^1} \geq T_U] = \frac{c_7}{1 - c_7} P_z[\widetilde{H}_{T^1} \geq T_U], \tag{7.9}$$

using the simple Markov property in the last step. By choice of U in (4.1), one readily infers that $\inf_{\xi \notin U} P_\xi[H_{T^1} = \infty] \geq c$, and applying the strong Markov property at time T_U, this is straightforwardly seen to imply that $P_z[\widetilde{H}_{T^1} \geq T_U] \leq c^{-1} e_{T^1}(z)$ for all $z \in \partial T^1$. Feeding this into (7.9) and the resulting bound on $V^-_{f_0}(z)$ into (7.7), and using that $\sum_{z\in\partial T^1} e_{T^1}(z)g(y,z) \le 1$, (7.6) follows for $f = f_0$.

Now we deal with the case $f = f_1$, see (4.4) for its definition. Using harmonicity, the formula (7.7) remains correct for $f = f_1$ but the sum on the right now ranges over $z \in \partial T^2 \cup \partial^{\text{out}} T^3 \cup \partial T^6 \cup \partial^{\text{out}} U$. Abbreviating $c_8 = (\frac{u_*+\varepsilon}{v})^{1/2} - 1$ and $c_9 = c_7 + c_8$, so that $f_1 = 1 + c_8 h_2 - c_8 h_1$, the analogue of (7.8) reads

$$V_{f_1}(z) = -\frac{\Delta f_1(z)}{f_1(z)} = \frac{c_8\Delta h_1(z) - c_9 \Delta h_2(z)}{1 + c_9 h_2(z) - c_8 h_1(z)},\ z \in \mathbb{Z}^d. \tag{7.10}$$

Recalling that $h_1(z) = P_z[H_{T^2} < T_{T^3}]$, $h_2(z) = P_z[H_{T^6} < T_U]$, it follows that for $z \in \partial^{\text{out}} T^3$, we have $h_2(z) = 1, \Delta h_2(z) = 0, h_1(z) = 0$, so that $V_{f_1}(z) \ge 0$ by inspection of (7.10), whence $V^-_{f_1}(z) = 0$. The same fate applies to $z \in \partial T^6$, using now that $h_1(z) = 0, \Delta h_1(z) = 0, h_2(z) = 1$. Thus, the relevant sum over z in the analogue of (7.7) boils down to $z \in \partial T^2 \cup \partial^{\text{out}} U$. One takes care of $z \in \partial T^2$ in much the same way as $z \in \partial T^1$ in the case of f_0, deducing this time that $V^-_{f_1}(z) \le C e_{T^2}(z)$ for $z \in \partial T^2$. Finally, for $z \in \partial^{\text{out}} U$, we have $h_1(z) = 0, \Delta h_1(z) = 0, h_2(z) = 0$, such that

$$V^-_{f_1}(z) = \frac{c_9}{2d} \sum_{z'\sim z} P_{z'}[H_{T^6} < T_U] = c_9 P_z[H_{T^6} < \widetilde{H}_{U^c}]. \tag{7.11}$$

By construction, see (3.1) and (4.1), $T^6 \subset U/2$, where with hopefully obvious notation $U/2$ is the box concentric to U with half the radius. Thus bounding $P_z[H_{T^6} < \widetilde{H}_{U^c}] \le P_z[H_{U/2} < \widetilde{H}_{U^c}]$ and by considering the projection of the walk in the direction orthogonal to the face to which $z \in \partial^{\text{out}} U$ belongs, a straightforward Gambler's ruin argument readily yields that $P_z[H_{T^6} < \widetilde{H}_{U^c}] \le C|x|^{-1}$.

