STRONG LOCAL UNIQUENESS FOR THE VACANT SET OF RANDOM INTERLACEMENTS

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Abstract

We consider the the vacant set $\mathcal{V}^u$ of random interlacements on $\mathbb{Z}^d$ in dimension $d \geq 3$. For varying intensity $u > 0$, the connectivity properties of $\mathcal{V}^u$ undergo a percolation phase transition across a non-degenerate critical parameter $u_* \in (0, \infty)$. It was established in a series of recent works [20, 21, 22] that this phase transition is sharp in the following sense. The one-arm probability decays stretched exponentially fast throughout the sub-critical regime, i.e. for $u > u_*$. In the super-critical regime, i.e. when $u < u_*$, there is a cluster of positive density inside any finite ball with a probability stretched exponentially close to 1 in its radius. Furthermore, with similar probability, any two large clusters are connected to each other in a configuration with a *strictly* smaller intensity. This last property falls short of the classical local uniqueness where they are required to be connected in the *same* configuration. In this article we resolve this question by proving a stronger property, namely that local uniqueness holds simultaneously for all configurations $\mathcal{V}^v$ with $v \leq u$. The degeneracies in the conditional law of $\mathcal{V}^u$ including the lack of any finite-energy property offer a major challenge to prove any such result. One main novelty of our work, among others, is to view the relevant events as functions of 'packets' of random walk excursions rather than just vacant sets, which has implications beyond the scope of this paper. Our strong local uniqueness property yields several important results characterizing the super-critical phase of $\mathcal{V}^u$, among which are sharp upper bounds on connectivity functions.

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1 Introduction

We study the percolation of the vacant set of random interlacements in its super-critical phase. Introduced in a seminal paper by Sznitman [44], the random interlacements describe the local limit of the trace of random walk on a torus and is an important example of a model displaying long-range dependence. Informally, the random interlacements correspond to a Poissonian realization of bi-infinite transient $\mathbb{Z}^d$-valued ($d \geq 3$) trajectories modulo time-shift carrying a time-like label $u \geq 0$ called the level or intensity (see §2.1 for precise definitions). The interlacement set $\mathcal{I}^u$ at level u is then defined as the range of all trajectories in this Poisson cloud with label at most u whereas $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$ is the corresponding vacant set. The law of the (random) set $\mathcal{V}^u$ is characterized by

$$\mathbb{P}[K \subset \mathcal{V}^u] = \exp\{-u \operatorname{cap}(K)\}, \tag{1.1}$$

for all $K \subset \mathbb{Z}^d$ finite where $\operatorname{cap}(K)$ refers to the capacity of K, see §2.1. As such, $\mathcal{V} = (\mathcal{V}^u)_{u>0}$ form a decreasing family of random subsets of $\mathbb{Z}^d$ in the parameter u which undergoes a percolation transition across a critical threshold $u_* = u_*(d) \in (0, \infty)$ (see [48, 42, 37]). This phase transition is intrinsically linked to various geometric properties of random walk (or Brownian motion) in transient regime; see, e.g. [10, 3, 43, 51, 28, 48, 33].

In view of the recent series of works [20, 21, 22] by Duminil-Copin, Goswami, Rodriguez, Severo and Teixeira, we now know that the phase transition of $\mathcal{V}^u$ around u_* is sharp in the following sense. For all $u > u_*$, there exist $c = c(d)$ and $C = C(u, d)$ in $(0, \infty)$ such that

$$\mathbb{P}\big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_L\big] \leq Ce^{-L^c} \tag{1.2}$$

for all $L \geq 1$, where, with hopefully obvious notation, the event in (1.2) refers to a (nearest-neighbor) path in $\mathcal{V}^u$ connecting 0 and ∂B_L with $B_L = [-L, L]^d \cap \mathbb{Z}^d$ and ∂B_L as the inner (vertex) boundary of B_L. On the other hand, for all $0 < v < u < u_*$, there exist $c = c(d)$ and $C = C(u, v, d)$ in $(0, \infty)$ such that

$$\mathbb{P}[\mathrm{Exist}(L, u)] \wedge \mathbb{P}[\mathrm{Unique}(L, u, v)] \geq 1 - Ce^{-L^c} \tag{1.3}$$

for all $L \geq 1$, where the two events in (1.3) are defined as

$$\begin{aligned} \mathrm{Exist}(L, u) &= \left\{ \begin{array}{c} \text{there exists a cluster in} \\ \mathcal{V}^u \cap B_L \text{ with diameter at least } \frac{L}{5} \end{array} \right\}, \text{ and} \\ \mathrm{Unique}(L, u, v) &= \left\{ \begin{array}{l} \text{any two clusters in } \mathcal{V}^u \cap B_L \text{ having diameter at} \\ \text{least } \frac{L}{10} \text{ are connected to each other in } \mathcal{V}^v \cap B_{2L} \end{array} \right\}. \end{aligned} \tag{1.4}$$

While (1.3) is good enough to pin down the precise large-deviation behavior of several events of interest in the super-critical phase $u < u_*$ (see §1.2 in [20] for a detailed discussion), it nevertheless leaves the question open as to whether $\mathcal{V}^u$ satisfies the so-called 'local uniquness' property, i.e. if u and v can be chosen to be *same*. The importance of this property is paramount for unravelling the geometric properties of the infinite cluster, see §1.1 below, not only for $\mathcal{V}^u$ but other models of interest as well. In the case of Bernoulli percolation on $\mathbb{Z}^d$ ($d \geq 3$), this follows from the classical work of Grimmett and Marstrand [26]. On more general graphs, questions of similar flavor were studied very recently by Contreas, Martineau and Tassion in [7] and by Easo and Hutchcroft in [23] (the latter in the context of locality of critical percolation probability). For the closely related FK-Ising model, Bodineau [5] proved an analogous result to [26]; see the recent paper [40] by Severo for a shorter proof and [18] for the corresponding result on Ising models.

In all of the above models, however, the correlations between local observables decay exponentially with distance. In [19], the authors established the local uniqueness for a model with slow, algebraic decay of correlations, namely the level-set percolation of the Gaussian free field throughout the super-critical regime. One of the main contributions of the current article is the discovery that the vacant set of random interlacements, a model with structural rigidities much more severe than the level-sets of Gaussian free field (see §1.2 below and also §1.4 in [20] for comparative discussions), possesses a *stronger* version of the local uniqueness property.

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1.1. Main results. We now describe our results more precisely. Our first theorem, which forms the foundation of other results in the paper, shows that a strengthened version of the local uniqueness event holds with a probability stretched exponentially close to 1. This also implies the purported equality between u_* and another critical parameter $\widehat{u}$ introduced previously in the literature in connection with uniform bound on the two-point function (see (1.8) below).

Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote the canonical law of the interlacement process; we refer e.g. to the original article [44] or [20, Section 2.1] for precise definitions. Following [49, (0.2)-(0.3)], we say that NLF(u), the *no large finite cluster* property in $[0, u]$, holds when

$$\begin{array}{l}\text{there exist } c_1(u) > 0,\, C_1(u) < \infty \text{ and } \gamma(u) \in (0,1], \text{ such that}\\ \text{for all } v \in [0,u] \text{ and } x, y \in \mathbb{Z}^d,\, \tau_v^{\text{tr}}(x,y) \le C_1 e^{-c_1|x-y|^{\gamma}}\end{array} \tag{1.5}$$

(all of c_1, C_1, γ may implicitly also depend on the dimension d), where

$$\tau_v^{\text{tr}}(x,y) = \mathbb{P}\big[x \overset{\mathcal{V}^v}{\longleftrightarrow} y,\, x \overset{\mathcal{V}^v}{\not\longleftrightarrow} \infty\big], \quad x, y \in \mathbb{Z}^d. \tag{1.6}$$

The (truncated) two-point function defined by (1.6) is symmetric in x and y and $\tau_v^{\text{tr}}(x,y) = \tau_v^{\text{tr}}(0, y - x) \equiv \tau_v^{\text{tr}}(y-x)$ by translation invariance of $\mathcal{V}^v$. Note that [49] employs a slightly different quantity than (1.6) to define NLF(u) as in (1.5) but the two can be related by a straightforward union bound, and the resulting NLF-properties are in fact identical. Noting that NLF(u) is a monotone property, let

$$\widehat{u} = \widehat{u}(d) \overset{\text{def.}}{=} \sup\{u \in [0, u_*) : \text{NLF}(u) \text{ holds}\}. \tag{1.7}$$

Recall that $u_* = u_*(d)$ denotes the critical parameter describing the percolation phase transition of $\mathcal{V}^u$. One deduces from [16] that $\widehat{u} \ge c_2$ for some $c_2 = c_2(d) \in (0,1)$ and $d \ge 3$.

Theorem 1.1. *For all $d \ge 3$,*

$$\widehat{u}(d) = u_*(d). \tag{1.8}$$

Moreover, defining for $u \ge 0$ and $L \in \mathbb{N}$ the 'strong local uniqueness' event

$$\text{SLU}_L(u) = \left\{ \begin{array}{l} \textit{for all } v \in [0,u], \textit{ any two clusters in } \mathcal{V}^v \cap B_L \textit{ having dia-}\\ \textit{meter at least } \tfrac{L}{10} \textit{ are connected to each other in } \mathcal{V}^v \cap B_{2L} \end{array} \right\}, \tag{1.9}$$

there exist $C = C(d,u)$, $c = c(d)$ in $(0, \infty)$ such that, for all $L \ge 1$ and $u < u_*$,

$$\mathbb{P}[\text{SLU}_L(u)] \ge 1 - Ce^{-L^c}. \tag{1.10}$$

The proofs of both (1.8) and (1.10) have a common root. In particular, the event in (1.9) specialized to the single value $v = u$ corresponds to $\text{Unique}(L, u, u)$ in the language of (1.4), and (1.10) thus implies a strengthening of the conclusion (1.14) in [20, Theorem 1.2] by which $u = v$, i.e. the sprinkling is removed. The presence of the quantifier 'for all $v \in (0, u]$' inside the probability in (1.10) hints at the strength of our methods, which in fact allows to prove a version of (1.10) not just involving $\mathcal{V}^v$ uniformly in $v \in [0, u]$ but any not too 'degenerate' subset $\mathcal{V} \subset \mathcal{V}^u$ formed out of excursions at scale L (of which $\mathcal{V}^v$ for $0 < v \leq u$). We return to this below, see (1.21) below. This change of perspective, by which we consider a much larger class of events (not at all measurable only with respect to $(\mathcal{V}^u)_{u>0}$ alone) is a novel and conceptually important part of our methods. In fact, this provides a new way to couple non-monotone events involving $(\mathcal{V}^u)_{u>0}$ without introducing any sprinkling which might eventually lead to a solution to the problem of determining the growth speed (with N) and rate (with u) of the diameter of the second largest cluster in the super-critical phase (see the main results in [51]). We will return to this question in a future work.

In the same vein as (1.10), (1.8) can be viewed as extending the string of equalities $\bar{u} = u_* = u_{**}$ between various critical parameters established as part of [20]; cf. §1.3 and (1.21) therein (see also Theorem 1.1 in [21]).

This extension is of independent interest. Together with Theorem 1 in [49], (1.8) implies regularity of the percolation function throughout the super-critical regime.

Corollary 1.2 (C^1 property of percolation function)**.** *The percolation function $\theta_\infty(u) = \mathbb{P}[0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} \infty]$ is C^1 with $\theta'_\infty < 0$ on $[0, u_*)$.*

Perhaps more importantly, Theorem 1.1 gives us access to intrinsic 'quenched' properties of $\mathcal{V}^u$ valid in the entire super-critical phase $u < u_*$. We illustrate this with the following application. A classical way to test the geometry of the super-critical phase is to probe the large-scale behaviour of the random walk on the infinite cluster. Questions of homogenisation in porous media have a long tradition, see for instance [2, 41, 29, 4, 36, 1] for sample results. Theorem 1.1 supplies the missing ingredient to prove an invariance principle on the infinite cluster $\mathcal{C}_\infty^u$ of $\mathcal{V}^u$ for all $u < u_*$, which is stated next.

For $\omega \in \Omega$ a realization of the interlacement point process, let $P_{x,\omega}$, $x \in \mathcal{V}^u(\omega)$ denote the law of the discrete-time Markov chain $(Y_n)_{n\geq 0}$ with $Y_0 = x$ and such that, for all e unit vector in $\mathbb{Z}^d$, one has

$$P_{\omega,x}[Y_{n+1} = z | Y_n = y] = \begin{cases} \frac{1}{2d}, & \text{if } z = y + e \in \mathcal{C}_\infty^u(\omega) \\ 1 - \frac{\deg_\omega(y)}{2d}, & \text{if } z = y \end{cases} \tag{1.11}$$

with $\deg_\omega(y) = |\{e : y + e \in \mathcal{C}_\infty^u(\omega)\}|$, where $|K|$ denotes the cardinality of $K \subset \mathbb{Z}^d$. Observe that (1.11) fully determines the law of $Y = (Y_n)_{n\geq 0}$, sometimes called the *blind ant in the labyrinth* in reference to de Gennes [8]. It represents one of several natural ways to define a Markov chain on $\mathcal{C}_\infty^u$. In the sequel for $n \geq 0$ we denote by $B^n = (B^n(t) : t \geq 0)$ the rescaled polygonal interpolation of $\frac{k}{n} \mapsto \frac{X_k}{\sqrt{n}}$, $k \geq 0$. Thus B^n has values in $C(\mathbb{R}_+, \mathbb{R}^d)$, which we endow in the sequel with the topology of uniform convergence on compact intervals.

Corollary 1.3 (Quenched invariance principle for B^n)**.** *Let $\mathbb{P}_0^u[\cdot] = \mathbb{P}[\cdot | 0 \in \mathcal{C}_\infty^u]$. For all $u \in (0, u_*)$ and $\mathbb{P}_0^u$-a.e. ω, the process B^n under $P_{0,\omega}$ converges in law as $n \to \infty$ to a Brownian motion with covariance matrix $\sigma^2 I$, with $\sigma^2 = \sigma^2(u) > 0$.*

Corollary 1.3 follows immediately from the general result [36, Theorem 1.1] and Theorem 1.1. Indeed, with the exception of **S1**, all the conditions **P1**-**P3** and **S2** appearing in [36] can be checked in the same way as in [17, Section 2.3], and the outstanding condition **S1** which postulates an effective criterion for the uniqueness of $\mathcal{C}_\infty^u$ is implied by (1.10). Previous results of the kind described by Corollary 1.3 for $\mathcal{V}^u$ were restricted to the regime $u \ll 1$, see [50, 16, 36].

Corollary 1.3 is but one emblematic example of the realm of Theorem 1.1 as concerns the geometry of the infinite cluster, which feeds into various other works that successfully exploited the general framework of conditions **P1**-**P3** and **S1**-**S2** to derive meaningful results. These results now apply to $(\mathcal{V}^u)_{u\in(0,u_*)}$ as a consequence of Theorem 1.1. They include, among others, the validity of a shape theorem [17], as well as quenched (Gaussian) heat kernel estimates for Y in (1.11), see [39], which are obtained by verifying a criterion of Barlow [2] on balls in the infinite cluster. Underlying these results is an important structural result on $\mathcal{C}_\infty^u$, namely, the validity of a certain *isoperimetric inequality*, which in its currently strongest available form is stated in [39, Theorem 5.10]. The proof of this result employs a delicate coarse-graining scheme which utilises (1.10) as a crucial ingredient.

With Theorem 1.1 at hand, we proceed to our second main result, which is concerned with the decay behaviour of the truncated two-point function $\tau_u^{\mathrm{tr}}(x) = \tau_u^{\mathrm{tr}}(0,x)$ defined in (1.6) at large distances $|x|$, where $|\cdot|$ denotes the Euclidean distance. Addressing this question is particularly challenging in the super-critical regime $u < u_*$, in which the decay crucially hinges on the truncation in the form of the additional disconnection from infinity, giving rise to a non-monotone event (on the contrary note that for $u > u_*$, this condition can be safely ignored). In view of (1.5) and (1.8), one knows that the decay is at least stretched exponential when $u < u_*$.

In the recent series of works [20, 21, 22] involving the first two authors, it was shown as a consequence of one of its main results that $\tau_u^{\mathrm{tr}}(x)$ decays stretched exponentially fast in $|x|$ when $u > u_*$. The analogous result in the super-critical regime, i.e. when $u < u_*$ is related to Theorem 1.1 as already noted above. From [20] one also obtains a stretched-exponential bound on a *sprinkled* version of $\tau_u^{\mathrm{tr}}(\cdot)$ for $u < u_*$ where the disconnection in (1.6) happens at a *strictly* lower intensity $v < u$ (cf. the event $\mathrm{Unique}(L,u,u)$). Our next theorem proves the rapid decay of $\tau_u^{\mathrm{tr}}(\cdot,\cdot)$ and pins down the right rate of decay for $\tau_u^{\mathrm{tr}}(\cdot,\cdot)$ at all non-critical values of u.

Theorem 1.4. *For $u \neq u_*$, with $|\cdot|$ denoting the Euclidean distance on $\mathbb{Z}^d$,*

$$\text{(1.12)}\qquad \textit{when } d \geq 4,\ \sup_{x\in\mathbb{Z}^d} \frac{1}{|x|}\log \tau_u^{\mathrm{tr}}(x) \leq -c(u,d)\ (\in(0,1));$$

$$\text{(1.13)}\qquad \textit{when } d = 3,\ \limsup_{|x|\to\infty} \frac{\log|x|}{|x|}\log \tau_u^{\mathrm{tr}}(x) \leq -\frac{\pi}{3}(\sqrt{u}-\sqrt{u_*})^2.$$

Remark 1.5. Our proof of Theorem 1.4 in §6.2 reveals that, in fact, the bounds (1.12) and (1.13) remain true when the event underlying the definition of $\tau_u^{\mathrm{tr}}(x)$ in (1.6) is enlarged in the same spirit as (1.9) for $u < u_*$. We record it here for potential applications in future work. For $u \in (0, u_*)$,

$$\text{(1.14)}\qquad \text{when } d \geq 4,\ \sup_{x\in\mathbb{Z}^d} \frac{1}{|x|}\log \mathbb{P}\Big[\bigcup_{v\leq u}\{x \overset{\mathcal{V}^v}{\longleftrightarrow} y, x \overset{\mathcal{V}^v}{\not\longleftrightarrow} \infty\}\Big] \leq -c(u,d) < 0;$$

$$\text{(1.15)}\qquad \text{when } d = 3,\ \limsup_{|x|\to\infty} \frac{\log|x|}{|x|}\log \mathbb{P}\Big[\bigcup_{v\leq u}\{x \overset{\mathcal{V}^v}{\longleftrightarrow} y, x \overset{\mathcal{V}^v}{\not\longleftrightarrow} \infty\}\Big] \leq -\frac{\pi}{3}(\sqrt{u}-\sqrt{u_*})^2.$$

Matching lower bounds, exhibiting the exponential decay of $\tau_u^{\mathrm{tr}}(x)$ in (1.12) and effectively allowing to replace the $\limsup$ by a $\lim$ in (1.13), are derived in [25, Theorem 1.1 and Corollary 1.2]. Both upper bounds in Theorem 1.4 will follow from corresponding estimates for a truncated radius observable, whereby the point x in the event defining $\tau_u^{\mathrm{tr}}(0,x)$ in (1.6) is replaced by the inner vertex boundary of the Euclidean ball B_r^2 centered at 0 of radius $r = |x|$, see Theorems 6.3 and 6.1 below. These theorems also supply pertinent bounds on other quantities of interest, such as the two-arms event (see around (6.13) in §6.2 for definitions). Incidentally, in the sub-critical regime $u > u_*$, the tail of the radius observable was derived independently in [35], building on the sharpness result of [20]. We further refer to [24] (building on [19]) for similar results in the context of level-set percolation for the Gaussian free field, and to [15, 14] for related results on the cable system; see also [32, 31] for similar results in the *sub-critical* regime concerning a more general class of Gaussian fields. In fact, one can 'guess' the asymptotic behavior in (1.13) by drawing inspiration from an isomorphism theorem [13, 45] and formally substituting $h = \sqrt{2u}$ on the right-hand side [24, (1.7)]. Bounds like (1.12) in the super-critical regime have been proved recently in [9] for Voronoi percolation.

In comparison with the above works, the results of Theorem 1.4, especially in the super-critical regime, face up to severe additional challenges, which already permeated earlier works [50, 16, 51] valid in perturbative regimes $u \ll 1$ (also excluding $d = 3, 4$ in the case of [50]). In essence, the reasons are the same as those mentioned in [20, Section 1.4]. Above all, a key difficulty is to circumvent rigidity features in the model, manifest for instance in the absence of finite energy. We come back to this below when outlining the proof.

1.2. Proof overview. (1.8) is a straightforward consequence of (1.10) combined with the disconnection estimate from [48] and so let us focus on (1.10) to start with. One natural approach to prove a statement like (1.10) (even for the *weaker* event $\mathrm{Unique}(L, u, u)$) would be to explore the clusters of B_L inside $\mathcal{V}^u \cap B_{2L}$ and inspect whether they connect to an *ambient cluster* every time they come near to it. If there is a uniformly positive conditional probability for the component being explored to connect to the ambient cluster at each of these encounters, then we can get a lower bound like in (1.10).

A new insertion tolerance property. Ensuring a uniformly positive conditional probability on connection events such as above along the exploration process is typically a daunting task for models with *strong* long-range correlations like $\mathcal{V}^u$. In the context of level-set percolation of the Gaussian free field (GFF), a model which has a similar decay of correlations as $\mathcal{V}^u$, the works [19, 24] achieve this by *laying aside* a part of the randomness in the form of a nondegenerate white noise process originating from an orthogonal decomposition of the GFF. Not only does this rest heavily on the Gaussian nature of the problem, the severe degeneracies in the law of $\mathcal{V}^u$, caused by the structural constraints imposed by its complement which consists of bi-infinite random walk trajectories, preclude separation of randomness like in the case of GFF. We refer the reader to the beginning of §1.4 in [20] for a more detailed discussion. Let us remark in passing that such a decomposition of randomness is also available for other dependent models of interest like the Voronoi percolation (see the work [9] mentioned above).

Instead we start by proving a *restricted* insertion tolerance property for $\mathcal{V}^u$ (see Section 3) to the effect that

$$\mathbb{P}[B_r \subset \mathcal{V}^u \,|\, \mathcal{F}_{B_r}(u)] \geq c(d,r) 1_{F_{B_r}(u)} \tag{1.16}$$

where $\mathcal{F}_{B_r}(u)$ is a σ-algebra *containing* $\sigma\big(\mathcal{V}^u_{|(B_{2r})^c}, \phi(\mathcal{V}^u_{|B_{2r}\setminus B_r}), \text{ambient cluster}\big)$ with $\phi : \{0,1\}^{B_{2r}} \mapsto \{0,1\}^{B_{2r}}$, $F_{B_r}(u)$ is a 'good' event measurable w.r.t. $\mathcal{F}_{B_r}(u)$ and $c(d,r) > 0$. The purpose of the functional ϕ is to *hide* some information in $\mathcal{V}^u_{|B_r}$ to facilitate a lower bound like above and needs to be chosen carefully (see (1.20) below). Let us reiterate that the model $\mathcal{V}^u$ does not possess the insertion tolerance property (at least not uniformly), i.e. (1.16) without restriction to any event on the right-hand side and $\phi \equiv \mathrm{id}$, owing to the reasons discussed in the previous paragraph.

Property (1.16) is a significant improvement over Proposition 3.1 in [21] where a *sprinkled* version was proved with $\phi(\mathcal{V}^u_{|B_{2r}\setminus B_r})$ replaced by the *smaller* configuration $\mathcal{V}^{u+\varepsilon}$ for some sprinkling parameter $\varepsilon > 0$. Read the first two paragraphs of Section 3 for more on this.

Monotonization of non-monotone events and renormalization. In order to apply (1.16) repeatedly during an exploration process unless one connects to an ambient cluster, one needs the event $F_B(u)$ (translate of $F_{B_r}(u)$) to occur for each r-box B that intersects it. To show that such a cluster exists with high probability like in (1.10), we can use a renormalization argument in the spirit of the proof of Proposition 1.5 in [19] starting from the bounds (1.3) given by Theorem 1.2 in [20] along with similar bounds for $\mathbb{P}[F_{B_r}(u)]$, which one needs to prove, as the corresponding a-priori estimates (see also Lemma 5.11 in [24]). However, to implement this plan, we also need to *decouple* the simultaneous occurrence of such events (or their complements) when they are separated in space. In the case of $\mathcal{V}^u$, we can decouple events via *monotone* couplings between sequences of random walk excursions $Z_z = (Z_k)_{1\leq k\leq n_z}$; $z \in \mathbb{Z}^d$ from the underlying interlacement point process (see (2.9) and (2.20) in Section 2) and sequences of i.i.d. excursions $\widetilde{Z}_z = (\widetilde{Z}_k)_{1\leq k\leq \tilde{n}_z}$; $z \in \mathbb{Z}^d$ which are *independent* over z (see (2.15) and (2.20)). For non-monotone events, like $\mathrm{Unique}(L,u,v)$ above, such couplings invariably lead the corresponding parameters to move in different directions depending on how they affect the event (see (6.48) in §6.2 and also (5.47) in [24] for an analogous statement in the context of GFF level-set percolation).

This opposite movement of parameters makes it difficult to renormalize non-monotone events *like* $\mathrm{Unique}(L,u,v)$ that *does not* involve any sprinkling, i.e. $u = v$. This difficulty is already visible in [21, Proposition 3.1] where the good event analogous to $F_{B_r}(u)$ involves an event similar to $\mathrm{Unique}(r,u,v)$ (the event $\widehat{F}^2_B$ in [21, (3.8)]) with the vacant sets replaced by their corresponding interlacement sets and, very importantly, $u \neq v$. We refer to this event as $\mathrm{LU}(r,u,v)$ in the sequel (cf. (6.31) in §6.2).

It is not surprising in view of the definition of the event $\mathrm{SLU}_L(u)$ in (1.9), which is bereft of any sprinkling, that we can *not* allow u' and v' to be different in $\mathrm{LU}(r,u',v')$ as it forms a part of the event $F_{B_r}(u)$. To get out of this apparent impasse, we adopt a new perspective to non-monotone events. Let us illustrate this with the example of $\mathrm{LU}(r,u) = \mathrm{LU}(r,u,u)$. As hinted above, we can view the event $\mathrm{LU}(r,u)$ as a function of a (finite) sequence of excursions $Z^u = (Z_k)_{1\leq k\leq N^u}$ (see the first display in (2.20)) *factoring* through its corresponding interlacement set $\mathcal{I}(Z^u) = \cup_{1\leq k\leq N^u}\mathrm{range}(Z_k)$. This enables us to define $\mathrm{LU}(r,u)$ for any sequence of excursions Z which we call $\mathrm{LU}(r,Z)$ with a slight abuse of notation. We refer to Z as a *packet* (of excursions). Now let ζ denote a collection of subsequences of Z^u containing *any* sequence $(Z_k)_{1\leq k\leq n}$ where $N^v \leq n \leq N^u$ for some $v \in (0,u)$ and consider a *strengthening* of the event $\mathrm{LU}(r,u)$ as follows:

$$\mathrm{LU}(r,\zeta) \stackrel{\text{def.}}{=} \bigcap_{Z\in\zeta} \mathrm{LU}(r,Z) \subset \mathrm{LU}(r,u). \tag{1.17}$$

It is clear from (1.17) that the event $\mathrm{LU}(r,\zeta)$ is *monotonically decreasing* in the collection ζ and one can also verify that this event is 'well-behaved' *across* the couplings mentioned previously (revisit (2.15)). This construction can (and will) be applied to other non-monotone events as well (see (6.23) and §6.2 for further details). Indeed, we prove an extended version of (1.16) in Section 3, namely

$$\mathbb{P}[B_r \subset \mathcal{V}(Z)\,|\,\mathcal{F}_{B_r}(Z)] \geq c(d,r)1_{F_{B_r}(Z)} \tag{1.18}$$

for *any* sequence of excursions Z from a *certain* collection ζ containing $(Z_k)_{1\leq k\leq n}$ for *all* $0 \leq n \leq N^u$ (cf. the definition (1.9) of $\mathrm{SLU}_L(u)$) where $\mathcal{V}(Z) = \mathbb{Z}^d\setminus\mathcal{I}(Z)$, $\mathcal{F}_{B_r}(Z)$ is defined analogously to $\mathcal{F}_{B_r}(u)$ with $\mathcal{V}(Z)$ in place of $\mathcal{V}^u$ and $F_{B_r}(Z) \supset F_{B_r}$.

While (1.17) renders the event $\mathrm{LU}(r,u)$ and eventually $F_{B_r}(u)$ amenable to decoupling and thus suitable for being fed into renormalization, it becomes clearly harder to show that the probability of this event is sufficiently large to trigger the process. We prove this estimate in Section 8. After carrying out the renormalization, we obtain

$$\mathbb{P}[\mathscr{G}_L(\zeta)] \geq 1 - C(d,u)e^{-L^{c(d)}} \tag{1.19}$$

(cf. (6.57) in §6.2) where $\mathscr{G}_L(\zeta)$ is the event that the renormalization is successful with $\mathcal{F}_{B_r}(\zeta)$ in place of $\mathcal{F}_{B_r}(u)$ defined as in (1.17).

Exploration and gluing of large clusters. Having ensured in (1.19) the existence of an ambient cluster strewn with copies of the 'insertion-tolerance good' event $F_B(\zeta)$ with very high probability via renormalization, we can now proceed with the exploration of large clusters as sketched in the beginning of this subsection. However, it is still a delicate business to connect the cluster(s) being explored to the ambient one as the σ-algebra $\mathcal{F}_B(Z)$ in (1.18) does not always allow us to *reveal* the points in $\mathcal{V}(Z)$ (where $Z \in \zeta$) *all the way* up to the (outer) boundary of B owing to the partial information provided by the functional ϕ in the vicinity of B (see below (1.16)). The implication is that we can *not* follow the conventional wisdom of trying to connect to the ambient cluster inside a good box B as the exploring cluster arrives at its boundary. This makes the exploration schemes used in some previous works, like the proof of Lemma 5.1 in [24] or Proposition 1.5 in [19], inadequate for our purpose.

In §7.2, we design an elaborate exploration stratagem that is suited to work in this terrain when

$$\phi(\mathcal{V}(Z)_{|B_{2r}\setminus B_r}) = \text{the cluster of } \partial B_{2r} \text{ inside } \mathcal{I}(Z) \cap (B_{2r} \setminus B_r) \tag{1.20}$$

(cf. (3.1) in Section 3). This exploration method is one of the novel contributions of this work. Performing exploration for *each* $Z \in \zeta$ and applying (1.18) repeatedly along each such exploration leads us to the bound

$$\mathbb{P}[(\mathrm{SLU}_L(\zeta))^c] \leq \mathbb{P}[\mathscr{G}_L(\zeta)^c] + e^{-c(u)L^c} + e^{-c(u)\frac{L}{\log L \vee 1} + C\log|\zeta|_*} \tag{1.21}$$

(recall (1.17) and (1.19)) for all $u \in (0, u_*)$ where $|\zeta|_*$ is the $L^\infty(\mathbb{P})$ norm of $|\zeta|$ (the cardinality of ζ) *on the (likely) event* that the number of excursions with 'label' at most u_* (see §2.2) between B_{2L} and B_{CL}^c is close to its expected value and $\mathscr{G}$ is the event that the renormalization from the previous part is successful. We refer the reader to Propositions 6.6 and 6.7 in §6.2 for a precise formulation.

We deduce (1.10) from (1.19) and (1.21) with the choice of ζ as consisting only of excursions $(Z_k)_{1\leq k\leq n}$ for $0 \leq n \leq N^u$ whence $|\zeta|_* \leq CL^{d-2}$ by (2.21) and (2.22) in §2.2 and (A.8).

Bootstrapping and precise asymptotics. Finally we come to the proofs of precise speed and rate for the two-point functions as suggested by Theorem 1.4. We achieve this via *bootstrapping* arguments inspired by the ideas in [24] and [48]. In Sections 4 and 5 of this paper, we develop a *unified* bootstrapping framework for the sub- and the super-critical regime which is interesting on its own and is subsequently applied in Section 6 to prove Theorem 1.4. In the super-critical regime, which is the main focus of the current article, we use (1.21) as the triggering estimate for bootstrapping. In fact, the *monotonized version* $\mathrm{SLU}_L(\zeta)$ of the event $\mathrm{SLU}_L(u)$ is primarily responsible as to why we are able to bootstrap *into* a non-monotone event like in (1.6) unlike in the previous works.

1.3. Organization. In Section 2, we collect some preliminary facts about random interlacements and introduce decompositions of the underlying ranom walk trajectories into excursions as a way to 'localize' the interlacement set. In this connection we record a coupling with independent excursions which will be used in the forthcoming sections for decoupling. The companion Appendix Section A collects some necessary results from the potential theory of simple random walk on $\mathbb{Z}^d$. Section 2 also contains the notation concerning excursions and packets thereof that will be routinely used in later sections.

In Section 3 we give two results related to finite-energy properties of the model, the first of which, Proposition 3.1, corresponds to the precise version of (1.18) and constitutes a first major contribution of this paper. Sections 4 and 5 contain a generic framework for bootstrapping which is eventually used (often several times in a row) to derive each of our bounds. The probabilistic estimates are given in Section 4 while the deterministic part that entails the propagation of an ambient cluster of 'good events' *across* a bootstrapping round is described in Section 5, aided by a topological result proved in Appendix C, which may be of independent interest. As part of the setup for bootstrapping of events in Section 4, we borrow a coarse-graining mechanism from [24] with an extension to the case of Euclidean balls (as required by Theorem 1.4), for which we include a proof in Appendix B. To track the dependence with enough precision, we also rely on certain (surrogate) harmonic averages inspired by ideas from [48], which have been streamlined along the way.

Sections 6 and 7 are the cornerstones of the proof. In Section 6, we provide the proofs of our main results as well as two other related results, namely Theorem 6.1 (proved in §6.1) and Theorem 6.3 (proved in §6.2). These two theorems deal with truncated one-arm probabilities in the sub- and super-critical regimes, respectively, and both results have the same proof structure that employs the same framework developed in the previous two sections. The results in the supercritical phase are proved *conditionally* on Propositions 6.6 and 6.7 and the seed estimates contained in Lemma 6.9. Needless to say that the proofs for the super-critical phase, carried out in §6.2, occupies most of Section 6. Propositions 6.6 and 6.7, which are in the same spirit as (1.21) above, are proved in Section 7. Finally in Section 8, we give the proof of Lemma 6.9.

Our convention regarding constants is as follows. Throughout the article $c, c', C, C', \dots$ etc. denote finite, positive constants which are allowed to change from place to place. All constants may implicitly depend on the dimension $d \geq 3$. Their dependence on other parameters will be made explicit. Numbered constants remain fixed after their first appearance within the text.

2 Setup and localization

We now gather preliminary facts that will be used throughout. In §2.1, we introduce some notation and recall a few useful facts concerning random walks and random interlacements. In §2.2 we discuss excursion decompositions and couplings with independent excursions, see in particular Lemma 2.1, which will fit all our purposes. This leads to threee important notions of vacant sets that are increasingly 'localized' (i.e. stripped of their long-range dependence), namely $\overline{\mathcal{V}}_z^u$, $\widetilde{\mathcal{V}}_z^u$ and $\widetilde{\mathcal{V}}_z^u$, see (2.20), and corresponding systems of excursions, that will play a central role in the sequel.

Let us begin with some preliminary notations which would be used repeatedly in this section as well as in the rest of the article. We consider the lattice $\mathbb{Z}^d, d \geq 3$, equipped with the usual nearest-neighbor graph structure. We denote by $|\cdot|$ and $|\cdot|_\infty$ the ℓ^2 and ℓ^∞-norms on $\mathbb{Z}^d$. We write $U \subset\subset \mathbb{Z}^d$ to denote a finite subset, $U^c = \mathbb{Z}^d \setminus U$ and $|U|$ is the cardinality of U. A *box* is a $\mathbb{Z}^d$-translate of $([0, L) \cap \mathbb{Z})^d$ for some $L \geq 1$. For two sets U, V with $U \subset V \subset \mathbb{Z}^d$, we denote by $\partial_V U := \{x \in U : \exists y \in (V \setminus U) \text{ s.t. } |y - x| = 1\}$ the *inner (vertex) boundary* of U relative to V. We write $\partial_V^{\text{out}} U := \partial(V \setminus U)$ for its *outer boundary* (rel. to V). The relative boundaries defined above correspond to the usual inner and outer vertex boundaries on the induced subgraph of $\mathbb{Z}^d$ with vertex set V (and edges between neighbouring pairs of vertices in V). We omit the subscript V when $V = \mathbb{Z}^d$. We write $\overline{U} := U \cup \partial^{\text{out}} U$ for the *closure* of $U \subset \mathbb{Z}^d$.

2.1. Setup. We consider the continuous-time simple random walk on $\mathbb{Z}^d$ with unit jump rate which we view as a discrete-time simple random walk coupled with an independent sequence of i.i.d. exponential random variables with parameter 1. We write P_x for the law of this process under which $Y = (Y_n)_{n\geq 0}$ has the law of the discrete-time simple random walk on $\mathbb{Z}^d$, starting from x, and $(\zeta_n)_{n\geq 0}$ are i.i.d. exponential variables with parameter 1. The continuous-time random walk $X = (X_t)_{t\geq 0}$ attached to this sequence is defined via

$$X_t = Y_k, \text{ for } t \geq 0 \text{ such that } \sum_{i=0}^{k-1} \zeta_i \leq t < \sum_{i=0}^{k} \zeta_i \tag{2.1}$$

where the empty summation is interpreted as 0. For any positive measure μ on $\mathbb{Z}^d$ we write $P_\mu = \sum_{x\in\mathbb{Z}^d} \mu(x)P_x$. We use E_x for the expectation with respect to P_x and similarly E_μ (although P_μ is not necessarily a probability measure). To a set $K \subset \mathbb{Z}^d$ we associate the stopping times $H_K, \widetilde{H}_K$, where $H_K = \inf\{t \geq 0 : X_t \in K\}$ and $\widetilde{H}_K = \inf\{t \geq \zeta_0 : X_t \in K\}$.

We briefly set up some notation for potential theoretic quantities associated to X and refer to Appendix A for details. We write $g(x, y)$, $x, y \in \mathbb{Z}^d$, for the Green's function associated to X; see (A.1), where $g = g_{\mathbb{Z}^d}$. For a (finite) set $K \subset \mathbb{Z}^d$, we denote by $e_K = e_{K,\mathbb{Z}^d}$ (cf. (A.3)) the equilibrium measure of K, which is supported on ∂K. Its total mass $\text{cap}(K) = \text{cap}_{\mathbb{Z}^d}(K)$ (cf. (A.4)) is the capacity of K, and $\bar{e}_K = \frac{e_K}{\text{cap}(K)}$ is the normalized equilibrium measure.

Throughout this article, we will work with the continuous-time interlacement point process, defined on its canonical space $(\Omega, \mathcal{A}, \mathbb{P})$. We only review a few salient features of the construction and refer to [45] for details. Let $\widehat{W}$ denote the set of doubly-infinite, $\mathbb{Z}^d \times (0, \infty)$-valued sequences, such that the first coordinate of the sequence forms a doubly infinite, nearest-neighbor transient trajectory in $\mathbb{Z}^d$, and the sequence of second coordinates has infinite 'forward' and 'backward' sums, that is

$$\widehat{W} \stackrel{\text{def.}}{=} \left\{ \begin{array}{c} \widehat{w} = (w_n, \zeta_n)_{n\in\mathbb{Z}} \in (\mathbb{Z}^d \times (0, \infty))^{\mathbb{Z}} : \, |w_n - w_{n+1}| = 1, \, n \in \mathbb{Z}, \\ (w_\cdot)^{-1}(\{x\}) \text{ is finite for all } x \in \mathbb{Z}^d, \text{ and } \sum_{n\leq 0} \zeta_n = \sum_{n\geq 0} \zeta_n = \infty \end{array} \right\}, \tag{2.2}$$

endowed with its canonical σ-algebra $\widehat{\mathcal{W}}$, generated by the evaluation maps $(X_t, \sigma_t)_{t\in\mathbb{R}}$, defined (in line with (2.1)) by setting $X_t(\widehat{w}) = w_n$, $\sigma_t(\widehat{w}) = \zeta_n$ with $n \in \mathbb{Z}$ uniquely determined such that $\sum_{i<n} \zeta_i < t \leq \sum_{i\leq n} \zeta_i$. The discrete time-shifts θ_n, $n \in \mathbb{N}$ naturally act on $\widehat{W}$ and we denote by $\widehat{W}^* = \widehat{W}/\sim$, where $\widehat{w} \sim \widehat{w}'$ if $\widehat{w} = \theta_n\widehat{w}'$ for some $n \in \mathbb{Z}$. We write $\pi^* : \widehat{W} \to \widehat{W}^*$ for the corresponding canonical projection and $\widehat{W}^*_K \subset \widehat{W}^*$ is the set of trajectories modulo time-shift whose first coordinate visits $K \subset \mathbb{Z}^d$. The space $\widehat{W}_+$ is defined analogously as in (2.2) but comprising one-sided trajectories $(w_n, \zeta_n)_{n\geq 0}$ instead. The laws P_x, $x \in \mathbb{Z}^d$, are defined on $\widehat{W}_+$.

The measure $\mathbb{P}$ is the probability governing a Poisson point process ω on $\widehat{W}^* \times \mathbb{R}_+$ with intensity measure $\nu(\mathrm{d}\widehat{w}^*)\mathrm{d}u$, where $\mathbb{R}_+$ is the set of all non-negative real numbers, $\mathrm{d}u$ denotes the Lebesgue measure and for all finite $K \subset \mathbb{Z}^d$,

$$\begin{aligned} &1_{\widehat{W}^*_K}\nu = \pi^* \circ \widehat{Q}_K, \text{ where } \widehat{Q}_K \text{ is a finite measure on } \widehat{W}, \, \widehat{Q}_K[X_0 = x] = e_K(x) \\ &\text{and, under } \widehat{Q}_K, \text{ conditionally on } X_0 = x, \, (X_t)_{t>0} \text{ and the right-continuous} \\ &\text{regularization of } (X_{-t})_{t>0} \text{ are independent, and distributed as } X = (X_t)_{t>0} \\ &\text{under } P_x, \text{ resp. } P_x[\,\cdot\,|\widetilde{H}_K = \infty] \text{ after its first jump time } \zeta_0 \end{aligned} \tag{2.3}$$

(extended in the latter case by setting $X_t = x$ for $0 \le t < \zeta_0$). Given any $u \ge 0$, the interlacement set is defined as

$$\mathcal{I}^u = \mathcal{I}^u(\omega) = \bigcup_{(\widehat{w}^*, v) \in \omega,\, v \le u} \text{range}(w^*), \tag{2.4}$$

where $\widehat{w}^* = (w^*, \zeta^*)$, thus explaining the meaning of w^* on the right-hand side of (2.4), and slightly abusing the notation, in writing $(\widehat{w}^*, v) \in \omega$ we implicitly identify the point measure ω with its support, a collection of points in $\widehat{W}^* \times \mathbb{R}_+$.

When only interested in $\mathcal{I}^u$ within a region $\Sigma \subset \mathbb{Z}^d$, it is convenient to project ω onto the effective (Poisson) measure $\mu_{\Sigma,u}(\omega)$ on $\widehat{W}_+$, defined as the push-forward of ω obtained by retaining only the points $(\widehat{w}^*, v) \in \omega$ such that $v \le u$ and $\widehat{w}^* \in \widehat{W}^*_\Sigma$ and mapping them to the onward trajectory ($\in \widehat{W}_+$) upon their first entrance in Σ. By (2.3), it follows that

$$\text{under } \mathbb{P},\ \mu_{\Sigma,u} \text{ is a Poisson process on } \widehat{W}_+ \text{ with intensity } uP_{e_\Sigma}[\,\cdot\,] \tag{2.5}$$

and on account of (2.4), one sees that

$$\mathcal{I}^u \cap \Sigma = \bigcup_{\widehat{w} \in \mu_{\Sigma,u}} \text{range}(w) \cap \Sigma. \tag{2.6}$$

With hopefully obvious notation, we write $\mu_\Sigma(\omega)$ for the pushforward measure on $\widehat{W}_+ \times (0,\infty)$ defined similarly as $\mu_{\Sigma,u}(\omega)$, but which retains the labels u. For a measurable functions $f : \widehat{W}_+ \to \mathbb{R}_+$ we write

$$\langle \mu_{\Sigma,u}, f\rangle \overset{\text{def.}}{=} \int_{\widehat{W}_+} f d\mu_{\Sigma,u} = \sum_{\widehat{w} \in \mu_{\Sigma,u}} f(\widehat{w}) \tag{2.7}$$

for its canonical pairing with $\mu_{\Sigma,u}$ (again identifying the point measure $\mu_{\Sigma,u}$ with the elements in its support in the sum). The sum on the right-hand side of (2.7) is finite $\mathbb{P}$-a.s. when Σ is a finite set on account of (2.5). We will be interested in the case where Σ is the union of well-separated boxes, cf. (4.2) below. The set Σ will typically arise in the context of certain coarse-graining arguments presented in Section 4.

2.2. Localization and couplings. We now set up the framework to decompose trajectories into excursions between a pair of nested sets, denoted as D and U below, which later will allow us to split certain 'local' connectivity events into two parts — one possessing very good decoupling properties and the other involving an atypical number of excursions (see Sections 5 and 6).

Recall from above (2.2) that $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the canonical interlacement space. From here on we assume that for any realization $\omega = \sum_{i \ge 0} \delta_{(\widehat{w}_i^*, u_i)} \in \Omega$, the labels u_i, $i \ge 0$ are pairwise distinct and that $\omega(\widehat{W}^*_K \times \mathbb{R}_+) = \infty$ and $\omega(\widehat{W}^*_K \times [0,u]) < \infty$ for all $u \ge 0$ and $K \subset\subset \mathbb{Z}^d$. We do not incur any loss of generality with these assumptions since these sets have full $\mathbb{P}$-measure.