Hence, returning to (7.10), which feeds into (the analogue of) (7.7), yields the desired estimate towards obtaining (7.6), since for all y as in (7.6),

$$\sum_{z\in\partial^{\text{out}} U} V^-_{f_1}(z) g(y,z) \le \frac{C}{|x|} \sum_{z\in\partial^{\text{out}} U} g(y,z) \le \frac{C'}{|x|} |\partial^{\text{out}} U| |x|^{2-d} \le C'', \tag{7.12}$$

where we have used that $|y-z| \ge c|x|$ and a standard estimate on the Green's function. Overall, this completes the proof of (7.10) for all $f \in \mathcal{F}$, and with it the verification of (7.2), thus concluding the proof. □

As an immediate consequence of Lemma 7.1, we obtain the following comparison between tilted and untilted entrance laws. Extending the notation $h_B(\cdot,\cdot)$, $\tilde{h}_B(\cdot,\cdot)$, from (6.10), we define for $B \subset B' \subset \mathbb{Z}^d$ the quantity $\tilde{h}_{B,B'}(x,y) = \widetilde{P}^f_x[H_B < T_{B'}, X_{H_B} = y]$ for $x, y \in \mathbb{Z}^d$, and similarly $h_{B,B'}(x,y)$ with P_x in place of $\widetilde{P}^f_x$. Thus, $\tilde{h}_B = \tilde{h}_{B,\mathbb{Z}^d}$. We will sometimes write $\tilde{h}^f_{B,B'} = \tilde{h}_{B,B'}$ or $\tilde{h}^f_B = \tilde{h}_B$ to insist on the dependence on f.

Corollary 7.2. *For all $f \in \mathcal{F}$, $\epsilon_0 > 0$, $\delta \in (0, \frac{1}{6})$, $|x| \ge C(\epsilon_0, \delta)$, $z \in \Gamma_f(x)$, $y \in \partial B^z_2$ and $y' \in \mathbb{Z}^d$, abbreviating $B_i = B^z_i$,*

$$h_{B_1,B_3}(y,y') \le \tilde{h}^f_{B_1}(y,y') \le (1+\epsilon_0) \min_{\tilde{y}\in\partial B_2} h_{B_1,B_3}(\tilde{y},y'). \tag{7.13}$$

Proof. The first bound in (7.13) is immediate since $\tilde{h}^f_{B_1}(y,y') \geq \tilde{h}^f_{B_1,B_3}(y,y') = \tilde{h}^f_{B_1}(y,y')$ for $y \in \partial B_2^z$, using that $X_{\cdot \wedge T_{B_4}}$ has the same law under P_y and $\widetilde{P}^f_y$, cf. below (7.4). For the second bound in (7.13), applying a similar reasoning as used to deduce (7.5) yields that

$$\tilde{h}^f_{B_1}(y,y') \leq \frac{1}{1-\beta^f(x)} \max_{\tilde{y} \in \partial B_2} \tilde{h}^f_{B_1,B_3}(\tilde{y},y'), \tag{7.14}$$

and the latter quantity equals $\max_{\tilde{y}} h_{B_1,B_3}(\tilde{y},y')$. To conclude (7.13), one applies Lemma 7.1 and uses the fact that $\max_{\tilde{y}} h_{B_1,B_3}(\tilde{y},y') \leq (1+|x|^{-c\delta}) \min_{\tilde{y}} h_{B_1,B_3}(\tilde{y},y')$, which follows by similar considerations as e.g. in [14, Lemma 3.5]. $\square$

Corollary 7.2 will readily yield the part of Lemma 6.2 concerning $\frac{\tilde{\pi}}{\pi}$. Dealing with $\frac{\tilde{\pi}_0}{\pi_0}$, cf. (6.12) requires a control on the tilted equilibrium measure, which is the object of the next lemma. This result will also be needed in the course of proving Lemma 6.3. In the sequel it will be convenient to introduce "effective" levels

$$\begin{aligned} &\tilde{u}^z_f = \begin{cases} u & \text{if } f \in \mathcal{F} \text{ and } z \in \Gamma_{\text{int}} \\ v & \text{if } f = f_1 \text{ and } z \in \Gamma_{\text{ext}} \end{cases} \\ &u^z_f = f^2(z)\tilde{u}^z_f \overset{(4.5),(4.6)}{=} \begin{cases} u_* - \varepsilon & \text{if } f \in \mathcal{F} \text{ and } z \in \Gamma_{\text{int}} \\ u_* + \varepsilon & \text{if } f = f_1 \text{ and } z \in \Gamma_{\text{ext}}. \end{cases} \end{aligned} \tag{7.15}$$

Note that (7.15) defines $\tilde{u}^z_f$ and u^z_f for any $z \in \Gamma_f$ since $\Gamma_{f_0} = \Gamma_{\text{int}}$ and $\Gamma_{f_1} = \Gamma_{\text{int}} \cup \Gamma_{\text{ext}}$ according to our definition at the beginning of this section.