Now let D, U be finite subsets of $\mathbb{Z}^d$ with $\emptyset \ne D \subset U$ and denote by $W^+_{D,U}$ the set of all excursions between D and $\partial^{\text{out}} U$, i.e. all finite $\mathbb{Z}^d$-valued nearest-neighbor piecewise constant right-continuous trajectories parametrized by $[0,\infty)$ starting in ∂D, ending in $\partial^{\text{out}} U$ and not exiting U in between (see (2.2) and below). Here by 'finite' we mean that the trajectory has only finitely many jumps. The interlacement point measure ω naturally gives rise to a sequence of excursions $W^+_{D,U}$, as follows. For $\widehat{w} \in \text{supp}(\mu_D)$

(see below (2.6) for notation) the infinite transient trajectory $(X_t(\widehat{w}) : t \geq 0)$ induces excursions between D and $\partial^{\text{out}}U$ according to the successive return times R_k and departure times T_k between these sets. The latter are defined recursively as $T_0 = 0$ and

$$R_k = T_{k-1} + H_D \circ \theta_{T_{k-1}}, \qquad T_k = R_k + T_U \circ \theta_{R_k}, \tag{2.8}$$

for $k \geq 1$, where $T_U = H_{U^c}$ and all of $T_k, R_j, T_j, j > k$ are understood to be $= \infty$ whenever $R_k = \infty$ for some $k \geq 0$. Given $\mu_D(\omega) = \sum_{i \geq 0} \delta_{(\widehat{w}_i, u_i)}$, we order all the excursions from D to $\partial^{\text{out}}U$, first by the increasing value of $\{u_i : \widehat{w}_i \in \widehat{W}^+\}$, then by the order of appearance within a given trajectory $\widehat{w}_i \in \widehat{W}^+$. This yields the sequence $Z^{D,U}(\omega) = (Z_k^{D,U}(\omega))_{k \geq 1}$ given by

$$\big(Z_k^{D,U}(\omega)\big)_{k \geq 1} \stackrel{\text{def.}}{=} \big(X^0[R_1, T_1], \dots, X^0[R_{N_{D,U}}, T_{N_{D,U}}],\ X^1[R_1, T_1], \dots\big), \tag{2.9}$$

of $W^+_{D,U}$-valued random variables under $\mathbb{P}$, encoding the successive excursions of ω; here, with hopefully obvious notation $X^i = X(\widehat{w}_i)$, $X^i[t_1, t_2]$ is the trajectory given by $X^i[t_1, t_2](s) = X^i((s+t_1) \wedge t_2)$ for $s \in [0, \infty)$ and $N_{D,U} = N_{D,U}(\widehat{w}_0)$ is the total number of excursions from D to $\partial^{\text{out}}U$ in $\widehat{w}_0$, i.e. $N_{D,U}(\widehat{w}_0) = \sup\{j : T_j(\widehat{w}_0) < \infty\}$. We further define (see (2.5) and (2.7) for notation)

$$N^u_{D,U} = N^u_{D,U}(\omega) = \langle \mu_{D,u}, N_{D,U} \rangle(\omega), \tag{2.10}$$

the total number of excursions from D to $\partial^{\text{out}}U$. By construction of the excursions $Z^{D,U}$ and on account of (2.6), it follows that $\mathcal{I}^u \cap D = \bigcup_{1 \leq k \leq N^u_{D,U}} \text{range}(Z_k) \cap D$.

Now suppose that $D \subset U \subset\subset \mathbb{Z}^d$ are such that $D \subset \check{D}$ and $U \subset \check{U}$. We can then define the successive return and departure times $(R^\ell_k, T^\ell_k)_{k \geq 1}$ between D and $\partial^{\text{out}}U$ as in (2.8) for any excursion $Z^{\check{D},\check{U}}_\ell$. Further since $D \subset \check{D}$ and $U \subset \check{U}$, any excursion $X[R_k, T_k]$ of a (bi-infinite, transient) trajectory X between D and $\partial^{\text{out}}U$ is a segment of a unique excursion $X[\check{R}_{k'}, \check{T}_{k'}]$ of X between $\check{D}$ and $\check{U}$. Therefore, letting $N^\ell_{D,U} = \sup\{j : T^\ell_j < \infty\}$ (cf. the definition of $N_{D,U}(\widehat{w}_0)$ above (2.10)), we have in view of (2.9) that

$$\begin{aligned} &\big(Z_1^{\check{D},\check{U}}[R^1_1, T^1_1], \dots, Z_1^{\check{D},\check{U}}[R^1_{N^1_{D,U}}, T^1_{N^1_{D,U}}], \dots, Z^{\check{D},\check{U}}_{N^u_{D,U}}[R^\ell_{N^1_{D,U}}, T^\ell_{N^1_{D,U}}]\big) \\ &= \big(Z_1^{D,U}, \dots, Z^{D,U}_{N^u_{D,U}}\big) \end{aligned} \tag{2.11}$$

for all $u \geq 0$. We call the sequence of excursions $(Z_\ell^{\check{D},\check{U}}[R^\ell_k, T^\ell_k])_{1 \leq k \leq N^\ell_{D,U}}$ between D and U as the sequence of excursions *induced* by $Z_\ell^{\check{D},\check{U}}$. (2.11) will be useful for comparing events defined using excursions between two different sets of nested boxes (see Section 6.2).

We now proceed to couple the excursions (2.9), which are highly correlated, with a suitable family of i.i.d. excursions between D and $\partial^{\text{out}}U$, in such a way that certain desirable inclusions hold. This is what we refer to as *localization*. The relevant technical result that will be sufficient for all our purposes is stated as Lemma 2.1 below. In applications, we will have to localize systems of excursions as in (2.9) jointly for several choices of sets (D, U) that are sufficiently well spread-out. Thus let $\mathcal{C}$ be a finite set and $\emptyset \neq D_z \subset U_z \subset\subset \mathbb{Z}^d$ be pairs of sets indexed by $z \in \mathcal{C}$ satisfying

$$\overline{U}_z \cap \overline{U}_{z'} = \emptyset, \text{ for all } z \neq z' \in \mathcal{C}, \tag{2.12}$$

where $\overline{U}_z$ denotes the closure of U_z (see the beginning of this section). In practice, we will choose D_z, U_z as in (2.19) in Section 4, with $z \in \mathcal{C} \subset \mathbb{L}$ satisfying (4.1), for which the condition (2.12) plainly holds.

For a given collection $(D_z, U_z : z \in \mathcal{C})$ satisfying (2.12), the desired coupling will be between $\mathbb{P}$ and an auxiliary probability $\widetilde{\mathbb{P}}_{\mathcal{C}}$ governing a collection of independent right-continuous, Poisson counting functions $(n_z(0,t))_{t\geq 0}$, $z \in \mathcal{C}$, with unit intensity, vanishing at 0, along with independent collections of i.i.d. excursions $(\widetilde{Z}_k^{D_z,U_z})_{k\geq 1}$, $z \in \mathcal{C}$, having for each $z \in \mathcal{C}$ the same law as $X_{\cdot\wedge T_{U_z}}$ under $P_{\bar{e}_{D_z}}$ (see below (2.1) for notation). We simply write $\widetilde{\mathbb{P}}_z = \widetilde{\mathbb{P}}_{\{z\}}$ when $\mathcal{C} = \{z\}$ is a singleton. For $m_0 \geq 1$ and $\varepsilon \in (0,1)$, the event (defined on the probability space underlying $\widetilde{\mathbb{P}}_{\mathcal{C}}$)

$$\mathcal{U}_z^{\varepsilon,m_0} \stackrel{\text{def.}}{=} \{n_z(m,(1+\varepsilon)m)) < 2\varepsilon m, (1-\varepsilon)m < n_z(0,m) < (1+\varepsilon)m, \, \forall\, m \geq m_0\}\,, \tag{2.13}$$

for $z \in \mathcal{C}$, will play a central role in the sequel. For later reference, we record the following estimate, valid for any $z \in \mathcal{C}$, $\varepsilon \in (0,1)$ and $m_0 \geq 1$,

$$\widetilde{\mathbb{P}}_{\mathcal{C}}[\mathcal{U}_z^{\varepsilon,m_0}] \geq 1 - C\varepsilon^{-2}e^{-cm_0\varepsilon^2}, \tag{2.14}$$

which follows from standard Poisson tail bounds. For $\mathbb{Q}$ any coupling of $\mathbb{P}$ and $(\widetilde{Z}_k^{D_z,U_z})_{k\geq 1}$ we define the inclusion event $\mathrm{Incl}_z^{\varepsilon,m_0}$ as

$$\begin{aligned} &\Big\{\{\widetilde{Z}_1^{D_z,U_z},\dots,\widetilde{Z}_{\lfloor(1-\varepsilon)m\rfloor}^{D_z,U_z}\} \subset \{Z_1^{D_z,U_z},\dots,Z_{\lfloor(1+3\varepsilon)m\rfloor}^{D_z,U_z}\}, \text{ and} \\ &\{Z_1^{D_z,U_z},\dots,Z_{\lfloor(1-\varepsilon)m\rfloor}^{D_z,U_z}\} \subset \{\widetilde{Z}_1^{D_z,U_z},\dots,\widetilde{Z}_{\lfloor(1+3\varepsilon)m\rfloor}^{D_z,U_z}\} \text{ for all } m \geq m_0\Big\}. \end{aligned} \tag{2.15}$$

In (2.15) and throughout the remainder of this article, with a slight abuse of notation, inclusions involving sets of excursions of the form $\{Z_1,\dots,Z_n\} \subset \{Z'_1,\dots Z'_{n'}\}$ are understood as inclusions between *mutlisets*; that is, the plain inclusion of sets holds and moreover if $Z_k = Z'_{k'}$ then $\mathrm{mult}(Z_k) \leq \mathrm{mult}(Z'_{k'})$, where $\mathrm{mult}(Z_k) = |\{j \in \{1,\dots,n\} : Z_j = Z_k\}|$ is the multiplicity of Z_k in the sequence $(Z_j)_{1\leq j\leq n}$.

Our aim is to devise a coupling $\mathbb{Q}_{\mathcal{C}}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_{\mathcal{C}}$ rendering (2.15) likely for all $z \in \mathcal{C}$ (when m_0 is large). The next lemma asserts in essence that a coupling with this property exists, provided the sets (D_z, U_z) satisfy certain harmonicity requirements for the random walk, see (2.16) below. These requirements are conveniently stated in terms of an auxiliary random variable Y defined as follows. For $x \in \mathbb{Z}^d$, let Q_x be the joint law of two independent simple random walks X^1, X^2 on $\mathbb{Z}^d$, respectively sampled from P_x and from $P_{\bar{e}_D}$, and define

$$Y = \begin{cases} X^1_{H^1_D}, & \text{if } H^1_D \stackrel{\text{def.}}{=} \inf\{t \geq 0 : X^1_t \in A\} < \infty \\ X^2_0, & \text{otherwise.} \end{cases}$$

The following result is a restatement of Lemma 2.1 in [6] adapted to our context and uses the soft local time technique from [34] (see also [48, Section 5]). Although the event $\mathrm{Incl}_z^{\varepsilon,m_0}$ does not include multiplicities in the context of [6], the inclusion (2.17) below continues to hold for this stronger notion of $\mathrm{Incl}_z^{\varepsilon,m_0}$, as follows directly from the soft local time technique, which entails a domination of point measures (that account for multiplicities). We omit the proof.

Lemma 2.1 (Coupling Z and $\widetilde{Z}$). *For any finite collection $D_z \subset U_z \subset\subset \mathbb{Z}^d$, $z \in \mathcal{C}$, satisfying (2.12), there exists a coupling $\mathbb{Q}_{\mathcal{C}}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_{\mathcal{C}}$ with the following property. If, for some $\varepsilon \in (0,1)$,*

$$\big(1 - \tfrac{\varepsilon}{3}\big)\bar{e}_{D_z}(y) \leq Q_x\left[Y = y \,|\, Y \in D_z\right] \leq \big(1 + \tfrac{\varepsilon}{3}\big)\bar{e}_{D_z}(y), \text{ for all } z \in \mathcal{C} \text{ and } x \in \partial^{\mathrm{out}} U_z, \tag{2.16}$$

then in the probability space underlying $\mathbb{Q}_{\mathcal{C}}$,

$$\mathcal{U}_z^{\varepsilon,m_0} \subset \mathrm{Incl}_z^{\varepsilon,m_0}, \text{ for all } z \in \mathcal{C} \text{ and } m_0 \geq 1. \tag{2.17}$$

We will use Lemma 2.1 to switch back and forth between interlacement sets comprising different sets of excursions, Three such interlacement sets, which we now introduce, will play a central role. The following notation will be convenient. If $Z = (Z_k)_{1\le k\le n_Z}$ is a sequence of excursions (i.e. $Z_k \in W^+_{D,U}$ for some $D \subset U$) and $n_Z \in \mathbb{N}$, which we sometimes call a *packet*, we write

$$\mathcal{I}(Z) \stackrel{\text{def.}}{=} \bigcup_{1\le k\le n_Z} \text{range}(Z_k) \text{ and } \mathcal{V}(Z) \stackrel{\text{def.}}{=} \mathbb{Z}^d \setminus \mathcal{I}(Z) \tag{2.18}$$

to denote the *interlacement* and *vacant* set corresponding to the Z, respectively. The number n_Z may or may not be random, i.e. *vary* with ω, and the Z_k's will typically be excursions between boxes which we now introduce,. Given a length scale $L \ge 1$ and a rescaling parameter $K \ge 100$, both integer, we consider boxes $C_z \subset \tilde{C}_z \subset \tilde{D}_z \subset D_z \subset U_z$ attached to points $z \in \mathbb{Z}^d$, where

$$\begin{aligned} &C_z \stackrel{\text{def.}}{=} z + [0, L)^d, \ \ \tilde{C}_z \stackrel{\text{def.}}{=} z + [-L, 2L)^d, \\ &\tilde{D}_z \stackrel{\text{def.}}{=} z + [-2L, 3L)^d, \ \ D_z \stackrel{\text{def.}}{=} z + [-3L, 4L)^d, \text{ and} \\ &U_z \stackrel{\text{def.}}{=} z + [-KL + 1, L + KL - 1)^d \end{aligned} \tag{2.19}$$

(all tacitly viewed as subsets of $\mathbb{Z}^d$). In case we work with more than one scale in a given context we sometimes explicitly refer to the associated length scale L by writing $C_{z,L} = C_z, \tilde{C}_{z,L} = \tilde{C}_z$ etc. We use capital letters L, L_0, N throughout the article to denote various length scales.

For $z \in \mathbb{L} = L\mathbb{Z}^d$ and $u > 0$, abbreviating $N^u_{D,U}$ in (2.10) as $N^u_z = N^u_{z,L}$ when $(D,U) = (D_z, U_z)$ as in (2.19), we introduce in the notation of (2.18),

$$\begin{aligned} &\overline{Z}^u_z = (Z_k^{D_z,U_z})_{1\le k\le N^u_z}, && \overline{\mathcal{V}}^u_z = \mathcal{V}(\overline{Z}^u_z) \, \big(= \mathcal{V}^u \text{ on } D_z \big), \\ &Z^u_z = (Z_k^{D_z,U_z})_{1\le k\le u\,\text{cap}(D_z)}, && \mathcal{V}^u_z = \mathcal{V}(Z^u_z) \\ &\widetilde{Z}^u_z = (\widetilde{Z}_k^{D_z,U_z})_{1\le k\le u\,\text{cap}(D_z)}, && \widetilde{\mathcal{V}}^u_z = \mathcal{V}(\widetilde{Z}^u_z); \end{aligned} \tag{2.20}$$

The quantities $\overline{Z}^u_z, Z^u_z$ and $\overline{\mathcal{V}}^u_z, \mathcal{V}^u_z$ are a priori defined under $\mathbb{P}$ (see (2.9)) and $\widetilde{Z}^u_z, \widetilde{\mathcal{V}}^u_z$ under $\widetilde{\mathbb{P}}_{\mathcal{C}}$, and all quantities in (2.20) are naturally declared under $\mathbb{Q}_{\mathcal{C}}$ any coupling of $(\mathbb{P}, \widetilde{\mathbb{P}}_{\mathcal{C}})$ (such as the one from Lemma 2.1). We seldom add subscripts L, e.g. write $Z^u_{z,L}$ or $\mathcal{V}^u_{z,L}$ instead of Z^u_z or $\mathcal{V}^u_z$, to insist on the scale L used in defining $\mathbb{L}$ and the sets D_z and U_z.

Given a sequence $Z = (Z_k)_{1\le k\le n_Z}$ of excursions and x in $\mathbb{Z}^d$, we denote by $\ell_x(Z)$ the *discrete* occupation time at x relative to the trajectories in Z, i.e. the total number of times x is visited by the (discrete-time) path underlying any trajectory in Z. Note that $\mathcal{V}(Z) = \{x : \ell_x(Z) = 0\}$. We use the simpler notation ℓ^u_x when $Z = \overline{Z}^u_z$ and $x \in D_z$.

We use $\overline{\mathcal{V}}_{\mathbb{L}}$, $\mathcal{V}_{\mathbb{L}}$ and $\widetilde{\mathcal{V}}_{\mathbb{L}}$ to refer collectively to the configurations $\{\overline{\mathcal{V}}^u_z : z \in \mathbb{L}, u > 0\}$, $\{\mathcal{V}^u_z : z \in \mathbb{L}, u > 0\}$ and $\{\widetilde{\mathcal{V}}^u_z : z \in \mathbb{L}, u > 0\}$, respectively, and use $\widehat{\mathcal{V}}_{\mathbb{L}}$ when referring to any one of them. Accordingly, we use $\widehat{\mathcal{V}}^u_z$ to denote either $\overline{\mathcal{V}}^u_z$, $\mathcal{V}^u_z$ or $\widetilde{\mathcal{V}}^u_z$ depending on whether $\widehat{\mathcal{V}}_{\mathbb{L}} = \overline{\mathcal{V}}_{\mathbb{L}}$, $\mathcal{V}_{\mathbb{L}}$ or $\widetilde{\mathcal{V}}_{\mathbb{L}}$.

We now define a very important class of events that will facilitate switching back and forth between the configurations appearing in (2.20). For any $u, v \ge 0$ and $z \in \mathbb{Z}^d$, let

$$\mathcal{F}^{u,v}_z = \mathcal{F}^{u,v}_{z,L} \stackrel{\text{def.}}{=} \begin{cases} \{N^u_z \le v\,\text{cap}(D_z)\}, & \text{if } u \le v \\ \{N^u_z \ge v\,\text{cap}(D_z)\}, & \text{if } u > v \end{cases} \tag{2.21}$$

As with $\mathcal{U}^{\varepsilon,m_0}_z$ in (2.13), the events $\mathcal{F}^{u,v}_z$ in (2.21) have been set up in a way so that they will in practice always be likely. To have an idea of how the afore mentioned switching will operate, notice for instance that, if for some $u > v$, $\varepsilon \in (0,1)$ and $\lfloor m_0(1+3\varepsilon)\rfloor = \lfloor v\,\mathrm{cap}(D_0)\rfloor$, the event $\mathcal{F}_z^{u,v} \cap \mathcal{U}_z^{\varepsilon,m_0}$ occurs under the coupling $\mathbb{Q}_{\mathcal{C}}$ from Lemma 2.1, then on account of (2.20), (2.18), (2.17) and (2.15), one has the chain of inclusions $(\mathcal{V}^u \cap D_z) = \overline{\mathcal{V}}_z^u \subset \mathcal{V}_z^v \subset \widetilde{\mathcal{V}}_z^{v'}$ for all v' such that $\lfloor v'\mathrm{cap}(D_0)\rfloor \leq \lfloor (1+\varepsilon)m_0\rfloor$. Analogous inclusions in the opposite direction can also be obtained.

For later reference, we record the following tail bounds from [48] (see displays (3.18) and (3.22) in the proof of Theorem 3.3, therein), valid for any $\varepsilon \in (0,1)$, if one chooses D_z and U_z as in (2.19), then

(2.22) $$\mathbb{P}\left[(\mathcal{F}_z^{u,v})^c\right] \leq e^{-c(\varepsilon)uN^{d-2}} \text{ for all } u,v>0 \text{ such that } \tfrac{u\vee v}{u\wedge v} \geq 1+\varepsilon \text{ and } K \geq C(\varepsilon).$$

Finally, we shall often work with a certain *noised* version of the configurations $\widehat{\mathcal{V}}_z^u$. To this effect we assume by suitable extension that $\mathbb{Q}_{\mathcal{C}}$ coupling of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_{\mathcal{C}}$ (and a fortiori also $\mathbb{P}$) carries independent i.i.d uniform random variables $\mathsf{U} = \{\mathsf{U}_x : x \in \mathbb{Z}^d\}$. For $\delta \in [0,1)$ and $\mathcal{V} \subset \mathbb{Z}^d$, let $(\mathcal{V})_\delta \subset \mathbb{Z}^d$ denote the set with the occupation variables

(2.23) $$1_{\{x \in (\mathcal{V})_\delta\}} = 1_{\{x\in\mathcal{V}\}} 1_{\{\mathsf{U}_x \geq \delta\}}.$$

See [19] and [20] for similar constructions albeit in different contexts. In plain words, the operator $(\cdot)_\delta$ thins down its argument set $\mathcal{V}$ by an independent percolation configuration of density δ. One immediate but important consequence of this definition is that

(2.24) $$(\mathcal{V})_\delta \text{ is increasing in } \mathcal{V} \text{ and decreasing in } \delta \text{ w.r.t. set inclusion.}$$

3 Insertion tolerance property

In this section we present two results bearing on the insertion tolerance property of the vacant set $\mathcal{V}(Z)$ for certain *suitable* sequences of excursions Z (recall around (2.18) for notation and (2.9) for the relevant notion of excursions). As already noted in the discussion leading to (1.16), a naive insertion tolerance property does not hold for such vacant sets owing to their structural rigidity. Consequently, the results we are alluding to can at best be valid *on* certain special 'good' events that occur with high probability. The two results in this section pertain to two such good events and are relevant when the underlying sequence Z is either sufficiently *l*arge (i.e. comprising sufficiently many excursions) or sufficiently *s*mall. We refer to Remark 3.4 below regarding the necessity to consider the case of small sequences (roughly corresponding to the case $u \ll 1$) separately.

Our first result, Proposition 3.1 below, is the main result of this section. It is reminiscent of Proposition 3.1 in [21] where the authors prove a *sprinkled* insertion tolerance property for $\mathcal{V}^u$. More precisely, in [21, Proposition 3.1], it is shown that a box B can be opened in $\mathcal{V}^{u-\varepsilon}$ with probability $c = c(\mathrm{rad}(B),\varepsilon) > 0$ conditionally on $\mathcal{V}^u$ (everywhere) and $\mathcal{V}^{u-\varepsilon}$ outside a *strictly larger* box $\widehat{B}$ provided some good event $\widetilde{F}_B$ occurs. In order to remove the sprinkling inherent in the main result of [21], which is one of the main objectives of the present paper, one would ideally want such a bound to hold with $\varepsilon = 0$ conditionally on $\mathcal{V}^u$ outside B *itself* rather than a larger box $\widehat{B}$. We accomplish this goal partly in Proposition 3.1 in the sense that we can retain partial information on $\mathcal{V}^u \cap (\widehat{B} \setminus B)$ in the conditioning. However, this information — captured in terms of boundary clusters of the interlacement set in the annulus $(\widehat{B} \setminus B)$ (see (3.1) below) — turns out to be sufficient for gluing large clusters in the proof of Proposition 6.6 in Section 7.

We now introduce some terminology and a few events that are necessary to state Proposition 3.1. Their joint occurrence will constitute the good event mentioned above.

A (nearest-neighbor) *path* γ in $\mathbb{Z}^d$ is a map $\gamma : \{0,\dots,k\} \to \mathbb{Z}^d$ for some integer $k \geq 0$ such that $|\gamma(i+1)-\gamma(i)| = 1$ for all $0 \leq i < k$. A *connected* set $U \subset \mathbb{Z}^d$ is a set such that any points $x,y \in U$

can be joined by a path whose range is contained in U. We call a *component of* U a maximal connected subset of U, and omit the attribute "of U" when $U = \mathbb{Z}^d$. We use the words *cluster* and *component* interchangeably in this article and their particular choice in a context is purely informed by linguistic considerations.

Now given any sequence $Z = (Z_j)_{1\le j\le n_Z}$ of excursions (see above (2.18) for the precise definition; the reference sets D and U are arbitrary at this point) and $y \in \mathbb{L}_0 = L_0\mathbb{Z}^d$ for some $L_0 \ge 10$, employing the notation from (2.19) we let

(3.1) $$\mathscr{C}_{\partial D_{y,L_0}}(Z) \stackrel{\text{def.}}{=} \text{the union of components of points in } \partial D_{y,L_0} \text{ inside } \mathcal{I}(Z) \cap (D_{y,L_0} \setminus C_{y,L_0}).$$

Using this set, we define a 'local uniqueness' event in a smaller annulus as

(3.2) $$\widetilde{\mathrm{LU}}_{y,L_0}(Z) = \bigcap_{x,x' \in (\tilde{D}_{y,L_0}\setminus \tilde{C}_{y,L_0}) \cap \mathscr{C}_{\partial D_{y,L_0}}(Z)} \{x \xleftrightarrow{\mathscr{C}_{\partial D_{y,L_0}}(Z)\setminus \partial D_{y,L_0}} x'\}.$$

In words, $\widetilde{\mathrm{LU}}_{y,L_0}(Z)$ is the event that the set $\mathscr{C}_{\partial D_{y,L_0}}(Z)\setminus\partial D_{y,L_0}$ has at most one component intersecting the annulus $\tilde{D}_{y,L_0}\setminus\tilde{C}_{y,L_0}$. Recalling the discrete occupation times $(\ell_x(Z))_{x\in\mathbb{Z}^d}$ from the paragraph below (2.20), we let

(3.3) $$\mathrm{O}_{y,L_0}(Z) \stackrel{\text{def.}}{=} \bigcap_{x\in\partial D_{y,L_0}} \{\ell_x(Z) \le L_0\}.$$

Next, for any (finite) $J \subset \mathbb{N}^*$ and $z \in N\mathbb{Z}^d$ where $N \ge 1$ is integer, using the notation from (2.9) and (2.19) we let $Z_J^{D_{z,N},U_{z,N}}$ denote the sequence of excursions $\big(Z_j^{D_{z,N},U_{z,N}}\big)_{j\in J}$. Also let $\delta \in (0,\frac{1}{2})$ denote a noise parameter as in (2.23) and $u' \ge u \in (0,\infty)$. Using this data, we can define the σ-algebra that we will use for conditioning in Proposition 3.1 (cf. the discussion preceding (3.1) above). With $Z_J = Z_J^{D_{z,N},U_{z,N}}$ and $y \in \mathbb{L}_0$ below,

(3.4) $$\mathcal{F}_{y,L_0}(Z_J,\delta,u,u') \stackrel{\text{def.}}{=} \sigma\big((\mathcal{V}^u)_\delta, (\mathcal{V}^{u'})_{2\delta}, N^u_{z,N}, \mathcal{I}(Z_J)_{|(\mathring{D}_{y,L_0})^c}, \mathscr{C}_{\partial D_{y,L_0}}(Z_J), \{\ell^u_x : x \in (\mathring{D}_{y,L_0})^c\}\big),$$

where $\mathring{D}_{y,L_0} \stackrel{\text{def.}}{=} D_{y,L_0}\setminus\partial D_{y,L_0}$ and $(\mathring{D}_{y,L_0})^c = D^c_{y,L_0}\cup\partial D_{y,L_0}$ denotes its complement, and $\sigma(\cdot)$ denotes the σ-algebra generated by a set of random variables. In (3.4), as in the rest of the article, we tacitly identify $\mathcal{I}(Z_J)$ (or any other subset of $\mathbb{Z}^d$, such as $\mathscr{C}_{\partial D_{y,L_0}}(Z_J)$ for instance) with its occupation function, i.e. $x \mapsto 1_{\{x\in\mathcal{I}(Z)\}}$, $x \in \mathbb{Z}^d$. Observe that the 'good' events $\widetilde{\mathrm{LU}}_{y,L_0}(Z_J^{D_{z,N},U_{z,N}})$ and $\mathrm{O}_{y,L_0}(\overline{Z}^u_{z,N})$ (see (2.20) regarding $\overline{Z}^u_{z,N}$) are both measurable relative to $\mathcal{F}_{y,L_0}(Z_J^{D_{z,N},U_{z,N}},\delta,u,u')$, as is the event $\{J\subset[1,N^u_{z,N}]\}$ *when* J is measurable relative to $N^u_{z,N}$ (like when it is *deterministic*). We are now ready to state the main result of this section, which entails a certain form of insertion tolerance.

Proposition 3.1. *Let $L_0 \ge 100$ and $\delta \in (0,\frac{1}{2})$. There exists $c = c(\delta,L_0) \in (0,1)$ such that for all $u' \ge u \in (0,\infty)$, $K \ge 100$, $z \in N\mathbb{Z}^d$ for some $N \ge 10^3L_0$, $y \in L_0\mathbb{Z}^d$ such that $D_{y,L_0} \subset D_{z,N}$ and any $J \subset \mathbb{N}^*$ measurable relative to $N^u_{z,N}$, abbreviating $Z_J = Z_J^{D_{z,N},U_{z,N}}$ we have*

(3.5) $$\mathbb{P}\text{-a.s.},\ \mathbb{P}\left[C_{y,L_0} \subset \mathcal{V}(Z_J)\,\middle|\,\mathcal{F}_{y,L_0}(Z_J,\delta,u,u')\right] \ge c\,1_G,$$

with the "good" event $G = \widetilde{\mathrm{LU}}_{y,L_0}(Z_J) \cap \mathrm{O}_{y,L_0}(\overline{Z}^u_{z,N}) \cap \{J\subset[1,N^u_{z,N}]\}$.

Remark 3.2 (Variations of (3.5)). 1) Although insufficient for our purposes, the choice $J = [1, N^u_{z,N}]$ is measurable w.r.t. $N^u_{z,N}$ and thus perfectly valid, whence $Z_J = \overline{Z}^u_{z,N}$ in the notation from below (2.20), and (3.5) yields a conditional lower bound on the probability that $C_{y,L_0} \subset \mathcal{V}^u$.

2) The presence of $(\mathcal{V}^{u'})_{2\delta}$ in the conditioning (see (3.4)) is tailored to our purposes, but (3.5) remains meaningful in its absence and there is flexibility in its choice. For instance $(\mathcal{V}^{u'})_{2\delta}$ could be replaced by conditioning on finitely many vacant configurations at levels different from u and appropriate noise parameters. This follows by minor adaptations to the proof given below.

Proof. We first introduce a certain sigma-algebra which corresponds to the 'right' conditioning outside $\mathring{D}_{y,L_0}$, see $\widehat{\mathcal{F}}$ in (3.7) below. For a discrete-time nearest-neighbor bi-infinitely transient trajectory $w = (w(n))_{n\in\mathbb{Z}}$ in $\mathbb{Z}^d$ and a pair of sets $\emptyset \neq D \subset U \subset\subset \mathbb{Z}^d$, let $-\infty < R_1(w) < T_1(w) < R_2(w) < \dots$ denote the successive return and departure times of w between D and $\partial^{\text{out}}U$, in an analogous way as in (2.8) but with $\mathbb{Z}$ in place of $[0,\infty)$ as the underlying parameter space. Note that possibly $R_1(w) = \infty$. Let $w^-_{D,U}$ denote the sequence of segments $(w(-\infty, R_1 - 1], w[T_1, R_2 - 1], \dots)$ where $w[n_1, n_2](n) = w(n_1 + n)$ for $0 \leq n \leq n_2 - n_1$, $w(-\infty, n_2](n) = w(n_2 - n)$ for $-\infty < n \leq n_2$ and $w(-\infty, \infty]$ $(= w(-\infty, \infty))$ is understood as $(w(n))_{n\in\mathbb{Z}}$. In words, $w^-_{D,U}$ is the sequence of segments of w obtained after deleting all its excursions between D and $\partial^{\text{out}}U$ *minus* their endpoints, i.e. the paths $w[R_k, T_k - 1]$ for $k \geq 1$. Since we forget the endpoints $R_k, T_k - 1$ in defining the segments $w[R_k, T_k - 1]$, it is clear from this definition that $w^-_{D,U} = \tilde{w}^-_{D,U}$ if $\tilde{w} = \theta_n w$ for some $n \in \mathbb{Z}$ and thus $w^-_{D,U}$ is well-defined as a function of the equivalence class w^* of any trajectory w under (discrete) time-shift (cf. the definition of the equivalence class $\widehat{w}^*$ below (2.2)).

Our case of interest is

$$D = U = \mathring{D}_{y,L_0} \; (= D_{y,L_0} \setminus \partial D_{y,L_0}), \tag{3.6}$$

in which we use the shorthand notation w^-_y for $w^-_{D,U}$. Using this, we introduce the point process (see above (2.3) regarding the process ω)

$$\omega^-_y = \sum_{(\widehat{w}^*, v)\in\omega} \delta_{(w^-_y, v)}$$

where, w^-_y is uniquely defined as a function of $\widehat{w}^*$ in view of our previous observation. Then, abbreviating $\mathcal{I} = \mathcal{I}(Z_J)$, $\mathcal{V} = \mathcal{V}(Z_J)$ and $D = \mathring{D}_{y,L_0}$, as above, we claim that

$$\begin{aligned} &\big((\mathcal{V}^u)_{\delta|D^c},\, (\mathcal{V}^{u'})_{2\delta|D^c},\, N^u_{z,N},\, \mathcal{I}_{|D^c},\, \{\ell^u_x : x \in D^c\}\big) \text{ are} \\ &\text{measurable relative to } \widehat{\mathcal{F}} \overset{\text{def.}}{=} \sigma\big(\omega^-_y, \{\mathsf{U}_x : x \in D^c\}\big) \end{aligned} \tag{3.7}$$

(see (3.4) to compare with the definition of $\mathcal{F}_{y,L_0}(Z_J, \delta, u, u')$). We now explain (3.7). Clearly, the discrete occupation times $\{\ell^u_x : x \in D^c\}$ are measurable relative to ω^-_y. Since $D_{y,L_0} \subset D_{z,N}$, $K \geq 100$ and $N \geq 10^3 L_0$ by our assumptions, it follows from the definitions of the underlying boxes in (2.19) that $U = D = \mathring{D}_{y,L_0} \subset U_{z,N}$ and that no excursion between D and $\partial^{\text{out}}U$ can intersect $U^c_{z,N}$. This implies that for any $v > 0$, the (finite, possibly empty) set of labels $\{v_1 < \ldots < v_k\} \subset (0, v)$ of any $(\widehat{w}, v') \in \mu_{D_{z,N}}(\omega)$ with label at most v (note that any two such labels are distinct with probability 1) is measurable relative to ω^-_y. In particular, the random variable $N^u_{z,N}$ and for any (random) $J \subset\subset \mathbb{N}^*$ measurable w.r.t. $N^u_{z,N}$, the set of labels $\{v_j : j \in J\}$ are both measurable relative to ω^-_y. Since ω^-_y retains all pieces of trajectories in the support of ω outside D, it is also clear from the respective definitions of the sets

$\mathcal{I}$ and $(\mathcal{V}^v)_{\delta'}$ that $\mathcal{I}_{|D^c}$ is measurable relative to ω_y^- whereas $(\mathcal{V}^v)_{\delta'|D^c}$ is measurable with respect to $\big(\omega_y^-, \{\mathsf{U}_x : x \in D^c\}\big)$ for any $v \geq 0$ and $\delta' \in [0,1)$. Overall, (3.7) follows.

We denote by $\Gamma^v(\omega_y^-)$ the *multiset* of all pairs of points $(w(R_k(w)-1), w(T_k(w))) \in (\partial D_{y,L_0})^2; k \geq 1$ such that $(\widehat{w}, v') \in \omega$ for some $v' \leq v$, i.e. we take into account the number of times each such pair appears over all pairs $(\widehat{w}, v')$. It is clear that $\Gamma^v(\omega_y^-)$ is indeed a (measurable) function of ω_y^-. The (multi-)set $\Gamma^v(\omega_y^-)$ may well be empty, in case none of the relevant trajectories $\widehat{w}$ visits D. In analogous manner, we define $\Gamma_J(\omega_y^-)$ for any (finite) $J \subset \mathbb{N}^*$ measurable w.r.t. $N_{z,N}$ by restricting v' to lie in the set $\{v_j : j \in J\}$, where v_j are the (ordered) labels of trajectories in the support of $\mu_{D_{z,N}}(\omega)$. Notice that $\Gamma_J(\omega_y^-)$ is a measurable *function* of ω_y^- in view of our discussion in the previous paragraph.

With $\widehat{\mathcal{F}} = \sigma(\omega_y^-, \{\mathsf{U}_x : x \in D^c\})$ from (3.7) in mind, due to the Markov property of random walks as well as the definition of the excursions $Z_j^{D_{z,N},U_{z,N}}$ (revisit displays (2.8)–(2.9) in Section 2.2), we have the following precise description of the law of excursions of trajectories underlying $\mu_{D_{z,N}}(\omega)$ as well as the variables $\{\mathsf{U}_x : x \in D\}$ conditionally on $\widehat{\mathcal{F}}$:

(3.8) the excursions $\{\boldsymbol{\gamma}_{x,x'} : (x,x') \in \Gamma^{u'}(\omega_y^-)\}$ under the (regular) conditional law $\mathbb{P}\big[\,\cdot\,|\,\widehat{\mathcal{F}}\big]$ are distributed as independent random walk bridges where $\boldsymbol{\gamma}_{x,x'}$ is conditioned to start at x, end at x' and lie inside D except at the final point independently of the i.i.d. uniform random variables $\{\mathsf{U}_x : x \in D\}$.

In view of (3.7) and the definition of $\mathcal{F} = \mathcal{F}_{y,L_0}(Z_J, \delta, u, u')$ from (3.4), we have

$$\mathcal{F} = \sigma(\widehat{\mathcal{F}}, (\mathcal{V}^u)_\delta \cap D, (\mathcal{V}^{u'})_{2\delta} \cap D, \mathscr{C}_{\partial D_{y,L_0}}(Z_J)).$$

Since the last three random objects in the list take values in a finite set, given any realization ζ_y of $(\omega_y^-, \{\mathsf{U}_x : x \in D^c\})$ as well as possible realizations η, η' and ξ of $(\mathcal{V}^u)_\delta \cap D$, $(\mathcal{V}^{u'})_{2\delta} \cap D$ and $\mathscr{C}_{\partial D_{y,L_0}} (= \mathscr{C}_{\partial D_{y,L_0}}(Z_J))$ respectively, we can write a regular conditional law $\mathbb{P}\big[\,\cdot\,|\,\mathcal{F}\big](\zeta_y, \eta, \eta', \xi)$ as follows:

$$\mathbb{P}\big[\,\cdot\,|\,\mathcal{F}\big](\zeta_y, \eta, \eta', \xi) = \mathbb{Q}_{\zeta_y}\big[\,\cdot\,|\,\mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi)\big]$$

where $\frac{0}{0}$ is interpreted as 0, $\mathbb{Q}_{\zeta_y}$ is the conditional law $\mathbb{P}\big[\,\cdot\,|\,\widehat{\mathcal{F}}\big]$ described in (3.8) evaluated at $\zeta_y \stackrel{\text{def.}}{=} (\omega_y^-, \{\mathsf{U}_x : x \in D^c\})$, and

$$\mathsf{V}(\eta,\eta') = \{(\mathcal{V}^u)_\delta \cap D = \eta, (\mathcal{V}^{u'})_{2\delta} \cap D = \eta'\}, \quad \mathsf{C}(\xi) = \{\mathscr{C}_{\partial D_{y,L_0}} = \xi\}.$$

With this, (3.5) follows if we can prove,

$$\mathbb{Q}_{\zeta_y}\big[C_{y,L_0} \subset \mathcal{V}\,|\,\mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi)\big] \geq c(\delta, L_0)\ (> 0) \tag{3.9}$$

for each (η, η', ξ), almost every ζ_y such that $\mathbb{Q}_{\zeta_y}\big[\mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi)\big] > 0$ and $(\zeta_y, \eta, \eta', \xi)$ belonging to the event G. In fact, we will prove an even stronger inequality than (3.9). Given any collection of excursions $\{\gamma_{x,x'} : (x,x') \in \Gamma^{u'}(\omega_y^-) \setminus \Gamma_J(\omega_y^-)\}$, where each $\gamma_{x,x'}$ is an excursion between D and $\partial^{\text{out}}U$ (with D, U as in (3.6)) starting and ending at x and x' respectively, we show that

$$\mathbb{Q}_{\tilde{\zeta}_y}\big[C_{y,L_0} \subset \mathcal{V}\,|\,\mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi)\big] \geq c(\delta, L_0) \tag{3.10}$$

for almost every $\tilde{\zeta}_y \coloneqq \big(\zeta_y, \big\{\boldsymbol{\gamma}_{x,x'} : (x,x') \in \Gamma^{u'}(\omega_y^-)\setminus\Gamma_J(\omega_y^-)\big\}\big)$ and all (η, η', ξ) such that $(\zeta_y, \eta, \eta', \xi)$ belongs to the event G and $\mathbb{Q}_{\tilde{\zeta}_y}\big[\mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi)\big] > 0$. The bound (3.9) follows immediately from (3.10) by integrating the latter over all realizations of $\{\boldsymbol{\gamma}_{x,x'} : (x,x') \in \Gamma^{u'}(\omega_y^-) \setminus \Gamma_J(\omega_y^-)\}$. The remainder of the proof is devoted to proving (3.10).

Since ζ_y (and hence $\tilde{\zeta}_y$) satisfies the event $\{J \subset [1, N^u_{z,N}]\}$ and $u \leq u'$, the events $\mathsf{V}(\eta, \eta')$ and $\mathsf{C}(\xi)$ are measurable relative to the excursions $\{\boldsymbol{\gamma}_{x,x'} : (x,x') \in \Gamma_J(\omega_y^-)\}$ and the noise variables $(\mathsf{U}_x)_{x\in D}$ given $\tilde{\zeta}_y$. By the definition of conditional probability, then there exist choices of excursions $\{\gamma_{x,x'} : (x,x') \in \Gamma_J(\omega_y^-)\}$ as well as occupation configurations $(b_x)_{x\in D}, (b'_x)_{x\in D} \in \{0,1\}^D$ for almost all realizations of $\tilde{\zeta}_y$ with $\mathbb{Q}_{\tilde{\zeta}_y}[\mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi)] > 0$ (henceforth tacitly assumed) such that

$$\bigcap_{(x,x')\in\Gamma_J(\omega_y^-)} \{\boldsymbol{\gamma}_{x,x'} = \gamma_{x,x'}\} \cap \bigcap_{x\in D} \big(\{1_{\{\mathsf{U}_x \geq \delta\}} = b_x\} \cap \{1_{\{\mathsf{U}_x \geq 2\delta\}} = b'_x\}\big) \subset \mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi)$$

and the left-hand side has positive probability under $\mathbb{Q}_{\tilde{\zeta}_y}$. We can specify the values of b_x and b'_x, which are informed only by η, η', using the properties of the noised sets in (2.23), whereby the occupied (i.e. non-vacant) vertices can be explained by triggering suitable noise variables. More precisely, letting

$$\mathcal{U}(\eta,\eta') \overset{\text{def.}}{=} \bigcap_{x\in D\setminus\eta} \{\mathsf{U}_x < \delta\} \cap \bigcap_{x\in\eta\setminus\eta'} \{\mathsf{U}_x \in [\delta, 2\delta)\} \cap \bigcap_{x\in\eta'} \{\mathsf{U}_x \geq 2\delta\}, \tag{3.11}$$

we have

$$\bigcap_{(x,x')\in\Gamma_J(\omega_y^-)} \{\boldsymbol{\gamma}_{x,x'} = \gamma_{x,x'}\} \cap \mathcal{U}(\eta,\eta') \subset \mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi)$$

(given $\tilde{\zeta}_y$). In fact, as we now explain, the same argument yields that for *any* choice of excursions $\{\bar{\gamma}_{x,x'} : (x,x') \in \Gamma_J(\omega_y^-)\}$ with

$$\begin{aligned} &\bigcup_{(x,x')\in\Gamma_J(\omega_y^-)} \text{range}(\bar{\gamma}_{x,x'}) \subset \bigcup_{(x,x')\in\Gamma_J(\omega_y^-)} \text{range}(\gamma_{x,x'}) \text{ and} \\ &\bigcap_{(x,x')\in\Gamma_J(\omega_y^-)} \{\boldsymbol{\gamma}_{x,x'} = \bar{\gamma}_{x,x'}\} \subset \mathsf{C}(\xi), \end{aligned} \tag{3.12}$$

one has

$$\bigcap_{(x,x')\in\Gamma_J(\omega_y^-)} \{\boldsymbol{\gamma}_{x,x'} = \bar{\gamma}_{x,x'}\} \cap \mathcal{U}(\eta,\eta') \subset \mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi). \tag{3.13}$$

To see (3.13), simply note that the inclusion in (3.12) entails that the choice of $\bar{\gamma}_{x,x'}$ over $\gamma_{x,x'}$ can only possibly increase the vacant sets in the event $\mathsf{V}(\eta,\eta')$, but the occurrence of $\mathcal{U}(\eta,\eta')$ precludes this on account of (2.23).

Our goal in the rest of the proof to reroute the trajectories $\gamma_{x,x'}$ into suitably chosen $\bar{\gamma}_{x,x'}$ so that (3.12) holds and $C_{y,L_0} \subset \mathcal{V}$. To this end, let us call $\gamma_{x,x'}$

crossing if it intersects C_{y,L_0}.

If none of the excursions in $\{\gamma_{x,x'} : (x,x') \in \Gamma_J(\omega_y^-)\}$ (as appearing in the display above (3.11)) is crossing, we have

$$\bigcap_{(x,x')\in\Gamma_J(\omega_y^-)} \{\boldsymbol{\gamma}_{x,x'} = \gamma_{x,x'}\} \subset \{C_{y,L_0} \subset \mathcal{V}\}$$

given ζ_y. Together with (3.12) and (3.13), this implies (3.10) in this case.

So let us suppose that at least one $\gamma_{x,x'}$ with $(x,x') \in \Gamma_J(\omega_y^-)$ is crossing. Our strategy is to 'reroute' the crossing excursions into non-crossing ones, thereby vacating C_{y,L_0}, all the while explaining the events $\mathsf{V}(\eta,\eta')$, $\mathsf{C}(\xi)$ as well as the configuration $\tilde{\zeta}_y$.

Recall from (3.1) that ξ, the realisation of $\mathscr{C}_{\partial D_{y,L_0}}$, is a disjoint union of connected subsets of $D_{y,L_0} \setminus C_{y,L_0}$ each of which intersects $\partial D_{y,L_0}$. Since $(\tilde{\zeta}_y, \eta, \eta', \xi)$ satisfies $\widetilde{\mathrm{LU}}_{y,L_0}$, it follows from (3.2) that there exists exactly one component of $\xi \setminus \partial D_{y,L_0}$ intersecting $\tilde{D}_{y,L_0}$, say $\mathcal{C}(\xi)$, which also contains $\xi \cap (\tilde{D}_{y,L_0} \setminus \tilde{C}_{y,L_0})$. But because any crossing excursion $\gamma_{x,x'}$ with $(x,x') \in \Gamma_J(\omega_y^-)$ belongs to a component of $\mathscr{C}_{\partial D_{y,L_0}}$ that also intersects $\tilde{D}_{y,L_0}$, it follows that all such crossing excursions are part of the same component, namely $\mathcal{C}(\xi)$. Based on this observation and using that the event $\widetilde{\mathrm{LU}}_{y,L_0}$ from (3.2) is in force, we can for each crossing excursion $\gamma_{x,x'}$ with $(x,x') \in \Gamma_J(\omega_y^-)$ find a non-crossing excursion $\bar{\gamma}_{x,x'}$ having the same endpoints as $\gamma_{x,x'}$ and such that

$$|\bar{\gamma}_{x,x'}| \leq (10L_0)^d \text{ and } \mathcal{C}(\xi) = \mathrm{range}(\bar{\gamma}_{x,x'}) \cap D \tag{3.14}$$

(simply by making $\bar{\gamma}_{x,x'}$ exhaust all of $\mathcal{C}(\xi)$ while reaching the desired endpoints. Any other component of $\xi \setminus \partial D_{y,L_0}$, i.e. any component $\mathcal{C}$ that does not intersect $\tilde{D}_{y,L_0}$ necessarily satisfies

$$\mathcal{C} \subset (D \setminus \tilde{D}_{y,L_0}) \cap \bigcup_{(x,x') \in \Gamma_J(\omega_y^-) : \gamma_{x,x'} \text{ is non-crossing}} \mathrm{range}(\gamma_{x,x'}).$$

For convenience, let us define $\bar{\gamma}_{x,x'} = \gamma_{x,x'}$ when $(x,x') \in \Gamma_J(\omega_y^-)$ is non-crossing. Putting the previous observation together with the second item in (3.14), and using the fact that all excursions $\bar{\gamma}_{x,x'}$ are non-crossing, it follows that (given $\tilde{\zeta}_y$)

$$\begin{aligned} &\bigcap_{(x,x') \in \Gamma_J(\omega_y^-)} \{\gamma_{x,x'} = \bar{\gamma}_{x,x'}\} \subset \big(\mathsf{C}(\xi) \cap \{C_{y,L_0} \subset \mathcal{V}\}\big), \text{ and} \\ &\bigcup_{(x,x') \in \Gamma_J(\omega_y^-)} \mathrm{range}(\bar{\gamma}_{x,x'}) \subset \bigcup_{(x,x') \in \Gamma_J(\omega_y^-)} \mathrm{range}(\gamma_{x,x'}). \end{aligned} \tag{3.15}$$

We can now deduce

$$\begin{aligned} \mathbb{Q}_{\tilde{\zeta}_y}\big[\{C_{y,L_0} \subset \mathcal{V}\} \cap \mathsf{V}(\eta,\eta') \cap \mathsf{C}(\xi)\big] &\overset{(3.12)+(3.13)+(3.15)}{\geq} \mathbb{Q}_{\tilde{\zeta}_y}\Big[\bigcap_{(x,x') \in \Gamma_J(\omega_y^-)} \{\gamma_{x,x'} = \bar{\gamma}_{x,x'}\} \cap \mathcal{U}(\eta,\eta')\Big] \\ &\overset{(3.8)+(3.11)}{\geq} \mathbb{Q}_{\tilde{\zeta}_y}\Big[\bigcap_{(x,x') \in \Gamma_J(\omega_y^-)} \{\gamma_{x,x'} = \bar{\gamma}_{x,x'}\}\Big] (\delta(1-2\delta))^{|D_{y,L_0}|} \overset{(3.8)}{\geq} (2d)^{-CL_0^{2d}} (\delta(1-2\delta))^{CL_0^d}, \end{aligned}$$

where the last step also uses the bound from (3.14) and the fact that $|\Gamma_J(\omega_y^-)| \leq |\Gamma^u(\omega_y^-)| \leq CL_0^{d-1} \cdot L_0$, since ζ_y satisfies $\mathrm{O}_{y,L_0}(\overline{Z}^u_{z,N})$ (recall (3.3)) and $\{J \subset [1, N^u_{z,N}]\}$. Overall, this yields (3.10). □

The second result, i.e. Lemma 3.3, is a synthesis of some of the results in [16] and is instrumental in the proof of Proposition 6.7, which deals with small packets Z of excursions; see also Remark 3.4 below. As we will see, this result allows us to condition on the configuration immediately outside a box while opening up its vertices unlike the partial conditioning in the case of Proposition 3.1. But this comes at the (serious) cost of holding on a good event which is only likely when the underlying interlacement set is very small.

In the sequel we let $\square(0,L)$ denote the set of all points in $C_{0,L}$ (see (2.19) for notation) such that at least two of their coordinates lie in the set $\{0,1,2,L-3,L-2,L-1\}$ and let $\square(z,L) = z + \square(0,L)$ for any $z \in \mathbb{Z}^d$. Now for any $y \in \mathbb{L}_0 = L_0\mathbb{Z}^d$ and a sequence of excursions $Z = (Z_j)_{1\le j\le n_Z}$, let us consider the event

$$\text{(3.16)} \qquad \mathrm{W}^-_{y,L_0}(Z) \overset{\text{def.}}{=} \left\{ \begin{array}{c} \square(z,L_0) \subset \mathcal{V}(Z) \text{ and } \sum_{x\in\partial C_{z,L_0}} \ell_x(Z) \le (L_0)^{d-1} \\ \text{for all } z \in \mathbb{L}_0 \text{ satisfying } |z-y|_\infty \le L_0 \end{array} \right\}.$$

Using this we can define the (random) set

$$\text{(3.17)} \qquad \mathcal{O}^-_0(Z) \overset{\text{def.}}{=} \{y \in \mathbb{L}_0 : \mathrm{W}^-_{y,L_0}(Z) \text{ occurs}\}.$$

Clearly the event $\mathrm{W}^-_{y,L_0}(Z_J)$, with Z_J as before, is measurable relative to the σ-algebra

$$\text{(3.18)} \qquad \mathcal{F}^-_{y,L_0}(Z_J) = \sigma\big(\mathcal{O}^-_0(Z_J), \mathcal{I}(Z_J)_{|C^c_{y,L_0}}\big).$$

Lemma 3.3. *Let $L_0 \ge 100$. There exists $c' = c'(L_0) \in (0,1)$ such that for all $K \ge 100$, $z \in N\mathbb{Z}^d$ for some $N \ge 10^3 L_0$, $y \in L_0\mathbb{Z}^d$ such that $D_{y,L_0} \subset D_{z,N}$, $x \in \partial^{\text{out}} C_{y,L_0}$ and any $J \subset \mathbb{N}^*$ deterministic and finite, abbreviating $Z_J = Z_J^{D_{z,N},U_{z,N}}$ we have*

$$\text{(3.19)} \qquad \mathbb{P}\big[x \overset{\mathcal{V}(Z_J)}{\longleftrightarrow} \square(y,L_0) \,\big|\, \mathcal{F}^-_{y,L_0}(Z_J)\big] \ge c'\,1_{G'},$$

with the 'good' event $G' = \{x \in \mathcal{V}(Z_J)\} \cap \mathrm{W}^-_{y,L_0}(Z_J)$.

To appreciate the utility of (3.19), one should imagine the sets $\square(y,L)$ being contained in $\mathcal{V}(Z_J)$ for many neighboring points $y \in \mathbb{L}_0$, thus forming an ambient cluster, and (3.19) gives a conditional probability on a point x at the doorstep of the box C_{y,L_0} to connect locally to this ambient cluster.

Proof. (3.19) follows from a straighforward adaptation of the argument underlying the proof of Lemma 5.10 in [16], using a similar computation as in the proof of Lemma 5.13 therein. □

Remark 3.4 (On the use of Lemma 3.3).

1) As opposed to Proposition 3.1, Lemma 3.3 entails implicitly that Z has to be sufficiently small, since otherwise one cannot expect the set $\square(z,L_0)$ in (3.16) to be fully comprised in $\mathcal{V}(Z)$ with high probability. The distinction between two different types of sequences Z (large and small) will be reflected later, cf. in particular the definitions in (6.19) and (6.20) in §6.2. In Section 7, we use Proposition 3.1 and Lemma 3.3 for the proofs of Propositions 6.6 and 6.7, respectively, which are crucial to derive our main results in the entire super-critical regime. The two cases we distinguish will thereby lead to so-called type-I and type-II estimates in that context.

2) Although we need Lemma 3.3 to account for the condition "for all $v \in [0,u]$" in (1.9) in the proof of Theorem 1.1 (see also (1.14)–(1.15)), or the sharp bound (1.13) in Theorem 1.4 for $d=3$ and all $u < u_*$, we can nevertheless prove a weaker version of Theorem 1.1, with the interval $[0,u]$ in (1.9) replaced by $[u',u]$ for any $u' \in (0,u)$ (e.g. $u' = u/2$) and the prefactor C in (1.10) depending on u' in addition to u, using only Proposition 3.1 as the relevant insertion tolerance bound. The same thing is also true with the upper bound on the truncated two-point function in (1.12) for $d=4$ and even the sharp bound for $d=3$ in (1.13) when u is larger than some constant level $u_0 < u_*$. See also Remark 6.8 in §6.2.

4 The observable h^u and coarse-graining

We now lay the foundation for the upper bounds in the upcoming sections. In Section 4.1, we introduce the important scalar random variable h^u, see (4.4), which is attached to the interlacement in a system Σ of well-separated boxes. In Proposition 4.2, we give bounds on the probabilities that h^u is atypical. An important feature of these bounds is that they involve exponentials of $\mathrm{cap}(\Sigma)$ with the *correct* dependence on u which is crucial to get sharp results in dimension 3. The system Σ of boxes will be drawn from a set of admissible coarsenings that arise from a coarse-graining scheme presented in §4.2. The coarse-graining leads to a certain 'good' event $\mathscr{G}$, introduced in (4.16), which plays a central role in the rest of the paper. As will become clear in the next section, the event $\mathscr{G}$ will be used to propagate certain bounds from a (base) scale L to scale $N \gg L$. The event $\mathscr{G}$ is sufficiently generic to fit all our later purposes (i.e. both sub- and supercritical regimes). The main result then comes in §4.3, see Proposition 4.5, which yields a deviation estimate on the probability of $\mathscr{G}^c$. Crucially, this estimate involves the afore bounds on h^u, which control its leading-order decay.

We start by introducing the relevant framework, which involves two parameters, a length scale $L \geq 1$ and a rescaling parameter $K \geq 100$, both integers. The scale L induces the renormalized lattice $\mathbb{L} \stackrel{\text{def.}}{=} L\mathbb{Z}^d$ and we consider the boxes $C_z \subset \tilde{C}_z \subset \tilde{D}_z \subset D_z \subset U_z$ attached to points $z \in \mathbb{L}$ (or $\mathbb{Z}^d$) as defined in (2.19). In case we work with more than one scale in a given context we sometimes explicitly refer to the associated length scale L by writing $C_{z,L} = C_z, \tilde{C}_{z,L} = \tilde{C}_z$ etc. Now, let

(4.1) $\quad \mathcal{C} \subset \mathbb{L}$ be a non-empty collection of sites with mutual $|\cdot|_\infty$-distance at least $10KL$

and define

$$\Sigma = \Sigma(\mathcal{C}) = \bigcup_{z \in \mathcal{C}} D_z. \tag{4.2}$$

In view of (4.1), the parameter K thus controls the separation between boxes D_z comprising the set Σ in (4.2).

4.1. Deviation estimate for h^u. We now introduce a scalar random variable $h^u = h^u(\mathcal{C})$ that will play a central role in the sequel. Consider the function V on $\mathbb{Z}^d$ defined as

$$V(x) = \sum_{D \in \mathcal{C}} e_\Sigma(D) \bar{e}_D(x), \quad x \in \mathbb{Z}^d, \tag{4.3}$$

where $e_\Sigma(D) = \sum_{y \in D} e_\Sigma(y)$ (cf. (A.3) for notation) and the sum ranges over all D such that $D = D_z$ for some $z \in \mathcal{C}$. Notice that V in (4.3) has the same support as e_Σ. Moreover, V is well-approximated by e_Σ (uniformly on $\mathbb{Z}^d$) as K becomes large, in a sense made precise in Lemma A.2. Now define h^u by (see (2.7) for notation)

$$h^u = h^u(\mathcal{C}) \stackrel{\text{def.}}{=} \left\langle \mu_{\Sigma,u}, \textstyle\int_0^\infty V(X_s)ds \right\rangle \tag{4.4}$$

with $\Sigma = \Sigma(\mathcal{C})$ as in (4.2), and where $\int_0^\infty V(X_s)ds$ is short for the map $\widehat{w} \mapsto \int_0^\infty V(X_s(\widehat{w}))ds$ ($\widehat{w} \in \widehat{W}_+$). In case $\mathcal{C} = \{z\}$ is a singleton, we write $h^u(z) = h^u(\{z\})$. A quantity akin to (4.4) was already implicit at play in the work [48], see for instance (3.13) therein. Our presentation is somewhat streamlined (compare Proposition 4.2 below and [48, Theorem 4.2]) and it includes two-sided deviation estimates. This is intimately related to the non-monotone nature of the events we consider, as opposed to the disconnection event investigated in [48].The following result prepares the ground to obtain suitable tail bounds on h^u.

Lemma 4.1. *Let $u > 0$, $L \geq 1$ and $\mathcal{C}$ as in (4.1). Then for all $a < 1$, one has that*

$$\mathbb{E}\left[\exp\left(a\left\langle \mu_{\Sigma,u}, \int_0^\infty e_\Sigma(X_s)ds\right\rangle\right)\right] = \exp\Big(\frac{u\,a\,\text{cap}(\Sigma)}{1-a}\Big). \tag{4.5}$$

Proof. In view of (2.5) and by a classical formula for Laplace transforms of Poisson functionals, see for instance [46, display (2.5)], we can write for any $a \in \mathbb{R}$,

$$\mathbb{E}\Big[\exp\Big(a\left\langle \mu_{\Sigma,u}, \textstyle\int_0^\infty e_\Sigma(X_s)ds\right\rangle\Big)\Big] = \exp\Big(uE_{e_\Sigma}\big[\exp\big(a\textstyle\int_0^\infty e_\Sigma(X_s)\,ds\big) - 1\big]\Big). \tag{4.6}$$

However, owing to the beautiful observation made in [48, Lemma 2.1], which is an application of Kac's moment formula, the quantity $\int_0^\infty e_\Sigma(X_s)\,ds$ is distributed under $P_{\bar{e}_\Sigma}$ as an exponential variable with mean 1. Consequently, for all $a < 1$,

$$\mathbb{E}_{e_\Sigma}\big[\exp\big(a\textstyle\int_0^\infty e_\Sigma(X_s)\,ds\big)\big] = \frac{\text{cap}(\Sigma)}{1-a}.$$

Substituting this into the right hand side of (4.6) gives (4.5). □

Following are the announced deviation estimates for h^u in (4.4).

Proposition 4.2. *Let $\varepsilon \in (0,1)$ and $0 < \frac{u_-}{1-\varepsilon} < u < \frac{u_+}{1+\varepsilon}$. Then, for any $K \geq \frac{C_8}{\varepsilon}$, $L \geq 1$, and $\mathcal{C}$ as in (4.1), one has that*

$$\mathbb{P}[h^u \geq u_+\,\text{cap}(\Sigma)] \leq \exp\bigg(-\Big(\sqrt{\tfrac{u_+}{1+\varepsilon}} - \sqrt{u}\Big)^2\,\text{cap}(\Sigma)\bigg) \tag{4.7}$$

and

$$\mathbb{P}[h^u \leq u_-\,\text{cap}(\Sigma)] \leq \exp\bigg(-\Big(\sqrt{u} - \sqrt{\tfrac{u_-}{1-\varepsilon}}\Big)^2\,\text{cap}(\Sigma)\bigg). \tag{4.8}$$

Proof. Using Lemma A.2 and subsequently applying Chebychev's inequality along with Lemma 4.1 yields that

$$\begin{aligned}\mathbb{P}[h^u \geq u_+\,\text{cap}(\Sigma)] &\leq \mathbb{P}\Big[(1+\varepsilon)\langle \mu_{\Sigma,u}, \textstyle\int_0^\infty e_\Sigma(X_s)ds\rangle \geq u_+\,\text{cap}(\Sigma)\Big] \\ &\leq \exp(-au_+\,\text{cap}(\Sigma))\exp\Big(\frac{u\,a(1+\varepsilon)\,\text{cap}(\Sigma)}{1-a(1+\varepsilon)}\Big),\end{aligned}$$

whenever $a(1+\varepsilon) \in (0,1)$. Now setting $a = \frac{1}{1+\varepsilon}\big(1 - \sqrt{\frac{(1+\varepsilon)u}{u_+}}\big) \in (0,1)$ (recall that $u < \frac{u_+}{1+\varepsilon}$) gives (4.7). For the second inequality, one similarly starts with

$$\begin{aligned}\mathbb{P}[h^u \leq u_-\,\text{cap}(\Sigma)] &\leq \mathbb{P}\Big[-a(1-\varepsilon)\langle \mu_{\Sigma,u}, \textstyle\int_0^\infty e_\Sigma(X_s)ds\rangle \geq -au_-\,\text{cap}(\Sigma)\Big] \\ &\leq \exp(au_-\,\text{cap}(\Sigma))\exp\Big(\frac{-u\,a(1-\varepsilon)\,\text{cap}(\Sigma)}{1+a(1-\varepsilon)}\Big),\end{aligned}$$

where $a > 0$. From this one deduces (4.8) by substituting $a = \frac{1}{1-\varepsilon}\big(\sqrt{\frac{u(1-\varepsilon)}{u_-}} - 1\big) > 0$ (recall that $u > \frac{u_-}{1-\varepsilon}$). □

4.2. Admissible coarsenings. A certain event $\mathscr{G}$, introduced below in §4.3, will play a central role in the remainder of this article. The definition of $\mathscr{G}$ involves a coarse-graining of paths that will give rise to collections $\mathcal{C}$ as in (4.1). We now present this coarse-graining method. On the one hand, an 'admissible' coarsening is designed so as to ensure that the capacity of the coarse-grained path lies above a required threshold while, on the other hand, provides good control on the entropy factor for its possible choices. We start by introducing this coarse-graining method. Its main features are summarized in Proposition 4.3 below, which is essentially a reproduction of Proposition 4.3 in [24] except that here we consider the more general case of paths crossing Euclidean balls (for our finer results in dimension 3) and that unlike [24], we need quantitative control on the range of certain parameters, in particular L and K below; see (4.14). A proof of Proposition 4.3 is included in Appendix B.

Below and in the rest of the paper, we write $B^2_r(x) \subset \mathbb{Z}^d$ for the closed ℓ^2-ball of radius $r \geq 0$ around $x \in \mathbb{Z}^d$ whereas we write $B_r(x) \subset \mathbb{Z}^d$ for the corresponding ℓ^∞-ball. We abbreviate $B^2_r = B^2_r(0)$ and $B_r = B_r(0)$. We will frequently identify a path γ with its range $\bigcup_{0\leq i\leq k}\{\gamma(i)\}$ (a subset of $\mathbb{Z}^d$). This identification will always be clear from the context.

The set B^2_r is *simply connected* in $\mathbb{Z}^d$, i.e. both B^2_r and its complement are connected in $\mathbb{Z}^d$ for any $r \geq 0$ and hence $\partial^{\text{ext}} B^2_r = \partial^{\text{out}} B^2_r$ (see the beginning of Section 2 and the paragraph above (C.1) in Appendix C for definitions). To see this simply observe that given any non-zero $x \in \mathbb{Z}^d$, there are neighbors y and z of x in $\mathbb{Z}^d$ such that $|y| < |x| < |z|$. For $U \subset V \subset\subset \mathbb{Z}^d$ where V is simply connected in $\mathbb{Z}^d$, we say that a path γ in $\mathbb{Z}^d$ *crosses* $V \setminus U$, or equivalently that γ is a *crossing* of $V \setminus U$, if it intersects both U and ∂V. If $U = \{0\}$, we omit the reference to U; e.g. when γ crosses B^2_r we mean that γ intersects both 0 and ∂B^2_r. In what follows, we always tacitly assume that $\Lambda_N \subset \mathbb{Z}^d$ is of the form (see (2.19) for notation)

$$\Lambda_N \in \mathcal{S}_N \stackrel{\text{def.}}{=} \{B^2_N,\, B^2_N \setminus B^2_{\sigma N}, \sigma \in (0, \tfrac{1}{3}),\, B^2_{2N} \setminus B^2_N,\, \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}\}. \tag{4.9}$$

To allow for a uniform presentation it will be convenient to define $\sigma = \sigma(\Lambda_N)$ for all choices in (4.9) by setting $\sigma(B^2_N \setminus B^2_{\sigma N}) = \sigma$ and $\sigma(\Lambda_N) = 0$ otherwise. Note that $\sigma \in [0, \frac{1}{3})$ and that for any choice of Λ_N, the quantity $(1-\sigma)N$ frequently encountered below roughly corresponds to the ℓ^2-diameter (ℓ^∞-diameter in the case of $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$}) of Λ_N. The upper bound on $\sigma (< \frac{1}{3})$ imposed by (4.9) is for convenience and could be relaxed.

We now introduce certain families of collections $\mathcal{C}$ satisfying (4.1) with additional useful properties. We will reuse the notion of coarsenings from [24, Definition 4.2] that are well-behaved with respect to a given entropy rate Γ. Let $\Gamma : [1, \infty) \mapsto [0, \infty)$ be an increasing function and $a \in (0, 1)$. For $L \geq 1$, $K \geq 100$ and $N \geq h(KL)$, where $h(x) = x(1 + (\log x)^2 1_{d\geq 4})$, a family $\mathcal{A} = \mathcal{A}^K_L(\Lambda_N)$ of collections $\mathcal{C} \subset \mathbb{L}$ satisfying (4.1) with K, L as above is (a, Γ)-*admissible* if,

$$\log |\mathcal{A}| \leq \Gamma(N/L), \tag{4.10}$$

$$D_z = D_{z,L} \subset \Lambda_N \text{ for all } z \in \mathcal{C}, \tag{4.11}$$

(4.12) all $\mathcal{C} \in \mathcal{A}$ have equal cardinality $n = |\mathcal{C}|$, which lies in the interval $\left[\frac{a(1-\sigma)N}{h(KL)}, \frac{(1-\sigma)N}{h(KL)}\right]$, and

(4.13) for any crossing γ of Λ_N, there exists $\mathcal{C} \in \mathcal{A}$ such that γ crosses $D_z \setminus C_z$ (recall (2.19)) for all $z \in \mathcal{C}$.

We are now ready to state our result on the existence of coarsenings with good capacity bound. In the sequel, we let T_N denote the line segment $([0, N] \cap \mathbb{Z}) \times \{0\}^{d-1}$.

Proposition 4.3 (Admissible coarsenings)**.** *There exist $C_2 \in [1,\infty)$ and $c_3 \in (0, \frac{1}{100d})$ such that, for all $K \geq 100$, $L \geq 1$, $N \geq c_3^{-1}\, h(KL)$ and $\Lambda_N \in \mathcal{S}_N$ (see (4.9)), there exists a (c_3, Γ)-admissible collection $\mathcal{A} = \mathcal{A}_L^K(\Lambda_N)$ with the following properties.*

i) If $d = 3$, one has for all $\rho \in (0,1)$ and with $\Gamma(x) = C_2 K^{-1} x \log ex$,

$$\liminf_{N\to\infty} \inf_{\substack{K\in[K_-,K_+],\\ L\in[L_-,L_+]}} \inf_{\mathcal{C}\in\mathcal{A}} \inf_{\substack{\tilde{\mathcal{C}}\subset\mathcal{C}\\ |\tilde{\mathcal{C}}|\geq(1-\rho)|\mathcal{C}|}} \frac{\mathrm{cap}(\Sigma(\tilde{\mathcal{C}}))}{(1-\frac{C_3}{K})\mathrm{cap}(T_{(1-\sigma)N})} \geq (1-\rho), \tag{4.14}$$

where $C_3 \in [200,\infty)$, $\Sigma(\tilde{\mathcal{C}})$ is as in (4.1) and $K_\pm = K_\pm(N)$, $L_\pm = L_\pm(N)$ satisfy

$$\begin{aligned} &K_-(N) = 100,\ \lim_N L_-(N) = \infty, \text{ and} \\ &\lim_N \Big(\tfrac{\log(K_+(N)L_+(N))}{\log N}\Big)^{1/K_+(N)} = 0, \quad L_+(N) \leq c_3 N/K_+(N). \end{aligned} \tag{4.15}$$

ii) If $d \geq 4$, choosing instead $\Gamma(x) = C_2\, x$, the bound (4.14) remains valid with $L_-(N) = 1$, any fixed $K \geq 100$, $L_+(N) = c_3 N/K$ and $\big(1 - \frac{C_3}{K}\big)$ replaced by some $c(K) \in (0,1]$.

A proof of Proposition 4.3 is given in Appendix B.

Remark 4.4. 1) By translation, the definition of $\mathcal{A}_L^K(\Lambda)$ can be extended to include any $\Lambda = z + \Lambda_N, z \in \mathbb{Z}^d$ so that it satisfies properties i) and ii) above except that the coarsenings would comprise points from $z + \mathbb{L}$ unless, of course, $z \in \mathbb{L}$.

2) As can be seen by inspection of the proof, the conclusions for $d = 3$ remain true if in (4.15) one replaces the last condition by the weaker requirement that $L_+ = L_+(N,K) \leq c_3 N/K$, in which case the infima in (4.14) are to be taken first over L (in a manner depending on K via L_+) and then over K.

3) The discrepancy between the entropy rates Γ in dimension 3 and higher dimensions arises from a simpler scheme that we adopt in dimension 3. Very recently, a more general scheme was obtained in [35, Proposition 3.4] which eliminates this difference for a wider class of graphs. We nevertheless use the paradigm from [24] because of its simplicity and quantitative dependence on K which will be crucial in our applications.

From here onwards until the end of this article, we will refer to *admissible* collections $\mathcal{C} \in \mathcal{A} = \mathcal{A}_L^K(\Lambda_N)$ without mention of (a,Γ), which are set to $a = c_3$ and Γ as supplied by Proposition 4.3.

4.3. The event $\mathscr{G}$. In the sequel, we work under $\mathbb{P}$ (see above (2.2)) and extensions thereof. The specification of the event $\mathscr{G}$ which plays a key role in our proofs involves two families of events $\mathcal{F} = \{\mathcal{F}_{z,L} : z \in \mathbb{L}\}$ and $\mathcal{G} = \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}$. Whereas $\mathcal{F}$ will be specified shortly (up to the choice a few parameters), see (4.18), $\mathcal{G}$ will be carefully chosen depending on the specific application. We comment further on the role of $\mathcal{F}$ and $\mathcal{G}$ in Remark 4.6 at the end of this section. Given families $\mathcal{F},\mathcal{G}$ and for any $\rho \in (0,1)$ and $\Lambda_N \in \mathcal{S}_N$ as in (4.9), let

$$\mathscr{G} = \mathscr{G}(\Lambda_N, \mathcal{G}, \mathcal{F}; \rho) \overset{\text{def.}}{=} \bigcap_{\mathcal{C}\in\mathcal{A}} \bigcup_{\substack{\tilde{\mathcal{C}}\subset\mathcal{C},\\ |\tilde{\mathcal{C}}|\geq\rho|\mathcal{C}|}} \bigcap_{z\in\tilde{\mathcal{C}}} (\mathcal{F}_{z,L} \cap \mathcal{G}_{z,L}), \tag{4.16}$$

where $\mathcal{A} = \mathcal{A}_L^K(\Lambda_N)$ is the family of admissible coarsenings supplied by Proposition 4.3, which implicitly requires that $L \geq 1, K \geq 100$ and $N \geq c_3^{-1}\, h(KL)$.

The (good) event $\mathscr{G}$ will typically be likely in upcoming applications. Following is an 'umbrella bound' in this direction which subsumes all the events of our interest in this paper. Whereas $\mathcal{F}_{z,L}$ will be specifically built using events as in (2.21) and will serve as a means for localization (see the discussion following (2.21)), the events $\mathcal{G}_{z,L}$ will be fairly generic and sufficiently versatile to fit all our purposes.

We start by specifying the events $\mathcal{F} = \{\mathcal{F}_{z,L} : z \in \mathbb{L}\}$ which will be relevant to us. Let $k \geq 1$ and $u_i, v_i \in (0, \infty)$ with $u_i \neq v_i$ for all $1 \leq i \leq k$. The parameters $\boldsymbol{u} = (u_1, \dots, u_k)$ and $\boldsymbol{v} = (v_1, \dots, v_k)$ represent various levels of the interlacements that will be involved in our construction. The dependence of constants etc. on $\boldsymbol{u}, \boldsymbol{v}$ will often factor through the quantity

$$\Delta^{\boldsymbol{u},\boldsymbol{v}} \stackrel{\text{def.}}{=} \big(\min_{1\leq i\leq k} |\sqrt{u_i} - \sqrt{v_i}|, \ \max_{1\leq i\leq k} \max(u_i, v_i)\big). \tag{4.17}$$

Extending (2.21) we now introduce

$$\mathcal{F}_{z,L} = \mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}} \stackrel{\text{def.}}{=} \bigcap_{1\leq i\leq k} \mathcal{F}_{z,L}^{u_i,v_i}, \tag{4.18}$$

so that $\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}} = \mathcal{F}_{z,L}^{u_1,v_1}$ in case $k = 1$, i.e. (4.18) boils down to (2.21) in this case. The events in (4.18) comprise the family $\mathcal{F} = \mathcal{F}_L^{\boldsymbol{u},\boldsymbol{v}} = \{\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}} : z \in \mathbb{L}\}$.

As to the events forming the family $\mathcal{G} = \mathcal{G}_L = \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}$, we assume that there exists an event $\widetilde{\mathcal{G}}_{z,L}$ for each $z \in \mathbb{L}$ measurable relative to the i.i.d. excursions $\widetilde{Z}^{D_z,U_z} = (\widetilde{Z}_k^{D_z,U_z})_{k\geq 1}$ governed by the law $\widetilde{\mathbb{P}}_z = \widetilde{\mathbb{P}}_{\{z\}}$ (see above (2.13) for notation) and an independent collection of i.i.d. uniform random variables $\mathsf{U} = \{\mathsf{U}_x : x \in D_{z,L}\}$, an integer $m_L > 0$ and $\varepsilon_L \in (0, 1)$ such that the inclusion

$$(\widetilde{\mathcal{G}}_{z,L} \cap \mathrm{Incl}_z^{\varepsilon_L,m_L}) \subset \mathcal{G}_{z,L} \text{ holds under any coupling } \mathbb{Q} \text{ of } \mathbb{P} \text{ and } \widetilde{\mathbb{P}}_z \tag{4.19}$$

(recall the event $\mathrm{Incl}_z^{\varepsilon_L,m_L}$ from (2.15)). Note that the above condition on $\mathcal{G}$ depends implicitly on the scale parameter K via U_z; see (2.19).

Proposition 4.5 (Estimate for $\mathscr{G}^c$)**.** *For any choice of $\Lambda_N \in \mathcal{S}_N$ (see (4.9)), $\rho \in (0, 1]$, $k \geq 1$ and families $\mathcal{F} = \mathcal{F}_L^{\boldsymbol{u},\boldsymbol{v}}$, $\mathcal{G} = \mathcal{G}_L$ as above, the following hold. If, for some $K_0, L_0 \geq 1$ and $\beta' \in (0, 1)$, one has $\sup_{z\in\mathbb{L}} \mathbb{P}\big[\widetilde{\mathcal{G}}_{z,L}^c\big] \leq p_L$ for all $K \geq K_0$ and $L \geq L_0$ with*

$$\sup_{L\geq L_0} L^{-\beta'} \log \big(p_L \vee \mathbb{P}\big[(\mathcal{U}_0^{\varepsilon_L,m_L})^c\big]\big) \leq -1 \tag{4.20}$$

(see (2.13) for notation), then:

i) for $d = 3$, there exists $\alpha = \alpha(\beta') \in (0, \infty)$ such that with $L(N) = \lfloor (\log N)^\alpha \rfloor$, one has for all $\delta \in (0, 1)$ and $N \geq C(\beta', \Delta^{\boldsymbol{u},\boldsymbol{v}}, k, \rho, K_0, L_0, \delta)$,

$$\begin{aligned} \sup_{K\in[K_-,K_+]} \big(1 - \tfrac{C_3}{K}\big)^{-1} &\log \mathbb{P}\big[(\mathscr{G}(\Lambda_N, \mathcal{G}_{L(N)}, \mathcal{F}_{L(N)}^{\boldsymbol{u},\boldsymbol{v}}; \rho))^c\big] \\ &\leq -(1-\delta)(1-\sigma)(1-C_4\rho)\frac{\pi}{3k}\Big[\min_{1\leq i\leq k}(\sqrt{u_i} - \sqrt{v_i})^2\Big]\frac{N}{(\log N)^\beta} \end{aligned} \tag{4.21}$$

for some $\beta = \beta(\beta') \in (1, \infty)$ if $\beta' \leq \frac{1}{2}$ and $\beta = 1$ otherwise, where $K_- = \frac{3C_8}{\varepsilon_{L(N)}} \vee C(\delta, \boldsymbol{u}, \boldsymbol{v}) \vee K_0$, $K_+ = \sqrt{\log\log e^2 N}$, C_8 and C_3 are from Propositions A.1 and 4.3, respectively and $C_4 \in (1, \infty)$.

ii) For $d \geq 4$ and $\varepsilon_L = \varepsilon \in (0,1)$ uniformly in L, we have for any fixed $L \geq C(\boldsymbol{u},\boldsymbol{v},\varepsilon,L_0)$, $K = C(\boldsymbol{u},\boldsymbol{v},\varepsilon) \vee K_0$ and $N \geq C(\boldsymbol{u},\boldsymbol{v},\varepsilon,K_0,L_0)$,

$$N^{-1}\log\mathbb{P}\big[(\mathscr{G}(\Lambda_N,\mathcal{G}_L,\mathcal{F}_L^{\boldsymbol{u},\boldsymbol{v}};\tfrac{1}{2}))^c\big] \leq -c(\boldsymbol{u},\boldsymbol{v},\varepsilon,K_0,L_0)\,(<0). \tag{4.22}$$

The proof of Proposition 4.5 will exhibit the observable $h^u(\mathcal{C})$ introduced in Section 4.1, for certain subsets $\mathcal{C}$ of admissible collections (in $\mathcal{A}$). This is not obvious at all (the generic event $\mathscr{G}$ does not involve h^u) and will require some work. A key step is a certain dichotomy, see (4.30) below, which will make h^u appear (cf. the event $\widetilde{E}_{2.2}$ below). The crucial properties of collections in $\mathcal{A}$ gathered in Proposition 4.3 (cf. in particular (4.14)) then come into play to produce the leading-order decay in (4.21) when combined with Proposition 4.2, which in particular requires a lower bound on $\text{cap}(\Sigma)$ for $\Sigma = \Sigma(\mathcal{C})$. The bound provided by (4.14) thus ensures, in a loose sense, that the coarse-grained path $\mathcal{C}$ does not "loose too much" capacity.

Proof. We will treat both the case $d \geq 3$ and $d = 4$ simultaneously. For now, assume that $K \geq 100$, $L \geq 1$, and $N \geq c_3^{-1}\,h(KL)$ so that the conclusions of Proposition 4.3 hold. In particular, this entails that a (c_3,Γ)-admissible collection $\mathcal{A} = \mathcal{A}_L^K(\Lambda_N)$ with the properties listed in Proposition 4.3 exists, and $\mathscr{G} = \mathscr{G}(\Lambda_N,\mathcal{G}_L,\mathcal{F}_L^{\boldsymbol{u},\boldsymbol{v}};\rho)$ is well-defined. For all such K, L, N, applying a union bound over $\mathcal{C}$ in (4.16) and using (4.10) yields that

$$\log\mathbb{P}[\mathscr{G}^c] \leq \Gamma(N/L) + \sup_{\mathcal{C}\in\mathcal{A}}\log\mathbb{P}\bigg[\Big(\bigcup_{\tilde{\mathcal{C}}}\ \bigcap_{z\in\tilde{\mathcal{C}}}\ (\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}}\cap\mathcal{G}_{z,L})\Big)^c\bigg], \tag{4.23}$$

where the union is over $\tilde{\mathcal{C}} \subset \mathcal{C}$ having cardinality $|\tilde{\mathcal{C}}| \geq \rho|\mathcal{C}|$. In the sequel we always tacitly assume that K, L, N satisfy the requirements above (4.23). Additional conditions on any of these parameters will be mentioned explicitly. We will deal with the term $\Gamma(\cdot)$ at the end of the proof and focus on the probability appearing on the right-hand side of (4.23), which is wherein most of the work relies. All subsequent considerations tacitly hold uniformly for all choices of $\mathcal{C} \in \mathcal{A}$. Let us call $z \in \mathbb{L}$ a *good* point if the event $\mathcal{G}_{z,L} \cap \mathcal{F}_{z,L}$ occurs, and bad otherwise. Then for a given admissible collection $\mathcal{C} \in \mathcal{A}$, the event appearing on the right of (4.23) asserts that there is no sub-collection $\tilde{\mathcal{C}} \subset \mathcal{C}$ of cardinality at least $\rho|\mathcal{C}|$ consisting of only good points. Thus, on this event $\mathcal{C}$ contains at least $(1-\rho)|\mathcal{C}|$ bad points. It follows that for any $\mathcal{C} \in \mathcal{A}$ (and K, L, N as above (4.23)),

$$\mathbb{P}\bigg[\Big(\bigcup_{\tilde{\mathcal{C}}}\ \bigcap_{z\in\tilde{\mathcal{C}}}\ (\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}}\cap\mathcal{G}_{z,L})\Big)^c\bigg] \leq \mathbb{Q}_{\mathcal{C}}[E_1] + \mathbb{Q}_{\mathcal{C}}[E_2], \tag{4.24}$$

where $\mathbb{Q}_{\mathcal{C}}$ is the extension of $\mathbb{P}$ supplied by Lemma 2.1 and

$$E_1 = \big\{\exists\mathcal{C}_1 \subset \mathcal{C},\ |\mathcal{C}_1| \geq \rho|\mathcal{C}| :\ (\mathcal{G}_{z,L})^c \text{ occurs for all } z \in C_1\big\},$$
$$E_2 = \big\{\exists\mathcal{C}_2 \subset \mathcal{C},\ |\mathcal{C}_2| \geq (1-2\rho)|\mathcal{C}| :\ (\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}})^c \text{ occurs for all } z \in C_2\big\}.$$

We will bound the two probabilities on the right-hand side of (4.24) individually. We start by observing that the inclusion (2.17) obtained in Lemma 2.1 holds for the choices $\varepsilon = \varepsilon_L$ and $m_0 = m_L$ whenever $K \geq C_8(\varepsilon_L)^{-1}$; indeed the relevant condition (2.16) holds on account of Proposition A.1 (see (A.9)). Using the inclusion (2.17), recalling that the events $\mathcal{U}_z^{\varepsilon_L,m_L}$ are independent as $z \in \mathbb{L}$ varies, and applying the relevant bound from (4.20), it follows that for all $L \geq L_0$ and $K \geq K_0 \vee C_8(\varepsilon_L)^{-1}$,

$$\mathbb{Q}_{\mathcal{C}}[(\text{Incl}_z^{\varepsilon_L,m_L})^c, z \in \mathcal{D}] \leq e^{-L^{\beta'}|\mathcal{D}|},\ \text{for all } \mathcal{D} \subset \mathbb{L}. \tag{4.25}$$

To bound $\mathbb{Q}_{\mathcal{C}}[E_1]$, one then proceeds as follows. First one applies a union bound over the choice of $\mathcal{C}_1$, using the elementary estimate $\binom{n}{k} \leq (\frac{en}{k})^k$ for all $1 \leq k \leq n$ (implied by the bound $\frac{k^k}{k!} \leq e^k$), applied with $n = |\mathcal{C}|$ and $k = \lfloor \rho n \rfloor$. Then one employs the inclusion $(\mathcal{G}_{z,L})^c \subset (\widetilde{\mathcal{G}}_{z,L})^c \cup (\mathrm{Incl}_z^{\varepsilon_L, m_L})^c$ implied by (4.19) together with a union bound, (4.25) and the control on the decay of p_L from (4.20). All in all, this yields, for $L \geq L_0 \vee C(\beta')$ and $K \geq K_0 \vee C_8(\varepsilon_L)^{-1}$,

$$\mathbb{Q}_{\mathcal{C}}[E_1] \leq \exp\big\{ -\rho|\mathcal{C}|\big(cL^{\beta'} - C|\log \rho\,|\big)\big\}. \tag{4.26}$$

The case of E_2 is more involved, and, as will turn out, produces the leading-order contribution to right-hand side of (4.23). We start by modifying the event E_2 to make it easier to handle. Recall from (4.18) that the event $(\mathcal{F}_{z,L}^{\boldsymbol{u},\boldsymbol{v}})^c$ involves a union over events at $k \geq 1$ different pairs of levels (u_i, v_i), $1 \leq i \leq k$. The collection $\mathcal{C}_2$ involved in E_2 must therefore contain a sub-collection of cardinality at least $|\mathcal{C}_2|/k = (1-2\rho)|\mathcal{C}|/k$ for which $(\mathcal{F}_{z,L}^{u,v})^c$ occurs for some $(u,v) \in \{(u_i,v_i) : 1 \leq i \leq k\}$ (the choice of (u,v) depends on $\mathcal{C}_2$ of course). By further sacrificing a fraction $\rho|\mathcal{C}|/k$ from this new collection one may further assume that for each point z, the event $\mathrm{Incl}_z^{\tilde{\varepsilon},\tilde{m}}$ occurs, where the parameters $\tilde{\varepsilon}$, $\tilde{m}$ are chosen for a given $\delta \in (0,1)$ as

$$\tilde{\varepsilon} = \tfrac{\delta}{100} \min_i |u_i - v_i|, \quad \tilde{m} = \tilde{\varepsilon}^{-3} \vee 2^{-1} \min\{u_i, v_i, 1 \leq i \leq k\}\mathrm{cap}(D_0). \tag{4.27}$$

Thus, defining the event

$$\widetilde{E}_2(\mathcal{D}) = \big\{(\mathcal{F}_{z,L}^{u,v})^c \cap \mathrm{Incl}_z^{\tilde{\varepsilon},\tilde{m}} \text{ occurs for all } z \in \mathcal{D}\big\},$$

it follows from the above discussion by means of appropriate union bounds that

$$\mathbb{Q}_{\mathcal{C}}[E_2] \leq (C(\rho^{-1} \vee k))^{\rho|\mathcal{C}|}\Big(\sup_{(u,v),\,\mathcal{D}} \mathbb{Q}_{\mathcal{C}}\big[\widetilde{E}_2(\mathcal{D})\big] + e^{-c\tilde{\varepsilon}^2 \tilde{m}\rho|\mathcal{C}|/k}\Big) \tag{4.28}$$

for $K \geq C_8\tilde{\varepsilon}^{-1}$, where the supremum ranges over all k choices for (u,v) and all subsets $\mathcal{D} \subset \mathcal{C}$ having cardinality $|\mathcal{D}| \geq (1-3\rho)|\mathcal{C}|/k$, and the last term in (4.28) is a bound for probability of jointly occurring events of type $(\mathrm{Incl}_z^{\tilde{\varepsilon},\tilde{m}})^c$, for z ranging over a given collection of cardinality $\rho|\mathcal{C}|/k$; this bound is obtained similarly as in (4.25), exploiting (2.17), using independence of the events $\mathcal{U}_z^{\tilde{\varepsilon},\tilde{m}}$ over z and applying a classical Poisson tail estimate (cf. (2.13)).

It remains to deal with $\widetilde{E}_2(\mathcal{D})$, for $\mathcal{D}$ as above. To this effect, let us introduce the following events. For a given sequence of excursions $Z^u = (Z_k^u)_{1 \leq i \leq n_Z}$ with $Z^u \in \{\overline{Z}_z^u, Z_z^u, \widetilde{Z}_z^u\}$ for some $z \in \mathbb{L}$ (see (2.20) for notation), let

$$\mathcal{E}^v(Z^u) \stackrel{\text{def.}}{=} \Big\{ \sum_{1 \leq i \leq n_Z} \int_0^{T_U} e_{D_z}\big(Z_i(s)\big)ds \leq v\,\mathrm{cap}(D_z)\Big\} \tag{4.29}$$

if $u \leq v$ and with opposite inequality when $u > v$. Notice in particular that $\mathcal{E}^v(\overline{Z}_z^u) = \{h^u(z) \leq v\,\mathrm{cap}(D_z)\}$ when $u \leq v$ (and similarly when $u > v$) on account of (4.4) and the first line of (2.20). As with the example from the previous sentence, the events $\mathcal{E}^v(Z^u)$ are defined in such a way that they are typical in practice, i.e. likely to occur.

Following is a crucial dichotomy, which brings into play deviations of the type considered in Proposition 4.2. If $u \leq v$, we claim that for any collection $\mathcal{D}$ with $|\mathcal{D}| \geq (1-3\rho)|\mathcal{C}|/k$ and $v' \in (u,v)\,(= (u \wedge v, u \vee v))$,

$$\widetilde{E}_2(\mathcal{D}) \subset \widetilde{E}_{2.1}(\mathcal{D}) \cup \widetilde{E}_{2.2}(\mathcal{D}) \tag{4.30}$$

where

$$\widetilde{E}_{2.1}(\mathcal{D}) = \bigcup_{\substack{\mathcal{D}'\subset\mathcal{D}:\\ |\mathcal{D}'|\geq \rho|\mathcal{C}|/k}} \bigcap_{z\in\mathcal{D}'} (\mathcal{E}^{v'}(Z^v_z))^c,$$

$$\widetilde{E}_{2.2}(\mathcal{D}) = \bigcup_{\substack{\mathcal{D}'\subset\mathcal{D}:\\ |\mathcal{D}'|\geq (1-4\rho)|\mathcal{C}|/k}} \big\{h^u(\mathcal{D}') \geq v'\text{cap}(\Sigma(\mathcal{D}'))\big\}.$$

The inclusion (4.30) continues to hold in case $u > v$ with our above convention on $\mathcal{E}^v(Z^u)$, but now for any $v' \in (v, u)$ and provided one defines $\widetilde{E}_{2.2}(\mathcal{D})$ with the inequality reverted in case $u > v'$.

Let us now explain (4.30). We focus on the case $u < v$ for concreteness, the remaining case is obtained by an analogous argument. Suppose $\widetilde{E}_2(\mathcal{D})$ occurs but $\widetilde{E}_{2.1}(\mathcal{D})$ doesn't. Define $\mathcal{D}' \subset \mathcal{D}$ to be the collection of $z \in \mathcal{D}$ such that $\mathcal{E}^{v'}(Z^v_z)$ occurs. We will show that with this choice of $\mathcal{D}'$, one has i) $|\mathcal{D}'| \geq (1-4\rho)|\mathcal{C}|/k$, and ii) $h^u(\mathcal{D}') \geq v'\text{cap}(\Sigma(\mathcal{D}')$. Thus, $\widetilde{E}_{2.2}(\mathcal{D})$ occurs and (4.30) follows.