Proposition 7.3. *For all $\epsilon_0 \in (0,1)$, $f \in \mathcal{F}$, $x \in \mathbb{Z}^d$, $z \in \Gamma_f(x)$ and $|x| \geq C(\epsilon_0, \delta)$,*

$$(1-\epsilon_0) u^z_f \cdot \text{cap}(B^z_2) \leq \tilde{u}^z_f \cdot \widetilde{\text{cap}}_f(B^z_2) \leq (1+\epsilon_0) u^z_f \cdot \text{cap}(B^z_2), \ \textit{and} \tag{7.16}$$

$$(1-\epsilon_0) u^z_f \cdot e_{B^z_1}(y) \leq \tilde{u}^z_f \cdot \tilde{e}_{B^z_1}(y) \leq (1+\epsilon_0) u^z_f \cdot e_{B^z_1}(y), \ y \in \mathbb{Z}^d. \tag{7.17}$$

Proof. A bound similar to the first inequality in (7.16) was proved in [14, Proposition 3.1], and can be obtained within the present setup via similar arguments, essentially with Lemma 7.1 now playing the role of [14, Lemma 3.3].

We now focus on the second inequality in (7.16), which requires additional arguments. We begin by noting that, with the effective levels defined in (7.15) and the tilted Green's function $\tilde{g}$ as introduced in (4.13), one has the identity

$$\tilde{u}^z_f \sum_{y,y' \in B_2} \tilde{e}_{B_2}(y) \tilde{g}(y,y') f^2(y') = u^z_f \sum_{y,y' \in B_2} e_{B_2}(y) g(y,y'), \tag{7.18}$$

where $g = g_{\mathbb{Z}^d}$ is the usual Green's function given by (2.1). To see (7.18), one simply applies (4.14) to the left-hand side and notices that $f^2(y') = f^2(z)$ for all $y' \in B^z_2$ and similarly (2.3) to the right-hand side, to conclude that both sides equal $u^z_f |B_2|$. Roughly speaking, we aim to argue that the left-hand side is an upper bound for $\tilde{u}^z_f \cdot \widetilde{\text{cap}}_f(B^z_2)$, while the right-hand side is a lower bound for $u^z_f \cdot \text{cap}(B^z_2)$. The desired inequality will then follow by means of (7.18).

To this end, using the fact that $\tilde{g}_{B_3}(y,y') = g_{B_3}(y,y')$ for any $y, y' \in B_3$ we bound

$$\begin{aligned} \sum_{y,y' \in B_2} \tilde{e}_{B_2}(y) \tilde{g}(y,y') \geq \widetilde{\text{cap}}_f(B_2) \inf_{y \in B_2} \sum_{y' \in B_2} g_{B_3}(y,y') = \widetilde{\text{cap}}_f(B_2) \times \\ \inf_{y \in B_2} \sum_{y' \in B_2} \big(g(y,y') - E_y[g(X_{T_{B_3}}, y')]\big) \geq \widetilde{\text{cap}}_f(B_2) \inf_{y \in B_2} \sum_{y' \in B_2} \big(g(y,y') - C|x|^{-c\delta} \sup_{y'' \in B_2} g(y'',y')\big), \end{aligned} \tag{7.19}$$

for all $|x| \geq C(\delta)$. Using the fact that uniformly in $y \in B_2$,

$$\sum_{y' \in B_2} g(y, y') = c_{10} |B_2|^{\frac{2}{d}} (1 + o(1)), \text{ as } |x| \to \infty, \tag{7.20}$$