To see i), recall that $|\mathcal{D}| \geq (1-3\rho)|\mathcal{C}|/k$ so if i) were not to hold then the set of points $z \in \mathcal{D}$ such that $(\mathcal{E}^{v'}(Z^v_z))^c$ occurs would have cardinality exceeding $\rho|\mathcal{C}|/k$, implying $\widetilde{E}_{2.1}(\mathcal{D})$, which is precluded. To see ii), notice that by joint occurrence for each $z \in \mathcal{D}'$ of the event $(\mathcal{F}^{u,v}_{z,L})^c$ (as implied by $\widetilde{E}_2(\mathcal{D})$) and $\mathcal{E}^{v'}(Z^v_z)$, one has, abbreviating $\Sigma = \Sigma(\mathcal{D}') = \bigcup_{z\in\mathcal{D}'} D_z$ (see (4.2) for notation), that

$$\begin{aligned} h^u(\mathcal{D}') &\overset{(4.4),(4.3)}{=} \sum_{z\in\mathcal{D}'} \frac{e_\Sigma(D_z)}{\text{cap}(D_0)} \sum_{1\leq i\leq N^u_z} \int_0^{T_U} e_{D_z}\big(Z_i^{D_z,U_z}(s)\big)ds \\ &\overset{(2.21)}{\geq} \sum_{z\in\mathcal{D}'} \frac{e_\Sigma(D_z)}{\text{cap}(D_0)} \sum_{1\leq i\leq v\text{cap}(D_0)} \int_0^{T_U} e_{D_z}\big(Z_i^{D_z,U_z}(s)\big)ds \overset{(4.29),(2.20)}{\geq} v'\text{cap}(\Sigma); \end{aligned}$$

in the last line, when using occurrence of $\mathcal{E}^{v'}(Z^v_z)$, recall that $v' < v$ since we are in the case $u < v$, so the event corresponds to the one defined below (4.29) with opposite inequality. Overall, ii) thus holds and the verification of (4.30) is complete.

We now use (4.30) to bound $\mathbb{Q}_\mathcal{C}[\widetilde{E}_2(\mathcal{D})]$ uniformly in (u, v) and $\mathcal{D}$ as appearing in (4.28). From here onwards we choose

$$v' = v(1 + 3\tilde{\varepsilon}(1_{u>v} - 1_{u<v})). \tag{4.31}$$

(see (4.27) regarding $\tilde{\varepsilon}$). For concreteness let us assume again that $u < v$, so $v' = v(1-\tilde{\varepsilon})$, the other case being treated similarly. We first deal with $\widetilde{E}_2(\mathcal{D}) \cap \widetilde{E}_{2.1}(\mathcal{D})$, and aim to decouple the (unlikely) events $(\mathcal{E}^{v'}(Z^v_z))^c$ as z varies in $\mathcal{D}' \subset \mathcal{D}$. To this effect, we use the occurrence of $\text{Incl}^{\tilde{\varepsilon},\tilde{m}}_z$ implied by $\widetilde{E}_2(\mathcal{D})$ and a localization argument similar to the one below (2.21) (with the difference that an event of the type (2.21) is not presently required because the number of excursions involved in Z^v_z is already deterministic, cf. (2.20)). By monotonicity of (4.29) in u, recalling (2.15) and the choices of parameters in (4.27) and (4.31), it thus follows that for all $L \geq C(u, v)$, when $u < v$ the inclusion

$$\big(\text{Incl}^{\tilde{\varepsilon},\tilde{m}}_z \cap (\mathcal{E}^{v'}(Z^v_z))^c\big) \subset (\mathcal{E}^{v'}(\widetilde{Z}^{v(1-\tilde{\varepsilon})}_z))^c \tag{4.32}$$

holds $\mathbb{Q}_\mathcal{C}$-a.s. The events on the right-hand side of (4.32) are independent as z varies by construction (see above (2.13)). For a single z, the probability of the event in question is best bounded by restituting $h^{v(1-2\tilde{\varepsilon})}(z)$ from the functional entering $\mathcal{E}^{v'}(\widetilde{Z}^{v(1-\tilde{\varepsilon})}_z)$. This is achieved by de-localizing, i.e. suitably coupling tilted with untilted trajectories and controlling the relevant number $N_z^{v(1-2\tilde{\varepsilon})}$ to effectively replace $\widetilde{Z}_z^{v(1-\tilde{\varepsilon})}$ by $\overline{Z}_z^{v(1-2\tilde{\varepsilon})}$. It follows that for all $L \geq C(u,v)$, $K \geq C\tilde{\varepsilon}^{-1}$ and $z \in \mathbb{L}$ (when $u < v$),

$$\mathbb{Q}_{\mathcal{C}}[(\mathcal{E}^{v'}(\widetilde{Z}_z^{v(1-\tilde{\varepsilon})}))^c] \leq e^{-c\tilde{\varepsilon}^2 v \mathrm{cap}(D_0)} + \mathbb{P}[h^{v(1-2\tilde{\varepsilon})}(z) \leq v'\mathrm{cap}(D_0)] \overset{(4.8),(4.31)}{\leq} e^{-c'\tilde{\varepsilon}^2 v\mathrm{cap}(D_0)}.$$

Combining this with (4.32) and applying a union bound over $\mathcal{D}'$ (cf. below (4.30)) yields the desired bound on $\mathbb{Q}_{\mathcal{C}}[\widetilde{E}_2(\mathcal{D}) \cap \widetilde{E}_{2.1}(\mathcal{D})]$. With regards to $\mathbb{Q}_{\mathcal{C}}[\widetilde{E}_{2.2}(\mathcal{D})]$, one performs a similar union bound and applies Proposition 4.2 with $\Sigma = \Sigma(\mathcal{D}')$ with $\mathcal{D}' \subset \mathcal{C}$ satisfying $|\mathcal{D}'| \geq (1-4\rho)|\mathcal{C}|/k$. Altogether, this gives, for all $u, v, \mathcal{D}$ as appearing in the sup of (4.28), all $L \geq C(u,v,\delta)$ (recall $\delta \in (0,1)$ is implicit in $\tilde{\varepsilon}$) and $K \geq C\tilde{\varepsilon}^{-1}$,

$$\text{(4.33)}\quad \mathbb{Q}_{\mathcal{C}}[\widetilde{E}_2(\mathcal{D})] \overset{(4.30)}{\leq} \mathbb{Q}_{\mathcal{C}}[\widetilde{E}_2(\mathcal{D}) \cap \widetilde{E}_{2.1}(\mathcal{D})] + \mathbb{Q}_{\mathcal{C}}[\widetilde{E}_{2.2}(\mathcal{D})]$$
$$\leq (C\rho^{-1})^{k^{-1}\rho|\mathcal{C}|}\Big(\exp\big\{-ck^{-1}\rho|\mathcal{C}|v\tilde{\varepsilon}^2\mathrm{cap}(D_0)\big\} + \sup_{\mathcal{D}'}\exp\big\{-(1-2\tilde{\varepsilon})(\sqrt{u}-\sqrt{v})^2\mathrm{cap}(\Sigma(\mathcal{D}'))\big\}\Big)$$

with the supremum ranging over $\mathcal{D}' \subset \mathcal{C}$ satisfying $|\mathcal{D}'| \geq (1-4\rho)|\mathcal{C}|/k$ in the second line.

We now assemble the various pieces, and in the process aim to apply (4.14) to control the term involving $\mathrm{cap}(\Sigma(\mathcal{D}'))$ in (4.33). We first focus on the case $d=3$, which is more intricate. In that case recall from (4.12) that $c(1-\sigma)N/KL \leq |\mathcal{C}| \leq (1-\sigma)N/KL$ and that $\sigma \leq \frac{1}{2}$. Define $L = L(N) = \lfloor (\log N)^\alpha \rfloor$ for $\alpha > 0$ to be determined. Then (4.26) yields that for all $N \geq C(\beta', \rho)$ (so that in particular $cL^{\beta'} - C'|\log\rho|$) and $K \geq K_0 \vee C_8(\varepsilon_L)^{-1}$,

$$\text{(4.34)}\qquad \log \mathbb{Q}_{\mathcal{C}}[E_1] \leq -\frac{c_4 N}{K(\log N)^{\alpha(1-\beta')}}.$$

As to $\mathbb{Q}_{\mathcal{C}}[E_2]$, we now examine the individual sizes of the various terms involved in (4.28) and (4.33). Owing to the fact that $\mathrm{cap}(D_0) \geq cL^{d-2} = cL$ when $d=3$ and the presence of $\tilde{m}$ together with the choice of $\tilde{\varepsilon}$ in (4.27), one readily finds that the last term in (4.28) decays faster than (4.34) (to leading exponential order as $cN/(\log L)^\theta$, with $L = L(N)$). The same conclusions can be reached of the first term in the last line of (4.33). All in all, these two terms are negligible relative to the decay in (4.34) as soon as $N \geq C(\beta', \rho, k, \delta, \Delta^{\boldsymbol{u},\boldsymbol{v}}, L_0)$ and $K \geq K_0 \vee C\tilde{\varepsilon}^{-1}$.

Lastly, the second term in (4.33) is bounded using (4.14). Note to this effect that (4.15) is satisfied for the choice $L = L(N)(= L_- = L_+)$ with $K_+ = \sqrt{\log\log e^2 N}$. Overall, this yields, for all $\delta \in (0,1)$, $N \geq C(\beta', \rho, k, \delta, \Delta^{\boldsymbol{u},\boldsymbol{v}}, L_0)$ and $K_0 \vee C\tilde{\varepsilon}^{-1} \leq K \leq K_+$

$$\text{(4.35)}\qquad \log \mathbb{Q}_{\mathcal{C}}[E_2] \leq \frac{(1-\frac{\delta}{2})\gamma N}{\log N}, \quad \gamma = (1-\sigma)(1-C\rho)\frac{\pi}{3k}\Big[\min_{(u,v)}(\sqrt{u}-\sqrt{v})^2\Big],$$

with (u,v) ranging among (u_i, v_i), $1 \leq i \leq k$. Returning to (4.23) and in view of (4.24), the bounds in (4.34) and (4.35) are now pitted against $\Gamma(N/L(N))$. Since $\Gamma(N/L) \leq C_2 N(\log N)/KL$, see Proposition 4.3, item i), it follows from (4.23) that for all $N \geq C(\beta', \rho, k, \delta, \Delta^{\boldsymbol{u},\boldsymbol{v}}, L_0)$ and $K \geq K_0 \vee C\tilde{\varepsilon}^{-1} \vee C_8(\varepsilon_{L(N)})^{-1}$,

$$\text{(4.36)}\qquad N^{-1}\log \mathbb{P}[\mathscr{G}^c] \leq \frac{C_2}{K_-(N)(\log N)^{\alpha-1}} - \bigg(\frac{c_4}{K_+(N)(\log N)^{\alpha(1-\beta')}} \vee \frac{(1-\frac{\delta}{2})\gamma}{\log N}\bigg).$$

In order for the term in parenthesis to be larger than the first term on the right of (4.36), and because $K_- \geq 1$ whereas $K_+ \leq C \vee \sqrt{\log\log N}$, it is sufficient that $\alpha - 1 > \alpha(1-\beta') \vee 1$. In particular this requires $\alpha > 2$. There are now two cases to consider, depending on the value of $\beta' \in (0,1)$. If $0 < \beta' \le \frac{1}{2}$, one simply picks any $\alpha > \frac{1}{\beta'}(> 2)$, for instance $\alpha = \frac{1}{\beta'} + 1$. Since for such β', one has $\alpha(1-\beta') \ge 1$, the right-hand side of (4.36) is thus bounded by $-c_4/(2K_+(N)(\log N)^{\alpha(1-\beta')})$ for sufficiently large N. By choosing any β ever so slightly larger than $\alpha(1-\beta')$, one can easily absorb for sufficiently large N the factor $1/2K_+(N)$ and produces instead the desired pre-factor, leading to (4.21) in this case. If instead $\frac{1}{2} < \beta' \le \frac{1}{2}$, one picks a value of α satisfying $2 < \alpha < (1-\beta')^{-1}$, for instance, $\alpha = 1 + \frac{1}{2}(1-\beta')^{-1}$. The decay in (4.36) is then governed by the second term in parenthesis (since now $\alpha(1-\beta') < 1$), and for suitably large N one ensures that the right-hand side of (4.36) is at most $\frac{(1-\delta)\gamma}{\log N}$, yielding (4.21).

The case $d \ge 4$ is simpler, notably because the complexity $\Gamma(N/L) \le CN/L$ never requires fine-tuning of L beyond choosing L large (in a manner depending on the various parameters). For instance, in the case of $\mathbb{Q}_{\mathcal{C}}[E_1]$, recalling that $|\mathcal{C}| \ge cN/L\log(KL)^2$ from (4.12), one obtains a bound on $\log \mathbb{Q}_{\mathcal{C}}[E_1]$ effectively of the form $c(N/L)\frac{L^{\beta'}}{\log(KL)^2}$ and the second fraction is more than enough for large L to produce a decay of exponential order N/L with arbitrary large pre-factor. The case of $\mathbb{Q}_{\mathcal{C}}[E_2]$ is handled similarly, using that the capacity of a box of side-length L grows at least quadratically when $d > 3$ to handle both the second term in (4.28) and the first term in (4.33), and appealing to item ii) of Proposition 4.3 for the remaining one in (4.33). Notice in particular that Proposition 4.3 yields an exponential decay in N rather than N/L in this case. The few remaining details to conclude (4.22) are left to the reader. □

Remark 4.6 (The events $\mathcal{F}$ and $\mathcal{G}$). We briefly return to the different roles played by the events $\mathcal{F}$ and $\mathcal{G}$ in defining the (good) event $\mathscr{G}$ in (4.16). The family $\mathcal{G}$ will be further specified in the next section, but remains largely flexible. In the simplest cases of interest $\mathcal{G}_{z,L}$ will correspond to a (dis-)connection event inside $\widetilde{D}_{z,L}$, see for instance (6.4) below, but more complex choices for $\mathcal{G}_{z,L}$ will also be required. The events $\mathcal{F}$ specified in (4.18) may superficially look like a mere means to localization (cf. §2.2), but inspection of the proof of Proposition 4.5 (in particular the bounds on the events E_2 defined below (4.24), and later on $\widetilde{E}_{2.2}$, cf. (4.28), (4.30), (4.33) and (4.35)) reveals that they generate the leading-order contribution to (4.21).

5 Bootstrapping

In the previous section, we introduced an event $\mathscr{G}$, see (4.16), which is at the center of our coarse-graining mechanism. Roughly speaking, the event $\mathscr{G} = \mathscr{G}(\Lambda_N, \mathcal{G}, \mathcal{F}; \rho)$, which lives at scale $N \gg L$, ensures that, for any choice of Λ_N in (4.9), any (admissible) coarse-grained path at scale L crossing Λ_N will meet a large number of good L-boxes (as parametrized by ρ), in the sense that the corresponding event $\mathcal{G}_{z,L}$ occurs (we are willfully ignoring the occurrence of $\mathcal{F}_{z,L}$ to simplify the discussion; these are however instrumental, see Remark 4.6 above). Whereas the events $\mathcal{F}_{z,L}$ are essentially specified up to the choice of parameters, see (4.18), so far the family $\mathcal{G} = \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}$ was completely generic, save for some localization properties (see (4.19)).

In the present section, we give more structure to the events defining $\mathcal{G}$, and show that if $\mathcal{G}$ is chosen from a suitable class, specified by Definition 5.1 below, the associated event $\mathscr{G} = \mathscr{G}(\Lambda_N, \mathcal{G}, \mathcal{F}; \rho)$ implies an event of type $\mathcal{G}$ at scale N. This is the content of Proposition 5.2 below, see in particular (5.8), which is entirely deterministic, and constitutes the main result of this section. The event $\mathscr{G}$ thus acts as a vehicle to propagate estimates for the events $\mathcal{G}$ from scale L to scale N, which is the *bootstrapping* alluded to in the header. The probability for this bootstrapping mechanism to fail will eventually be controlled by the previously derived Proposition 4.5.

We now add one more scale $L_0 \ll L(\ll N)$ to our setup. Thus, for the remainder of Section 5 we work with three concurrent scales $N, L, L_0 \in \mathbb{N}^* = \{1, 2, \dots\}$ and an integer scaling factor K which are always (tacitly) assumed to satisfy

$$K \geq 100, N \geq c_3^{-1} 10^d \rho^{-1}\, h(KL) \text{ and } L > 100 L_0, \tag{5.1}$$

cf. above (4.10) regarding the function $h(\cdot)$, the statement of Proposition 4.3 regarding c_3 and (4.21) regarding C_4. We also introduce $\mathbb{L}_0 = L_0\mathbb{Z}^d$ and for $A \subset \mathbb{Z}^d$ the set $\mathbb{L}_0(A) = \{z \in \mathbb{L}_0 : A \cap C_{z,L_0} \neq \emptyset\}$; see below (2.19) for notation. If γ is a path in $\mathbb{Z}^d$ we abbreviate $\mathbb{L}_0(\gamma) = \mathbb{L}_0(\text{Range}(\gamma))$. In bootstrapping from scale L to N, the parameters L_0 and K will remain fixed. For this reason, the dependence of quantities on L_0 and K will be implicit in our notation.

We now introduce the family of (likely) events $\mathcal{G}_L = \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}$ of interest. Their definition will also depend on the scale L_0, which, in accordance with the previous paragraph, will not appear explicitly in our notation. We seize this opportunity to emphasize that below, when passing from one scale L to another for the family $\mathcal{G}_L$, thus varying L, the (base) scale L_0 will not change. Probability won't enter the picture until the next section. For the purposes of the present section, it is sufficient to assume that all events appearing below are implicitly defined on a joint measurable space. The events in $\mathcal{G}_L$ are specified in terms of a 'data'

$$(\mathrm{V}, \mathrm{W}, \mathscr{C}), \tag{5.2}$$

where $\mathrm{V} = \{\mathrm{V}_z : z \in \mathbb{L}\}$ and $\mathrm{W} = \{\mathrm{W}_{z,y} : z \in \mathbb{L}, y \in \mathbb{L}_0\}$ are two families of events indexed by $\mathbb{L}$ and $\mathbb{L} \times \mathbb{L}_0$, respectively, and $\mathscr{C} = \{\mathscr{C}_z : z \in \mathbb{L}\}$ is a family of finite subsets of $\mathbb{Z}^d$. In practice, $\mathscr{C}_z \subset \mathbb{Z}^d$ will be random, and the events V_z, $\mathrm{W}_{z,y}$ and the sets $\mathscr{C}_z$ will be chosen suitably depending on whether we work in the sub or supercritical phase.

Definition 5.1 (The events $\mathcal{G} = \mathcal{G}_L = \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}$). For $a \geq 0$ and $(\mathrm{V}, \mathrm{W}, \mathscr{C})$ as in (5.2), let $\mathcal{G}_z = \mathcal{G}_{z,L} (= \mathcal{G}_{z,L,L_0})$ be given by

$$\mathcal{G}_z(\mathrm{V}, \mathrm{W}, \mathscr{C}; a) = G_z(\mathrm{W}, \mathscr{C}; a) \cap \mathrm{V}_z \tag{5.3}$$

where the event $G_z = G_{z,L}$ is defined as

$$\left\{ \begin{array}{c} \text{for any crossing } \gamma \text{ of } \tilde{D}_z \setminus \tilde{C}_z, \text{ there exists a collection of points } S_\gamma \subset \mathbb{L}_0(\gamma) \\ \text{such that } |S_\gamma| \geq a \text{ and for each } y \in S_\gamma, \mathrm{W}_{z,y} \text{ occurs and } C_{y,L_0} \cap \mathscr{C}_z \neq \emptyset \end{array} \right\}. \tag{5.4}$$

Notice $\mathcal{G}_z$ depends on V, W and $\mathscr{C}$ only through V_z, $\mathrm{W}_{z,\cdot}$ and $\mathscr{C}_z$. Moreover, in the case $a = 0$, all of W, $\mathscr{C}$ and G_z become superfluous in view of (5.4), and $\mathcal{G}_z$ coincides with V_z. This simplified setup is already non-trivial and will be pertinent in the (simpler) subcritical regime; cf. §6.1.

The effectiveness of the set-up of Definition 5.1 is demonstrated in our next result, which shows that one can relate events $\mathcal{G}$ of the above form at two different scales L and N using the event $\mathscr{G}$, defined by (4.16). Even though the events $\mathcal{F}_{z,L}$ appearing in (4.16) will in practice be of the form (4.18), see also (2.21), and facilitate the switching between configurations, for the purposes of the present section it is sufficient that the inclusion appearing as condition (5.5) below holds. Thus, the reader need not think beyond (4.16) about a specific space on which the events $\mathcal{F}$ and $\mathcal{G}$ are realized for the purposes of the next result.

The switching between configurations alluded to above is reflected in the next proposition by the presence of two sets of data $(\mathrm{V}^1, \mathrm{W}^1, \mathscr{C}^1)$ and $(\tilde{\mathrm{V}}^1, \tilde{\mathrm{W}}^1, \tilde{\mathscr{C}}^1)$. The reader could however choose to *omit* this layer of complexity at first reading, i.e. assume that the events $\mathcal{F}_{z,L}$ in (4.16) are trivial (i.e. the full space), whence (5.5) below plainly holds with identical configurations $(\tilde{\mathrm{V}}^1, \tilde{\mathrm{W}}^1, \tilde{\mathscr{C}}^1) = (\mathrm{V}^1, \mathrm{W}^1, \mathscr{C}^1)$.

For the topological component of the result (see (5.6) below), we need to consider a different graph structure on $\mathbb{Z}^d$, namely $x, y \in \mathbb{Z}^d$ are called $*$-*neighbors* if $|x-y|_\infty = 1$. We can define a $*$-path, $*$-connected set, $*$-clusters in $\mathbb{Z}^d$ etc. exactly as before using this new graph structure.

Proposition 5.2 (Bootstrap of the events $\mathcal{G}_z$)**.** *Under* (5.1) *and for any choice of* $(\mathrm{V}^1, \mathrm{W}^1, \mathscr{C}^1)$, $(\tilde{\mathrm{V}}^1, \tilde{\mathrm{W}}^1, \tilde{\mathscr{C}}^1)$ *as in* (5.2)*, all* $\Lambda_N \in \mathcal{S}_N$ *(see* (4.9)*),* $a^{(1)} \geq 0$ *and* $\rho \in (0,1]$*, the following hold. If*

$$(5.5) \qquad \big(\mathcal{G}_{z,L}(\mathrm{V}^1, \mathrm{W}^1, \mathscr{C}^1; a^{(1)}) \cap \mathcal{F}_{z,L}\big) \subset \mathcal{G}_{z,L}(\tilde{\mathrm{V}}^1, \tilde{\mathrm{W}}^1, \tilde{\mathscr{C}}^1; a^{(1)}), \text{ for each } z \in \Lambda_N \cap \mathbb{L},$$

then there exists a non-empty set $\mathcal{O} \subset \mathbb{L}$ *defined measurably in* $\{1_{\mathcal{G}_{z,L}(\tilde{\mathrm{V}}^1, \tilde{\mathrm{W}}^1, \tilde{\mathscr{C}}^1; a^{(1)})} : z \in \Lambda_N \cap \mathbb{L}\}$*, such that*

(5.6) *$D_{z,L} \subset \Lambda_N$ for each $z \in \mathcal{O}$ and, writing $\Lambda_N = V_N \setminus U_N$, each $*$-component $\mathcal{O}'$ of $\mathcal{O}$ satisfies $\{0\} \cup (U_N \cap \mathbb{L}) \preceq \mathcal{O}' \preceq \partial_{\mathbb{L}}(V_N \cap \mathbb{L})$ as subsets of the coarse-grained lattice $\mathbb{L}$ (see Appendix C for notation)*

and, abbreviating $\mathcal{G}^1 = \{\mathcal{G}_{z,L}(\mathrm{V}^1, \mathrm{W}^1, \mathscr{C}^1; a^{(1)}) : z \in \mathbb{L}\}$*, one has the inclusion*

$$(5.7) \qquad \mathscr{G}(\Lambda_N, \mathcal{G}^1, \mathcal{F}; \rho) \subset \bigcap_{z \in \mathcal{O}} \mathcal{G}_{z,L}(\tilde{\mathrm{V}}^1, \tilde{\mathrm{W}}^1, \tilde{\mathscr{C}}^1; a^{(1)}) \ \Big(\subset \bigcap_{z \in \mathcal{O}} \tilde{\mathrm{V}}^1_z \Big),$$

where $\bigcap_{z \in \mathcal{O}} A_z \overset{\text{def.}}{=} \bigcap_{z \in \mathbb{L}} (A_z \cup \{\mathcal{O} \not\ni z\})$*. Furthermore, if* $\tilde{\mathrm{W}}^1_{z,\cdot} = \tilde{\mathrm{W}}^1_{0,\cdot}$ *for all* $z \in \mathbb{L}$*, then*

$$(5.8) \qquad \mathscr{G}(\Lambda_N, \mathcal{G}^1, \mathcal{F}; \rho) \subset \mathcal{G}_{0,N}(\mathrm{V}^2, \mathrm{W}^2, \mathscr{C}^2; a^{(2)}),$$

where $a^{(2)} \overset{\text{def.}}{=} 10^{-d} \lfloor \frac{\rho(1-\sigma)c_3 N}{h(KL)} \rfloor \cdot a^{(1)}$, $\mathrm{V}^2_0 \overset{\text{def.}}{=} \bigcap_{z \in \mathcal{O}} \tilde{\mathrm{V}}^1_z$, $\mathrm{W}^2_{0,y} = \tilde{\mathrm{W}}^1_{0,y}$ *for all* $y \in \mathbb{L}_0$ *and* $\mathscr{C}^2_0 \overset{\text{def.}}{=} \bigcup_{z \in \mathcal{O}} \tilde{\mathscr{C}}^1_z$ *(note that only* $\mathrm{V}^2_0, \mathrm{W}^2_{0,\cdot}$ *and* $\mathscr{C}^2_0$ *are required to define* $\mathcal{G}_{0,N}$ *in* (5.8)*; see below* (5.4)*).*

The inclusion (5.8) is precisely expressing the announced *bootstrap* mechanism: on the event $\mathscr{G}$ defined in (4.16), which is a certain composite of events of type $\mathcal{G}$ as in Definition 5.1 at scale L (along with the events $\mathcal{F}$ but let us forego this point), one witnesses an event of the same type at scale N.

Proof. Write $\Lambda_N = V_N \setminus U_N$, where Λ_N ranges among any of the choices in $\mathcal{S}_N$. Throughout the proof, we always tacitly assume that the event $\mathscr{G} = \mathscr{G}(\Lambda_N, \mathcal{G}^1, \mathcal{F}; \rho)$ occurs. Let $\Sigma \subset (\mathbb{L} \cap \Lambda_N)$ be defined as

$$(5.9) \qquad \Sigma = \{z \in \mathbb{L} : D_{z,L} \subset \Lambda_N \text{ and } \mathcal{G}_{z,L}(\tilde{\mathrm{V}}^1, \tilde{\mathrm{W}}^1, \tilde{\mathscr{C}}^1; a^{(1)}) \text{ occurs}\}.$$

We claim that any path $\gamma \subset \mathbb{L}$ connecting $\{0\} \cup (U_N \cap \mathbb{L})$ to $V_N \cap \mathbb{L}$ intersects Σ in at least

$$(5.10) \qquad k \overset{\text{def.}}{=} \lfloor \rho(1-\sigma)c_3 N / h(KL) \rfloor$$

points. Indeed consider $\bar{\gamma}$ (path in $\mathbb{Z}^d$) any extension of γ to a crossing of Λ_N. We choose $\bar{\gamma}$ in such a way that $\text{Range}(\bar{\gamma}) \subset \bigcup_{z \in \gamma} C_{z,L}$, which can always be arranged. By (4.13), there exists an admissible coarsening $\mathcal{C} = \mathcal{C}(\bar{\gamma}) \in \mathcal{A}_L^K(\Lambda_N)$ such that $\bar{\gamma}$ crosses $\tilde{D}_z \setminus C_z$ for all $z \in \mathcal{C}$. In particular it intersects C_z whence $z \in \text{Range}(\gamma)$. But by definition of $\mathscr{G}$ (which is in force), see (4.16), and owing to (4.12), one can extract from $\mathcal{C}$ a sub-collection $\tilde{\mathcal{C}}$ of cardinality at least k as given by (5.10) such that $\mathcal{G}_{z,L}(\mathrm{V}^1, \mathrm{W}^1, \mathscr{C}^1; a^{(1)}) \cap \mathcal{F}_{z,L} \subset \mathcal{G}_{z,L}(\tilde{\mathrm{V}}^1, \tilde{\mathrm{W}}^1, \tilde{\mathscr{C}}^1; a^{(1)})$ (see (5.5)) occurs for all $z \in \tilde{\mathcal{C}}$. The claim follows. In light of the previous paragraph, Proposition C.1 applies on $\mathbb{L}$ (rather than $\mathbb{Z}^d$) with Σ as in (5.9), $U = U_N \cap \mathbb{L}$, $V = V_N \cap \mathbb{L}$ and k as in (5.10), yielding $*$-connected sets $O_1, \dots, O_\ell$ satisfying items (a)-(c). Letting $\mathcal{O} \stackrel{\text{def.}}{=} \bigcup_i O_i$, it then immediately follows from (a) that any component $\mathcal{O}'$ satisfies $U_N \cap \mathbb{L} \preceq \mathcal{O}' \preceq \partial_{\mathbb{L}}(V_N \cap \mathbb{L})$. The other properties required in (5.6) (including the measurability requirements on $\mathcal{O}$ above (5.6)) plainly hold. Moreover, since $\mathcal{O} \subset \Sigma$, the first inclusion in (5.7) is immediate on account of (5.9). The second inclusion in (5.7) follows plainly from (5.3).

It remains to prove (5.8). The fact that V_0^2 as defined below (5.8) occurs on $\mathscr{G}$ is already implied by (5.7), hence in view of (5.3) it remains to show that $\mathscr{G}$ implies the occurrence of $G_{0,N} = G_{0,N}(\mathrm{W}^2, \mathscr{C}^2; a^{(2)})$ as defined in (5.4). Thus let γ now be a crossing of $\tilde{D}_{0,N} \setminus \tilde{C}_{0,N} (= \Lambda_N)$. By definition of $\mathrm{W}^2, \mathscr{C}^2$, see below (5.8), the proof is complete once we extract a collection of points $S_\gamma \subset \mathbb{L}_0(\gamma)$ such that

$$|S_\gamma| \geq a^{(2)} \text{ and for each } y \in S_\gamma,\, \tilde{\mathrm{W}}^1_{0,y} \text{ occurs and } C_{y,L_0} \cap \Big(\bigcup_{z \in \mathcal{O}} \tilde{\mathscr{C}}^1_z \Big) \neq \emptyset. \tag{5.11}$$

Consider $\gamma' \subset \mathbb{L}$, the $*$-path obtained from γ by retaining the sequence of all z's intersected by Range(γ), in the order visited by γ. By construction of $\mathcal{O}$ and item (b) in Proposition C.1, γ' intersects $\mathcal{O}$ in at least k points, with k as in (5.10). If $z \in \mathcal{O} \cap \text{Range}(\gamma')$ is any such point, using the fact that $\tilde{D}_{z,L}$ is contained in Λ_N (see (5.9) and recall that $\mathcal{O} \subset \Sigma$), it follows that the path γ must cross $\tilde{D}_{z,L} \setminus \tilde{C}_{z,L}$. Moreover, still using that $\mathcal{O} \subset \Sigma$, (5.9), (5.5) and (5.3) yield that $G_{z,L}(\tilde{\mathrm{W}}^1, \tilde{\mathscr{C}}^1; a^{(1)})$ occurs. By definition, see (5.4), this implies that there exists a set $S_\gamma(z) \subset \mathbb{L}_0(\gamma)$ of cardinality at least $a^{(1)}$ and such that for each $y \in S_\gamma(z)$, the event $\mathrm{W}_{z,y} = \mathrm{W}_{0,y}$ occurs and $C_{y,L_0} \cap \tilde{\mathscr{C}}^1_z$ is not empty.

The claim (5.11) now follows immediately by extracting S_γ from $\bigcup_{z \in \mathcal{O} \cap \text{Range}(\gamma')} S_\gamma(z)$, by retaining at least a fraction 10^{-d} of points z in the union, thereby ensuring that the sets $\tilde{D}_{z,L}$ are disjoint. It follows that the cardinalities of $S_\gamma(z)$ as z varies over this thinning of $\mathcal{O} \cap \text{Range}(\gamma')$ are additive, yielding an overall cardinality for S_γ at least $10^{-d} k a^{(1)} = a^{(2)}$, as required. The other requirements on S_γ in (5.11) are immediate from the previous paragraph. □

6 Upper bounds for $\mathcal{V}^u$

In this section, we prove two results for random interlacements, Theorems 6.1 and 6.3 below, using the framework laid out in Sections 4 and 5, involving in particular, Propositions 4.5 and 5.2. Theorem 6.1, which is proved in §6.1, deals with the sub-critical regime $u > u_*$ and represents a first (simple) *illustration* of these methods. This result is already known from Theorem 3.1 in [34] for $d \geq 4$ and more recently from Theorem 1.3 in [35] for $d = 3$. The main focus of the section, however, is Theorem 6.3, proved in §6.2, which concerns the super-critical regime $u < u_*$ and is considerably more involved. Indeed, the both sub- and super-critical results share a similar proof architecture, but the *complete* proof of Theorem 6.3 rests on several intermediate results in addition to Propositions 4.5 and 5.2, the proofs of which span the Sections 3, 7 and 8. One of the main results of the introduction concerning $\mathcal{V}^u$, namely Theorem 1.4, will immediately follow from these results, as shown at the end of this section. The other result, i.e. Theorem 1.1, will be derived along the way.

6.1. Sub-critical phase. Recall that B^2_N denotes the ball of radius N around 0 in the ℓ^2 (Euclidean) norm. We return to the choice of Euclidean norm in Remark 6.4 below. The aim of this short section is to prove the following result.

Theorem 6.1 (Sub-critical regime)**.** *For all $u > u_*$,*

$$\sup_{N\geq 1} N^{-1} \log \mathbb{P}\big[0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} \partial B_N^2\big] \leq -c(u), \quad \textit{if } d \geq 4; \tag{6.1}$$

$$\limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}\big[0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} \partial B_N^2\big] \leq -\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*})^2, \quad \textit{if } d = 3. \tag{6.2}$$

Proof. We use the framework of Sections 4–5 albeit not harnessing its full strength. Let $u > 0$ and $d \geq 3$. We start by specifying the collection $\mathrm{V} = \{\mathrm{V}_z : z \in \mathbb{L}\}$ from (5.2) by setting $\mathrm{V}_z = \mathrm{Dis}_z(\widehat{\mathcal{V}}_{\mathbb{L}}, u)$, where, for any $z \in \mathbb{L}$ and $\widehat{\mathcal{V}}_{\mathbb{L}} \in \{\overline{\mathcal{V}}_{\mathbb{L}}, \mathcal{V}_{\mathbb{L}}, \widetilde{\mathcal{V}}_{\mathbb{L}}\}$ (see (2.20) for notation),

$$\mathrm{Dis}_z(\widehat{\mathcal{V}}_{\mathbb{L}}, u) \stackrel{\text{def.}}{=} \big\{\tilde{C}_z \stackrel{\widehat{\mathcal{V}}_z^u}{\longleftrightarrow\!\!\!\!\!\!/\;\;} \partial\tilde{D}_z\big\}. \tag{6.3}$$

We refer to $\mathrm{V} = \mathrm{Dis}(\widehat{\mathcal{V}}_{\mathbb{L}}, u)$ as the collection of events thereby obtained. In writing $\widehat{\mathcal{V}}_{\mathbb{L}}$ in the sequel, we tacitly imply that the conclusions hold for any choice of $\widehat{\mathcal{V}}_{\mathbb{L}} \in \{\overline{\mathcal{V}}_{\mathbb{L}}, \mathcal{V}_{\mathbb{L}}, \widetilde{\mathcal{V}}_{\mathbb{L}}\}$. Recalling Definition 5.1, it follows that

$$\mathcal{G}_z(\mathrm{Dis}(\widehat{\mathcal{V}}_{\mathbb{L}}, u), a = 0, \mathrm{W}, \mathscr{C}) \stackrel{(5.3),(5.4)}{=} \mathrm{Dis}_z(\widehat{\mathcal{V}}_{\mathbb{L}}, u) \tag{6.4}$$

regardless of the choice of families W and $\mathscr{C}$, cf. also (5.2). These will play no role in the sequel and are therefore omitted from all notation. This also makes superfluous the scale L_0 involved in the definition of $(\mathrm{W}, \mathscr{C})$ (and the event G_z in (5.4)). In the sequel we always assume that the scales N, L and the scaling factor K satisfy (5.1) with $L_0 = 1$.

Observe that (6.4) asserts that $\mathcal{G} = \mathrm{Dis}(\widehat{\mathcal{V}}_{\mathbb{L}}, u)(= \mathrm{V})$, which feeds into the definition of the event $\mathscr{G} = \mathscr{G}(\Lambda_N, \mathcal{G}, \mathcal{F}; \rho)$ from (4.16). We proceed to explain why, for suitable choice of levels u_i, v_i for $\mathcal{F} = \{\mathcal{F}_{z,L} : z \in \mathbb{L}\}$ in (4.18), the event $\mathscr{G}$ relates to the task of bounding the connection probability appearing in (6.1)–(6.2) (the relation will be given by the inclusion (6.6) below). To this effect, we start by observing that, for any $0 < u < v$, the inclusions

$$\begin{aligned} &\mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u) \cap \mathcal{F}_z^{v,u} \subset \mathrm{Dis}_z(\overline{\mathcal{V}}_{\mathbb{L}}, v), \text{ and} \\ &\mathrm{Dis}_z(\overline{\mathcal{V}}_{\mathbb{L}}, u) \cap \mathcal{F}_z^{u,v} \subset \mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, v) \end{aligned} \tag{6.5}$$

hold: indeed these follow from the observation that the event $\mathrm{Dis}_z(\widehat{\mathcal{V}}_{\mathbb{L}}, u)$ in (6.3) is decreasing in the configuration $\widehat{\mathcal{V}}_z^u$, along with the definitions of $\overline{\mathcal{V}}_z^u$ and $\mathcal{V}_z^u$ in (2.20) and of the relevant event $\mathcal{F}_z^{u,v}$ in (2.21).

Focusing on the first inclusion (6.5) (the second one will be used only later), we now set $\mathcal{F}_{z,L} = \mathcal{F}_z^{v,u}$, which is of the form (4.18) with $k = 1$ (and $u_1 = v$, $v_1 = u$). With this choice, the first inclusion in (6.5), in combination with (6.4), tells us that the condition (5.5) of Proposition 5.2 is satisfied by $\mathrm{V}_z^1 = \mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u)$, $\tilde{\mathrm{V}}_z^1 = \mathrm{Dis}_z(\overline{\mathcal{V}}_{\mathbb{L}}, v)$, $a^1 = 0$ (omitting references to W^1, $\mathscr{C}^1$,$\tilde{\mathrm{W}}^1$, $\tilde{\mathscr{C}}^1$ and a^1), which are all declared under $\mathbb{P}$. Thus Proposition 5.2 applies (on the probability space carrying $\mathbb{P}$) and we obtain from (5.6)–(5.7) that, for any N, L, K satisfying (5.1) with $L_0 = 1$, any $\rho \in (0, \frac{1}{2}]$, any $0 < u < v$ and any choice of $\Lambda_N \in \mathcal{S}_N$ (recall (4.9)), writing $\Lambda_N = V_N \setminus U_N$,

> there exists a (random) $*$-connected $\mathcal{O}' \subset \mathbb{L}$ such that $D_z = D_{z,L} \subset \Lambda_N$ for each $z \in \mathcal{O}'$, $\{0\} \cup U_N \cap \mathbb{L} \preceq \mathcal{O}' \preceq \partial_{\mathbb{L}}(V_N \cap \mathbb{L})$ (see (C.1) for notation) and on $\mathscr{G}(\Lambda_N, \mathrm{Dis}(\mathcal{V}_{\mathbb{L}}, u), \mathcal{F}_L^{v,u}; \rho)$, the event $\mathrm{Dis}_z(\overline{\mathcal{V}}_{\mathbb{L}}, v)$ occurs for each $z \in \mathcal{O}'$.

Let γ now be a crossing of Λ_N, i.e. a (nearest neighbor) path on $\mathbb{Z}^d$ intersecting both U_N and ∂V_N (see above (4.9)). Its coarse-graining $\gamma_{\mathbb{L}}$ obtained as the (ordered) sequence of points $z \in \mathbb{L}$ such that γ visits C_z is a connected set in $\mathbb{L}$, which owing to the above must intersect $\mathcal{O}'$ on the event $\mathscr{G}(\Lambda_N, \mathrm{Dis}(\mathcal{V}_{\mathbb{L}}, u, \mathcal{F}_L^{v,u}; \rho)$. Thus, let $z \in \mathrm{range}(\gamma_{\mathbb{L}}) \cap \mathcal{O}'$. It follows that γ must cross $\tilde{D}_z \setminus \tilde{C}_z$ and that $\mathrm{Dis}_z(\overline{\mathcal{V}}_{\mathbb{L}}, v)$ occurs. In particular, this implies that γ cannot lie inside $\mathcal{V}^v$. All in all,

$$\mathscr{G}(\Lambda_N, \mathrm{Dis}(\mathcal{V}_{\mathbb{L}}, u), \mathcal{F}_L^{v,u}; \rho) \subset \big\{U_N^0 \overset{\mathcal{V}^v}{\not\longleftrightarrow} V_N\big\}, \quad U_N^0 = U_N \cup \{0\}. \tag{6.6}$$

In view of (6.6), we now apply Proposition 4.5, from which the desired bounds (6.1) and (6.2) will eventually follow. Let $u > u_*$ and consider any $\varepsilon \in \big(0, ((\frac{u}{u_*})^{\frac{1}{10}} - 1) \wedge \frac{1}{10}\big)$. We proceed to verify conditions (4.19)–(4.20) inherent to Proposition 4.5. To this effect, from [20, Theorem 1.2-(i)] and a straightforward union bound, we know that for any $z \in \mathbb{L}$ and $L \geq 1$,

$$\mathbb{P}[\mathrm{Dis}_z(\overline{\mathcal{V}}_{\mathbb{L}}, u_*(1+\varepsilon))] \geq 1 - C(\varepsilon)e^{-L^c}. \tag{6.7}$$

We aim to transfer the bound (6.7) to the configuration $\widetilde{\mathcal{V}}_{\mathbb{L}}$, cf. (2.20) at a slightly different level than $u_*(1+\varepsilon)$. We do this in two steps, using the intermediate configuration $\mathcal{V}_{\mathbb{L}}$. In view of the second inclusion in (6.5), we obtain from (6.7) that for any $K \geq C(\varepsilon)$,

$$\mathbb{P}[\mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_*(1+\varepsilon)^2)] \geq \mathbb{P}[\mathrm{Dis}_z(\overline{\mathcal{V}}_{\mathbb{L}}, u_*(1+\varepsilon))] - \mathbb{P}[(\mathcal{F}_z^{u_*(1+\varepsilon), u_*(1+\varepsilon)^2})^c] \overset{(2.22)}{\geq} 1 - C(\varepsilon)e^{-L^c}. \tag{6.8}$$

To proceed, we obain from the definition of the event $\mathrm{Incl}_z^{\varepsilon,m}$ in (2.15) and the previously alluded monotonicity of $\mathrm{Dis}_z(\widehat{\mathcal{V}}_{\mathbb{L}}, u)$ in $\widehat{\mathcal{V}}_z^u$ that under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_z$, any $v > 0$, $\epsilon \in (0, \frac{1}{2})$ and $L \geq C(v, \epsilon)$, the inclusions

$$\begin{aligned} &\mathrm{Dis}_z(\widetilde{\mathcal{V}}_{\mathbb{L}}, v) \cap \mathrm{Incl}_z^{\frac{\epsilon}{6}, \lfloor v\,\mathrm{cap}(D_z)\rfloor} \subset \mathrm{Dis}_z\big(\mathcal{V}_{\mathbb{L}}, (1+\epsilon)v\big) \text{ and} \\ &\mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, v) \cap \mathrm{Incl}_z^{\frac{\epsilon}{6}, \lfloor v\,\mathrm{cap}(D_z)\rfloor} \subset \mathrm{Dis}_z\big(\widetilde{\mathcal{V}}_{\mathbb{L}}, (1+\epsilon)v\big) \end{aligned} \tag{6.9}$$

hold. Now using the second inclusion in (6.9) with $v = u_*(1+\varepsilon)^2$ and $\delta = \varepsilon$, we can use the coupling $\mathbb{Q}_{\{z\}}$ from Lemma 2.1 to deduce that for all $L \geq C$ and $K \geq \frac{18C_8}{\varepsilon}$ (so that condition (2.16) is satisfied in view of Proposition A.1),

$$\begin{aligned} \mathbb{P}[\mathrm{Dis}_z(\widetilde{\mathcal{V}}_{\mathbb{L}}, u_*(1+\varepsilon)^3)] &\overset{(2.17)}{\geq} \mathbb{P}[\mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_*(1+\varepsilon)^2)] - \widetilde{\mathbb{P}}_z\big[\big(\mathcal{U}_z^{\frac{\varepsilon}{6}, \lfloor u_*(1+\varepsilon)^2\,\mathrm{cap}(D_z)\rfloor}\big)^c\big] \\ &\overset{(6.8)+(2.14)}{\geq} 1 - C(\varepsilon)e^{-L^c}, \end{aligned} \tag{6.10}$$

where in the last step we also used the lower bound $\mathrm{cap}(D_z) \geq cL$ (see (A.8)) valid in all dimensions $d \geq 3$.

We have now gathered all the ingredients to apply Proposition 4.5 and conclude the proof. We choose $\rho = \frac{1}{2C_4}$ where $C_4(d) = 1$ for $d \geq 4$ (see (4.21)–(4.22)), $k = 1$, $\boldsymbol{u} = u_1 = u_*(1+\varepsilon)^5$, $\boldsymbol{v} = v_1 = u_*(1+\varepsilon)^4$ (cf. (4.18)) and $\widetilde{\mathcal{G}}_{z,L} = \mathrm{Dis}_z(\widetilde{\mathcal{V}}_{\mathbb{L}}, u_*(1+\varepsilon)^3)$ and $\mathcal{G}_{z,L} = \mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_*(1+\varepsilon)^4) = \mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, v_1)$. With these choices, applying (6.6) with $v = u_1$ and $u = v_1$, the associated event $\mathscr{G} = \mathscr{G}(\Lambda_N, \mathcal{G}_L, \mathcal{F}_L^{u_1,v_1}; \rho)$ of concern in Proposition 4.5 satisfies $\big\{U_N^0 \overset{\mathcal{V}^{u_1}}{\longleftrightarrow} V_N\big\} \subset \mathcal{G}^c$. Thus, provided the conditions (4.19)–(4.20) are met, the bounds (4.21) and (4.22) apply and yield an upper bound on the former connection event. The fact that (4.19) holds for $\widetilde{\mathcal{G}}_{z,L}$, $\mathcal{G}_{z,L}$ as above and with the choices $\varepsilon_L = \frac{\varepsilon}{6}$, $m_L = \lfloor u_*(1+\varepsilon)^3\,\mathrm{cap}(D_z)\rfloor$ is an immediate consequence of the first inclusion in (6.9) with $\delta = \varepsilon$ and $v = u_*(1+\varepsilon)^3$. Having fixed, ε_L, m_L, the condition (4.20) holds by virtue of (6.10) and (2.14) for $p_L = C(\varepsilon)e^{-L^c}$, $K_0 = C(\varepsilon)$ and $L_0 = C(\varepsilon)$ and suitable $\beta' = c > 0$ uniform in ε.

In dimensions $d \geq 4$, choosing $K = K(\varepsilon)$, $L = L(\varepsilon)$ sufficiently large and $\Lambda_N = B_N^2$ (recall (4.9)), (4.22) immediately yields (6.1) for $u = u_1 = u_*(1+\varepsilon)^5$. Since $\varepsilon \downarrow 0$ as $u \downarrow u_*$ and by monotonicity, this concludes the verification of (6.1). For $d = 3$, (4.21) yields instead for the choice $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$ that for all $N \geq 1$,

$$\mathbb{P}[\tilde{C}_{0,N} \overset{\mathcal{V}^{u_1}}{\not\longleftrightarrow} \tilde{D}_{0,N}] \overset{(6.3)}{=} \mathbb{P}[\mathrm{Dis}_z(\overline{\mathcal{V}}_{\mathbb{L}}, u_1)] \geq 1 - C(\varepsilon) e^{-\frac{N}{(\log N)^\beta}} \tag{6.11}$$

for some absolute constant $\beta = \beta(\beta') \in (1, \infty)$. To deduce (6.2), we apply Proposition 4.5 again, now starting with this *improved* estimate instead of (6.7) and running the same procedure as above. We get that the conditions (4.19)–(4.20) of Proposition 4.5 are now satisfied by the events $\widetilde{\mathcal{G}}_{z,L} = \mathrm{Dis}_z(\widetilde{\mathcal{V}}_{\mathbb{L}}, u_*(1+\varepsilon)^7)$ and $\mathcal{G}_{z,L} = \mathrm{Dis}_z(\mathcal{V}_{\mathbb{L}}, u_*(1+\varepsilon)^8)$ for $\varepsilon_L = \frac{\varepsilon}{6}$, $m_L = \lfloor u_*(1+\varepsilon)^7 \,\mathrm{cap}(D_z) \rfloor$, $p_L = C(\varepsilon)\exp\{-\frac{L}{(\log L)^\beta}\}$ with β as in (6.11) and $K_0 = C(\varepsilon)$. In particular, this entails that (4.20) is now satisfied for any choice of $\beta' < 1$, say $\beta' = \frac{3}{4}$ (any number larger than $\frac{1}{2}$ will do). We then deduce from (4.21) and the inclusion in (6.6) with $\boldsymbol{u} = u$, $\boldsymbol{v} = u_*(1+\varepsilon)^8$ (by choice of ε, $u_*(1+\varepsilon)^8 < u$), any $\rho \in (0, \frac{1}{2C_4})$ and $\delta' \in (0,1)$, $K = \sqrt{\log\log e^2 N}$ and $\Lambda_N = B_N^2$ that

$$\limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_N^2] \leq -(1-\delta)(1-C_4\rho)\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*(1+\varepsilon)^8})^2.$$

Sending δ, ρ and ε to 0 yields (6.2). □

Remark 6.2. Although not optimal, the following result, which incorporates noise and is obtained by a variation of the above argument, will be useful below. Recall $\mathsf{N}_\delta(\mathcal{V})$ from (2.23)–(2.24). There exists $C_5 < \infty$ such that, for all $u < u_*$, $\delta \leq c_5(u) (> 0)$, and $N \geq 1$

$$\mathbb{P}\big[C_{0,N} \overset{\mathsf{N}_\delta(\mathcal{V}^u)}{\longleftrightarrow} \partial D_{0,N}\big] \geq 1 - C(u) e^{-c(u)N/(1\vee \log N)^{C_5}} \tag{6.12}$$

(in fact, the error in (6.12) should be an exponential in N when $d = 3$ and super-exponential when $d \geq 4$, as in the main result in [48], which concerns the case $\delta = 0$, but we will not need this stronger result). Further we can choose $C_5 = 0$ when $d \geq 4$. The estimate (6.12) can be readily deduced using the above framework, as we now briefly explain.

To obtain (6.12), it is in fact enough to show an analogue of the *a-priori estimate* (6.7) but replacing the event $\widehat{\mathcal{G}}_{z,L} = \mathcal{G}_{z,L}(\mathrm{Dis}(\widehat{\mathcal{V}}_{\mathbb{L}}), \dots)$ in (6.4) by an analogue of the local uniqueness event [24, (5.43)], involving configurations in $\mathsf{N}_\delta(\widehat{\mathcal{V}}_{\mathbb{L}}^{\cdot})$ at levels close to u instead of $(\{\chi^z \geq \cdot\})_{z\in\mathbb{L}}$ (see the event V_z defined in (6.18) in the next subsection). As we now briefly explain, once this a-priori estimate is shown, (6.12) follows in exactly the same way as (6.11). Indeed, by design of $\overline{\mathcal{G}}_{z,L}$ (call $z \in \mathbb{L}$ *good* if $\overline{\mathcal{G}}_{z,L}$ occurs), the complement of the event on the left of (6.12) implies the *absence* of a path of good vertices in $\mathbb{L}$ joining $C_{0,N}$ and $\partial D_{0,N}$. By a standard duality argument, this implies the existence of a macroscopic $*$-path of *bad* vertices in the annulus $D_{0,N} \setminus C_{0,N}$ (in fact even a 'surface'), which in turn implies $\mathscr{G}^c(\Lambda_{cN}, \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}, \mathcal{F}_L^{u,v}; \rho)$ for some $c > 0$ and $u, v > 0$ up to an inconsequential spatial shift. This replaces (6.6) and leads overall to (6.12) with $C_5 = \beta$ as supplied by (4.21) when $d = 3$. When $d \geq 4$ one can even choose $C_5 = 0$ on account of (4.22).

We will not give a full proof of the a-priori estimate, and instead refer to [24, Lemma 5.16] and its proof for a similar argument. We now highlight the necessary changes. In essence, one applies the same renormalisation argument as in that proof but replaces the (seed) events A_x^1 and A_x^2 in [24, (5.60), (5.61)] at scale $L_0 \geq 1$ by obvious analogues involving configurations in $\mathsf{N}_\delta(\mathcal{V}^{\cdot})$ at levels close to u, and A_x^3 in [24, (5.62)] by the event $\{\mathsf{U}_y > \delta, y \in D_{x,L_0}\}$. When it occurs, the latter event implies in view of (2.23) that $\mathsf{N}_\delta(\mathcal{V}) \cap D_{x,L_0} = \mathcal{V} \cap D_{x,L_0}$ for any configuration $\mathcal{V}$. Upon choosing $\delta < L_0^{C(\lambda)}$ with $\lambda = \lambda(d)$ as below [24, (5.62)], one obtains that $\lim_{L_0\to\infty} \mathbb{P}[\widetilde{A}^k_{x,0}] = 1$ for $k = 3$ with $\widetilde{A}^3_{x,0}$ as in [24, (5.63)]. The cases of $k = 1, 2$ follow similarly as in [24] using the bounds (1.13) and (1.14) from [20, Theorem 1.3]. One then still applies [12, Proposition 7.1] (which also concerns interlacements) to deduce (analogues of) the bounds [24, (5.64)] upon choosing $L_0 = L_0(u)$ large enough, thereby also fixing $c_5(u) = L_0^{-C}$. The desired a-priori estimate for the event $\mathcal{G}_0 = \mathcal{G}_{0,L}$ thus follows as in [24, Lemma 5.16] at scales $L = L_n = L_0\ell_0^n$, $n \geq 1$, with ℓ_0 as chosen in [24]. The case of general $L \in [L_n, L_{n+1}]$ is taken care of by straightforward union bounds using (a bounded number of) spatially shifted copies of the event $\mathcal{G}_{0,L_n}$.

6.2. Super-critical phase. We now turn to matters when $u < u_*$. This will also involve the *local uniqueness* event

$$\text{LocUniq}(N, u) \overset{\text{def.}}{=} \{\mathcal{V}^u \text{ has a unique cluster crossing } B^2_{2N} \setminus B^2_N\} \tag{6.13}$$

as well as $2\text{-arms}(N, u)$, the two arms event in $B^2_{2N} \setminus B^2_N$, which refers to the existence of (at least) two crossings of $(B^2_{2N} \setminus B^2_N)$ in $\mathcal{V}^u$ that are not connected in $\mathcal{V}^u \cap (B^2_{2N} \setminus B^2_N)$. The two-arms event is a subset of $\text{LocUniq}(N, u)^c$ – the latter does not preclude the absence of a crossing cluster. We can of course use any annulus Λ_N listed in (4.9) in place of $B^2_{2N} \setminus B^2_N$ for these two events and denote them as $\text{LocUniq}(\Lambda_N, u)$ and $2\text{-arms}(\Lambda_N, u)$ respectively. The analogue of Theorem 6.1 in the case $u < u_*$ is the following result.

Theorem 6.3 (Super-critical regime)**.** *For all* $u \in (0, u_*)$,

$$\sup_{N\geq 1} N^{-1} \log \mathbb{P}\big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B^2_N, 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\big] \leq -c(u), \quad \textit{if } d \geq 4; \tag{6.14}$$

$$\limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}\big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B^2_N, 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\big] \leq -\frac{\pi}{3}(\sqrt{u} - \sqrt{u_*})^2, \quad \textit{if } d = 3. \tag{6.15}$$

Moreover, the bounds (6.14) *and* (6.15) *also hold for the events* $\text{LocUniq}(N, u)^c$ *and* $2\text{-arms}(N, u)$.

Remark 6.4 (Norms). The bounds (6.1) and (6.14) valid for $d \geq 4$ continue to hold (up to possibly modifying the value of $c(u)$) for any ℓ^p-ball B^p_N of radius $N \geq 1$, for all values of $p \in [1, \infty]$. This follows immediately by inclusion using equivalence of norms. The limits (6.2) and (6.15) valid in dimension three (including those concerning $\text{LocUniq}(N, u)^c$ and $2\text{-arms}(N, u)$) remain unchanged if B^2_N is replaced by B^p_N for any $p \in [2, \infty]$ and in particular for the ℓ^∞-ball B_N, but not when $p \in [1, 2)$, in which case the value $\frac{\pi}{3}$ decreases, as can be seen by inspecting our proofs. The choice of B^2_N stands out as it immediately implies corresponding bounds on $\tau^{\text{tr}}_u(x)$ for $N = |x|$ by inclusion, cf. Theorem 1.4 and its proof later in this subsection. This is important in view of the desired (as will turn out, sharp(!)) bound we aim to obtain in (1.13), which involves the *Euclidean* distance, cf. [25].

The remainder of this section is concerned with the proof of Theorem 6.3. This will harness the full strength of our setup from Sections 4–5, and in particular of the event $\mathcal{G}_z$ from Definition 5.1 (in contrast, see (6.4)).

We now prepare the ground for the proof of Theorem 6.3. We proceed by specifying the data $(\mathrm{V}, \mathrm{W}, \mathscr{C})$ in (5.2) that will play a role in the sequel. This fully determines the family of events $\mathcal{G}$ in Definition 5.1 via (5.3). In fact we will have to work with two sets $(\mathrm{V}, \mathrm{W}^{\mathrm{I}}, \mathscr{C})$ and $(\mathrm{V}, \mathrm{W}^{\mathrm{II}}, \mathscr{C})$ of data, roughly speaking in order to deal with small and large numbers of excursions; see Remark 6.8 below for more on this. We will refer to I and II as *types*.

We start by collecting here all the parameters that will appear in the sequel. These are:

(6.16) $u_0 \in (0,\infty)$, $u < u_1 < u_2 < u_3 \in (0,u_*)$ and $u_4 \in (0,\infty)$, $a \in \mathbb{N}^*$, $\nu \geq 0$, $\delta \in [0,\frac{1}{2})$, scales $N > L > L_0$, K and ρ satisfying (5.1), a scale $L_0^- \in \mathbb{N}^*$ with $L_0 > 100L_0^-$.

To keep notations reasonable, we will henceforth routinely suppress the dependence of parametrized quantities (in particular, events or sets) on parameters which stay fixed in a given context or are otherwise clear.

Recall that $\mathbb{L} = L\mathbb{Z}^d$, that $\widehat{\mathcal{V}}_{\mathbb{L}} \in \{\overline{\mathcal{V}}_{\mathbb{L}}, \mathcal{V}_{\mathbb{L}}, \widetilde{\mathcal{V}}_{\mathbb{L}}\}$ (see below (2.20) for notation), that $(\mathcal{V})_\delta$ refers to a noised configuration (see below (2.24)) and the boxes $C_z, \tilde{C}_z, D_z, \dots$ from (2.19). We will also use $\widehat{Z} = \widehat{Z}_{\mathbb{L}} = \{\widehat{Z}_z^u : z \in \mathbb{L}, u > 0\}$ to refer *collectively* to any one of the three sequences defined in (2.20).

In anticipation of the arguments to follow, we parametrize events below in terms of (finite) sequences $Z = (Z_j)_{1\leq j\leq n_Z}$ (and sometimes $Z', Z'', \dots$) of excursions rather than through their associated vacant sets $\mathcal{V} = \mathcal{V}(Z)$, cf. (2.18)–(2.20). Even though this could in some cases be dispensed with, this shift in perspective will be instrumental to our arguments.

- **The sets $\mathscr{C} = \{\mathscr{C}_z : z \in \mathbb{L}\}$.** For Z an arbitrary (finite) sequence of excursions, the set $\mathscr{C}_z(Z,Z',\delta)$ ($= \mathscr{C}_{z,L}(Z,Z',\delta)$) is the subset of $\mathbb{Z}^d$ obtained as the union of all clusters in $D_z \cap (\mathcal{V}(Z))_\delta$ containing a crossing of $\tilde{D}_z \setminus \tilde{C}_z$ in $(\mathcal{V}(Z'))_{2\delta}$. We also set $\mathscr{C}_z(Z,\delta) = \mathscr{C}_z(Z,Z,\delta)$. With this notation, we define

$$\mathscr{C}_z = \mathscr{C}_{z,L} = \mathscr{C}_z(\widehat{Z},\delta,u_1,u_3) = \mathscr{C}_z(\widehat{Z}_z^{u_1},\widehat{Z}_z^{u_3},\delta). \tag{6.17}$$

The events comprising V, which we introduce next, are of "local uniqueness" flavor, cf. (6.13).

- **The events $\mathrm{V} = \{\mathrm{V}_z : z \in \mathbb{L}\}$.** For Z, Z', Z'' any finite sequences of excursions, set

$$\mathrm{V}_{z,L}(Z,Z',Z'',\delta) = \left\{ \begin{array}{c} C_z \text{ is connected to } \partial D_z \text{ in } (\mathcal{V}(Z''))_{2\delta} \text{ and all clusters of} \\ D_z \cap (\mathcal{V}(Z'))_{2\delta} \text{ crossing } \tilde{D}_z \setminus \tilde{C}_z \text{ are connected} \\ \text{inside } D_z \cap (\mathcal{V}(Z))_\delta \end{array} \right\}$$

  and abbreviate $\mathrm{V}_z(Z,\delta) = \mathrm{V}_z(Z,Z,Z,\delta) = \mathrm{V}_{z,L}(Z,Z,Z,\delta)$. We then set

$$\mathrm{V}_z = \mathrm{V}_{z,L} = \mathrm{V}_z(\widehat{Z},\delta,u_1,u_2,u_3) \stackrel{\text{def.}}{=} \mathrm{V}_z(\widehat{Z}_z^{u_1},\widehat{Z}_z^{u_2},\widehat{Z}_z^{u_3},\delta) \tag{6.18}$$

  and abbreviate $\mathrm{V}_z(\widehat{Z},\delta,u) = \mathrm{V}_z(\widehat{Z},\delta,u,u,u)$. We remove δ from all notation when $\delta = 0$.

It remains to specify the events W, which are more involved and will be introduced shortly (and can be of one of two types). For the time being, notice that the set $\mathscr{C}_z$ is in fact a cluster, i.e. a connected set, on the event V_z. We are ultimately interested in the special case $\mathrm{V}_z(\overline{Z},u) = \mathrm{V}_z(\overline{Z},\delta = 0,u)$, which deals with the actual vacant set of random interlacements (cf. (2.20)) and entails that there is *no* sprinkling.

So far one could have afforded to express the quantities $\mathscr{C}_z$ and V_z directly in terms of vacant sets $\widehat{\mathcal{V}}_{\mathbb{L}}$ (see below (2.20) for notation) via the identification $\widehat{\mathcal{V}}_z^u = \mathcal{V}(\widehat{Z}_z^u)$. In the sequel we will deal with a more general class of events involving subsets of excursions for which this factorization property no longer holds. We now lay out the ground for this.

We will consider two basic collections of (sub-)sequences of excursions. Given a (finite) sequence $Z = (Z_j)_{1\leq j\leq n_Z}$ of excursions (see above (2.18)) and any $\nu \in [0,\infty]$, we introduce

$$Z_+(\nu) \stackrel{\text{def.}}{=} \left\{ \begin{array}{l} \text{the collection of all sequences } (Z_j)_{j\in J} \text{ with } J \subset \\ \{1,\dots,n_Z\} \text{ such that } \{1,\dots,\lfloor\nu\rfloor\} \subset J \end{array} \right. \tag{6.19}$$

and

$$Z_-(\nu) \overset{\text{def.}}{=} \begin{cases} \text{the collection of all sequences } (Z_j)_{j\in J} \text{ with } J \subset \\ \{1,\dots,n_Z\} \text{ such that } J \subset \{1,\dots,\lfloor \nu \rfloor\}. \end{cases} \tag{6.20}$$

In words, the sequence Z' belongs to $Z_{\pm}(\nu)$ if the elements of Z' contain/are contained in the first $\lfloor \nu \rfloor$ excursions from Z. By convention, $Z_+(\nu = 0)$ comprises all subsequences of Z whereas $Z_-(\nu = 0)$ consists only of the empty sequence.

Rather than dealing directly with $\mathrm{V}_z(\widehat{Z}^u_z)$, we will bound the complement of a stronger (i.e. smaller) event involving subsequences of the excursions forming $\mathrm{V}_z(\widehat{Z}^u_z)$, which we set out to introduce. Given a (finite) sequence $Z = (Z_j)_{1\le j\le n_Z}$ of excursions and any $\nu \in [0,\infty]$, we let

$$Z(\nu) \overset{\text{def.}}{=} \bigcup_{j\ge 0} \big(Z_+(j) \cap Z_-(j + \lfloor \nu \rfloor)\big). \tag{6.21}$$

In words, $Z(\nu)$ denotes the collection of all sequences $(Z_j)_{j\in J}$ such that $J \subset \{1,\dots,n_Z\}$ satisfies $\{1,\dots,j\} \subset J \subset \{1,\dots,j+\lfloor \nu \rfloor\}$ for some integer $j \ge 0$. Roughly speaking, $(Z_j)_{j\in J} \in Z(\nu)$ if J is 'almost' an interval. Note that $Z(\nu)$ is increasing in ν and that $Z \in Z(\nu)$ (pick $j = n_Z$) for any $\nu \in [0,\infty]$. Now, for $\nu \in [0,\infty]$, with the notation from above (6.18), we introduce

$$\mathrm{V}_z(\widehat{Z}^u_z(\nu)) \overset{\text{def.}}{=} \bigcap_{Z \in \widehat{Z}^u_z(\nu)} \mathrm{V}_z(Z) \ (\subset \mathrm{V}_z(\widehat{Z}, u)). \tag{6.22}$$

In accordance with above convention, the choice $\delta = 0$ is implicit in (6.22). In what follows, much like in (6.22), given an event $E(Z)$ and ζ a collection of subsequences of Z, the event $E(\zeta)$ is declared by setting

$$E(\zeta) = \bigcap_{Z' \in \zeta} E(Z') \tag{6.23}$$

(measurability is never an issue since Z is always a finite sequence).

As an illustration of the relevance of the event $\mathrm{V}_z(\widehat{Z}^u_z(\nu))$ or the efficacy of constructs like $Z(\nu)$ and $E(\zeta)$, let us present a lemma which will later form the starting point of our proof of Theorem 1.1.

Lemma 6.5. *For any $L \ge 1$ and $u > 0$, with $\mathrm{V}_{z,L} \equiv \mathrm{V}_z(\overline{Z}^u_{z,L}(\nu = 0))$ as defined by (6.22), one has the inclusion (recall (1.9) regarding the event $\mathrm{SLU}_L(u)$)*

$$\bigcap_{z \in B_{2L}} \mathrm{V}_{z,L/100} \subset \mathrm{SLU}_L(u). \tag{6.24}$$

Proof. Notice that any cluster of $\mathcal{V}^v$ for some $v \in [0,u]$ in B_L having diameter at least $L/10$, as appearing in (1.9), will in fact cross $\tilde{D}_{z,L/100} \setminus \tilde{C}_{z,L/100}$ for some (nearby) $z \in B_{2L}$. The occurrence of $\mathrm{V}_{z,L/100}$, cf. above (6.18) and recall that $\delta = 0$, along with that of other z's in B_{2L}, will thus allow to connect any two such clusters inside B_{2L}, provided one can identify $\mathcal{V}^v$ as the vacant set $\mathcal{V}(Z)$ for some sequence $Z \in \overline{Z}^u_{z,L/100}(\nu = 0)$. But by definition, see (6.21) and (6.19)-(6.20), $\overline{Z}^u_{z,L}(\nu = 0)$ comprises all collections of excursions of the form $(Z_1, \dots Z_j)$ for some $j \le N^u_{z,L/100}$, where $(Z_1, Z_2, \dots)$ refer to the excursions in (2.9) with $D = D_{z,L/100}$ and $U = U_{z,L/100}$ (see also (2.20)), exactly one of which (when $j = N^v_{z,L/100}$) corresponds to the excursions underlying $\mathcal{V}^v \cap D_{z,L/100}$ between D and $\partial^{\text{out}} U$. The inclusion (6.24) follows. □

The goal of our next two propositions is to bound the probability of $\mathrm{V}_z(\overline{Z}_z^u(\nu))$ (recall (2.20) concerning $\overline{Z}$). We will split this task into two parts, which brings into play the *types* I and II alluded to above (each proposition deals with one of the types). Roughly speaking, type I, resp. II, corresponds to the cases that J is 'sizeable,' resp. 'small.' More precisely, we let (see (6.22) and (6.19)–(6.19) for notation and recall $N_z^u = N_{z,L}^u$ from above (2.20))

$$\mathrm{V}_z^{\mathrm{I}} = \mathrm{V}_{z,L}^{\mathrm{I}}(\nu; u_0, u) = \mathrm{V}_z\Big(\overline{Z}_z^u(\nu) \cap \big((\overline{Z}_z^u)_+(N_z^{u_0/2})\big)\Big), \tag{6.25}$$

$$\mathrm{V}_z^{\mathrm{II}} = \mathrm{V}_{z,L}^{\mathrm{II}}(\nu; u_0, u) = \mathrm{V}_z\Big(\overline{Z}_z^u(\nu) \cap \big((\overline{Z}_z^u)_-(N_z^{3u_0/2})\big)\Big). \tag{6.26}$$

Proposition 6.6 deals with $\mathrm{V}_z^{\mathrm{I}}$ and Proposition 6.7 with $\mathrm{V}_z^{\mathrm{II}}$ As we now explain, abbreviating $\mathrm{V}_z = \mathrm{V}_{z,L}(\overline{Z}_{z,L}^u(\nu))$, the event of interest, for all $\nu \in [1, \infty]$,

$$\mathrm{V}_z^c \cap \big\{N_z^{\frac{3u_0}{2}} - N_z^{\frac{u_0}{2}} > \nu\big\} \subset (\mathrm{V}_z^{\mathrm{I}})^c \cup (\mathrm{V}_z^{\mathrm{II}})^c. \tag{6.27}$$

Indeed, in view of (6.22), let $Z \in \overline{Z}_z^u(\nu) = \overline{Z}_{z,L}^u(\nu)$ be such that the event $(V_z(Z))^c$ occurs. If $Z \in (\overline{Z}_z^u)_+(N_z^{u_0/2})$, then $(\mathrm{V}_z^{\mathrm{I}})^c$ occurs. Otherwise, by (6.20) and (6.21), $Z = (Z_j)_{j\in J}$ is such that $\{1, \dots, j\} \subset \{1, \dots, j + \lfloor \nu \rfloor\}$ for some $j \geq 0$ and $j < N_z^{u_0/2}$. In particular, on the additional event appearing on the left of (6.27), J is contained in an interval of length at most $j+\nu < N_z^{u_0/2}+\nu \leq N_z^{\frac{3u_0}{2}}$, i.e. $Z \in (\overline{Z}_z^u)_-(N_z^{3u_0/2})$, and $(\mathrm{V}_z^{\mathrm{II}})^c$ occurs.

We now focus on the event $\mathrm{V}_z^{\mathrm{I}}$. Its occurrence will be bounded in terms of a (good) event $\mathscr{G}_z^{\mathrm{I}}$ of the form (4.16), whose constituent family $\mathcal{G}^{\mathrm{I}}$ will be of the form given by Definition 5.1 for suitable data $(\mathrm{V}, \mathrm{W}^{\mathrm{I}}, \mathscr{C})$, with $\mathscr{C}$, V as in (6.17)-(6.18), and events W^{I} that we now introduce. For what follows it will be convenient to declare $\hat{\nu}_{z,L}(u)$ for $u > 0$, $z \in \mathbb{L}$ (where, as with $\widehat{Z}$, the hat is a placeholder for three possibilities; cf. (2.20)) as

$$\nu_{z,L}(u) = \tilde{\nu}_{z,L}(u) = u \,\mathrm{cap}(D_{z,L}), \quad \bar{\nu}_{z,L}(u) = N_{z,L}^u. \tag{6.28}$$

We typically abbreviate $\hat{\nu}_z(u) = \hat{\nu}_{z,L}(u)$ when the scale L is clear from the context.

- **The events** $\mathrm{W}^{\mathrm{I}} = \{\mathrm{W}_{z,y}^{\mathrm{I}} : z \in \mathbb{L}, y \in \mathbb{L}_0\}$**.** Let (see below (6.22) for notation)

$$\mathrm{W}_{z,y}^{\mathrm{I}} \equiv \mathrm{W}_{z,y}^{\mathrm{I}}(\widehat{Z}, u_0, u_1) \overset{\text{def.}}{=} \mathrm{FE}_y\big((\widehat{Z}_z^{u_1})_+(\hat{\nu}_z(\tfrac{u_0}{8}))\big), \tag{6.29}$$

where, for any sequence Z of excursions we define the event $\mathrm{FE}_y(Z)$ as follows:

$$\mathrm{FE}_y(Z) = \mathrm{FE}_{y,L_0}(Z) = \mathrm{LU}_y(Z) \cap \mathrm{O}_y(Z) \tag{6.30}$$

with $\mathrm{O}_y(Z) = \mathrm{O}_{y,L_0}(Z)$ as in (3.3) and (cf. (3.2) for a related event)

$$\mathrm{LU}_y(Z) = \mathrm{LU}_{y,L_0}(Z) \overset{\text{def.}}{=} \bigcap_{x,x' \in (\tilde{D}_y \setminus \tilde{C}_y) \cap \mathcal{I}(Z)} \{x \xleftrightarrow{\mathcal{I}(Z) \cap (D_y \setminus (\partial D_y \cup C_y))} x'\}. \tag{6.31}$$

The above data set $(\mathrm{V}, \mathrm{W}^{\mathrm{I}}, \mathscr{C})$ leads to the well-defined event

$$\mathcal{G}_z^{\mathrm{I}}(\widehat{Z}, \delta, u_0, u_1, u_2, u_3; a) \overset{\text{def.}}{=} \mathcal{G}_z(\mathrm{V}, \mathrm{W}^{\mathrm{I}}, \mathscr{C}; a), \tag{6.32}$$

(recall (5.3) for the right-hand side), with $\mathrm{V}_z = \mathrm{V}_z(\widehat{Z}, \delta, u_1, u_2, u_3)$ given by (6.18), $\mathrm{W}^{\mathrm{I}} = \mathrm{W}^{\mathrm{I}}(\widehat{Z}, u_0, u_1)$ given by (6.29), and $\mathscr{C}_z = \mathscr{C}_z(\widehat{Z}, \delta, u_1, u_3)$ given by (6.17). Finally, the relevant event $\mathscr{G}^{\mathrm{I}}_{z,N}(\widehat{Z}, \delta, u_0, u_1, u_2, u_3; a)$, $z \in N\mathbb{Z}^d$, is defined as

$$\text{(6.33)}\quad \mathscr{G}^{\mathrm{I}}_{z,N}(\widehat{Z}, \delta, u_0, u_1, u_2, u_3; a) \stackrel{\text{def.}}{=} \mathscr{G}\big(\tilde{D}_{z,N} \setminus \tilde{C}_{z,N}, \mathcal{G}^{\mathrm{I}} = \Big\{ \mathcal{G}^{\mathrm{I}}_{z'}(\widehat{Z}, \delta, u_0, u_1, u_2, u_3; a) : z' \in \mathbb{L} \Big\}, \mathcal{F}^{\boldsymbol{u}_1, \boldsymbol{u}_2}_L; \rho = \tfrac{1}{2C_4}\big)$$

(recall (4.16) and (6.32)), where $\mathcal{F}^{\boldsymbol{u}_1, \boldsymbol{u}_2}_L$ is given by (4.18) and (4.22)) and $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ are as follows:

$$\text{(6.34)}\qquad \boldsymbol{u}_1 = (u, u_{2,3} \stackrel{\text{def.}}{=} \tfrac{u_2+u_3}{2}, u_{2,3}, \tfrac{u_0}{2}) \text{ and } \boldsymbol{u}_2 = (u_1, u_2, u_3, \tfrac{u_0}{8}).$$

The next proposition is the announced estimate for $\mathbb{P}[(\mathrm{V}^{\mathrm{I}}_{z,N})^c]$, which will eventually be used to take care of the first event in the union on the right-hand side of (6.27). The bound is expressed in terms of $\mathbb{P}\big[(\mathscr{G}^{\mathrm{I}}_{z,N})^c\big]$, where (see (6.33) for notation)

$$\text{(6.35)}\qquad \mathscr{G}^{\mathrm{I}}_{z,N} \stackrel{\text{def.}}{=} \mathscr{G}^{\mathrm{I}}_{z,N}(Z_{\mathbb{L}}, \delta, u_0, u_1, u_2, u_3; a), \quad z \in N\mathbb{Z}^d$$

and $Z_{\mathbb{L}} = \{Z^u_{z',\mathbb{L}} : u > 0, z' \in \mathbb{L}\}$ refers to the sequence from (2.20). The probability $\mathbb{P}\big[(\mathscr{G}^{\mathrm{I}}_{z,N})^c\big]$ will later be controlled separately by means of Proposition 4.5.

Proposition 6.6 (Type I estimate)**.** *With the choice of parameters as in* (6.16) *and* $\delta > 0$*, as well as* $\nu \geq 0$*,* $2u_0 < u_*$*, and* $L \geq C(u_0)$*, there exists* $c = c(\delta, L_0) > 0$ *such that, for* $z \in N\mathbb{Z}^d$*,*

$$\text{(6.36)}\qquad \mathbb{P}[(\mathrm{V}^{\mathrm{I}}_{z,N})^c] \leq \mathbb{P}\big[(\mathscr{G}^{\mathrm{I}}_{z,N})^c\big] + \mathbb{P}\big[C_{z,N} \stackrel{(\mathcal{V}^{u_{2,3}})_{2\delta}}{\nleftrightarrow} \partial D_{z,N}\big] + e^{-c(a\frac{N}{h(KL)} \wedge N) + C(\nu + \log N)},$$

where $h(x) = x(1 + (\log x)^2 1_{d \geq 4})$ *as in §4.2.*

Let us briefly pause to emphasize that, for the event $\mathrm{V}^{\mathrm{I}}_{z,N}$ on the left-hand side of (6.36), the relevant excursions are $\overline{Z}^u_{z,N}$ (i.e. at scale N) in view of (6.25) and the event comprises no noise $\delta(=0)$, which is absent from the notation (see (6.22) and (6.25)-(6.26), where δ does not appear), whereas the event $\mathscr{G}^{\mathrm{I}}_{z,N}$ declared by (6.35) involves excursions $Z_{\mathbb{L}}$ (i.e., at scale L) and the noise δ (appearing in (6.35)) is strictly positive for (6.36) to hold.

We now present an analogue of Proposition 6.6 for $\mathbb{P}[(\mathrm{V}^{\mathrm{II}}_{z,N})^c]$, as needed in view of (6.27). Similarly, the estimate will bring into play an event $\mathscr{G}^{\mathrm{II}}_z$ built using different events W^{II} which we now introduce. Recall the scale L_0^- from (6.16) as well as the event $\mathrm{W}^-_{y^-}(Z) = \mathrm{W}^-_{y^-, L_0^-}(Z)$ from (3.16) in Section 3. As with L_0, to keep notations reasonable and because L_0^- will not change as we operate our bootstrap argument, we keep its dependence implicit. Recall $\hat{\nu}_z$ from (6.28).

- **The events** $\mathrm{W}^{\mathrm{II}} = \{\mathrm{W}^{\mathrm{II}}_{z,y} : z \in \mathbb{L}, y \in \mathbb{L}_0\}$**.** Let

$$\text{(6.37)}\qquad \mathrm{W}^{\mathrm{II}}_{z,y} \equiv \mathrm{W}^{\mathrm{II}}_{z,y}(\widehat{Z}, u_4) \stackrel{\text{def.}}{=} \mathcal{G}^-_y(\widehat{Z}^{u_4}_z),$$

where (see Definition 5.1 for notation) $\mathcal{G}^-_y(Z) \stackrel{\text{def.}}{=} \mathcal{G}_{y, L_0, L_0^-}(\mathrm{V} = \{\Omega : y \in \mathbb{L}_0\}, \mathrm{W}^-(Z), \mathscr{C} = \{\mathbb{Z}^d : y \in \mathbb{L}_0\}; a = 1)$ and for $y \in \mathbb{L}_0, y^- \in \mathbb{L}_0^- = L_0^- \mathbb{Z}^d$, $\mathrm{W}^-_{y,y^-}(Z) \equiv \mathrm{W}^-_{y^-}(Z)$.

Somewhat in the same way as (6.32) and (6.33), this leads to events

$$\mathcal{G}_z^{\mathrm{II}}(\widehat{Z}, u_4; a) = \mathcal{G}_z(\mathrm{V}, \mathrm{W}^{\mathrm{II}}, \mathscr{C}; a) \tag{6.38}$$

with $\mathrm{V}_z = \Omega$, $\mathrm{W}^{\mathrm{II}} = \mathrm{W}^{\mathrm{II}}(\widehat{Z}, u_4)$ given by (6.37) and $\mathscr{C}_z = \mathbb{Z}^d$, and subsequently

$$\mathscr{G}_{z,N}^{\mathrm{II}}(\widehat{Z}, u_4; a) \stackrel{\text{def.}}{=} \mathscr{G}\big(\tilde{D}_{z,N} \setminus \tilde{C}_{z,N}, \mathcal{G}^{\mathrm{II}} = \Big\{\mathcal{G}_{z'}^{\mathrm{II}}(\widehat{Z}, u_4; a) : z' \in \mathbb{L}\Big\}, \mathcal{F}_L^{\frac{3u_0}{2}, u_4}; \rho = \tfrac{1}{2C_4}\big). \tag{6.39}$$

Finally we let

$$\mathscr{G}_{z,N}^{\mathrm{II}} \stackrel{\text{def.}}{=} \mathscr{G}_{z,N}^{\mathrm{II}}(Z_{\mathbb{L}}, u_0, u_4; a), \quad z \in N\mathbb{Z}^d \tag{6.40}$$

(cf. (6.35)). The analogue of Proposition 6.6 reads as follows.

Proposition 6.7 (Type II estimate)**.** *With the choice of parameters as in (6.16) as well as $\nu \geq 0$, $2u_0 < u_4 < u_*$ and $L \geq C(u_0)$, there exists $c = c(L_0^-) > 0$ such that*

$$\mathbb{P}[(\mathrm{V}_{z,N}^{\mathrm{II}})^c] \leq \mathbb{P}\big[(\mathscr{G}_{z,N}^{\mathrm{II}})^c\big] + \mathbb{P}\Big[(C_{z,N} \cap \mathbb{L}_0^-) \overset{\mathcal{O}_0^-(\overline{Z}_{z,N}^{\frac{3u_0}{2}})}{\not\leftrightarrow} \partial_{\mathbb{L}_0^-}(D_{z,N} \cap \mathbb{L}_0^-)\Big] + e^{-c(a\frac{N}{h(KL)} \wedge N) + C(\nu + \log N)}, \tag{6.41}$$

where the set $\mathcal{O}_0^-(Z)$ was defined in (3.17) in Section 3 and the connectivity in $\mathcal{O}_0^-(\overline{Z}_z^v)$ is w.r.t. the nearest-neighbor graph structure inherited from the coarse-grained lattice $\mathbb{L}_0^-$, i.e. two points $z_1, z_2 \in \mathbb{L}_0^-$ are neighbors if and only if $|z_1 - z_2| = L_0^-$.

Remark 6.8 (Types I and II). In view of (6.25)-(6.26) and (6.19)-(6.20), the types I and II respectively deal with typical and small numbers of excursions. An argument involving type I only would be sufficient to prove our main results with the desired degree of precision (in particular, with regards to the dependence on u) for $u < u_*$ sufficiently close to u_*; for instance, when $u > \frac{u_*}{2}$. In this sense, type I is more fundamental than type II. The fact that type II needs to be considered is due to the case of small u and the pathologies that arise. For instance, the probability for (6.31) with $\mathcal{I} = \mathcal{I}^u$ will degenerate as $u \downarrow 0$. The event in (3.16) is in fact inspired by an event introduced in [17] (see Definition 3.3 therein), which dealt precisely with the perturbative regime $u \ll 1$.

Propositions 6.6 and 6.7 will be proved separately in Section 7. For now we apply them freely to conclude the proof of Theorem 6.3. Their usefulness hinges on suitable bounds for $\mathbb{P}[(\mathscr{G}_{z,N}^{\mathrm{i}})^c]$, $i = \mathrm{I}, \mathrm{II}$, for which we will rely on Proposition 4.5. This in turn requires verifying the condition (4.20) and notably to exercise control (as parametrized by p_L) on a localized version of the events $\mathcal{G}_z^i$, $i = \mathrm{I}, \mathrm{II}$, see (4.19). The necessary control will be provided by the following triggering estimate. For the remainder of this section it is always implicit that all values of parameters satisfy (6.16). Any additional (or overriding) condition on the parameters listed in (6.16) will appear explicitly.

Lemma 6.9 (Seed estimates)**.** *There exist $c_6 \in (0, u_*)$, scales L_0^- and $L_0 = L_0(u_0, u_1, u_2, u_3)$, and $c_7(L_0) \in (0, \frac{1}{2})$ such that for all $\delta \in [0, c_7]$, $u_4 \in [0, c_6]$ and $L \geq 1$,*

$$\mathbb{P}\left[\mathcal{G}_{0,L}^{\mathrm{i}}\big(\overline{Z}_{\mathbb{L}}, \delta, \boldsymbol{u}; a = 1\big)\right] \geq 1 - Ce^{-L^c}, \quad \mathrm{i} = \mathrm{I}, \mathrm{II}, \tag{6.42}$$

for some $C = C(\boldsymbol{u}) \in (0, \infty)$, with $\overline{Z}_{\mathbb{L}} = \{\overline{Z}_z^u : z \in \mathbb{L}, u > 0\}$ as in (2.20) and $\boldsymbol{u} = (u_0, \ldots, u_4)$.

In (6.42) and in what follows, there is in fact no dependence of the events in question on u_4 when $\mathrm{i} = \mathrm{I}$, nor on $u_0, \dots, u_3$ and δ when $\mathrm{i} = \mathrm{II}$; this slight abuse of notation allows for a unified presentation and will be used for other events like $\mathscr{G}^{\mathrm{i}}_{z,N}$ etc., as well.

We postpone the proof of Lemma 6.9 to Section 8. Next, we collect important localization properties of the events $\mathcal{G}^{\mathrm{i}}_{0,L}$, by which we mean how the event behaves as one moves through the sequences of excursions in (2.20). By suitably tuning the parameters of $\mathcal{G}^{\mathrm{i}}_{0,L}$, we deduce certain inclusions that roughly play the same role as (6.5) and (6.9) in the sub-critical regime. Owing to the more involved nature of the events in question, these inclusions are now less straightforward.

Recall that (6.16) is in force, and in particular that the scales N, L, L_0 and K satisfy (5.1). The following conclusions all hold uniformly in $z \in \mathbb{L}$ and $\mathrm{i} \in \{\mathrm{I}, \mathrm{II}\}$ without further explicit mention.

Lemma 6.10 (Localization of $\mathcal{G}^{\mathrm{i}}_z$)**.** *For all $u_0, u_4, v_0, v_4 \in [0, \infty), u_1 < u_2 < u_3 \in (0, \infty)$ and $v_1 < v_2 < v_3 \in (0, \infty)$ such that $u_0 < v_0$, $u_2 < v_2$, and $u_1 > v_1$, $u_3 > v_3$, $u_4 > v_4$, abbreviating $\boldsymbol{u} = (u_0, \dots, u_4)$, $\boldsymbol{u}' = (u_0/8, u_1, \dots, u_4)$ and $\boldsymbol{v} = (v_0, \dots, v_4)$, $\boldsymbol{v}' = (v_0/8, v_1, \dots, u_4)$, one has*

$$\mathcal{G}^{\mathrm{i}}_z(\overline{Z}_{\mathbb{L}}, \delta, \boldsymbol{u}; a) \cap \mathcal{F}^{\boldsymbol{u}', \boldsymbol{v}'}_z \subset \mathcal{G}^{\mathrm{i}}_z(Z_{\mathbb{L}}, \delta, \boldsymbol{v}; a) \tag{6.43}$$

(see (2.21) and (4.18) regarding the definition of $\mathcal{F}^{\boldsymbol{u},\boldsymbol{v}}_z$). Moreover, under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_z$,

$$\begin{aligned} &\mathcal{G}^{\mathrm{i}}_z(\widetilde{Z}_{\mathbb{L}}, \delta, \boldsymbol{u}; a) \cap \mathrm{Incl}_z^{\frac{\varepsilon}{10}, \lfloor v\, \mathrm{cap}(D_z) \rfloor} \subset \mathcal{G}^{\mathrm{i}}_z(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(1, \varepsilon); a) \text{ and} \\ &\mathcal{G}^{\mathrm{i}}_z(Z_{\mathbb{L}}, \delta, \boldsymbol{u}; a) \cap \mathrm{Incl}_z^{\frac{\varepsilon}{10}, \lfloor v\, \mathrm{cap}(D_z) \rfloor} \subset \mathcal{G}^{\mathrm{i}}_z(\widetilde{Z}_{\mathbb{L}}, \delta, \boldsymbol{u}(1, \varepsilon); a) \end{aligned} \tag{6.44}$$

for $v = \frac{1}{20} \min(u_0, u_1, u_4)$, $\varepsilon \in (0, 1)$ and $L \geq C(v, \varepsilon)$, where $\boldsymbol{u}(1, \varepsilon)$ stands for the tuple $(u_{0,\varepsilon}, u_{1,-\varepsilon}, u_{2,\varepsilon}, u_{3,-\varepsilon}, u_{4,-\varepsilon})$ with $u_{k,\epsilon} = u_k(1 + \epsilon)$ for any $\epsilon \in (-1, 1)$.

Proof. To see (6.43) for the case $\mathrm{i} = \mathrm{I}$, one first goes back to (6.32) and recalls that $\mathcal{G}^{\mathrm{I}}_z$ is increasing in all of V_z, the events comprising W^{I} and $\mathscr{C}$ by (5.3)-(5.4). Let us inspect how each of these events moves across the two sides of (6.43) starting with the events $\mathrm{W}^{\mathrm{I}}_{z,y}$. Letting $u_0' = u_0/8$, $v_0' = v_0/8$, the occurrence of the event $\mathcal{F}^{u_0', v_0'}_z \cap \mathcal{F}^{u_1, v_1}_z (\supset \mathcal{F}^{\boldsymbol{u}', \boldsymbol{v}'}_z)$, see (2.21), guarantees that any set of indices J with

$$\{1, \dots, v_0' \mathrm{cap}(D_z)\} \subset J \subset \{1, \dots, v_1 \mathrm{cap}(D_z)\}$$

satisfies

$$\{1, \dots, N^{u_0'}_z\} \subset J \subset \{1, \dots, N^{u_1}_z\}.$$

In other words, in view of the definition of Z_+ in (6.19) and of (2.20), on the event $\mathcal{F}^{\boldsymbol{u}', \boldsymbol{v}'}_z$ we have the inclusion

$$(Z^{v_1}_z)_+(\nu_z(v_0')) \subset (\overline{Z}^{u_1}_z)_+(\bar{\nu}_z(u_0')) \tag{6.45}$$

(see also (6.28) regarding ν_z and $\bar{\nu}_z$). By (6.23) and the definition of W^{I} in (6.29), it then follows that

$$\mathrm{W}^{\mathrm{I}}_{z,y}(\overline{Z}_{\mathbb{L}}, u_0, u_1) \cap \mathcal{F}^{\boldsymbol{u}', \boldsymbol{v}'}_z \subset \mathrm{W}^{\mathrm{I}}_{z,y}(Z_{\mathbb{L}}, v_0, v_1) \tag{6.46}$$

for any $z \in \mathbb{L}$ and $y \in \mathbb{L}_0$. Next, we deal with the event $\mathrm{V}_z = \mathrm{V}_{z,L}$ defined above (6.18). It is clear from this definition that $\mathrm{V}_z(Z, Z', Z'', \delta)$ is decreasing in Z and Z'' and increasing in Z' (w.r.t. inclusion of the underlying sets). Now on the event $\mathcal{F}^{u_1, v_1}_z \cap \mathcal{F}^{u_2, v_2}_z \cap \mathcal{F}^{u_3, v_3}_z (\supset \mathcal{F}^{\boldsymbol{u}', \boldsymbol{v}'}_z)$, by (2.20) and (2.21) we have the inclusions

$$\{Z^{v_1}_z\} \subset \{\overline{Z}^{u_1}_z\},\ \{Z^{v_3}_z\} \subset \{\overline{Z}^{u_3}_z\} \text{ and } \{\overline{Z}^{u_2}_z\} \subset \{Z^{v_2}_z\}, \tag{6.47}$$

where $\{Z\} = \{Z_1, \dots, Z_{n_Z}\}$ for any sequence $Z = (Z_j)_{1\le j\le n_Z}$. Although we don't need this for the purpose of dealing with V_z (but we'll use it shortly), let us emphasize that the inclusions in (6.47) do in fact hold as multisets (as per our convention below (2.15)) on the event $\mathcal{F}_z^{\boldsymbol{u}',\boldsymbol{v}'}$. With (6.47) at hand, in view of (6.18), we have

$$\mathrm{V}_z(\overline{Z}_{\mathbb{L}}, \delta, u_1, u_2, u_3) \cap \mathcal{F}_z^{\boldsymbol{u}',\boldsymbol{v}'} \subset \mathrm{V}_z(Z_{\mathbb{L}}, \delta, v_1, v_2, v_3). \tag{6.48}$$

By similar arguments (see (6.17) regarding $\mathscr{C}_z$), we also obtain that

$$\mathscr{C}_z(\overline{Z}_{\mathbb{L}}, \delta, u_1, u_3) \cap \mathcal{F}_z^{\boldsymbol{u}',\boldsymbol{v}'} \subset \mathscr{C}_z(Z_{\mathbb{L}}, \delta, v_1, v_3). \tag{6.49}$$

Together with the observation made in the line below (6.43) and the definition of $\mathcal{G}_z^{\mathrm{I}}$ in (6.32), the displays (6.46), (6.48) and (6.49) yield (6.43) for $\mathrm{i} = 1$. The case $\mathrm{i} = \mathrm{II}$ is handled similarly with u_4,v_4 in place of u_0, v_0 (recall (6.37)).