for a suitable constant c_{10}, see e.g. [14, Lemma 1.1.] for a proof, one immediately deduces by inserting (7.20) into (7.19) and using that $f(y')$ is constant in B_2 that the left-hand side of (7.18) is bounded from below by $c_{10}\tilde{u}^z_f f^2(z) \cdot \widetilde{\text{cap}}_f(B_2)|B_2|^{\frac{2}{d}}(1-\epsilon_0)$, for all $|x| \geq C(\delta, \epsilon_0)$. Employing (7.20) allows to bound the right-hand side of (7.18) from above by $c_{10}u^z_f \cdot \widetilde{\text{cap}}_f(B_2)|B_2|^{\frac{2}{d}}(1+\epsilon_0)$. The claim now follows using (7.19) and the fact that $u^z_f = f^2(z)\tilde{u}^z_f$, see (7.15). This completes the proof of (7.16).

The inequalities (7.17) are a consequence of (7.16) and (7.13), as we now explain. Indeed, to obtain the desired upper bound for $u^z_f \cdot \tilde{e}_{B_1}(y)$, applying the sweeping identity to the sets $B_1 \subset B_2$ yields that for all $|x| \geq C'(\epsilon_0, \delta)$,

$$\begin{aligned} \tilde{e}_{B_1}(y) &= \sum_{y' \in \partial B_2} \tilde{e}_{B_2}(y')\tilde{h}_{B_1}(y'y) \overset{(7.13)}{\leq} (1+\epsilon_0/2)\widetilde{\text{cap}}_f(B_2) \min_{\tilde{y}' \in \partial B_2} h_{B_1,B_3}(\tilde{y}', y) \\ &\overset{(7.16)}{\leq} (1+\epsilon_0)\frac{u^z_f}{\tilde{u}^z_f}\text{cap}(B^z_2) \min_{\tilde{y}' \in \partial B_2} h_{B_1}(\tilde{y}', y) \leq (1+\epsilon_0)\frac{u^z_f}{\tilde{u}^z_f} \sum_{\tilde{y}' \in \partial B_2} e_{B_2}(y')h_{B_1}(\tilde{y}', y) = (1+\epsilon_0)\frac{u^z_f}{\tilde{u}^z_f} e_{B_1}(y), \end{aligned}$$

where the last step uses again the sweeping identity. The lower bound on $\tilde{e}_{B_1}$ in (7.17) is obtained similarly. □

Proof of Lemma 6.2. We first deal with the part of (6.13) concerning $\frac{\tilde{\pi}_0}{\pi_0}$. Summing (7.17) over B^z_1 and subsequently using this resulting inequality when dividing by the capacity of B^z_1 yields an analogue of (6.13) rgarding normalized (tilted and untilted) equilibrium measures. In view of the definition of π_0 in (6.12), the claim follows.

We now show the part of (6.13) concerning $\frac{\tilde{\pi}}{\pi}$. To this effect, we first observe that, for all $\epsilon_0 > 0$, $\delta \in (0, \frac{1}{6})$ and $|x| \geq C(\epsilon_0, \delta)$, uniformly in $f \in \mathcal{F}$, $z \in \Gamma_f(x)$, $y \in \partial B^z_2$ and $y' \in \partial B^z_1$,

$$(1-\epsilon_0)\bar{e}_{B_1}(y') \leq \frac{\tilde{h}^f_{B_1}(y, y')}{\widetilde{P}^f_y[H_{B_1} < \infty]} \leq (1+\epsilon_0)\bar{e}_{B_1}(y'); \tag{7.21}$$

indeed (7.21) follows readily from Corollary 7.2, as we now explain. By applying (7.13), both in its given form and when summing over y', one deduces that the ratio in (7.21) and its untilted analogue $\frac{h_{B_1}(y,y')}{P_y[H_{B_1}<\infty]}$ are comparable up to multiplicative errors of order $1+O(\epsilon_0)$ when $|x| \geq C(\epsilon_0, \delta)$. Then one uses the fact that the untilted analogue of (7.21), i.e. bounding $\frac{h_{B_1}(y,y')}{P_y[H_{B_1}<\infty]}$ from above and below by $(1 \pm \epsilon_0)\bar{e}_{B_1}(y')$ is classically known, see, for example, [13, Theorem 2.1.3]. Hence, overall, (7.21) follows.