Notice that, since the event $\mathrm{V}_{z,L}$ and the set $\mathscr{C}_{z,L}$ do *not* depend on the ordering of excursions in the sequences Z, Z' or Z'' (recall (6.17) and (6.18)), we only used inclusions of the sets underlying these sequences in (6.47) to derive (6.48) and (6.49). The same is also true for the events $\mathrm{W}_{z,y}^{\mathrm{I}}$ and $\mathrm{W}_{z,y}^{\mathrm{II}}$ (see (6.29) and (6.37)), although in the latter case, it is crucial for the inclusions in (6.47) to hold as multisets, since $\mathrm{W}_{z,y}^{\mathrm{I}}$ requires a control on the occupation times via (6.30). Accordingly, we could have used the following relaxed version of (6.45) to deduce (6.46):

$$\text{for any } Z \in (Z_z^{v_4})_+(\nu_z(\tfrac{v_0}{8})), \text{ there exists } Z' \in (\overline{Z}_z^{u_4})_+(\bar{\nu}_z(\tfrac{u_0}{8})) \text{ satisfying } \{Z\} = \{Z'\}$$

(with $\{Z\} = \{Z'\}$ following the convention below (2.15)). This small observation is particularly useful for passing to the $\widetilde{Z}_{\mathbb{L}}$-version of the events $\mathcal{G}_z^{\mathrm{i}}$ as the event $\mathrm{Incl}_z^{\varepsilon/10,m_0}$ in (2.15) with $m_0 = \lfloor v\,\mathrm{cap}(D_z)\rfloor$ and v as below (6.44) precisely ensures the desired inclusions between the relevant multisets of excursions belonging to $Z_{\mathbb{L}}$ and $\widetilde{Z}_{\mathbb{L}}$. With this in mind we straightforwardly obtain that (6.44) holds from the arguments leading to (6.43). □

We can now already conclude Theorem 1.1, and give a brief overview of the argument. We first apply Lemma 6.5 in combination with (6.27) and Propositions 6.6 and 6.7. This essentially reduces the task to deducing suitable bounds on the probabilities $\mathbb{P}\big[(\mathscr{G}_{z,N}^{\mathrm{i}})^c\big]$, $\mathrm{i} = \mathrm{I}, \mathrm{II}$, appearing in Propositions 6.6 and 6.7 (one must also deal with the disconnection probabilities present but let us forego this). To bound $\mathbb{P}\big[(\mathscr{G}_{z,N}^{\mathrm{i}})^c\big]$, we use the seed estimates of Lemma 6.9 along with Lemma 6.10 in order to apply Proposition 4.5. In this context, the parameter $a(=1)$ will play no role. Improving on a is only relevant to get the more refined estimates that constitute Theorems 6.3 and 1.4.

Proof of Theorem 1.1. We prove (1.10) and explain at the end of the proof how (1.8) is deduced. Let $\bar{u}_0 \in [\frac{c_6}{10}, \infty)$, where c_6 is supplied by Lemma 6.9. With Lemma 6.5 at hand, applying (6.27) with the choice $\nu = 0$ at scale $L/100$ together with a union bound over z and using that the probability that $\big\{N_{z,L/100}^{3\bar{u}_0/2} - N_{z,L/100}^{\bar{u}_0/2} = 0\big\}$ (recall that the difference on the left dominates a Poisson variable with mean $\bar{u}_0\,\mathrm{cap}(B_{L/100})$) is bounded from above by $\exp\{-c\bar{u}_0 L^{d-2}\} \le \exp\{-cL^{d-2}\}$, the task of proving (1.10) reduces to showing that for suitable $c = c(d) > 0$, all $u \in (0, u_*)$, $N \ge C(u)$ and $\mathrm{i} = \mathrm{I}, \mathrm{II}$,

$$\mathbb{P}[(\mathrm{V}_{0,N}^{\mathrm{i}}(\nu = 0; \bar{u}_0, u))^c] \le e^{-N^c}, \tag{6.50}$$

for some value $\bar{u}_0 = \bar{u}_0(u) \ge \frac{c_6}{10}$. We will prove (6.50) by application of Propositions 6.6 and 6.7 to deal with the case $\mathrm{i} = \mathrm{I}$ and $\mathrm{i} = \mathrm{II}$, respectively. In view of (6.36) and (6.41), this requires deriving a similar estimate as (6.50) but concerning the events $(\mathscr{G}^{\rm i}_{0,N})^c$, $\rm i = I, II$, for a suitable choice of the remaining parameters among $(\delta, u_0, u_1, u_2, u_3, u_4, a)$. The desired bounds will be obtained by means of Proposition 4.5.

We begin with some preparation that will be needed for the application of Proposition 4.5, which in particular entails that condition (4.20) must be verified, for suitable choice of events $\mathcal{G}_{z,L}$ and $\widetilde{\mathcal{G}}_{z,L}$ to be specified soon (see (6.54) below). As the notation indicates, these will basically consist of the events $\mathcal{G}^{\rm i}_z$ from (6.32) and (6.38) in case $\rm i = I$ and II, respectively, for a certain choice of excursions $\widehat{Z}$ and parameters.

Consider L_0^- and L_0 as given by Lemma 6.9 corresponding to some choice of parameters

$$\boldsymbol{u} = (u_0 = \tfrac{c_6}{10}, u_1, u_2, u_3, u_4 = c_6) \text{ and } \delta \in (0, c_7(L_0)] \text{ satisfying (6.16)}. \tag{6.51}$$

We will later tune $\boldsymbol{u}$ to the requirement of (6.50). Combining (6.42), which is in force, with the inclusion (6.43), we obtain that for any $z \in \mathbb{L}$, L_0^-, L_0 and δ as above, and for $\varepsilon \in (0,1)$, $K \geq C(\varepsilon)$,

$$\begin{aligned}\mathbb{P}[\mathcal{G}^{\rm i}_z(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(1,\varepsilon); a=1)] &\overset{(6.43)}{\geq} \mathbb{P}[\mathcal{G}^{\rm i}_z(\overline{Z}_{\mathbb{L}}, \delta, \boldsymbol{u}; a=1)] - \mathbb{P}[(\mathcal{F}_z^{\boldsymbol{u}', \boldsymbol{u}(1,\varepsilon)'})^c] \\ &\overset{(6.42),(2.22)}{\geq} 1 - C(\varepsilon, \boldsymbol{u}) e^{-L^c},\end{aligned} \tag{6.52}$$

where $\boldsymbol{u}(k,\varepsilon) = (u_0(1+\varepsilon)^k, u_1(1-\varepsilon)^k, u_2(1+\varepsilon)^k, u_3(1-\varepsilon)^k, u_4(1-\varepsilon)^k)$ for any $k \in \mathbb{N}$ (cf. $\boldsymbol{u}(1,\varepsilon)$ below (6.44)) and $\boldsymbol{u}'$ (or $\boldsymbol{u}(1,\varepsilon)'$) is as above (6.43). Now using the second inclusion in (6.44), we can use the coupling $\mathbb{Q}_{\{z\}}$ from Lemma 2.1 to deduce that

$$\begin{aligned}&\mathbb{P}[\mathcal{G}^{\rm i}_z(\widetilde{Z}_{\mathbb{L}}, \delta, \boldsymbol{u}(2,\varepsilon); a=1)] \\ &\quad\overset{(2.17)}{\geq} \mathbb{P}[\mathcal{G}^{\rm i}_z(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(1,\varepsilon); a=1)] \widetilde{\mathbb{P}}_z[(\mathcal{U}_z^{\frac{\varepsilon}{10}, \lfloor (1-\varepsilon) v \,\mathrm{cap}(D_{z,L}) \rfloor})^c] \\ &\quad\overset{(6.52),(2.14),(A.8)}{\geq} 1 - C(\varepsilon, \boldsymbol{u}) e^{-L^c} - C\varepsilon^2 e^{-c(\varepsilon,\boldsymbol{u}) L^{d-2}} \geq 1 - C(\varepsilon, \boldsymbol{u}) e^{-L^c},\end{aligned} \tag{6.53}$$

for all $L \geq C(\varepsilon)$ and $K \geq \frac{30 C_8}{\varepsilon} \vee C(\varepsilon)$ (ensures that condition (2.16) holds on account of Proposition A.1 and that (6.52) applies) where $v = \frac{1}{20} \min(u_0, u_1, u_4)$ as below (6.44).

Next, in view of the first inclusion in (6.44) together with Lemma 2.1 and Proposition A.1, and the probability bounds in (6.53) above and (2.14), we see that the conditions (4.19)–(4.20) of Proposition 4.5 are satisfied by the events

$$\widetilde{\mathcal{G}}_{z,L} = \mathcal{G}^{\rm i}_z(\widetilde{Z}_{\mathbb{L}}, \delta, \boldsymbol{u}(2,\varepsilon); a=1) \text{ and } \mathcal{G}_{z,L} = \mathcal{G}^{\rm i}_z(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(3,\varepsilon); a=1) \tag{6.54}$$

and for $\varepsilon_L = \frac{\varepsilon}{10}$, $m_L = \lfloor v(1-\varepsilon)^2 \,\mathrm{cap}(D_{z,L}) \rfloor$, $\beta' = c \in (0,\infty)$, $K_0 = C(\varepsilon)$ and $L_0 = C(\varepsilon, \boldsymbol{u})$ (as for (4.20) to hold). Let us suppose for the remainder of this proof that, on top of the conditions specified in (6.51), $\boldsymbol{u}$ and ε also satisfy

$$u < u_1(1-\varepsilon)^4,\ u_2(1+\varepsilon)^3 < u_3(1-\varepsilon)^4 \text{ and } 2u_0(1+\varepsilon)^3 < u_4(1-\varepsilon)^3 \tag{6.55}$$

(cf. (6.16) and also the assumptions underlying Propositions 6.6 and 6.7). Then in dimension $d \geq 4$, choosing $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}(3,\varepsilon)_1, \boldsymbol{u}(3,\varepsilon)_2)$ for $\rm i = I$ and $(\frac{3u_0}{2}(1+\varepsilon)^3, u_4(1-\varepsilon)^3)$ for $\rm i = II$ (recall (6.39) and our convention on the dependence of parameters below Lemma 6.9), where $\boldsymbol{u}(3,\varepsilon)_1$ and $\boldsymbol{u}(3,\varepsilon)_2$ are defined exactly as $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ with $\boldsymbol{u}(3,\varepsilon)$ replacing $\boldsymbol{u}$ in (6.34), $\widetilde{\mathcal{G}}_{z,L}, \mathcal{G}_{z,L}$ as in (6.54), $K = K(\varepsilon, \boldsymbol{u})$, $L = L(\varepsilon, \boldsymbol{u})$ large enough and $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$, we obtain from (4.22) that for all $N \geq 1$,

$$\mathbb{P}[\mathscr{G}^{\rm i}_{0,N}(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(3,\varepsilon); a=1)] \geq 1 - C(\varepsilon, \boldsymbol{u}) e^{-c(\varepsilon,\boldsymbol{u}) N}. \tag{6.56}$$

On the other hand for $d = 3$, (4.21) yields with the choice of $(\boldsymbol{u}, \boldsymbol{v})$, $\widetilde{\mathcal{G}}_{z,L}$ and $\mathcal{G}_{z,L}$ as above, $K = K(\varepsilon, \boldsymbol{u})$, $\delta = \frac{1}{2}$ for the parameter appearing in (4.21)), $\rho = \frac{1}{2C_4}$, $L = L(N) = \lfloor (\log N)^\alpha \rfloor$ for some absolute constant $\alpha \in (0,\infty)$ and $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$ that for all $N \geq 1$, with $\beta \in (0,\infty)$ an absolute constant (determined by the choice of $\beta' = c$ from below (6.54)),

$$\mathbb{P}[\mathscr{G}^{\mathrm{i}}_{0,N}(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(3,\varepsilon); a = 1)] \geq 1 - C(\varepsilon, \boldsymbol{u}) e^{-\frac{N}{(1\vee \log N)^\beta}}. \tag{6.57}$$

Now plugging the bounds (6.56) and (6.57) into the right-hand side of (6.36) and (6.41) in Propositions 6.6 and 6.7 respectively with $\nu = 0$ (the required conditions are ensured by (6.51) and (6.55)), we get that for all $N \geq 2$ and $d \geq 3$, with $\mathrm{V}^{\mathrm{i}}_{0,N} = \mathrm{V}^{\mathrm{i}}_{0,N}(\nu = 0; \bar{u}_0 = u_0(1+\varepsilon)^3, u)$,

$$\mathbb{P}[(\mathrm{V}^{\mathrm{i}}_{0,N})^c] \leq \mathbb{P}\big[\mathrm{Disc}^{\mathrm{i}}_{0,N}\big] + C(\delta, \varepsilon, \boldsymbol{u}) \times \begin{cases} e^{-\frac{N}{(\log N)^\beta}} & \text{if } d = 3 \\ e^{-c(\delta,\varepsilon,\boldsymbol{u})N} & \text{if } d \geq 4 \end{cases} \tag{6.58}$$

where

$$\mathrm{Disc}^{\mathrm{i}}_{0,N} = \begin{cases} \Big\{ C_{0,N} \overset{(\mathcal{V}^{u_{2,3}(\varepsilon)})_{2\delta}}{\not\longleftrightarrow} \partial D_{0,N} \Big\}, & \text{if i} = 1 \\ \Big\{ (C_{0,N} \cap \mathbb{L}_0^-) \overset{\mathcal{O}_0^-(\overline{Z}^{3\bar{u}_0/2}_{0,N})}{\not\longleftrightarrow} \partial_{\mathbb{L}_0^-}(D_{0,N} \cap \mathbb{L}_0^-) \Big\}, & \text{if i} = 2 \end{cases}$$

and $u_{2,3}(\varepsilon) = \frac{u_2(1+\varepsilon)^3 + u_3(1-\varepsilon)^3}{2}$ (recall (6.34)). Recall from (6.51) that $u_0 = \frac{c_6}{10}$ and hence $\bar{u}_0 = u_0(1+\varepsilon)^3 > \frac{c_6}{10}$ as required by (6.50).

In view of (6.58), and with a view towards our aim in (6.50), it remains to derive suitable bounds on the probability of the disconnection events $\mathrm{Disc}^{\mathrm{i}}_{0,N}$. We already have, from (6.12) in Remark 6.2, a bound on the disconnection probability in (6.58) when $\mathrm{i} = \mathrm{I}$ and $\delta \in (0, c_5(u_{2,3}(\varepsilon))]$. As to the case $\mathrm{i} = \mathrm{II}$, we employ the following analogous result: there exists $C_6 < \infty$ with $C_6(d) = 0$ for $d \geq 4$ such that for all $u \in (0, c_6]$ and $N \geq 2$,

$$\mathbb{P}\big[(C_{0,N} \cap \mathbb{L}_0^-) \xleftrightarrow{\mathcal{O}_0^-(\overline{Z}^u_{0,N})} \partial_{\mathbb{L}_0^-}(D_{0,N} \cap \mathbb{L}_0^-)\big] \geq 1 - Ce^{-\frac{cN}{(\log N)^{C_6}}}. \tag{6.59}$$

Like (6.12), the exponent in the bound above is suboptimal (see the discussion following (6.12) in the previous subsection). Also just as in (6.12), we can obtain (6.59) by adapting the proof of Theorem 6.1 in §6.1. We omit the details and only highlight the aspects that are specific to this case.

Instead of the events in (6.4) (see also the discussion in the paragraph after (6.12)), one works with

$$\overline{\mathcal{G}}_{z,L} \overset{\text{def.}}{=} \left\{ \begin{array}{c} \tilde{C}_z \text{ is \textit{not} connected to } \partial \tilde{D}_z \text{ by any path } \gamma \text{ in } \mathbb{L}_0^- \\ \text{such that } (\mathrm{W}^-_{y^-}(\overline{Z}^{c_6}_z))^c \text{ occurs for each } y^- \in \gamma \end{array} \right\}.$$

We define $\mathcal{G}_{z,L}$ and $\widetilde{\mathcal{G}}_{z,L}$ similarly with $\overline{Z}^{c_6}_z$ replaced by $Z_z^{\frac{3c_6}{2}}$ and $\widetilde{Z}^{2c_6}_z$ respectively. By construction of the events $\overline{\mathcal{G}}_{z,L}$ (call $z \in \mathbb{L}$ *good* if $\overline{\mathcal{G}}_{z,L}$ occurs) and $\mathrm{W}^-_{y^-}(Z) = \mathrm{W}^-_{y^-,L_0^-}(Z)$ (revisit (3.16)) and a standard duality argument just as in the case of (6.12), the complement of the event in (6.59) implies the existence of a macroscopic $*$-path of *bad* vertices in the annulus $\Lambda_N = D_{0,N} \setminus C_{0,N}$, which in turn implies the event $\mathscr{G}^c(\Lambda_{cN}, \{\mathcal{G}_{z,L} : z \in \mathbb{L}\}, \mathcal{F}_L^{c_6, 3c_6/2}; \rho)$ up to a translation in space (cf. (6.6)). The estimate (6.12) can now be readily deduced following the steps in the proof of Theorem 6.1 and using Lemma 3.6 in [16] as the a-priori bound on $\mathbb{P}[\mathcal{G}^c_{z,L}]$. The bound given by [16, Lemma 3.6] may in fact hold with a different absolute constant ${c_6}' > 0$ (say), but we can always set c_6 to be the smaller constant because the event in (6.59) and $\mathcal{G}^{\mathrm{II}}_{z,L}$ in (6.38) (see also (6.42)) are both *decreasing* w.r.t. their underlying parameter u or u_4. Overall, (6.59) follows.

To conclude the proof of (6.50), returning to (6.58) and setting, for a given $u \in (0, u_*)$,

$$
\begin{aligned}
u_0 = \tfrac{c_6}{10}, u_1 = u_*(1-\varepsilon)^{10}, u_2 = u_*(1-\varepsilon)^9, u_3 = u_*(1-\varepsilon) \text{ and } u_4 = c_6 \\
\varepsilon = ((1 - (\tfrac{u}{u_*})^{\frac{1}{20}}) \wedge \tfrac{1}{10}, \text{ as well as } \delta = \tfrac{c_5(u_{2,3}(\varepsilon))}{2} \wedge c_7(L_0(\boldsymbol{u}))\,(>0)
\end{aligned}
\tag{6.60}
$$

with $L_0(\cdot)$ provided by Lemma 6.9, we see that the conditions (6.51) and (6.55) are satisfied and the bounds (6.12) and (6.59) hold for the values $u = u_{2,3}(\varepsilon)$ and $u = 3\bar{u}_0/2$ respectively (see below (6.58) for definitions). Therefore we can plug the bounds from (6.12) when $\mathrm{i} = \mathrm{I}$ and from (6.59) when $\mathrm{i} = \mathrm{II}$ into the right-hand side of (6.58) to deduce (6.50), thus concluding the proof of (1.10).

It remains to argue that (1.8) holds. The following inclusion of events follows from the definition of $\mathrm{SLU}_L(u)$ in (1.9). For any $v \in [0, u]$ and $x \in \mathbb{Z}^d$ such that $|x|_\infty \geq 2$, one has

$$
\big\{0 \stackrel{\mathcal{V}^v}{\longleftrightarrow} x, x \stackrel{\mathcal{V}^v}{\not\longleftrightarrow} \infty\big\} \subset \big(\mathrm{SLU}_{|x|_\infty}(u) \cap \big\{B_{|x|_\infty/4} \stackrel{\mathcal{V}^v}{\longleftrightarrow} \infty\big\}\big)^c. \tag{6.61}
$$

Also following the derivation of (5.73) in [24], we obtain by combining (1.10), the disconnection estimate, i.e. the main result in [48] (see Theorem 7.3) which holds for all $u < \bar{u}$ and the equality of u_* and $\bar{u}$ in Theorem 1.2 of [20] that the connection event to infinity on the right of (6.61) has probability at least $1 - C(v)e^{-|x|^c}$, for all $v \in [0, u_*)$ and $x \in \mathbb{Z}^d$. Since the connection event in question is decreasing w.r.t. v, feeding the previous bound together with (1.10) into (6.61) yields that $\tau^{\mathrm{tr}}_v(x,y) = \tau^{\mathrm{tr}}_v(0, y-x) \leq C(u)e^{-|x-y|^c}$, *uniformly* over all $v \in [0,u]$ and $x, y \in \mathbb{Z}^d$ when $u < u_*$ (see (1.6)). But this is precisely the equality of $\widehat{u}$ and u_* in (1.8) in view of the definition of $\widehat{u}$ in (1.7). □

We now move on to the proof of Theorem 6.3 starting with the case $d \geq 4$, which is simpler.

Proof of (6.14). Plugging the bounds from (6.12) and (6.59) for $d \geq 4$ into the right-hand side of the inequalities in (6.58) with the choice of parameters $\boldsymbol{u}, \varepsilon$ and δ from (6.60), we obtain in view of (6.27) (see also the paragraph above (6.50) in the proof of Theorem 1.1) that

$$
\mathbb{P}\big[(\mathrm{V}_{0,N}(\overline{Z}^u_{0,N}(\nu = 0)))^c\big] \leq C(u)e^{-c(u)N} \tag{6.62}
$$

for all $u \in (0, u_*)$ and $N \geq 1$. Now it follows from (6.61) and Lemma 6.5 applied with $L = N/\sqrt{d}$ (using the inclusion $B_{N/\sqrt{d}} \subset B^2_N$) that for all $N \geq 10\sqrt{d}$,

$$
\big\{0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} \partial B^2_N, 0 \stackrel{\mathcal{V}^u}{\not\longleftrightarrow} \infty\big\} \subset \bigcup_{z \in B^2_{2N}} \big(\mathrm{V}_{z, \frac{N}{100\sqrt{d}}} \cap \big\{B_{\frac{N}{4\sqrt{d}}} \stackrel{\mathcal{V}^u}{\longleftrightarrow} \infty\big\}\big)^c \tag{6.63}
$$

where $\mathrm{V}_{z,L} = \mathrm{V}_{z,L}(\overline{Z}^u_{z,L}(\nu = 0))$. Following the steps leading to the bound (5.73) in [24], we deduce from (6.62) and [48, Theorem 7.3] that the complement of the connection probability $\big\{B_{N/4\sqrt{d}} \stackrel{\mathcal{V}^u}{\longleftrightarrow} \infty\big\}$ decays super-exponentially in N as $N \to \infty$. Together with (6.62), this implies (6.14) via (6.63) and a union bound. In fact, the inclusion (6.63) continues to hold with the event $\mathrm{LocUniq}(N,u)^c$ (and hence also $2\text{-arms}(N,u)$, see below (6.13)) on the left-hand side, as follows readily from definition (6.13). Therefore we get the same bound for the events $\mathrm{LocUniq}(N,u)^c$ and $2\text{-arms}(N,u)$ as well. □

The case $d = 3$ of Theorem 6.3, i.e. (6.15), requires several rounds of bootstrapping owing to the refined nature of the bounds involved. Content of the first round is summarized in our next lemma. Recall that $\mathbb{L}(L) = L\mathbb{Z}^d$.

Lemma 6.11 (Bootstrapping $\mathcal{G}_z^{\mathrm{i}}$; $d=3$)**.** *Suppose that (cf.* (6.42)*)*

$$\mathbb{P}\big[\mathcal{G}^{\mathrm{i}}_{0,L}\big(\overline{Z}_{\mathbb{L}(L)},\delta,\boldsymbol{u};a^{(1)}\big)\big]\geq 1-\theta' e^{-L^{\theta}}, \quad L\geq 1,\ \mathrm{i}=\mathrm{I},\mathrm{II} \tag{6.64}$$

for some $\theta\in(0,1)$, $\theta'\in(0,\infty)$, $a^{(1)}\geq 1$, L_0, L_0^- *and* $\boldsymbol{u}=(u_0,u_1,\dots,u_4)$ *satisfying* $2u_0<u_4\in(0,c_6]$ *in addition to* (6.16) *(our standing assumption) and all* $\delta\in[0,\delta']$ *for some* $\delta'\in(0,\frac12)$*. Then there exist* $\delta''=\delta''(\boldsymbol{u},\delta')\in(0,\frac12)$ *such that, with* $\boldsymbol{u}(k,\varepsilon)$ *as below* (6.52)*,*

$$\mathbb{P}\Big[\mathcal{G}^{\mathrm{i}}_{0,N}\Big(\overline{Z}_{\mathbb{L}(N)},\delta,\boldsymbol{u}(4,\varepsilon);a^{(2)}(N)=\tfrac{c'N}{(\log N)^{C(\theta)}}\cdot a^{(1)}\Big)\Big]\geq 1-C'e^{-\frac{N}{(1\vee\log N)^{C(\theta)}}}, \tag{6.65}$$

for all $N\geq 1$, $\mathrm{i}=\mathrm{I},\mathrm{II}$, $\delta\in[0,\delta'']$, $\varepsilon\in(0,1)$ *satisfying the last two of the three conditions in* (6.55) *and some* $C'=C'(\boldsymbol{u},L_0^-,L_0,\delta',\theta,\theta',\varepsilon)<\infty$ *and* $c'=c'(\boldsymbol{u},\theta,\theta',\varepsilon)>0$.

The proof of Lemma 6.11 is postponed for a few lines. From now on until the end of this section we assume that $d=3$. We start by explaining how Lemma 6.11 leads to a bound for the event $(\mathrm{V}^{\mathrm{i}}_{0,N}(\nu))^c$ similar to (6.50) but with a *larger* value of ν and a better error bound. The need for a larger value of ν arises from the change in the form of $\mathrm{V}_z=\mathrm{V}_{z,L}(\overline{Z}^u_{z,L}(\nu))$ (see above (6.27)) across any inclusion of the type (2.15) (see (6.82) below) which is essential for further improving the error bound in view of (4.19).

To the effect of improving over (6.50), for any given $u\in(0,u_*)$, let

$$\varepsilon=((1-(\tfrac{u}{u_*})^{\frac{1}{30}})\wedge\tfrac{1}{20}\ \text{and}\ \delta'=c_7(L_0(\boldsymbol{u})) \tag{6.66}$$

where $\boldsymbol{u}=(u_0,\dots,u_4)$ is given by

$$u_0=\tfrac{c_6}{10}, u_1=u_*(1-\varepsilon)^{21}, u_2=u_*(1-\varepsilon)^{20}, u_3=u_*(1-\varepsilon)\ \text{and}\ u_4=c_6 \tag{6.67}$$

(cf. (6.60)). In view of Lemma 6.9, we see that the conditions of Lemma 6.11 are satisfied with $\boldsymbol{u}$, ε and δ' as above, $a^{(1)}=1$ and $\theta=c$, $\theta'=C(u)$, $L_0=L_0(\boldsymbol{u})$ and L_0^- from Lemma 6.9. Thus (6.65) holds with ε as in (6.66) and the constants c', C depending effectively only on u with the above choices.

Now we notice from (6.66) and (6.67) that the conditions (6.51) and (6.55) are satisfied by $\boldsymbol{u}(4,\varepsilon)$ instead of $\boldsymbol{u}$ as well and consequently we can follow the steps leading to (6.58) in the proof of Theorem 1.1 starting from (6.65) in place of (6.42), which feeds into (6.52) and the subsequent estimates. In particular, when reaching the point in the argument leading to (6.58) at which Propositions 6.6 and 6.7 are applied, we can now afford to choose $a=a^{(2)}(L)$ owing to (6.65) when applying (6.36) and (6.41). Moreover, we are free to choose any value of ν for which these bounds remain meaningful; that is, with $K(u)=K(\varepsilon,\boldsymbol{u})$ as above (6.57) for the choices of $\varepsilon,\boldsymbol{u}$ from (6.66)-(6.67) and $L=L(N)$ as above (6.57), we pick

$$\nu\overset{\text{def.}}{=}\big(c(u)a^{(2)}(L)\,\tfrac{N}{h(K(u)L)}\big)\big|_{L=L(N)}\overset{(d=3)}{=}c(u)\,a^{(2)}(L(N))\,\tfrac{N}{K(u)L(N)}\overset{(6.65)}{\geq}c(u)\tfrac{N}{(1\vee\log\log N)^C}$$

when applying Propositions 6.6 and 6.7 (recall from §4.2 that $h(x)=x$ when $d=3$). All in all, we thus obtain, similarly as (6.50), that for all $u\in(0,u_*)$, $\mathrm{i}=\mathrm{I},\mathrm{II}$ and $N\geq 2$,

$$\mathbb{P}[(\mathrm{V}^{\mathrm{i}}_{0,N}(\nu;\bar{u}_0,u))^c]\leq C(u)e^{-N/(\log N)^{C'}}, \tag{6.68}$$

for some absolute constant $C'\in(0,\infty)$, with ν as above and $\bar{u}_0=u_0(1+\varepsilon)^3$ with ε and u_0 defined in (6.66)-(6.67).

The bound (6.68) brings us to the final round of bootstrapping where we derive the optimal upper bound on the probability of the 2-arms event. In fact (6.68) is more than we need, a stretched exponential bound in N with exponent close enough to 1 (cf. Proposition 4.5–i)) would have been sufficient (a similar comment applies to ν), as entailed by the following lemma.

Lemma 6.12 (Bootstrapping V_z to 2-arms; $d=3$)**.** *Suppose that for all $u \in (0,u_*)$, we have*

$$\sup_{L \geq C(u)} L^{-\frac{3}{4}} \log \mathbb{P}\big[(\mathrm{V}_{0,L}(\overline{Z}^u_{0,L}(\nu_L)))^c\big] \leq -1 \tag{6.69}$$

(see (6.22) for the event in question), for some $C(u) < \infty$ and $\nu_L \geq L(1 \vee \log L)^{-1/4}$. Then for any $\Lambda_N \in \mathcal{S}_N$ (recall (4.9)) and $u \in (0,u_*)$, we have (see below (6.13) for notation)

$$\limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[\text{2-arms}(\Lambda_N,u)] \leq -\frac{\pi}{3}(1-\sigma)(\sqrt{u}-\sqrt{u_*})^2. \tag{6.70}$$

Assuming Lemma 6.12 for a moment, we are now ready to conclude the proof of (6.15), thereby completing the proof of Theorem 6.3, contingent on Lemmas 6.11 and 6.12 which are proved below.

Proof of (6.15)*.* Combining the two estimates (6.68) for $\mathrm{i}=\mathrm{I},\mathrm{II}$ with (6.27) and using (2.22) to bound the Poisson deviation appearing in (6.27), we readily deduce that (6.69) is satisfied with ν_L as defined above (6.68) with L in place of N. This choice satisfies $\nu_L \geq L(1\vee \log L)^{-1/4}$ for $L \geq C(u)$ hence Lemma 6.12 is in force and thus (6.70) holds for all $u \in (0,u_*)$ and $\Lambda_N \in \mathcal{S}_N$. Now observe that,

$$\big\{0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B^2_N, 0 \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\big\} \cap \big\{B^2_{\sigma N} \overset{\mathcal{V}^u}{\longleftrightarrow} \infty\big\} \subset \text{2-arms}\left(B^2_N \setminus B^2_{\sigma N}, u\right) \tag{6.71}$$

for any $\sigma \in (0,1)$. Mimicking the proof of (5.73) in [24], we obtain from (6.70) applied to $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$, the disconnection estimate in [48, Theorem 7.3] which holds for all $u < \bar{u}$ and the equality of u_* and $\bar{u}$ in Theorem 1.2 of [20] that

$$\lim_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}\big[B^2_{\sigma N} \overset{\mathcal{V}^u}{\not\longleftrightarrow} \infty\big] = -\infty$$

for all $u \in [0,u_*)$ and $d=3$. Jointly with (6.71) this implies via a union bound that the left-hand side of (6.15) is bounded by $-\frac{\pi}{3}(1-\sigma)(\sqrt{u}-\sqrt{u_*})^2$for *any* $\sigma \in (0,1)$ and $u \in (0,u_*)$, and (6.15) follows upon letting $\sigma \downarrow 0$.

The corresponding bound for the event $\text{2-arms}(N,u)$ follows directly from (6.70). As to the event $\mathrm{LocUniq}(N,u)^c$, recall from (6.13) that

$$\mathrm{LocUniq}(N,u)^c \subset \text{2-arms}(N,u) \cup \big\{B^2_N \overset{\mathcal{V}^u}{\not\longleftrightarrow} B^2_{2N}\big\}.$$

The probability $\mathbb{P}[\text{2-arms}(N,u)]$ yields the desired contribution to the upper bound and similarly as before, one obtains by combining the results from [48] and [20] that for all $u \in (0,u_*)$, the above disconnection probability decays exponentially in $N^{d-2}=N$ as $N\to\infty$ when $d=3$. □

We now give the pending proofs of Lemma 6.11 and 6.12 starting with the:

Proof of Lemma 6.11. Let $\mathbb{L} = L\mathbb{Z}^d$. We will introduce slightly modified versions of the events $\mathscr{G}^{\mathrm{I}}_{z,N}$ and $\mathscr{G}^{\mathrm{II}}_{z,N}$. To this end we let, for any $\varepsilon \in (0,1)$, $\delta \in [0,\frac{1}{2})$, $\mathrm{i} \in \{\mathrm{I},\mathrm{II}\}$, and $\boldsymbol{u}(k,\varepsilon)$ as below (6.52),

$$\begin{aligned} &\overline{\mathscr{G}}^{\mathrm{i}}_{z,N}(Z_{\mathbb{L}},\delta,\boldsymbol{u},\varepsilon;a^{(1)}) \\ &\qquad \overset{\text{def.}}{=} \mathscr{G}\big(\tilde{D}_{z,N}\setminus\tilde{C}_{z,N}, \mathcal{G}^{\mathrm{i}} = \big\{\mathcal{G}^{\mathrm{i}}_{z'}(Z_{\mathbb{L}},\delta,\boldsymbol{u};a^{(1)}) : z' \in \mathbb{L}\big\}, \mathcal{F}^{\boldsymbol{u}(1,\varepsilon)',\boldsymbol{u}'}_L; \rho = \tfrac{1}{2C_4}\big) \end{aligned} \tag{6.72}$$

(cf. (6.33) and (6.39)) where $\boldsymbol{u}'$ is defined as above (6.43) for any $\boldsymbol{u}$ (recall from below the statement of Lemma 6.9 that we include the full list of parameters, including redundant ones, regardless of $\mathrm{i} \in \{\mathrm{I}, \mathrm{II}\}$). Now mimicking the derivation of (6.57) in the proof of Lemma 6.9 with (6.64) in lieu of (6.42) as the corresponding a-priori estimate, we obtain from an application of (4.21) at the final stage that with any $\varepsilon > 0$ small enough depending solely on $\boldsymbol{u}$ and $K = K(\boldsymbol{u}, \theta, \theta', \varepsilon)$ and $L(N) = \lfloor (\log N)^{C(\theta)} \rfloor$,

$$\mathbb{P}[\overline{\mathscr{G}}^{\mathrm{i}}_{0,N}(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(3,\varepsilon), \varepsilon; a^{(1)})] \geq 1 - C'e^{-\frac{N}{(1 \vee \log N)^{C(\theta)}}}, \quad \mathrm{i} = \mathrm{I}, \mathrm{II} \tag{6.73}$$

for any $\delta \in [0, \delta']$ and $N \geq 1$, and some C' with a dependence on parameters as specified below (6.65).

We also need to introduce versions of the events $\mathcal{G}^{\mathrm{i}}_z$ from (6.32) and (6.38) with excursions at scale N instead of L. These will carry a superscript "0." Thus, for $z \in \mathbb{L}$, recalling Definition 5.1, let

$$\mathcal{G}^{\mathrm{I},0}_{z,L}(\widehat{Z}_{0,N}, \delta, u_0, u_1, u_2, u_3; a) \stackrel{\text{def.}}{=} \mathcal{G}_z(\mathrm{V}^0, \mathrm{W}^{\mathrm{I},0}, \mathscr{C}^0; a), \tag{6.74}$$

where $\mathrm{V}^0_z = \mathrm{V}_z(\widehat{Z}^{u_1}_{0,N}, \widehat{Z}^{u_2}_{0,N}, \widehat{Z}^{u_3}_{0,N}, \delta)$ (see above (6.18)), $\mathrm{W}^{\mathrm{I},0}_{z,y} = \mathrm{FE}_y\big((\widehat{Z}^{u_1}_{0,N})_+(\hat{\nu}_{0,N}(\frac{u_0}{8}))\big)$ (cf. (6.29)), and $\mathscr{C}_z = \mathscr{C}_z(\widehat{Z}^{u_1}_{0,N}, \widehat{Z}^{u_3}_{0,N}, \delta)$ (see around (6.17)). In a similar vein, we let

$$\mathcal{G}^{\mathrm{II},0}_{z}(\widehat{Z}_{0,N}, \delta, u_4; a) \stackrel{\text{def.}}{=} \mathcal{G}_z(\mathrm{V}, \mathrm{W}^{\mathrm{II},0}, \mathscr{C}; a), \tag{6.75}$$

where $\mathrm{V}_z = \Omega$ and $\mathscr{C}_z = \mathbb{Z}^d$ as in (6.38) whereas $\mathrm{W}^{\mathrm{II},0}_{z,y} = \mathcal{G}^-_y(\widehat{Z}^{u_4}_{0,N})$ (cf. (6.37)). We now claim that

$$\mathcal{G}^{\mathrm{i}}_{z,L}(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(3,\varepsilon); a^{(1)}) \cap \mathcal{F}^{\boldsymbol{u}(4,\varepsilon)', \boldsymbol{u}(3,\varepsilon)'}_{z,L} \subset \mathcal{G}^{\mathrm{i},0}_{z,L}(\overline{Z}_{0,N}, \delta, \boldsymbol{u}(4,\varepsilon); a^{(1)}), \quad \mathrm{i} = \mathrm{I}, \mathrm{II} \tag{6.76}$$

for any $z \in \mathbb{L}$ such that $D_{z,L} \subset D_{0,N}$ and $U_{z,L} \subset U_{0,N}$; for later orientation, the event $\mathcal{G}^{\mathrm{i}}_{z,L}$ with arguments as on the left-hand side of (6.76) belongs precisely to the family used to declare the event $\overline{\mathscr{G}}^{\mathrm{i}}_{0,N}$ appearing in (6.73) in view of (6.72).

The inclusion (6.76) follows from similar arguments as those leading to (6.43) in the proof of Lemma 6.10, except that some caution is needed as the event on the right-hand side now involves excursions between $D_{0,N}$ and $U_{0,N}$ instead of $D_{z,L}$ and $U_{z,L}$. We now highlight these changes. Since $D_{z,L} \subset D_{0,N}$ and $U_{z,L} \subset U_{0,N}$, we get from (2.11) in §2.2 that on the event $\mathcal{F}^{\boldsymbol{u}(4,\varepsilon)', \boldsymbol{u}(3,\varepsilon)'}_{z,L}$,

$$\mathcal{I}\big(\overline{Z}^{u_k(1-\varepsilon)^4}_{0,N}\big) \cap D_{z,L} \subset \mathcal{I}\big(Z^{u_k(1-\varepsilon)^3}_{z,L}\big) \cap D_{z,L} \text{ for } k = 1,3, \quad \mathcal{I}\big(Z^{u_2(1+\varepsilon)^3}_{z,L}\big) \subset \mathcal{I}\big(\overline{Z}^{u_2(1+\varepsilon)^4}_{0,N}\big),$$

and moreover that for any $Z \in (Z^{u_1(1-\varepsilon)^4}_{0,N})_+\big(\bar{\nu}_{0,N}(\frac{u_0(1+\varepsilon)^4}{8})\big)$,

> there exists $Z' \in (Z^{u_1(1-\varepsilon)^3}_{z,L})_+\big(\nu_{z,L}\big(\frac{u_0(1+\varepsilon)^3}{8}\big)\big)$ satisfying $\mathcal{I}(Z) \cap D_{z,L} = \mathcal{I}(Z') \cap D_{z,L}$ and $\ell_x(Z) = \ell_x(Z')$ for all $x \in D_{z,L}$.

But these are enough to deduce (6.76) following the arguments in the proof of (6.43) owing to the defintions of the set $\mathscr{C}_{z,L}$ and the events $\mathrm{V}_{z,L}$ and $\mathrm{W}^{\mathrm{i}}_{z,y}$ (revisit (6.17), (6.18), (6.29) and (6.37)).

Now in view of (6.76), whereby condition (5.5) of Proposition 5.2 is satisfied for the pair of events $(\mathcal{G}^{\mathrm{i}}_{z,L}(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(3,\varepsilon); a^{(1)}), \mathcal{G}^{\mathrm{i},0}_{z,L}(\overline{Z}_{0,N}, \delta, \boldsymbol{u}(4,\varepsilon); a^{(1)}))$ and $\mathcal{F}_{z,L} = \mathcal{F}^{\boldsymbol{u}(4,\varepsilon)', \boldsymbol{u}(3,\varepsilon)'}_{z,L}$, we obtain by application of (5.8) that there exist (random) non-empty sets $\mathcal{O}^{\mathrm{I}}$ and $\mathcal{O}^{\mathrm{II}}$ satisfying (5.6) such that

$$\overline{\mathscr{G}}^{\mathrm{i}}_{0,N}(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(3,\varepsilon), \varepsilon; a^{(1)}) \subset \mathcal{G}_{0,N}(\mathrm{V}^{2,\mathrm{i}}, \mathrm{W}^{2,\mathrm{i}}, \mathscr{C}^{2,\mathrm{i}}; a^{(2)}), \quad \mathrm{i} = \mathrm{I}, \mathrm{II} \tag{6.77}$$

(recall (6.72) and that $\boldsymbol{u}(4,\varepsilon) = (\boldsymbol{u}(3,\varepsilon))(1,\varepsilon)$ in the notation from below (6.52)), where $a^{(2)} = \lfloor \frac{cN}{KL(N)} \rfloor \cdot a^{(1)}$ with K as above (6.73) (recall from §4.2 that $h(x) = x$ when $d = 3$), and the triplets $(\mathrm{V}^{2,\mathrm{i}}, \mathrm{W}^{2,\mathrm{i}}, \mathscr{C}^{2,\mathrm{i}})$ are specified as follows:

$$\mathrm{V}_0^{2,\mathrm{I}} = \bigcap_{z \in \mathcal{O}^{\mathrm{I}}} \mathrm{V}_{z,L}\big(\overline{Z}_{0,N}^{u_1(1-\varepsilon)^4}, \overline{Z}_{0,N}^{u_2(1+\varepsilon)^4}, \overline{Z}_{0,N}^{u_3(1-\varepsilon)^4}, \delta\big) \overset{(6.18)}{=} \bigcap_{z \in \mathcal{O}^{\mathrm{I}}} \mathrm{V}_{z,L}(\overline{Z}_{\mathbb{L}(N)}, \delta, \boldsymbol{u}(4,\varepsilon)),$$

$\mathrm{V}_0^{2,\mathrm{II}} = \Omega$, $\mathrm{W}_{0,y}^{2,\mathrm{I}} = \mathrm{W}_{0,y}^{\mathrm{I},0}$ and $\mathrm{W}_{0,y}^{2,\mathrm{II}} = \mathrm{W}_{0,y}^{\mathrm{II},0}$ for all $y \in \mathbb{L}_0$ (see below (6.74) and (6.75) respectively), $\mathscr{C}_0^{2,\mathrm{I}} = \bigcup_{z \in \mathcal{O}} \mathscr{C}_{z,L}(\overline{Z}_{0,N}^{u_1(1-\varepsilon)^4}, \overline{Z}_{0,N}^{u_3(1-\varepsilon)^4}, \delta)$ and $\mathscr{C}_0^{2,\mathrm{II}} = \mathbb{Z}^3$. In particular, these choices entail that (5.8) indeed applies in deducing (6.77).

It immediately follows from (6.77) and the definition of the event $\mathcal{G}_{0,N}^{\mathrm{II}}(\overline{Z}_{0,N}, \delta, \boldsymbol{u}; a)$ in (6.38) that

$$\overline{\mathscr{G}}_{0,N}^{\mathrm{II}}(Z_{\mathbb{L}}, \delta, \boldsymbol{u}(3,\varepsilon), \varepsilon; a) \subset \mathcal{G}_{0,N}^{\mathrm{II}}\big(\overline{Z}_{\mathbb{L}(N)}, \delta, \boldsymbol{u}(4,\varepsilon); a^{(2)}\big)$$

with $a^{(2)} = \lfloor \frac{c'N}{(\log N)^{C(\theta)}} \rfloor \cdot a^{(1)}$ for some $c' = c'(\boldsymbol{u}, \theta, \theta', \varepsilon) > 0$, whence (6.65) follows for $\mathrm{i} = \mathrm{II}$ by (6.73). For $\mathrm{i} = \mathrm{I}$, abbreviating $\mathrm{Conn} = \big\{C_{0,N} \leftrightarrow \partial D_{0,N} \text{ in } \mathsf{N}_{2\delta}(\mathcal{V}^{u_3(1-\varepsilon)^4})\big\}$ we have

$$\mathcal{G}_{0,N}\big(\mathrm{V}^{2,\mathrm{I}}, \mathrm{W}^{2,\mathrm{I}}, \mathscr{C}^{2,\mathrm{I}}; a^{(2)}\big) \cap \mathrm{Conn} \subset \mathcal{G}_{0,N}^{\mathrm{I}}\big(\overline{Z}_{\mathbb{L}(N)}, \delta, \boldsymbol{u}(4,\varepsilon); a^{(2)}\big), \tag{6.78}$$

which also follows readily from the definition of $\mathcal{G}_{0,N}^{\mathrm{I}}(\overline{Z}_{\mathbb{L}(N)}, \delta, \boldsymbol{u}; a^{(2)})$ in (6.32) provided one has $\mathrm{V}_0^{2,\mathrm{I}} \subset \mathrm{V}_{0,N}(\overline{Z}_{\mathbb{L}(N)}, \delta, \boldsymbol{u}(4,\varepsilon))$ and $\mathscr{C}_0^{2,\mathrm{I}} \subset \mathscr{C}_{0,N}(\overline{Z}_{\mathbb{L}(N)}, \delta, \boldsymbol{u}(4,\varepsilon))$ *on* the event Conn (as already noted, the event $\mathcal{G}_z(\mathrm{V}, \mathrm{W}, \mathscr{C}; a)$ is increasing in V_z and $\mathscr{C}_z$ which is evident from (5.3)–(5.4)). Both of these inclusions follow from standard gluing arguments inherent in the definition of the events $\mathrm{V}_{z,L}$ and already used in the proof of Lemma 6.5 above. For a precise verification, we refer the reader to the arguments used in §7.2 to derive (7.11) in the course of proving Proposition 6.6. Finally, (6.78), (6.73) and the upper bound on the disconnection probability from (6.12) for $\delta \le \frac{c_7(u_3)}{2}$ (recall (2.24)) together imply (6.65) for $\mathrm{i} = \mathrm{I}$ via a simple union bound. □

Next we give the:

Proof of Lemma 6.12. Let us start with an inclusion of events. For any $\Lambda_N \in \mathcal{S}_N$ as in (4.9), all $\rho \in (0,1)$, $0 < u < v < u_*$ and $\nu \ge 0$, we have

$$\mathscr{G}(\Lambda_N, \mathrm{V}_L, \mathcal{F}_L^{u,v}; \rho) \subset (\text{2-arms}(\Lambda_N, u))^c \tag{6.79}$$

where $\mathrm{V}_L = \{\mathrm{V}_z : z \in \mathbb{L}\}$ and $\mathrm{V}_z = \mathrm{V}_z(Z_z^v(\nu))$ for $z \in \mathbb{L}$. To see this, first note that the sequence $\overline{Z}_z^u$ lies in the family $Z_z^v(\nu)$ on the event $\mathcal{F}_z^{u,v}$ (revisit (6.21) and (2.21) for relevant definitions) and therefore by (6.22),

$$\mathrm{V}_z \cap \mathcal{F}_z^{u,v} \subset \mathrm{V}_z(\overline{Z}, u).$$

Thus condition (5.5) of Proposition 5.2 is satisfied for the pair of events $(\mathrm{V}_z, \mathrm{V}_z(\overline{Z}, u))$ and $\mathcal{F}_{z,L} = \mathcal{F}_z^{u,v}$, and hence by (5.7) there exists a (random) non-empty, $*$-connected set $\mathcal{O}'$ satisfying (5.6) such that

$$\mathscr{G}(\Lambda_N, \mathrm{V}_L, \mathcal{F}_L^{u,v}; \rho) \subset \bigcap_{z \in \mathcal{O}'} \mathrm{V}_z(\overline{Z}, u).$$

From this and the definition of the event $\mathrm{V}_z(\overline{Z}, u)$ given below (6.18) it follows by elementary gluing considerations (see also (7.12) and (7.13) in §7.1) that on the event $\mathscr{G}(\Lambda_N, \mathrm{V}, \mathcal{F}_L^{u,v}; \rho)$,

(6.80) there exists a component $\mathscr{C}_{\mathcal{O}'}$ of $\Lambda_N \cap \mathcal{V}^u$ which contains all crossing clusters of $\tilde{D}_z \setminus \tilde{C}_z$ in $D_z \cap \mathcal{V}^u$ for each $z \in \mathcal{O}'$.

Moreover, writing $\Lambda_N = V_N \setminus U_N$, since $\{0\} \cup U_N \cap \mathbb{L} \preceq \mathscr{O}' \preceq \partial_{\mathbb{L}}(V_N \cap \mathbb{L})$ by (5.6) (see (C.1) for definition), any crossing of Λ_N in $\mathcal{V}^u$ must lie in the *same* component of $\Lambda_N \cap \mathcal{V}^u$ as $\mathscr{C}_{\mathscr{O}'}$ on the event (6.80), thus yielding (6.79) (the definition of the 2-arms event appears below (6.13)).

In view of (6.79), it suffices to obtain (6.70) with 2-arms(Λ_N, u) replaced by the event $(\mathscr{G}(\Lambda_N, \mathrm{V}_L, \mathcal{F}_L^{u,v}; \rho))^c$ for 'suitable' values of the parameters v, ρ and K (recall (4.16)). Obviously, we will use Proposition 4.5 to this end. We start just like in the proof of Theorem 1.1 with the events $\mathrm{V}_z(\widehat{Z}_z^v(\nu))$ in place of $\mathcal{G}_z^{\mathrm{i}}(\widehat{Z}_{\mathbb{L}}, \delta, \boldsymbol{u}; a)$. By similar arguments as those yielding (6.43), we have that

$$\mathrm{V}_z(\overline{Z}_z^v(\nu)) \cap \mathcal{F}_z^{v,v(1-\varepsilon)} \subset \mathrm{V}_z(Z_z^{v(1-\varepsilon)}(\nu)) \tag{6.81}$$

where $\varepsilon \in (0,1)$. In place of (6.44), on the other hand, we have that under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_z$,

$$\begin{aligned} &\mathrm{V}_z(\widetilde{Z}_z^v(\nu)) \cap \mathrm{Incl}_z^{\frac{\varepsilon}{10}, \frac{1}{20}(\nu \wedge v\,\mathrm{cap}(D_z))} \subset \mathrm{V}_z(Z_z^{v(1-\varepsilon)}(\nu - 2u_*\,\mathrm{cap}(D_z)\,\varepsilon)) \text{ and} \\ &\mathrm{V}_z(Z_z^v(\nu)) \cap \mathrm{Incl}_z^{\frac{\varepsilon}{10}, \frac{1}{20}(\nu \wedge v\,\mathrm{cap}(D_z))} \subset \mathrm{V}_z(\widetilde{Z}_z^{v(1-\varepsilon)}(\nu - 2u_*\,\mathrm{cap}(D_z)\,\varepsilon)) \end{aligned} \tag{6.82}$$

whenever $\nu \geq 4u_*\,\mathrm{cap}(D_z)\,\varepsilon$, for $\varepsilon \in (0, \frac{1}{2})$ and $L \geq \frac{C}{v\varepsilon}$. In view of (6.23), (6.82) follows readily from the inclusions

$$\big\{Z_z^{v(1-\varepsilon)}(\nu - 2u_*\,\mathrm{cap}(D_z)\,\varepsilon)\big\} \subset \big\{\widetilde{Z}_z^v(\nu)\big\}, \quad \big\{\widetilde{Z}_z^{v(1-\varepsilon)}(\nu - 2u_*\,\mathrm{cap}(D_z)\,\varepsilon)\big\} \subset \big\{Z_z^v(\nu)\big\},$$

which hold on the event $\mathrm{Incl}_z^{\frac{\varepsilon}{10}, \frac{1}{20}(\nu \wedge v\,\mathrm{cap}(D_z))}$ owing to the definition of the latter in (2.15) and the family $Z(\nu)$ in (6.21). In fact, the family $Z(\nu)$ was defined in this way precisely so that the inclusions like those in (6.82) could hold on the event $\mathrm{Incl}_z^{\varepsilon, m_0}$.

Equipped with (6.81) and (6.82), we can now follow in the steps of the proof of Theorem 1.1 starting from (6.69) instead of (6.42) as the corresponding a-priori estimate and with $\nu = \nu_L$. In particular, we obtain that the conditions (4.19)–(4.20) of Proposition 4.5 are satisfied by the events

$$\widetilde{\mathcal{G}}_{z,L} = \mathrm{V}_z(Z_z^{v(1-\varepsilon)^2}(\nu_L - 2u_*\,\mathrm{cap}(D_z)\,\varepsilon_L)), \quad \mathcal{G}_{z,L} = \mathrm{V}_z(Z_z^{v(1-\varepsilon)^3}(\nu_L - 4u_*\,\mathrm{cap}(D_z)\,\varepsilon_L)) \tag{6.83}$$

for $\varepsilon_L = c(1 \vee \log L)^{-1/4}$, $m_L = cL(1 \vee \log L)^{-1/4}$, whence

$$\widetilde{\mathbb{P}}_z[(\mathcal{U}_z^{\varepsilon_L, m_L})^c] \overset{(2.14)}{\leq} C\varepsilon_L^{-2} e^{-cm_L \varepsilon_L^2} \leq Ce^{-\frac{L}{1 \vee \log L}}$$

(cf. (6.53) and (4.19)–(4.20)); $\beta' = \frac{3}{4} - \frac{1}{8}\,(> \frac{1}{2})$, $K_0 = \frac{C}{\varepsilon_L} = C(1 \vee \log L)^{\frac{1}{4}}$ and any $\varepsilon \in (0,1)$; L_0 (from (4.20)) and L sufficiently large depending only on v and ε (here we plugged the bound $cL \leq \mathrm{cap}(D_z) \leq CL$ from (A.8) into (6.82)). The estimate (4.21) now yields with the choice $(\boldsymbol{u}, \boldsymbol{v}) = (u, v(1-\varepsilon)^3)$, $\widetilde{\mathcal{G}}_{z,L}$ and $\mathcal{G}_{z,L}$ as in (6.83), $\delta \in (0,1)$, $\rho \in (0,1)$, $L(N) = \lfloor (\log N)^\alpha \rfloor$ for some absolute constant $\alpha \in (0, \infty)$ and $K = \sqrt{1 \vee \log\log N}$ that for all $v \in (0, u_*)$,

$$\begin{aligned} &\limsup_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}[\mathscr{G}^c(\Lambda_N, \mathrm{V}_{L(N)}, \mathcal{F}_{L(N)}^{u,v(1-\varepsilon)^3}; \rho)] \\ &\qquad \leq -(1-\delta)(1-\sigma)(1-C_4\rho)\frac{\pi}{3}(\sqrt{u} - \sqrt{v(1-\varepsilon)^3})^2. \end{aligned} \tag{6.84}$$

Sending δ, ρ and ε to 0 and subsequently v to u_*, we obtain (6.70) in view of (6.79). □

Proof of Theorem 1.4. Theorems 6.1 and 6.2 immediately imply Theorem 1.4 on account of the inclusion

$$\{0 \overset{\mathcal{V}^u}{\longleftrightarrow} x\} \subset \{0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B^2_{|x|}\}$$

(see (1.6) and below to recall the definition of $\tau_u^{\mathrm{tr}}(x)$). □

7 Gluing of large clusters

In this section we give the proofs of Propositions 6.6 and 6.7. The proof of Proposition 6.6 takes up most of this section and contains several novel ideas. It is split into two subsections. In §7.1, we deduce Proposition 6.6 from the presence of a large number of 'good' encounter points between any crossing cluster of $\tilde{D}_{z,N} \setminus \tilde{C}_{z,N}$ in $\mathcal{V}(Z)$ and an ambient cluster in $(\mathcal{V}^u)_\delta$ when the former is explored by a suitable algorithm. Here Z is any sequence of excursions included in the definition of the event $\mathrm{V}^{\mathrm{I}}_{z,N}$ in (6.25). A 'good' encounter point, in this context, is a point $y \in \mathbb{L}_0$ such that both the ambient cluster in $(\mathcal{V}^u)_\delta$ and the crossing cluster in $\mathcal{V}(Z)$ intersect the box D_{y,L_0} *during* the exploration of the latter in a way that there is a sizeable chance for them to connect to each other. The precise formulation of the exploration in question will allow us to leverage Proposition 3.1 and to accumulate on the event $\mathscr{G}^{\mathrm{i}}_{z,N}$ the cost of failure to connect in *many* good encounter points as the exponential term on the right-hand side of (6.36) a (6.41) (we are willfully neglecting the role of the disconnnection event which plays a minor role). In §7.2, we describe a delicate exploration scheme designed to ensure a large number of such good encounter points, thus supplying the last missing ingredient to the proof of Proposition 6.6. Finally, in §7.3, we prove Proposition 6.7 by adapting some parts of the proof of Proposition 6.6.

7.1. Gluing clusters using good encounter points. We introduce a sequence $\tau = (\tau_k)_{k\geq 1}$ of *encounter times* attached to the dynamics of a standard cluster exploration algorithm. These random times are carefully designed to signal the presence of encounter points in the above sense while retaining propitious measurability features; see (7.3)-(7.5) below. Their existence is guaranteed by Proposition 7.1, which plays a pivotal role, and from which Proposition 6.6 is derived in the present subsection. Throughout §7.1 and §7.2, we assume that the assumptions of Proposition 6.6 hold; in particular, this entails that all parameters (such as $N, L, \dots$) appearing below satisfy (6.16).

We start by introducing the relevant cluster exploration algorithm. Throughout, J denotes a fixed finite (possibly empty) subset of $\mathbb{N}^* = \{1, 2., \dots\}$ and we use Z_J as a shorthand for the sequence $Z_J^{D_{z,N},U_{z,N}} = (Z_j^{D_{z,N},U_{z,N}})_{j\in J}$ (see (2.11) and (2.19) for notation), where $z \in N\mathbb{Z}^d$ as in the statement of Proposition 6.6. We will omit N in all the subscripts involving z like $\tilde{C}_{z,N}$, $N^u_{z,N}$ etc. For a point $x \in \partial\tilde{C}_z$, we let $\mathscr{C}_J(x)$ denote the (possibly) cluster of x inside $D_z \cap \mathcal{V}(Z_J)$ with $\mathcal{V}(Z_J)$ given by (2.18). We refer to the beginning of Section 2 as to the definition of $\partial^{\mathrm{out}}_{D_z}$. By convention, we set $\partial^{\mathrm{out}}_{D_z}\mathscr{C}_J(x) = \{x\}$ in the sequel whenever $\mathscr{C}_J(x) = \emptyset$.

The algorithm in question consists of a sequence $(w_n)_{n\geq 1}$ of $\mathbb{Z}^d$-valued random variables on the space $(\Omega, \mathcal{A}, \mathbb{P})$ defined above (2.2), where $w_1 = x$ and

> if for some $n > 1$, the set $\bigcup_{1\leq i\leq n-1}\{w_i\}$ equals $\mathscr{C}_J(x) \cup \partial^{\mathrm{out}}_{D_z}\mathscr{C}_J(x)$ (an event measurable relative to the random variables $(w_1, 1_{\{w_1\in\mathcal{V}(Z_J)\}}), \dots, (w_{n-1}, 1_{\{w_{n-1}\in\mathcal{V}(Z_J)\}})$), then $w_n = w_{n-1}$. Otherwise, w_n is the smallest point (in a fixed deterministic ordering of $\mathbb{Z}^d$) in $\mathbb{Z}^d \setminus \bigcup_{1\leq i\leq n-1}\{w_i\}$ that lies on $\partial^{\mathrm{out}}_{D_z}\mathscr{C}_{J,n}(x)$ where $\mathscr{C}_{J,n}(x)$ is the cluster of x in $\bigcup_{1\leq i\leq n-1}\{w_i\} \cap \mathcal{V}(Z_J)$.

In plain words, $(w_n)_{n\geq 1}$ reveals $\mathscr{C}_J(x) \cup \partial^{\mathrm{out}}_{D_z}\mathscr{C}_J(x)$ vertex by vertex, starting from x, inspecting at each step the state of the smallest *unexplored* point in the outer boundary of the currently explored part of $\mathscr{C}_J(x)$ *provided* the set of such vertices is non-empty. The set $\bigcup_{1\leq i\leq n}\{w_i\}$ consists of the points explored up to *time* n with $\mathscr{C}_{J,n}(x)$ as the explored part of $\mathscr{C}_J(x)$ by this time. The set of explored vertices at any time is a *connected* subset of $\mathscr{C}_J(x) \cup \partial^{\mathrm{out}}_{D_z}\mathscr{C}_J(x)$, which follows via a straightforward induction argument. One has that $w_n \notin \bigcup_{1\leq i\leq n-1}\{w_i\}$ as long as $\bigcup_{1\leq i\leq n-1}\{w_i\}$ is a proper subset of $\mathscr{C}_J(x) \cup \partial^{\mathrm{out}}_{D_z}\mathscr{C}_J(x)$. Since the latter is a finite set, $\mathbb{P}$-a.s. $w_{n+1} = w_n$ for all sufficiently large n, i.e. the exploration is complete in finite time.

The aforementioned variables $(\tau_k)_{k\geq 1}$ will be coupled to the exploration algorithm $(w_n)_{n\geq 1}$. They roughly act as *stopping times* for the underlying process, thereby providing useful control on the exploration, as will be seen shortly. We start with some preparation. Recall that $\mathbb{L} = L\mathbb{Z}^d$ and the event V_z given by (6.18). For fixed z (as in the statement of Proposition 6.6), consider the (random) set

$$\Sigma \stackrel{\text{def.}}{=} \big\{z' \in \mathbb{L} : D_{z',L} \subset \tilde{D}_z \setminus \tilde{C}_z \text{ and } \mathrm{V}_{z',L}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3}, u_{2,3}) \text{ occurs}\big\}, \tag{7.1}$$

where, as for the remainder of this section, the parameters $u_0, u, u_{2,3}$ etc. carry the same meaning as in (6.16) and (6.34). We now apply Proposition C.1 on $\mathbb{L}$ instead of $\mathbb{Z}^d$ (identified e.g. via the canonical isomorphism $z \in \mathbb{L} \mapsto (L^{-1}z) \in \mathbb{Z}^d$) with the choices $U = \{z' \in \mathbb{L} : C_{z',L} \cap \tilde{C}_z \neq \emptyset\}$, $V = \{z' \in \mathbb{L} : C_{z',L} \cap \tilde{D}_z \neq \emptyset\}$ and Σ as in (7.1). We refer to $k(\Sigma)$ as the maximal value of $k \geq 0$ such that the assumptions of Proposition C.1 are met with these choices, and denote by $O_1, \dots, O_\ell \subset \Sigma$ the $*$-connected sets (as subsets of $\mathbb{L}$) thus obtained when choosing $k = k(\Sigma)$. Note that possibly $k(\Sigma) = 0$ in which case $\ell = 0$. Using the sets $O_1, \dots, O_\ell$, we can define a special property of a point $y \in \mathbb{L}_0 = L_0\mathbb{Z}^d$ (cf. (6.16)) as follows:

$$\begin{aligned}&C_{y,L_0} \text{ intersects } \mathscr{C}_{z',L} = \mathscr{C}_{z'}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3}) \text{ (see (6.17)) for some } z' \in \textstyle\bigcup_{1\leq j\leq \ell} O_j\\ &\text{and } \widetilde{\mathrm{FE}}_{y,L_0}(Z_J) \stackrel{\text{def.}}{=} \widetilde{\mathrm{LU}}_{y,L_0}(Z_J) \cap \mathrm{O}_{y,L_0}(\overline{Z}^u_z) \text{ occurs,}\end{aligned} \tag{7.2}$$

(recall (3.2) and (3.3) for $\widetilde{\mathrm{LU}}_{y,L_0}$ and O_{y,L_0} respectively and compare with the definition of the event $\mathrm{FE}_{y,L_0}(Z)$ in (6.30)).

Finally it will be convenient introduce a partition of the set $\mathbb{L}_0$. For each $y \in \mathbb{L}_0 \cap D_{0,L_0}$, we call $\mathbb{L}_{0,y} = y + 7L_0\mathbb{Z}^d$. In view of (2.19), the sets $\mathbb{L}_{0,y}$ partition $\mathbb{L}_0$ as y ranges over $\mathbb{L}_0 \cap D_{0,L_0}$. Furthermore, for any given y, the boxes D_{y',L_0} with $y' \in \mathbb{L}_{0,y}$ form a partition of $\mathbb{Z}^d$. Let $\widehat{\mathbb{L}}_{0,y}$ be obtained from $\mathbb{L}_{0,y}$ by removing all points y' such that D_{y',L_0} intersects $\widetilde{C}_z$.

We now have all the necessary terms to spell out the main result of this section. Fix $y \in \mathbb{L}_0 \cap D_{0,L_0}$, which acts as a reference point. We call $\tau = (\tau_k)_{k\geq 1}$ a sequence of *good encounter times (for $\mathscr{C}_J(x)$)* if $(\tau_k)_{k\geq 1}$ is a non-decreasing sequence of $\mathbb{N}^* \cup \{\infty\}$-valued random variables on the space $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying the following three conditions:

(7.3) If $\tau_k < \infty$, then $w_{\tau_k} \in \partial D_{y_k,L_0}$ for some (unique) $y_k \in \widehat{\mathbb{L}}_{0,y}$ such that y_k satisfies property (7.2). Conversely, if $y' \in \widehat{\mathbb{L}}_{0,y}$ satisfies property (7.2) and C_{y',L_0} intersects $\mathscr{C}_J(x)$, there exists $k \geq 1$ such that $\tau_k < \infty$ with $y' = y_k$.

(7.4) If $\tau_k < \infty$ and $C_{y_k,L_0} \subset \mathcal{V}(Z_J)$, then w_{τ_k} is connected to C_{y_k,L_0} in $D_{y_k,L_0} \cap \mathcal{V}(Z_J)$.

(7.5) For any $y' \in \widehat{\mathbb{L}}_{0,y}$ and $k \geq 0$, the event $\bigcap_{1\leq j\leq k}\{\tau_j < \infty, C_{y_j,L_0} \not\subset \mathcal{V}(Z_J)\} \cap \{\tau_{k+1} < \infty, y_{k+1} = y'\}$ is measurable rel. to the σ-algebra $\mathcal{F}_{y',L_0}(Z_J, \delta, u, u_{2,3})$ defined in (3.4).

Following is the main result of this section.

Proposition 7.1. *For all finite $J \subset \mathbb{N}^*$, $y \in \mathbb{L}_0 \cap D_{0,L_0}$ and $x \in \partial\tilde{C}_z$, there exists a sequence of good encounter times $(\tau_k)_{k\geq 1} \equiv (\tau_{k;J,y}(x))_{k\geq 1}$ for $\mathscr{C}_J(x)$.*

We will prove this result in the next subsection by an elaborate construction. For the time being we proceed with the proof of Proposition 6.6 assuming this. The *good encounter points* alluded to in the beginning of this section correspond to the points y_k in (7.3) when $\tau_k < \infty$. So far we have not said much on the events $\mathscr{G}^{\mathrm{I}}_z$ and $\{C_z \leftrightarrow \partial D_z \text{ in } (\mathcal{V}^{u_{2,3}})_{2\delta}\}$ whose complements appear on the right-hand side of (6.36). Our next lemma connects these two events to the finiteness of τ_k in Proposition 7.1 for a *large* value of k. As already mentioned, we are implicitly working under the assumptions appearing in the statement of Proposition 6.6 (in particular, (6.16) is in force) and $u_{2,3} = \frac{1}{2}(u_2 + u_3)$ (as in (6.34)).

Lemma 7.2. *There exists $c_8 \in (0,\infty)$ such that, for all J, y, x as above, letting*

$$A_{J,y}(x) \stackrel{\text{def.}}{=} \{\tau_{\lceil c_8 am\rceil;J,y}(x) < \infty\}, \tag{7.6}$$

where $a (\in \mathbb{N}^*)$ enters the definition of $\mathscr{G}^{\mathrm{I}}_{z,N}$ (see (6.35)) and m is the common cardinality of coarsenings in $\mathcal{A}_L^K(\tilde{D}_z \setminus \tilde{C}_z)$ (cf. Prop. 4.3), one has the inclusion, with y ranging over $\mathbb{L}_0 \cap D_{0,L_0}$ below,

$$\mathscr{G}^{\mathrm{I}}_z \cap \{x \xleftrightarrow{(\mathcal{V}^{u_{2,3}})_{2\delta}} \partial\tilde{D}_z\} \cap \{[1, N_z^{\frac{u_0}{2}}] \subset J \subset [1, N_z^u]\} \subset \bigcup_y A_{J,y}(x). \tag{7.7}$$

In (7.7) and below we tacitly identify an interval $[a,b] \subset \mathbb{R}$ with $[a,b]\cap\mathbb{Z}$ and $[1,0] = \emptyset$ by convention. We will prove Lemma 7.2 at the end of this subsection. We now proceed with the:

Proof of Proposition 6.6. For any (deterministic) finite $J \subset \mathbb{N}^*$, we claim that

$$\mathbb{P}\big[(\mathrm{V}_z(Z_J))^c,\, \mathscr{G}^{\mathrm{I}}_z,\, C_z \xleftrightarrow{(\mathcal{V}^{u_{2,3}})_{2\delta}} \partial D_z,\, [1, N_z^{\frac{u_0}{2}}] \subset J \subset [1, N_z^u]\big] \leq CN^{d-1}e^{-c(\delta,L_0)am} \tag{7.8}$$

with m as below (7.6). Let us quickly conclude the proof assuming this bound. Recalling the definition of the event $\mathrm{V}^{\mathrm{I}}_z$ from (6.25) which depends on the collection $\overline{Z}^u_z(\nu) \cap \big((\overline{Z}^u_z)_+(N_z^{u_0/2})\big)$ defined in (6.19)–(6.21) (see also (6.23)), we deduce the inclusion

$$\begin{aligned}(\mathrm{V}^{\mathrm{I}}_z)^c \subset \Big(&(\mathscr{G}^{\mathrm{I}}_z)^c \cup \{C_z \overset{(\mathcal{V}^{u_{2,3}})_{2\delta}}{\nleftrightarrow} \partial D_z\} \cup (\mathcal{F}^{u,2u_*}_z)^c \cup \\ &\bigcup_{\substack{\{1,\dots,j\}\subset J\subset\{1,\dots,j+\nu\}\\ j+\nu\leq 2u_*\mathrm{cap}(D_z)}} \big((\mathrm{V}_z(Z_J))^c \cap \mathscr{G}^{\mathrm{I}}_z \cap \{C_z \xleftrightarrow{(\mathcal{V}^{u_{2,3}})_{2\delta}} \partial D_z\} \cup \{[1, N_z^{u_0}] \subset J \subset [1, N_z^u]\}\big)\Big);\end{aligned} \tag{7.9}$$

indeed to obtain this it suffices to note that $N_z^u \leq 2u_*\mathrm{cap}(D_z)$ on the event $\mathcal{F}^{u,2u_*}_z$, see (2.21), whence any package $Z_J = (Z_j)_{j\in J}$ belonging to $\overline{Z}^u_z(\nu)$, which by definition comprises excursions drawn from $\overline{Z}^u_z$ in (2.20), have label at most $2u_*\mathrm{cap}(D_z)$. Taking expectations and applying a union bound in (7.9), the claim (6.36) readily follows upon using that $\mathbb{P}[(\mathcal{F}^{u,2u_*}_z)^c] \leq e^{-cN^{d-2}}$ by (2.22) (and since $u \in (0,u_*)$, see (6.16)), observing that the number of terms in the resulting summation over J is at most

$$C\,2^\nu \mathrm{cap}(D_z) \overset{(A.8)}{\leq} C2^\nu N^{d-2},$$

and combining this with the bound (7.8) to deduce that the probability of the event in the second line of (7.9) is bounded by

$$\sum_{\substack{\{1,\dots,j\}\subset J\subset\{1,\dots,j+\nu\}\\ j+\nu\leq 2u_*\mathrm{cap}(D_z)}} CN^{d-1}e^{-c(\delta,L_0)am} \leq \exp\{-c(\delta,L_0)(am\wedge N) + C(\nu + \log N)\},$$

which leads to the last term in (6.36) on account of (4.12), by which $m \geq cN/h(KL)$.

It remains to show (7.8). Towards this, we will show an intermediate statement which is formally the same as (7.8) but with the event $\mathrm{V}_z(Z_J)$ replaced by $\widetilde{\mathrm{V}}_z(Z_J) \stackrel{\text{def.}}{=} \bigcap_{x\in\partial\tilde{C}_z} \widetilde{\mathrm{V}}_{z,x}(Z_J)$, where

$$\widetilde{\mathrm{V}}_{z,x}(Z_J) \stackrel{\text{def.}}{=} \{x \overset{\mathcal{V}(Z_J)}{\nleftrightarrow} \partial\tilde{D}_z\} \cup \{x \xleftrightarrow{D_z\cap\mathcal{V}(Z_J)} \textstyle\bigcup_{z'} \mathscr{C}_{z',L}\}; \tag{7.10}$$

here $\mathscr{C}_{z',L} = \mathscr{C}_{z'}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3})$ and the union is over $z' \in \bigcup_{1\le j\le \ell} O_j$ with O_j as introduced below (7.1). The bound (7.8) follows from its version for $\widetilde{\mathrm{V}}_z(Z_J)$ and the following inclusion of events:

$$\widetilde{\mathrm{V}}_z(Z_J) \cap \{C_z \xleftrightarrow{(\mathcal{V}^{u_{2,3}})_{2\delta}} \partial D_z\} \cap \{J \subset [1, N_z^u]\} \subset \mathrm{V}_z(Z_J). \tag{7.11}$$

Let us first derive the inclusion (7.11). Recall the definition of the event $\mathrm{V}_{z',L} = \mathrm{V}_{z',L}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3}, u_{2,3})$ from (6.18) and also the set $\mathscr{C}_{z',L} = \mathscr{C}_{z'}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3})$ from (6.17) for $z' \in \mathbb{L}$. It follows from these two definitions combined with the observations (a) $\mathcal{V}(\overline{Z}^v_{z',L}) \cap D_{z',L} = \mathcal{V}^v \cap D_{z',L}$ for any $v \ge 0$ (see (2.20)), (b) $(\tilde{D}_{z',L} \setminus \tilde{C}_{z',L}) \subset (D_{z'',L} \setminus C_{z'',L})$ for any $|z' - z''|_\infty \le L$ (recall (2.19)) and (c) the inclusion $(\mathcal{V}^u)_\delta \subset \mathcal{V}^u$ (see (2.24)) that

$$\begin{aligned}&\mathscr{C}_{z',L} \text{ and } \mathscr{C}_{z'',L} \text{ are (non-empty and) connected in } (D_{z',L} \cup D_{z'',L}) \cap \mathcal{V}^u\\ &\text{whenever } |z' - z''|_\infty \le L \text{ and } \mathrm{V}_{z',L} \cap \mathrm{V}_{z'',L} \text{ occurs.}\end{aligned} \tag{7.12}$$

In particular, (7.12) applies by (7.1) when z, z' are $*$-neighbors in Σ. Recalling from the paragraph containing (7.1) that each O_j is a $*$-connected subset of Σ, (7.12) thus implies that

$$\text{the set } \bigcup_{z' \in O_j} \mathscr{C}_{z',L} \text{ is connected in } D_z \cap \mathcal{V}^u \text{ for each } 1 \le j \le \ell, \tag{7.13}$$

where we also used that $D_{z',L} \subset D_z$ for any $z' \in \Sigma$ (see (7.1)). The sets $O_1, \dots, O_\ell$ also satisfy property (a) in Proposition C.1 with $\mathbb{L}$ as the underlying lattice and hence any crossing of $\tilde{D}_z \setminus \tilde{C}_z$ must necessarily cross $\tilde{D}_{z',L} \setminus \tilde{C}_{z',L}$ for some $z' \in O_j$ and each $1 \le j \le \ell$. Furthermore, if this crossing lies in $(\mathcal{V}^{u_{2,3}})_{2\delta}$, then it must be connected to $\mathscr{C}_{z'}$ in $(\mathcal{V}^u)_\delta$ (and hence in $\mathcal{V}^u$) by the definition of $\mathscr{C}_{z',L}$ (revisit (6.17)). Combined with the last two displays, this yields that

$$\text{the set } \bigcup_{z' \in \bigcup_{1\le j\le \ell} O_j} \mathscr{C}_{z',L} \subset D_z \text{ is connected in } \mathcal{V}^u \text{ on the event } \{C_z \xleftrightarrow{(\mathcal{V}^{u_{2,3}})_{2\delta}} \partial D_z\}.$$

Now together with definitions of the events $\widetilde{\mathrm{V}}_{z,x}(Z_J)$ and $\widetilde{\mathrm{V}}_z(Z_J)$ in and above (7.10) and also $\mathrm{V}_z(Z_J)$ above (6.18), the previous display yields the inclusion (7.11).

It remains to prove (7.8) in its version for $\widetilde{\mathrm{V}}_z$. In view of the definition of the event $\widetilde{\mathrm{V}}_z(Z_J)$ above (7.10) and the inclusion (7.7) in Lemma 7.2, it suffices to show that for all $x \in \partial\tilde{C}_z$, $y \in \mathbb{L}_0 \cap D_{0,L_0}$,

$$\mathbb{P}\big[(\widetilde{\mathrm{V}}_{z,x}(Z_J))^c \cap A_{J,y}(x) \cap \big\{J \subset [1, N_z^u]\big\}\big] \le e^{-c(\delta, L_0)am}. \tag{7.14}$$

The desired bound (7.8) with $\widetilde{\mathrm{V}}_z$ instead of V_z follows from (7.14) via a union bound applied *first* over $y \in \mathbb{L}_0 \cap D_{0,L_0}$ for *given* $x \in \partial\tilde{C}_z$ and then over $x \in \partial\tilde{C}_z$ (recall (2.19) for the cardinality of these sets). To show (7.14), we first claim that for any $x \in \partial\tilde{C}_z$ and $y \in \mathbb{L}_0 \cap D_{0,L_0}$,

$$(\widetilde{\mathrm{V}}_{z,x}(Z_J))^c \cap \{J \subset [1, N_z^u]\} \subset \{C_{y_k,L_0} \not\subset \mathcal{V}(Z_J) \text{ for any } k \ge 1 \text{ such that } \tau_k < \infty\}, \tag{7.15}$$

where y_k is as in (7.3) and $\tau_k \equiv \tau_{k;J,y}(x)$ is supplied by Proposition 7.1. To verify this, first note that by (7.10), we can write

$$(\widetilde{\mathrm{V}}_{z,x}(Z_J))^c \subset \big\{x \overset{D_z \cap \mathcal{V}(Z_J)}{\not\leftrightarrow} \mathscr{C}_{z',L} \text{ for any } z' \in \textstyle\bigcup_{1\le j\le \ell} O_j\big\}. \tag{7.16}$$

Now if $\tau_k < \infty$ and $C_{y_k,L_0} \subset \mathcal{V}(Z_J)$, then by property (7.4), w_{τ_k} (part of the exploration process $(w_n)_{n\ge1}$ for $\mathscr{C}_J(x)$, the cluster of x in $D_z \cap \mathcal{V}(Z_J)$) is connected to C_{y_k,L_0} in $D_{y_k,L_0} \cap \mathcal{V}(Z_J)$. On the other hand, by property (7.3), y_k satisfies property (7.2) and therefore C_{y_k,L_0} intersects $\mathscr{C}_{z',L} \subset D_{z',L} \subset \tilde{D}_z = \tilde{D}_{z,N}$ for some $z' \in \bigcup_{1\leq j\leq \ell} O_j$ (see (6.17) and (7.1) respectively for the two inclusions). These two observations imply that

$$\text{(7.17)} \quad \{C_{y_k,L_0} \subset \mathcal{V}(Z_J) \text{ for some } k \geq 1 \text{ s.t. } \tau_k < \infty\} \cap \{J \subset [1, N_z^u]\} \subset \big\{x \xleftrightarrow{D_z \cap \mathcal{V}(Z_J)} \mathscr{C}_{z',L} \text{ for some } z' \in \textstyle\bigcup_{1\leq j\leq \ell} O_j\big\},$$

provided $D_{y_k,L_0} \subset D_z$ on the event $\{\tau_k < \infty, C_{y_k,L_0} \subset \mathcal{V}(Z_J)\}$. But the inclusion $D_{y_k,L_0} \subset D_z$ follows from our earlier observation that C_{y_k,L_0} intersects $\mathscr{C}_{z',L} \subset \tilde{D}_z$ together with the definitions of the boxes $\tilde{D}_z = \tilde{D}_{z,N}$, $D_z = D_{z,N}$, C_{y_k,L_0} and D_{y_k,L_0} in (2.19) and the fact that $N \geq 10^3 L_0$ which is a consequence of (5.1) as part of our standing assumption (6.16). Together, (7.16) and (7.17) imply (7.15).

With (7.15) at hand, recalling the definition of the event $A_{J,y}(x)$ from (7.6), we see that the intersection $\widetilde{\mathrm{V}}_{z,x}(Z_J))^c \cap A_{J,y}(x)$ of the first two events appearing in (7.14) implies $\mathcal{E}_{\lceil c_8 am\rceil}$, where

$$\text{(7.18)} \qquad \mathcal{E}_b \stackrel{\text{def.}}{=} \{\tau_k < \infty \text{ and } C_{y_k,L_0} \not\subset \mathcal{V}(Z_J) \text{ for all } k \leq b\}$$

for any integer $b \geq 1$. Hence (7.14) follows immediately from the bound

$$\text{(7.19)} \qquad \mathbb{P}\big[\mathcal{E}_{\lceil c_8 am\rceil} \cap \big\{J \subset [1, N_z^u]\big\}\big] \leq e^{-c(\delta,L_0)am}.$$

We will set up a recursive inequality in b for this probability (with $\lceil c_8 am\rceil$ replaced by b) using Proposition 3.1 along the way. Abbreviating $p_b \equiv \mathbb{P}[\mathcal{E}_b \cap \{J \subset [1, N_z^u]\}]$, we have

$$\begin{aligned}
p_{b+1} &\stackrel{(7.18)}{=} \sum_{y' \in \widehat{\mathbb{L}}_{0,y}} \mathbb{P}\big[\mathcal{E}_b \cap \big\{C_{y',L_0} \not\subset \mathcal{V}(Z_J)\big\} \cap \big\{J \subset [1, N_z^u], \tau_{b+1} < \infty, y_{b+1} = y'\big\}\big] \\
&\stackrel{(7.5)+(3.4)}{=} \sum_{y' \in \widehat{\mathbb{L}}_{0,y}} \mathbb{E}\left[\mathbb{P}\big[C_{y',L_0} \not\subset \mathcal{V}(Z_J) \,\big|\, \mathcal{F}_{y,L_0}(Z_J, \delta, u, u_{2,3})\big] 1_{\mathcal{E}_b \cap \{J \subset [1,N_z^u], \tau_{b+1}<\infty, y_{b+1}=y'\}}\right] \\
&\stackrel{(3.5)}{\leq} (1-c) \sum_{y' \in \widehat{\mathbb{L}}_{0,y}} \mathbb{P}\big[\mathcal{E}_b \cap \big\{\tau_{b+1} < \infty, y_{b+1} = y'\big\} \cap \big\{J \subset [1, N_z^u]\big\}\big] \leq (1-c)p_b,
\end{aligned}$$

where $c = c(\delta, L_0) \in (0, 1)$ is from Proposition 3.1. In the third step, we used that $\{\tau_{b+1} < \infty, y_{b+1} = y'\} \subset \widetilde{\mathrm{LU}}_{y'}(Z_J) \cap \mathrm{O}_{y'}(\overline{Z}_z^u)$ as well as the set inclusion $D_{y,L_0} \subset D_z$, as needed for (3.5) to apply. The inclusion of events is a consequence of (7.3) and property (7.2). The inclusions of the boxes, on the other hand, follow from an argument similar to that used at the end of the paragraph containing (7.17). Iterating the previous inequality then yields that the left-hand side of (7.19) is bounded by $(1-c)^{c_8 am}$. □

It remains to prove Lemma 7.2. The following result will be useful.

Lemma 7.3. *For any sequence $Z = (Z_j)_{1\leq j\leq n_Z}$ of excursions and $y \in \mathbb{L}_0$, one has the inclusion*

$$\text{(7.20)} \qquad \mathrm{LU}_{y,L_0}(Z) \subset \widetilde{\mathrm{LU}}_{y,L_0}(Z) \quad (\textit{see } (6.31) \textit{ and } (3.2)).$$

Proof. Suppose we are on the event $\mathrm{LU}_{y,L_0}(Z)$ and $x, x' \in \mathscr{C}_{\partial D_{y,L_0}}(Z) \cap (\tilde{D}_{y,L_0} \setminus \tilde{C}_{y,L_0})$. Since $\mathscr{C}_{\partial D_{y,L_0}}(Z) \subset \mathcal{I}(Z)$ (see (3.1)), it then follows from the definition of $\mathrm{LU}_y(Z)$ in (6.31) that x and x' lie in the same component of $\mathcal{I}(Z) \cap (D_{y,L_0} \setminus (\partial D_{y,L_0} \cup C_{y,L_0}))$. Recalling (3.1) and that $x, x' \in \mathscr{C}_{\partial D_{y,L_0}}(Z)$, we can therefore conclude that the aforementioned component must lie in the same cluster of $\mathscr{C}_{\partial D_{y,L_0}}(Z)$. Therefore x, x' are in fact connected in $\mathscr{C}_{\partial D_{y,L_0}}(Z) \cap (D_{y,L_0} \setminus (\partial D_{y,L_0} \cup C_{y,L_0}))$. Since x, x' are two arbitrary points inside $\mathscr{C}_{\partial D_{y,L_0}}(Z) \cap (\tilde{D}_{y,L_0} \setminus \tilde{C}_{y,L_0})$, the previous conclusion yields (7.20) in view of the definition (3.2). □

We are now ready to give the

Proof of Lemma 7.2. We will harness the *full* strength of Proposition C.1 in this proof, including item (c) therein. Let us define a slightly reformulated version of the event $A_{J,y}(x)$ in (7.6), namely

$$\widetilde{A}_{J,y}(x) = \Big\{ \sum_{y' \in \widehat{\mathbb{L}}_{0,y}} 1\left\{ y' \text{ satisfies property (7.2) and } C_{y',L_0} \text{ intersects } \mathscr{C}_J(x) \right\} \geq c_8 am \Big\}. \tag{7.21}$$

In view of the second part of (7.3), and since $k \mapsto \tau_{k;J,y}$ is non-decreasing, it follows that

$$\widetilde{A}_{J,y}(x) \subset A_{J,y}(x).$$

Also since $u_{2,3} > u$ (see (6.16) and (6.34)) and $(\mathcal{V})_\delta$ is decreasing in δ (see (2.24)), we have

$$\big\{ \tilde{C}_z \xleftrightarrow{(\mathcal{V}^{u_{2,3}})_{2\delta}} \partial \tilde{D}_z, [1, N_z^{\frac{u_0}{2}}] \subset J \subset [1, N_z^u] \big\} \subset \big\{ \tilde{C}_z \xleftrightarrow{\mathcal{V}(Z_J)} \partial \tilde{D}_z, [1, N_z^{\frac{u_0}{2}}] \subset J \subset [1, N_z^u] \big\}.$$

Hence it is enough to show

$$\mathscr{G}_z^{\mathrm{I}} \cap \{ x \xleftrightarrow{\mathcal{V}(Z_J)} \partial \tilde{D}_z \} \cap \big\{ [1, N_z^{\frac{u_0}{2}}] \subset J \subset [1, N_z^u] \big\} \subset \bigcup_{y \in \mathbb{L}_0 \cap D_{0,L_0}} \widetilde{A}_{J,y}(x) \tag{7.22}$$

(cf. (7.7)).

Towards showing (7.22), let us start with an inclusion of events which plays a crucial role. For any $z' \in \mathbb{L}$ satisfying $D_{z',L} \subset D_z (= D_{z,N})$ and $U_{z',L} \subset U_z$, and with $\boldsymbol{u}_1, \boldsymbol{u}_2$ are as in (6.34), we claim that

$$\begin{aligned} \mathcal{G}_{z'}^{\mathrm{I}}(Z_{\mathbb{L}}, \delta, u_0, u_1, u_2, u_3; a) \cap \mathcal{F}_{z',L}^{\boldsymbol{u}_1, \boldsymbol{u}_2} \cap \big\{ [1, N_z^{\frac{u_0}{2}}] \subset J \subset [1, N_z^u] \big\} \\ \subset \mathcal{G}_{z'}\big(\overline{Z}_{\mathbb{L}}, \mathrm{V}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3}, u_{2,3}), \widetilde{\mathrm{W}}^{\mathrm{I}}, \mathscr{C}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3}); a \big) \equiv \widetilde{\mathcal{G}}_{z'}^{\mathrm{I}} \end{aligned} \tag{7.23}$$

(see (6.32) and (5.3) for notation), where $\widetilde{\mathrm{W}}^{\mathrm{I}} = \{ \widetilde{\mathrm{W}}_{z',y'}^{\mathrm{I}} : z' \in \mathbb{L}, y' \in \mathbb{L}_0 \}$ with

$$\widetilde{\mathrm{W}}_{z',y'}^{\mathrm{I}} = \widetilde{\mathrm{FE}}_{y'}(Z_J) \tag{7.24}$$

(cf. (6.29)). Let us assume (7.23) for the moment and finish the proof of Lemma 7.2, i.e. deduce (7.22).

In essence, we will find many points satisfying (7.2), as required for $\widetilde{A}_{J,y}(x)$ to occur, along a path γ realizing the crossing event on the left-hand side of (7.22). More precisely, on the event $\{x \leftrightarrow \partial \tilde{D}_z \text{ in } \mathcal{V}(Z_J)\}$ appearing in (7.22), and since $x \in \partial \tilde{C}_z$ by assumption, the component $\mathscr{C}_J(x)$ contains a crossing γ of $\tilde{D}_z \setminus \tilde{C}_z$. Let $\gamma_{\mathbb{L}}$ denote the sequence of points $(z_1', z_2', \dots)$ in $\mathbb{L}$ such that γ visits the boxes $C_{z_1',L}, C_{z_2',L}$ etc. in that order. Since γ is a crossing of $\tilde{D}_z \setminus \tilde{C}_z$, it follows that $\gamma_{\mathbb{L}}$ is itself a crossing on the coarse-grained lattice $\mathbb{L}$ of $V \setminus U$, with U, V as introduced below (7.1). For later reference, note that conversely, given any crossing $\boldsymbol{\gamma}'$ of $V \setminus U$, one easily constructs a crossing γ' of $\tilde{D}_z \setminus \tilde{C}_z$ in $\mathbb{Z}^d$ such that $\boldsymbol{\gamma}' = \gamma'_{\mathbb{L}}$.