Now let $\zeta, \zeta' \in \Xi$ be two excursions between $B = B^z_1$ and U^c, where $U = B^z_2$, cf. around (6.9) for notation. Rather than dealing with the ratio $\frac{\tilde{\pi}(\zeta,\zeta')}{\pi(\zeta,\zeta')}$ directly, we will separately consider $\frac{\tilde{\pi}(\zeta,\zeta')}{\bar{e}_{B_1}(\zeta'_0)}$ and $\frac{\pi(\zeta,\zeta')}{\bar{e}_{B_1}(\zeta'_0)}$, with $\zeta'_0 \in \partial B^z_1$ denoting the starting point of ζ'. Recalling $\tilde{\pi}$ from (6.12), it follows, abbreviating $y = \zeta_n$, $y' = \zeta'_0$ and with the aid of (7.21) that

$$\frac{\tilde{\pi}(\zeta, \zeta')}{\bar{e}_{B_1}(y')} = \frac{\tilde{h}^f_{B_1}(y, y')}{\bar{e}_{B_1}(y')} + 1 - h_{B_1}(y) \leq 1 + \epsilon_0 h_{B_1}(y),$$

along with a similar lower bound, implying overall that

$$\left| \frac{\tilde{\pi}(\zeta,\zeta')}{\bar{e}_{B_1}(y')} - 1 \right| \leq \epsilon_0 h_{B_1}(y). \tag{7.22}$$

The same bound as (7.22) is obtained for π instead of $\tilde{\pi}$ using the (classical) untilted analogue of (7.21), see [13, Theorem 2.1.3]. From (7.22) and its version for π, the desired bound on $|\frac{\tilde{\pi}}{\pi} - 1|$ readily follows with $\epsilon_0 = 1$. □

The rest of this section is geared towards the proof of Lemma 6.3, which concerns the random variables $\mathcal{N}^u$ introduced in (6.16) and their tilted analogue. We first isolate the following result. Under P_x, define the successive return times to $B = B_1^z$ and departure times from $U = B_2^z$ as $R_1 = H_B$, and for $k \geq 1$, $D_k = R_k + T_U \circ R_k$, $R_{k+1} = D_k + H_B \circ D_k$ (assuming R_k is finite, and else set $D_k = R_{k+1} = \infty$). Now let

$$\tau = \tau_B = \sup\{k \geq 1 : D_k < \infty\}. \tag{7.23}$$

The random variable τ counts the number of excursions between B and U^c made by the walk. We write $\tilde{\tau}$ for its pendant defined under $\widetilde{P}_x^f$. Recall $\tilde{\pi}_0$ from (6.12).

Lemma 7.4. *For all $\lambda \in (-\infty, c)$, $f \in \mathcal{F}$, $x \in \mathbb{Z}^d$, $z \in \Gamma_f(x)$, with $\tau = \tau_{B_1^z}$,*

$$e^{\lambda}(1-\beta^f(x)) \leq \widetilde{E}_{\tilde{\pi}_0}^f\big[e^{\lambda\tau}\big] \leq \frac{e^{\lambda}(1-\beta^f(x))}{1-e^{\lambda}\beta^f(x)}, \quad \textit{(see (7.1) for } \beta^f(x)\textit{)} \tag{7.24}$$

Proof. The upper bound is proved in a similar way as [21, Lemma 2.7]. The lower bound is obtained by bounding $\widetilde{E}_{\tilde{\pi}_0}^f\big[e^{\lambda\tau}\big] \geq e^{\lambda}\widetilde{P}_{\tilde{\pi}_0}^f[\tau = 1]$. □

It remains to give the

Proof of Lemma 6.3. In view of (7.23), and by inspection of (6.15), one observes that if one defines recursively $\hat{\sigma}_1 = 1$ and for $k \geq 1$ $\hat{\sigma}_{k+1} = \inf\{k \geq \hat{\sigma}_k : \sigma_k = 1\}$, then $\hat{\sigma}_{k+1} - \hat{\sigma}_k$ has the same law as τ under P_{e_B}. Moreover, the random variables $\hat{\sigma}_{k+1} - \hat{\sigma}_k$, $k \geq 1$ are independent and analogous statements hold for tilted quantities. It follows in view of (6.16) that for all $v > 0$

$$\widetilde{\mathcal{N}}^v \overset{\text{law}}{=} \sum_{i=1}^{\widetilde{\Theta}(u)} \tilde{\tau}_i \tag{7.25}$$

where $\tilde{\tau}_i$, $i \geq 1$, are i.i.d. with same law as $\tilde{\tau}$ under $\widetilde{P}_{\tilde{\pi}_0}^f$, where $\tilde{\pi}_0$ is the normalized tilted equilibrium measure on B, cf. (6.12), and $\widetilde{\Theta}(u)$ is an independent Poisson variable with mean $u \cdot \widetilde{\text{cap}}(B)$. A representation similar to (7.25) can be derived for $\mathcal{N}^u$.

We now focus on (6.17); the remaining bounds are obtained similarly. Thus let $f = f_1^{v,u;\varepsilon}$, $v = u(1-\eta\sqrt{\delta})$ for some $u < u_*$ and $B = B_1^z$ for some $z \in \Gamma_{\text{ext}}$. From (7.25), Lemma 7.4 and Proposition 7.3, we infer that for $\lambda \in (-\infty, c)$, $\epsilon_0 \in (0,1)$ and $|x| \geq C(\epsilon_0)$,

$$\begin{aligned} \log \widetilde{\mathbb{E}}^f\big[e^{\lambda \cdot \widetilde{\mathcal{N}}^v}\big] = v \cdot \widetilde{\text{cap}}(B)\big(\widetilde{E}_{\tilde{\pi}_0}^f\big[e^{\lambda\tau}\big] - 1\big) &\overset{(7.24)}{\leq} v \cdot \widetilde{\text{cap}}(B)\frac{e^{\lambda}-1}{1-e^{\lambda}\beta^f(x)} \\ &\overset{(7.17)}{\leq} (u_*+\varepsilon)\text{cap}(B)\frac{(1+\epsilon_0)(e^{\lambda}-1)}{1-e^{\lambda}\beta^f(x)}, \end{aligned} \tag{7.26}$$

where, in applying (7.17), we have summed over $y \in \mathbb{Z}^d$ and used that $v \cdot \frac{u_f^z}{\tilde{u}_f^z} = u_* + \varepsilon$ by (7.15) (and using that $z \in \Gamma_{\text{ext}}$ with $f = f_1$). A lower bound corresponding to (7.26) can be derived similarly, along with similar estimates for $\log \mathbb{E}[e^{\lambda \cdot \mathcal{N}^w}]$, $w > 0$, obtained by means of an obvious analogue of Lemma 7.4 (but no longer requiring Proposition 7.3). Using that $(e^\lambda - 1) \vee (1 - e^{-\lambda}) \leq \lambda(1 + C\lambda)$ for $0 \leq \lambda \leq 1$ and applying Chebyshev's inequality separately to $\mathcal{N}^{u_* + \frac{1}{2}\varepsilon}$, $\widetilde{\mathcal{N}}^v$ and $\mathcal{N}^{u_* + \frac{3}{2}\varepsilon}$, selecting in each case $\lambda = \epsilon_0 = c\varepsilon$ in (7.26), while using Lemma 7.1 to control β^f in (7.26), one ensures that with probability at least $1 - e^{-\tilde{c} r_0^{\tilde{c}}}$ and for $|x| \geq C(\varepsilon, \delta)$, the inequalities

$$\begin{aligned}
&\mathcal{N}^{u_* + \frac{1}{2}\varepsilon} \leq \text{cap}(B)(u_* + \tfrac{5}{8}\varepsilon), \\
&\text{cap}(B)(u_* + \tfrac{7}{8}\varepsilon) \leq \widetilde{\mathcal{N}}^v \leq \text{cap}(B)(u_* + \tfrac{9}{8}\varepsilon), \\
&\mathcal{N}^{u_* + \frac{3}{2}\varepsilon} \geq \text{cap}(B)(u_* + \tfrac{11}{8}\varepsilon)
\end{aligned}$$

all hold. From this (6.17) immediately follows for sufficiently small choice of $c \in (0, 1)$. □

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