Now recall from the paragraph below (7.1) that the sets $O_1, \dots, O_\ell \subset \Sigma$ satisfy property (c) in Proposition C.1, with U, V as above, Σ as in (7.1) and $\mathbb{L}$ as the underlying lattice. We deduce from this property and the observations made in the previous paragraph that there exists a crossing γ' of $\tilde{D}_z \setminus \tilde{C}_z$ satisfying (with γ as above)

$$\mathrm{range}(\gamma'_{\mathbb{L}}) \cap \Sigma = \mathrm{range}(\gamma_{\mathbb{L}}) \cap O,$$

where $O = \bigcup_{1 \leq j \leq \ell} O_j$. By Proposition 4.3, there exists a coarsening $\mathcal{C}_{\gamma'} \in \mathcal{A}_L^K(\tilde{D}_z \setminus \tilde{C}_z)$ that satisfies (4.13) for γ'. Since $\mathcal{C}_{\gamma'}$ is necessarily a subset of $\gamma'_{\mathbb{L}}$ (this follows from (4.13) and the definition of *crossing* above (4.9)) it follows that $\mathcal{C}_{\gamma'}$ intersects Σ *only* in range$(\gamma_{\mathbb{L}}) \cap O$. Further, by property (4.11) of admissible coarsenings, the fact that $\widetilde{\mathcal{G}}^{\mathrm{I}}_{z'} \subset \mathrm{V}_{z'}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3}, u_{2,3})$, which is a consequence of (7.23) and (5.3), and by definition of the set Σ in (7.1), it follows that

$$\text{any } z' \in \mathcal{C}_{\gamma'} \text{ such that } \widetilde{\mathcal{G}}^{\mathrm{I}}_{z'} \text{ occurs is contained in } \Sigma \text{ and hence also in range}(\gamma_{\mathbb{L}}) \cap O.$$

However, on the event $\mathscr{G}^{\mathrm{I}}_{z}$ which appears on the left of (7.22) (recall the definition of the event $\mathscr{G}(\cdot)$ in (4.16) and also the specific choice of arguments fed into $\mathscr{G}^{\mathrm{I}}_{z}$ from the displays (6.32), (6.33) and (6.35)), the number of points $z' \in \mathcal{C}_{\gamma'}$ such that the event $\mathcal{G}^{\mathrm{I}}_{z'}(Z_{\mathbb{L}}, \delta, u_0, u_1, u_2, u_3; a) \cap \mathcal{F}^{\boldsymbol{u}_1,\boldsymbol{u}_2}_{z',L}$ occurs is at least $\rho m = \frac{m}{2C_4}$, and these two events together with the control over J (also present on the left of (7.22)) imply $\widetilde{\mathcal{G}}^{\mathrm{I}}_{z'}$ by (7.23). Also since any $z' \in \mathcal{C}_{\gamma'}$ satisfies $D_{z',L} \subset \tilde{D}_z$ (see property (4.11)), one has $D_{z',L} \subset D_z$ and $U_{z',L} \subset U_z$ by (2.19) and (5.1) (the latter holds as (6.16) is in force). All in all, it thus follows that on the event on the left-hand side of (7.22), with γ realizing the crossing in $\{x \leftrightarrow \partial \tilde{D}_z \text{ in } \mathcal{V}(Z_J)\}$ (thus in particular range$(\gamma) \subset \mathscr{C}_J(x)$),

$$\begin{aligned}&\text{there exists } \Sigma_\gamma \subset \text{range}(\gamma_{\mathbb{L}}) \cap O \text{ such that } |\Sigma_\gamma| \geq \tfrac{m}{2C_4}, \text{ and}\\ &\text{for each } z' \in \Sigma_\gamma, \text{ the event } \widetilde{\mathcal{G}}^{\mathrm{I}}_{z'} \text{ occurs and } \gamma \text{ crosses } \tilde{D}_{z'} \setminus C_{z'}\end{aligned}$$

(simply pick $\Sigma_\gamma = \mathcal{C}_{\gamma'}$ in the above construction). In view of Definition 5.1 of the events $\mathcal{G}_{z'}(\cdot)$ (see around (5.4)) and the definition of $\widetilde{\mathrm{W}}^{\mathrm{I}}$ in (7.24), it follows in this case that there exists a set $S_{z'} \subset \mathbb{L}_0$ with $|S_{z'}| \geq a$ for any $z' \in \Sigma_\gamma$ such that both range(γ) $(\subset \mathscr{C}_J(x))$ and $\mathscr{C}_{z',L} = \mathscr{C}_{z'}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3})$ intersect C_{y',L_0} as well as $\widetilde{\mathrm{FE}}_{y'}(Z_J)$ occurs for each $y' \in S_{z'}$. In other words, for each such y', (7.2) holds and C_{y',L_0} intersects $\mathscr{C}_J(x)$. Now observe that any C_{y',L_0} can intersect at most 10^d-many boxes $\tilde{D}_{z',L}$ with $z' \in \mathbb{L}$ (see (2.19) and (5.1)). Also note that any C_{y',L_0} with $y' \in \mathbb{L}_0$ and intersecting $\tilde{D}_{z',L}$ for some $z' \in \Sigma_\gamma \subset \Sigma$ must necessarily satisfy $D_{y',L_0} \subset D_{z',L} \subset (\tilde{C}_z)^c$ (see (7.1) and (5.1)), i.e. $y' \in \widehat{\mathbb{L}}_{0,y}$ for some $y \in \mathbb{L}_0 \cap D_{0,L_0}$ (recall the definition of $\widehat{\mathbb{L}}_{0,y}$ above Proposition 7.1). All in all we thus obtain, on the event on the left-hand side of (7.22)

$$\textstyle\sum_{y'} 1\left\{(7.2) \text{ holds with } y' \text{ in place of } y_k \text{ and } C_{y',L_0} \text{ intersects } \mathscr{C}_J(x)\right\} \geq \frac{1}{2\cdot 10^d C_4} am$$

where the sum ranges over $y' \in \widehat{\mathbb{L}}_0 = \bigcup_y \widehat{\mathbb{L}}_{0,y}$ with y ranging in $\mathbb{L}_0 \cap D_{0,L_0}$. From this (7.22) (and hence Lemma 7.2) follows immediately with $c_8 = (2 \cdot 10^d C_4 |\mathbb{L}_0 \cap D_{0,L_0}|)^{-1} = (2 \cdot 70^d)^{-1}$.

We still need to verify (7.23). Using our argument for (6.43) (see also the steps leading to (6.76)) for the second inclusion below, we deduce that

$$\begin{aligned}&\mathcal{G}^{\mathrm{I}}_{z'}(Z_{\mathbb{L}}, \delta, u_0, u_1, u_2, u_3; a) \cap \mathcal{F}^{\boldsymbol{u}_1,\boldsymbol{u}_2}_{z',L}\\ &\quad\overset{(4.18),(6.34)}{\subset} \mathcal{G}^{\mathrm{I}}_{z'}(Z_{\mathbb{L}}, \delta, u_0, u_1, u_2, u_3; a) \cap \mathcal{F}^{u,u_1}_{z',L} \cap \mathcal{F}^{u_{2,3},u_2}_{z',L} \cap \mathcal{F}^{u_{2,3},u_3}_{z',L}\\ &\qquad\overset{(6.32)}{\subset} \mathcal{G}_{z'}\big(\overline{Z}_{\mathbb{L}}, \mathrm{V}_{z'}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3}, u_{2,3}), \mathrm{W}^{\mathrm{I}}, \mathscr{C}_{z'}(\overline{Z}_{\mathbb{L}}, \delta, u, u_{2,3}); a\big)\end{aligned}$$

(note that the event $\mathcal{G}_{z'}(\cdot)$ in the last line involves W^{I} as opposed to $\widetilde{\mathrm{W}}^{\mathrm{I}}$ appearing on the right-hand side of (7.23)). Since the event $\mathcal{G}_{z'}(\cdot)$ from Definition 5.1 is increasing w.r.t. the events $\{\mathrm{W}_{z',y'} : z' \in \mathbb{L}, y' \in \mathbb{L}_0\}$, and because

$$\begin{aligned}\text{(7.25)}\quad \mathrm{W}^{\mathrm{I}}_{z',y'} \cap \mathcal{F}^{\boldsymbol{u}_1,\boldsymbol{u}_2}_{z',L} &\overset{(6.29)}{=} \mathrm{FE}_{y',L_0}((Z^{u_1}_{z',L})_+(\tfrac{u_0}{8}\,\mathrm{cap}(D_{z',L}))) \cap \mathcal{F}^{\boldsymbol{u}_1,\boldsymbol{u}_2}_{z',L}\\ &\overset{(4.18),(6.34)}{\subset} \mathrm{FE}_{y',L_0}((Z^{u_1}_{z',L})_+(\tfrac{u_0}{8}\,\mathrm{cap}(D_{z',L}))) \cap \mathcal{F}^{\frac{u_0}{2},\frac{u_0}{8}}_{z',L}\end{aligned}$$

all that we are therefore left to show towards proving (7.23) is to argue that the intersection of $\{[1, N_z^{u_0/2}] \subset J \subset [1, N_z^u]\}$ with the event in the second line of (7.25) implies $\widetilde{\mathrm{FE}}_{y',L}(Z_J)$. But by Lemma 7.3, the definitions of FE in (6.30) and of $\widetilde{\mathrm{FE}}$ in (7.2) and the monotonicity of $\mathrm{O}_{y'}(Z)$ in Z (with respect to inclusion of the underlying sets, see (3.3)), we have

$$\mathrm{FE}_{y',L_0}(Z_J) \cap \{J \subset [1, N_z^u]\} \subset \widetilde{\mathrm{FE}}_{y',L_0}(Z_J)$$

for any $y' \in \mathbb{L}_0$. Therefore it suffices to show that the intersection of $\{[1, N_z^{u_0/2}] \subset J \subset [1, N_z^u]\}$ with the event in the second line of (7.25) implies $\mathrm{FE}_{y',L}(Z_J)$ rather than $\widetilde{\mathrm{FE}}_{y',L}(Z_J)$.

Since $D_{z',L} \subset D_z$ and $U_{z',L} \subset U_z$ by our assumption above (7.23), it follows from (2.11) in §2.2 that the sequence of excursions Z_J between D_z and U_z induces a sequence of excursions $Z_{J',L} = (Z_j^{D_{z',L},U_{z',L}})_{j\in J'}$ between $D_{z',L}$ and $\partial_{\mathrm{out}} U_{z',L}$ such that $\mathcal{I}(Z_J) \cap D_{z',L} = \mathcal{I}(Z_{J',L})$ and $\ell_{x'}(Z_J) = \ell_x(Z_{J',L})$ for all $x' \in D_{z',L}$. In particular, if $J = [1, N_z^v] = [1, N_{z,N}^v]$ for some $v > 0$ then $J' = [1, N_{z',L}^v]$ on account of (2.10). Furthermore, on the event (recall $\mathcal{F}_{z',L}^{u,v}$ from (2.21)),

$$\mathcal{F}_{z',L}^{\frac{u_0}{2},\frac{u_0}{8}} \cap \mathcal{F}_{z',L}^{u,u_1} \cap \{[1, N_z^{\frac{u_0}{2}}] \subset J \subset [1, N_z^u]\},$$

we have $\{1, \dots, \frac{u_0}{8}\,\mathrm{cap}(D_{z',L})\} \subset J' \subset \{1, \dots, u_1\,\mathrm{cap}(D_{z',L})\}$. However, this means $Z_{J',L}$ lies in the family $(Z_{z',L}^{u_1})_+(\frac{u_0}{8}\,\mathrm{cap}(D_{z',L}))$ by (6.19), thus yielding the desired inclusion. □

7.2. Discovery of good encounter points. This subsection is devoted to the proof of Proposition 7.1. In the sequel, we drop J, y and x from the notations $\tau_{k;J,y}(x)$ etc. and abbreviate $(\mathcal{I}, \mathcal{V}) = (\mathcal{I}(Z_J), \mathcal{V}(Z_J))$. We thus proceed to construct a sequence of random times $(\tau_k)_{k\geq 1}$ satisfying properties (7.3)–(7.5). Recall the exploration sequence $(w_n)_{n\geq 1}$ of the cluster $\mathscr{C}_J(x)$ of x in $\mathcal{V} \cap D_z$ from the beginning of §7.1.

We start with the sequence of successive times $(\tilde{\tau}_k)_{k\geq 1}$ at which the exploration of $\mathscr{C}_J(x)$ visits

$$\partial \overset{\text{def.}}{=} \textstyle\bigcup_{y'\in\widehat{\mathbb{L}}_{0,y}} \partial D_{y',L_0},$$

and certain additional (good) properties are satisfied. Formally, with $\tilde{\tau}_0 = 1$, for $k \geq 1$ we let

$$\tilde{\tau}_k = \inf\{n > \tilde{\tau}_{k-1} : w_n \in \partial \cap \mathcal{V} \text{ and } (*) \text{ holds}\}, \quad \tilde{x}_k = w_{\tilde{\tau}_k} \text{ if } \tilde{\tau}_k < \infty \tag{7.26}$$

(with the convention $\inf \emptyset = \infty$), where $(*)$ refers to the property that if $\tilde{y}_k \in \widehat{\mathbb{L}}_{0,y}$ denotes the unique point such that $\tilde{x}_k \in D_{\tilde{y}_k,L_0}$ (when $\tilde{\tau}_k < \infty$), then $\tilde{y}_k$ satisfies property (7.2) (for Z_J). We will maintain, at each time n, three sets $\mathsf{B}_n, \mathsf{W}_n$ and G_n of so called *black*, *white* and *grey* vertices, whose key features are summarized in Lemma 7.4 below. When speaking of *revealing* a vertex $v \in \mathbb{Z}^d$ in the sequel, we mean disclosing the value of $1\{v \in \mathcal{I}\}$. It will always be the case that W_n and B_n are precisely the set of vertices in $\mathcal{V}$ and $\mathcal{I}$ respectively that have been revealed up until time n. We start by defining

$$\mathsf{B}_0 = \mathsf{W}_0 = \emptyset,\ \mathsf{G}_0 = \mathbb{Z}^d. \tag{7.27}$$

For each n such that $1 = \tilde{\tau}_0 \leq n < \tilde{\tau}_1$, we define the triplet $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$ inductively from $(\mathsf{B}_{n-1}, \mathsf{W}_{n-1}, \mathsf{G}_{n-1})$ as follows (note that the following simplifies for $n < \tilde{\tau}_1$ but the formulation will generalize immediately to $n \in (\tilde{\tau}_k, \tilde{\tau}_{k+1})$ for $k \geq 1$). If $w_n \notin \mathsf{G}_{n-1}$ (which cannot happen when $n < \tilde{\tau}_1$, as can be easily seen inductively) we set $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n) = (\mathsf{B}_{n-1}, \mathsf{W}_{n-1}, \mathsf{G}_{n-1})$. Otherwise, we reveal w_n, remove w_n from G_{n-1} and add it to B_{n-1} if $w_n \in \mathcal{I}$ and to W_{n-1} if $w_n \in \mathcal{V}$, thus yielding $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$. We call such a step of the exploration *generic*.

Assume now that $n = \tilde{\tau}_1 < \infty$, recall $\tilde{x}_1$ from (7.26) and denote by $\tilde{y}_1 \in \widehat{\mathbb{L}}_{0,y}$ the point such that $\tilde{x}_1 \in \partial D_{\tilde{y}_1,L_0}$. The sets $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$ are now obtained as follows. First, let

$$(\mathsf{B}_n \cap D^c_{\tilde{y}_1,L_0}, \mathsf{W}_n \cap D^c_{\tilde{y}_1,L_0}, \mathsf{G}_n \cap D^c_{\tilde{y}_1,L_0}) = (\mathsf{B}_{n-1} \cap D^c_{\tilde{y}_1,L_0}, \mathsf{W}_{n-1} \cap D^c_{\tilde{y}_1,L_0}, \mathsf{G}_{n-1} \cap D^c_{\tilde{y}_1,L_0}).$$

It thus remains to specify changes to the sets $\mathsf{B}_{n-1}, \mathsf{W}_{n-1}, \mathsf{G}_{n-1}$ in D_{y_1,L_0}. To this effect, we first reveal all the points in $\partial D_{\tilde{y}_1,L_0} \cap \mathsf{G}_{n-1}$. Then we explore the clusters of points in $\partial D_{\tilde{y}_1,L_0}$ inside $\mathcal{I} \cap (D_{\tilde{y}_1,L_0} \setminus C_{\tilde{y}_1,L_0})$, thereby revealing the points in these clusters and their outer boundary in $D_{\tilde{y}_1,L_0} \setminus C_{\tilde{y}_1,L_0}$. All the points in $D_{\tilde{y}_1,L_0} \setminus C_{\tilde{y}_1,L_0}$ thereby revealed are removed from G_{n-1} and constitute B'_n, resp. W'_n, depending on whether they are in $\mathcal{I}$, resp. $\mathcal{V}$. The remaining points in $\mathsf{G}_{n-1} \cap D_{\tilde{y}_1,L_0}$ define the set G'_n. Notice that $C_{\tilde{y}_1,L_0} \subset \mathsf{G}'_n$. Now define (with $n = \tilde{\tau}_1$)

(7.28) $\quad \mathsf{G}''_n =$ the points in the component of $C_{\tilde{y}_1,L_0}$ in G'_n,

(7.29) $\quad \mathsf{B}''_n = \mathsf{B}'_n$

(7.30) $\quad \mathsf{W}''_n = \mathsf{W}'_n \cup (\mathsf{G}'_n \setminus \mathsf{G}''_n).$

Notice that $\tilde{x}_1 \in \mathsf{W}'_n (\subset W''_n)$ since $\tilde{x}_1 \in \mathcal{V}$ on account of (7.26). Next we reveal the points in $\mathsf{G}'_n \setminus \mathsf{G}''_n$, thus revealing all the points in W''_n (see (7.30) above). As shown in Lemma 7.4 below, see (7.33), we have $\mathsf{W}''_n \subset \mathcal{V}$. We now inspect whether $\tilde{x}_1$ is connected to $C_{\tilde{y}_1,L_0}$ by a path in $\mathsf{G}''_n \cup \mathsf{W}''_n$.

If the answer is no (Case I), we set $\mathsf{G}_n \cap D_{\tilde{y}_1,L_0} = \mathsf{G}''_n$, $\mathsf{B}_n \cap D_{\tilde{y}_1,L_0} = \mathsf{B}''_n$, and $\mathsf{W}_n \cap D_{\tilde{y}_1,L_0} = \mathsf{W}''_n$, completing the specification of $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$ in that case. If the answer is yes (Case II), we reveal all the vertices in G''_n, add them to B''_n and W''_n depending on their state to obtain $\mathsf{B}_n \cap D_{\tilde{y}_1,L_0}$ and $\mathsf{W}_n \cap D_{\tilde{y}_1,L_0}$ respectively, and set $\mathsf{G}_n \cap D_{\tilde{y}_1,L_0} = \emptyset$. Notice that, since $\mathsf{W}''_n \in \mathcal{V}$ as already observed, we have $\mathsf{W}_n \subset \mathcal{V}$ and $\mathsf{B}_n \subset \mathcal{I}$ in all cases.

By induction, if $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)_{0 \le n \le \tilde{\tau}_k}$ has been specified on the event $\{\tilde{\tau}_k < \infty\}$ for some $k \ge 1$, we continue for times $\tilde{\tau}_k + 1 \le n < \tilde{\tau}_{k+1}$ by performing generic steps of the exploration, as defined above for $1 \le n < \tilde{\tau}_1$. When $n = \tilde{\tau}_{k+1}$, there are three possible scenarios based on which we determine the next course of action. If $\tilde{y}_{k+1} \neq \tilde{y}_l$ for *any* $l \le k$, then we follow exactly the same procedure as described for $n = \tilde{\tau}_1$. Otherwise, and if in addition $\mathsf{G}_{n-1} \cap D_{\tilde{y}_{k+1},L_0} \neq \emptyset$, we set

(7.31) $\quad \mathsf{G}''_n = \mathsf{G}_{n-1} \cap D_{\tilde{y}_{k+1},L_0}$, $\mathsf{B}''_n = \mathsf{B}_{n-1} \cap D_{\tilde{y}_{k+1},L_0}$ and $\mathsf{W}''_n = \mathsf{W}_{n-1} \cap D_{\tilde{y}_{k+1},L_0}$

and skip to the remaining steps for $n = \tilde{\tau}_1$ (starting with inspecting whether $\tilde{x}_{k+1}$ is connected to $C_{\tilde{y}_{k+1},L_0}$ by a path in $\mathsf{G}''_n \cup \mathsf{W}''_n$). Finally if $\mathsf{G}_{n-1} \cap D_{\tilde{y}_{k+1},L_0} = \emptyset$, we simply carry out a generic step, which in this case boils down to setting $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n) = (\mathsf{B}_{n-1}, \mathsf{W}_{n-1}, \mathsf{G}_{n-1})$. Overall, this defines $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)_{n \ge 0}$ and the exploration effectively stops when discovering the whole component $\mathscr{C}_J(x)$, from which time on the sets $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$ remain fixed at their terminal value. We now collect a few features of the exploration sequence $(w_n)_{n \ge 1}$ as well as the triplets $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$ and $(\mathsf{B}''_n, \mathsf{W}''_n, \mathsf{G}''_n)$.

Lemma 7.4. *For all $n, k \ge 1$, the following hold.*

(7.32) *The sets $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$ form a partition of $\mathbb{Z}^d$ and if $n = \tilde{\tau}_k$ is such that $\mathsf{G}_{n-1} \cap D_{\tilde{y}_k,L_0} \neq \emptyset$, then $(\mathsf{B}''_n, \mathsf{W}''_n, \mathsf{G}''_n)$ forms a partition of $D_{\tilde{y}_k,L_0}$ (see definitions (7.28)–(7.30) and (7.31)).*

(7.33) *If $n = \tilde{\tau}_k$ and $\mathsf{G}_{n-1} \cap D_{\tilde{y}_k,L_0} \neq \emptyset$, then $\mathsf{B}''_n \subset \mathcal{I}$ and $\mathsf{W}''_n \subset \mathcal{V}$.*

(7.34) *$\mathsf{S}_n \cap D_{\tilde{y}_k,L_0} = \mathsf{S}_{\tilde{\tau}_k} \cap D_{\tilde{y}_k,L_0}$ for all $\mathsf{S}_n \in \{\mathsf{W}_n, \mathsf{B}_n, \mathsf{G}_n\}$ provided $n \ge \tilde{\tau}_k$ and $\mathsf{G}_n \cap D_{\tilde{y}_k,L_0} \neq \emptyset$.*

(7.35) *If $\mathsf{G}_{n-1} \cap D_{\tilde{y}_k,L_0} \neq \emptyset$ whereas $\mathsf{G}_n \cap D_{\tilde{y}_k,L_0} = \emptyset$, then $n = \tilde{\tau}_l$ for some $l \ge 1$ such that $\tilde{y}_l = \tilde{y}_k$ and $\tilde{x}_l$ is connected to $C_{\tilde{y}_l,L_0}$ by a path in $\mathsf{G}''_{\tilde{\tau}_l} \cup \mathsf{W}''_{\tilde{\tau}_l}$.*

(7.36) *if $n = \tilde{\tau}_k$ such that $\mathsf{G}_{n-1} \cap D_{\tilde{y}_k,L_0} \neq \emptyset$ and $C_{\tilde{y}_k,L_0} \subset \mathcal{V}$, then $\mathsf{G}''_n \subset \mathcal{V}$ (see (7.28) and (7.31)).*

Let us first conclude the proof of Proposition 7.1 assuming Lemma 7.4.

Proof of Proposition 7.1. We define a sequence $(\tau_k)_{k\geq 0}$ inductively by setting $\tau_0 = 0$,

$$\tau_k = \inf\left\{\tilde{\tau}_l > \tau_{k-1} : \; l > 0, \mathsf{G}_{\tilde{\tau}_l - 1} \cap D_{\tilde{y}_l, L_0} \neq \emptyset \text{ and } \tilde{x}_l \leftrightarrow C_{\tilde{y}_l, L_0} \text{ in } \mathsf{G}''_{\tilde{\tau}_l} \cup \mathsf{W}''_{\tilde{\tau}_l} \right\} \tag{7.37}$$

for any $k \geq 1$, with the sequence $(\tilde{\tau}_k)_{k\geq 0}$ as in (7.26). We set $y_k = \tilde{y}_l$ if $\tau_k = \tilde{\tau}_l < \infty$. We will refer to the property of $\tilde{\tau}_l$ appearing in the infimum in (7.37) as $(**)$. We proceed to verify that $(\tau_k)_{k\geq 1}$ defines a sequence of good encounter times.

The first part of property (7.3) follows from the definition of $\tilde{\tau}_l$ in (7.26). Let us now deal with the converse part. Consider a point $y' \in \widehat{\mathbb{L}}_{0,y}$ satisfying property (7.2) and $\mathscr{C}_J(x) \cap C_{y',L_0} \neq \emptyset$. Since $D_{y',L_0} \subset (\tilde{C}_z)^c$ (see the definition of $\widehat{\mathbb{L}}_{0,y}$ above Proposition 7.1), it follows from the properties of y' above together with definition (7.26) that $y' = \tilde{y}_l$ for some $l \geq 1$ and $\tilde{x}_l = w_{\tilde{\tau}_l}$ is connected to C_{y',L_0} in $\mathcal{V} \cap D_{y',L_0}$. If $\mathsf{G}_{\tilde{\tau}_l - 1} \cap D_{y',L_0} \neq \emptyset$, then $\tilde{\tau}_l$ satisfies $(**)$ as $\mathcal{V} \cap D_{y',L_0} \subset \mathsf{G}''_{\tilde{\tau}_l} \cup \mathsf{W}''_{\tilde{\tau}_l}$ by (7.33) and the second part of (7.32) and hence $y' = y_k$ for some $k \geq 1$. Otherwise, we get the same conclusion from (7.35).

Property (7.4) is a consequence of (7.36) and (7.33) on account of property $(**)$ of τ_k as defined in (7.37).

We are left with proving property (7.5). To this end we need to make two important observations.

Firstly, It follows from the definitions of τ_k and $\tilde{\tau}_l$ in (7.37) and (7.26) respectively and subsequently the definitions in (7.2) and (3.1)–(3.3) that the event $\{\tau_k = n, y_k = y'\}$ is measurable w.r.t. the configurations $((\mathcal{V}^u)_\delta, (\mathcal{V}^{u_{2,3}})_{2\delta}, \mathcal{I}_{|(\mathring{D}_{y',L_0})^c})$, the occupation time profile $\{\ell^u_x : x \in (\mathring{D}_{y',L_0})^c\}$, the set $\mathscr{C}_{\partial D_{y',L_0}}(Z_J)$ as well as the sets $\overline{\mathsf{S}}''_n \cap D_{y',L_0}$ with $\mathsf{S} \in \{\mathsf{B}, \mathsf{W}, \mathsf{G}\}$ where $\overline{\mathsf{S}}''_n = \mathsf{S}''_n$ if $n = \tilde{\tau}_l$ for some $l \geq 1$ and $(\mathsf{G}_{n-1} \cap D_{\tilde{y}_l, L_0}) \neq \emptyset$ and $\overline{\mathsf{S}}''_n = \emptyset$ otherwise (see the paragraph containing the displays (7.28)–(7.30) and (7.31)).

Secondly, in view of (7.34) and the definitions of the sets S''_n in (7.28)–(7.30) and (7.31) where $\mathsf{S} \in \{\mathsf{B}, \mathsf{W}, \mathsf{G}\}$, we have either $\overline{\mathsf{S}}''_n \cap D_{y',L_0} = \emptyset$ or is determined by the set $\mathscr{C}_{\partial D_{y',L_0}}(Z_J)$ according to the rules described in the paragraph containing the displays (7.28)–(7.30).

Together, these two observations imply that the event

$$\begin{aligned}\mathcal{H}(y'_1, \ldots, y'_{k+1}; n_1, \ldots, n_{k+1}) \overset{\text{def.}}{=} &\bigcap_{1\leq j\leq k} \{\tau_j = n_j, y_j = y'_j, C_{y'_j, L_0} \not\subset \mathcal{V}(Z_J)\} \\ &\cap \{\tau_{k+1} = n_{k+1}, y_{k+1} = y'_{k+1}\},\end{aligned}$$

where $\{y'_1, \ldots, y'_{k+1}\} \subset \widehat{\mathbb{L}}_{0,y}$ with $y'_{k+1} = y'$ and $n_1 < \ldots < n_{k+1}$ are positive integers, is measurable relative to the triplet $((\mathcal{V}^u)_\delta, (\mathcal{V}^{u_{2,3}})_{2\delta}, \mathcal{I}_{|(\mathring{D}_{y',L_0})^c})$, the occupation times $\{\ell^u_x : x \in (\mathring{D}_{y',L_0})^c\}$ and the set $\mathscr{C}_{\partial D_{y',L_0}}(Z_J)$ (note that the boxes $D_{y'_j, L_0}; j \leq k$ are disjoint from D_{y',L_0} as $y'_j \neq y' \in \widehat{\mathbb{L}}_{0,y}$ for all $j \leq k$). From the definition of the σ-algebra $\mathcal{F}_{y,L_0}(Z_J, \delta, u, u_{2,3})$ in (3.4), we then obtain that $\mathcal{H}(y'_1, \ldots, y'_{k+1}; n_1, \ldots, n_{k+1})$ is measurable with respect to $\mathcal{F}_{y,L_0}(Z_J, \delta, u, u_{2,3})$. Now note that by our treatment of Case II and the definition of τ_k in (7.37), we have $y_k (= \tilde{y}_{\tau_k}) \neq y_{k'} (= \tilde{y}_{\tau'_k})$ whenever $k \neq k'$. So we can partition the event $\mathcal{H}_{y',k} \overset{\text{def.}}{=} \cap_{1\leq j\leq k} \{\tau_j < \infty, C_{y_j, L_0} \not\subset \mathcal{V}(Z_J)\} \cap \{\tau_{k+1} < \infty, y_{k+1} = y'\}$ as

$$\mathcal{H}_{y',k} = \bigcup_{\substack{y'_1, \ldots, y'_{k+1}, \\ n_1, \ldots, n_{k+1}}} \mathcal{H}(y'_1, \ldots, y'_{k+1}; n_1, \ldots, n_{k+1})$$

where the union is over all *sets* of points $\{y'_1, \ldots, y'_{k+1}\} \subset \widehat{\mathbb{L}}_{0,y}$ with $y'_{k+1} = y'$ and positive integers $n_1 < \ldots < n_{k+1}$. Since each such $\mathcal{H}(y'_1, \ldots, y'_{k+1}; n_1, \ldots, n_{k+1})$ lies in $\mathcal{F}_{y,L_0}(Z_J, \delta, u, u_{2,3})$, so does the event $\mathcal{H}_{y',k}$ on account of the partition above and thus property (7.5) follows. □

We now give the

Proof of Lemma 7.4. We show each of the above properties separately.

Property (7.32). For $n = 0$, $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$ is a partition of $\mathbb{Z}^d$ by (7.27). For general n, this is deduced inductively by following the update rule for the sets $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$. The second part is an immediate consequence of definitions (7.28)–(7.30) and (7.31), together with the definitions of the sets $\mathsf{B}'_n, \mathsf{W}'_n$ and G'_n in the paragraph preceding the display (7.28).

Property (7.33). It follows from the update rule for the triplet $(\mathsf{B}_n, \mathsf{W}_n, \mathsf{G}_n)$ and (7.31) that if $\tilde{y}_k = \tilde{y}_l$ for some $l < k$ and (7.33) holds for $n = \tau_l$, then it also holds for $n = \tau_k$. However, when $\tilde{y}_k \neq \tilde{y}_l$ for any $l < k$, we are in the same situation as $k = 1$ (see above (7.31)) and therefore it suffices to verify the property for $n = \tilde{\tau}_1$.

It is clear from (7.29) that $\mathsf{B}''_n \subset \mathcal{I}$. On the other hand, since $W'_n \subset \mathcal{V}$, we have $\mathsf{W}''_n \subset \mathcal{V}$ in view of (7.30) *provided* we also have $\mathsf{G}'_n \setminus \mathsf{G}''_n \subset \mathcal{V}$. To this end, let $x' \in \mathcal{I} \cap \mathsf{G}'_n$ and we will show $x' \in \mathsf{G}''_n$.

Let us first observe that the cluster $\mathscr{C}(x')$ (say) of x' in $\mathcal{I} \cap (D_{\tilde{y}_1,L_0} \setminus C_{\tilde{y}_1,L_0})$ is necessarily a subset of G'_n and is disjoint from $\partial D_{\tilde{y}_1,L_0}$. This is because, by definition, $(\mathsf{B}'_n, \mathsf{W}'_n, \mathsf{G}'_n)$ forms a partition of $D_{\tilde{y}_1,L_0}$ with $\mathsf{W}'_n \subset \mathcal{V}$ and B'_n comprising the clusters of $\partial D_{\tilde{y}_1,L_0}$ in $\mathcal{I} \cap (D_{\tilde{y}_1,L_0} \setminus C_{\tilde{y}_1,L_0})$. But any (non-empty) component of $\mathcal{I} \cap D_{\tilde{y}_1,L_0} = \mathcal{I}(Z_J) \cap D_{\tilde{y}_1,L_0}$ must intersect $\partial D_{\tilde{y}_1,L_0}$ as the sequence Z_J consists of excursions $Z_j^{D_z,U_z}$'s between D_z and $\partial^{\text{out}} U_z$ with $D_{\tilde{y}_1,L_0} \subset D_z$ (see the property (7.2) satisfied by $\tilde{y}_1$ as well as the definitions of the sets $\tilde{D}_z$ and D_z in (2.19) and the relation (5.1) between N, L and L_0). Since $\mathscr{C}(x')$ is a component of $\mathcal{I} \cap (D_{\tilde{y}_1,L_0} \setminus C_{\tilde{y}_1,L_0})$ disjoint from $\partial D_{\tilde{y}_1,L_0}$, the previous observation implies that $\mathscr{C}(x') \cap \partial^{\text{out}} C_{\tilde{y}_1,L_0} \neq \emptyset$. Together with the fact that both $\mathscr{C}(x')$ and $C_{\tilde{y}_1,L_0}$ are subsets of G'_n (see above (7.28) for the latter), this implies x' lies in the component of $C_{\tilde{y}_1,L_0}$ in G'_n, i.e. G''_n (7.28) and our proof is complete. For later use in the proof of (7.36), let us also note here that another important conclusion that we can draw from the arguments in this paragraph is that the cluster of x' in $\mathcal{I} \cap D_{\tilde{y}_1,L_0}$ (as opposed to $\mathcal{I} \cap (D_{\tilde{y}_1,L_0} \setminus C_{\tilde{y}_1,L_0})$ considered above) must intersect $C_{\tilde{y}_1,L_0}$.

Property (7.34). It clearly suffices to prove the property with k replaced by k_1 where $k_1 \stackrel{\text{def.}}{=} \inf\{l \geq 1 : \tilde{y}_l = \tilde{y}_k\}$ and therefore we can assume, without any loss of generality, that $\tilde{y}_k \neq \tilde{y}_l$ for any $l < k$. Now suppose that the statement, i.e. (7.34) holds for some $n \geq \tilde{\tau}_k$ such that $\mathsf{G}_{n+1} \cap D_{\tilde{y}_k,L_0} \neq \emptyset$. We will verify that the statement also holds at time $n+1$ and the proof will then follow by induction. To this end, let us first note that if $w_{n+1} \in \mathbb{Z}^d \setminus D_{\tilde{y}_k,L_0}$, then the statement clearly holds at time $n+1$ as no vertex in $D_{\tilde{y}_k,L_0}$ gets inspectd in this case. So we assume that $w_{n+1} \in D_{\tilde{y}_k,L_0}$. Here we need to consider two possibilities. Firstly, $w_{n+1} \in D_{\tilde{y}_k,L_0} \setminus \partial D_{\tilde{y}_k,L_0}$ in which case one performs a generic step of the exploration (see the start of the paragraph containing (7.31)). Since $\mathsf{W}_n \subset \mathcal{V}$ and $\mathsf{B}_n \subset \mathcal{I}$ by (7.33), the only way the statement may fail to hold in this case is if $w_{n+1} \in \mathsf{G}_n$. Since w_{n+1} is selected from the outer boundary of a connected set in W_n which is the explored part of $\mathscr{C}_J(x)$ at time n and this set intersects $\partial D_{\tilde{y}_k,L_0}$ as it contains the point $w_{\tilde{\tau}_k} \in \partial D_{\tilde{y}_k,L_0}$ (see (7.26)), it follows that w_{n+1} is adjacent to a boundary component of $\mathsf{W}_n \cap D_{\tilde{y}_k,L_0}$. The points in this component, in particular the ones that also lie in $\partial D_{\tilde{y}_k,L_0}$, were revealed at some time $m \leq n$. In view of (7.26), it then follows that that $\tilde{\tau}_l \leq n$ for some $l \geq 1$ such that $\tilde{y}_l = \tilde{y}_k$ and w_{n+1} has a neighbor connected to $\tilde{x}_l \in \partial D_{\tilde{y}_l,L_0}$ inside $\mathsf{W}_n \cap D_{\tilde{y}_k,L_0}$. Since $\tilde{y}_k \neq \tilde{y}_{l'}$ for any $l' < k$ by our assumption at the beginning of this part, we in fact have $l \geq k$. All in all we get that *either* (7.34) holds in this case at time $n+1$ *or*

$$w_{n+1} \in \mathsf{G}_n \text{ is connected to } \tilde{x}_l \text{ by a path in } (\mathsf{G}_n \cup \mathsf{W}_n) \cap D_{\tilde{y}_k, L_0} \\ \text{for some } l \geq 1 \text{ such that } \tilde{\tau}_l \in [\tilde{\tau}_k, n] \text{ and } \tilde{y}_l = \tilde{y}_k. \tag{7.38}$$

Next recall that $\mathsf{G}_{n+1} \cap D_{\tilde{y}_k, L_0} \neq \emptyset$ by our induction hypothesis which implies $\mathsf{G}_{n'} \cap D_{\tilde{y}_k, L_0} \neq \emptyset$ for any $n' \leq n$ as the set G_n is non-increasing with respect to n (revealed vertices are removed from the grey set at each step). Therefore, $\mathsf{G}_{\tilde{\tau}_l} \cap D_{\tilde{y}_k, L_0} = \mathsf{G}_{\tilde{\tau}_l} \cap D_{\tilde{y}_l, L_0} \neq \emptyset$ as $\tilde{\tau}_l \leq n$ and $\tilde{y}_l = \tilde{y}_k$ by (7.38). Consequently, the triplet $(\mathsf{B}''_{\tilde{\tau}_l}, \mathsf{W}''_{\tilde{\tau}_l}, \mathsf{G}''_{\tilde{\tau}_l})$ satisfies the condition considered under Case I below (7.30) for otherwise we would have $\mathsf{G}_{\tilde{\tau}_l} \cap D_{\tilde{y}_l, L_0} = \emptyset$ (see how we deal with Cases I and II below (7.30)). But in Case I, there exists *no* path in $\mathsf{G}''_{\tilde{\tau}_l} \cup \mathsf{W}''_{\tilde{\tau}_l} = (\mathsf{G}_{\tilde{\tau}_l} \cup \mathsf{W}_{\tilde{\tau}_l}) \cap D_{\tilde{y}_l, L_0}$ connecting $\tilde{x}_l$ to $C_{\tilde{y}_l, L_0}$. As $\mathsf{G}''_{\tilde{\tau}_l}$ is a connected set containing $C_{\tilde{y}_l, L_0}$ (see (7.28) and the line above that), the previous fact also implies that there is *no* path in $(\mathsf{G}_{\tilde{\tau}_l} \cup \mathsf{W}_{\tilde{\tau}_l}) \cap D_{\tilde{y}_l, L_0}$ connecting $\tilde{x}_l$ to any point in $\mathsf{G}_{\tilde{\tau}_l} \cap D_{\tilde{y}_l, L_0}$. However, this contradicts (7.38) as $\mathsf{G}_n \cap D_{\tilde{y}_k, L_0} = \mathsf{G}_{\tilde{\tau}_l} \cap D_{\tilde{y}_k, L_0}$ and $\mathsf{W}_n \cap D_{\tilde{y}_k, L_0} = \mathsf{W}_{\tilde{\tau}_l} \cap D_{\tilde{y}_k, L_0}$ for any $l \geq 1$ satisfying $\tilde{\tau}_l \in [\tilde{\tau}_k, n]$ according to our induction hypothesis. Thus property (7.34) holds in this case at time $n+1$.

The other possibility as to the choice of $w_{n+1} \in D_{\tilde{y}_k, L_0}$ is that $w_{n+1} \in \partial D_{\tilde{y}_k, L_0}$, i.e. $n+1 = \tilde{\tau}_l$ and $w_{n+1} = \tilde{x}_l$ for some $l > k$ with $\tilde{y}_l = \tilde{y}_k$. So we are in the situation considered in (7.31). As $\mathsf{G}_{n+1} \cap D_{\tilde{y}_k, L_0} = \mathsf{G}_{\tilde{\tau}_l} \cap D_{\tilde{y}_k, L_0} \neq \emptyset$ by our induction hypothesis, it is clear as in the previous case that $(\mathsf{B}''_{\tilde{\tau}_l}, \mathsf{W}''_{\tilde{\tau}_l}, \mathsf{G}''_{\tilde{\tau}_l})$ satisfies the condition of Case I at time $\tilde{\tau}_l = n+1$ and hence $\mathsf{S}_{n+1} \cap D_{\tilde{y}_l, L_0} = \mathsf{S}''_{n+1}$ for all $\mathsf{S}_{n+1} \in \{\mathsf{W}_{n+1}, \mathsf{B}_{n+1}, \mathsf{G}_{n+1}\}$ (revisit the update rule for Case I in the paragraph below (7.28)). But $\mathsf{S}''_{n+1} = \mathsf{S}_n \cap D_{\tilde{y}_l, L_0}$ for all $\mathsf{S}_n \in \{\mathsf{W}_n, \mathsf{B}_n, \mathsf{G}_n\}$ due to (7.31) and hence (7.34) holds in this case as well at time $n+1$ by our induction hypothesis.

Property (7.35). The proof of this property is contained in the proof of (7.34).

Property (7.36). In view of (7.34), we only need to verify this when $\tilde{y}_k \neq \tilde{y}_l$ for any $l < k$ which is similar to the case for $k = 1$. But then it is precisely the statement mentioned at the end of the proof of (7.33). □

7.3. Adaptations for Proposition 6.7. In this subsection we will show how we can prove Proposition 6.7 by adapting some *parts* of the proof of Proposition 6.6 which spans the previous two subsections. The proof is comparatively simpler in this case owing to the stronger conditioning permitted by Lemma 3.3 which is related to the fact that the sequences of excursions within the puview of this lemma are meant to be 'small'. We direct the readers to the discussions above Proposition 3.1 and Lemma 3.3 in Section 3 for more on this.

Proof of Proposition 6.7. The following inequality is analogous to (7.8). For any $J \subset \mathbb{N}^*$ deterministic and finite, we have

$$\mathbb{P}\big[(\mathrm{V}_z(Z_J))^c \cap \mathscr{G}_z^{\mathrm{II}} \cap \big\{(C_z \cap \mathbb{L}_0^-) \xleftrightarrow{\mathcal{O}_0^-(Z_z^{\frac{3u_0}{2}})} \partial_{\mathbb{L}_0^-}(D_z \cap \mathbb{L}_0^-), J \subset [1, N_z^{\frac{3u_0}{2}}]\big\}\big] \leq CN^{d-1}e^{-c(L_0^-)am} \tag{7.39}$$

(recall the set $\mathcal{O}_0^-(Z)$ from (3.17)). We can now derive Proposition 6.7 from this using the similar line of arguments that we used to deduce Proposition 6.6 from (7.8).

We will obtain (7.39) from a related statement. For any $k \geq 0$,

$$\mathbb{P}\big[(\mathrm{V}_z(Z_J))^c \cap \big\{(C_z \cap \mathbb{L}_0^-) \xleftrightarrow{\mathcal{O}_0^-(Z_J)} \partial_{\mathbb{L}_0^-}(D_z \cap \mathbb{L}_0^-), k(\mathcal{O}_0^-(Z_J)) \geq k\big\}\big] \leq CN^{d-1}e^{-c(L_0^-)k} \tag{7.40}$$

where the functional $k(\Sigma)$ is defined similarly as below (7.1) in §7.1 with L and $\mathbb{L}$ replaced by L_0^- and $\mathbb{L}_0^-$ respectively. (7.39) follows from this and the inclusion of events

$$\mathscr{G}_z^{\mathrm{II}} \subset \big\{k(\mathcal{O}_0^-(Z_z^{\frac{3u_0}{2}})) \geq cam\big\} \tag{7.41}$$

for some $c \in (0, \infty)$ in view of monotonicity of the set $\mathcal{O}_0^-(Z)$ in Z (see (3.17) and (3.16)) and also of the functional $k(\Sigma)$ in Σ (revisit its definition below (7.1)). We will show (7.41) at the end after giving a proof of (7.40). To this end, let $O_1', \dots, O_\ell' \subset \mathcal{O}_0^-(Z_z^{\frac{3u_0}{2}})$ denote the $*$-connected sets satisfying properties (a)–(b) in Proposition C.1 (as subsets of the coarse-grained lattice $\mathbb{L}_0^-$) with $V = \mathbb{L}_0^-(\tilde{D}_z)$, $U = \mathbb{L}_0^-(\tilde{C}_z)$ and $\Sigma = \mathcal{O}_0^-(Z_z^{\frac{3u_0}{2}}) \cap (V \setminus U)$. Note that $\ell = 0$ when $k(\mathcal{O}_0^-(Z_z^{\frac{3u_0}{2}})) = 0$. We can now reduce (7.40) to a statment analogous to (7.14) in the proof of Proposition 6.7.

$$\begin{aligned}&\mathbb{P}\big[(\overline{\mathrm{V}}_z(Z_J))^c \cap \big\{(C_z \cap \mathbb{L}_0^-) \xleftrightarrow{\mathcal{O}_0^-(Z_J)} \partial_{\mathbb{L}_0^-}(D_z \cap \mathbb{L}_0^-), k(\mathcal{O}_0^-(Z_J)) \geq k\big\}\big] \\ &\leq C N^{d-1} e^{-c(L_0^-)k}\end{aligned} \tag{7.42}$$

where the event $\overline{\mathrm{V}}_z(Z_J) \stackrel{\text{def.}}{=} \bigcap_{x \in \partial \tilde{C}_z} \overline{\mathrm{V}}_{z,x}(Z_J)$ with

$$\overline{\mathrm{V}}_{z,x}(Z_J) \stackrel{\text{def.}}{=} \big\{x \stackrel{\mathcal{V}(Z_J)}{\not\longleftrightarrow} \partial \tilde{D}_z\big\} \cup \Big\{x \xleftrightarrow{D_z \cap \mathcal{V}(Z_J)} \bigcup_{y^- \in \bigcup_{1 \leq j \leq \ell} O_j'} \square(y^-, L_0^-)\Big\}. \tag{7.43}$$

We can derive (7.40) from (7.42) and the inclusion of events,

$$\widetilde{\mathrm{V}}_z(Z_J) \cap \big\{(C_z \cap \mathbb{L}_0^-) \xleftrightarrow{\mathcal{O}_0^-(Z_J)} \partial_{\mathbb{L}_0^-}(D_z \cap \mathbb{L}_0^-)\big\} \subset \mathrm{V}_z(Z_J). \tag{7.44}$$

However, (7.44) can be obtained in a similar way as (7.11) in §7.1 in view of following analogue of (7.12) which is a direct consequence of the definition of the set $\square(y^-, L_0^-)$ above (3.16) and the event $\mathrm{W}_{y^-}^-(Z) = \mathrm{W}_{y^-, L_0^-}^-(Z)$ in (3.16).

$$\begin{aligned}&\square(y^-, L) \text{ and } \square(z^-, L) \text{ are connected in } (\tilde{C}_{y^-, L_0^-} \cup \tilde{C}_{z^-, L_0^-}) \cap \mathcal{V}(Z_J) \\ &\text{whenever } |y^- - z^-|_\infty \leq L_0^- \text{ and } \mathrm{W}_{y^-}^-(Z_J) \cap \mathrm{W}_{z^-}^-(Z_J) \text{ occurs.}\end{aligned} \tag{7.45}$$

Let us get back to (7.42). On account of definition of the event $\overline{\mathrm{V}}_z(Z_J)$ above (7.43), we can deduce this from

$$\mathbb{P}\big[(\overline{\mathrm{V}}_{z,x}(Z_J))^c \cap \big\{(C_z \cap \mathbb{L}_0^-) \xleftrightarrow{\mathcal{O}_0^-(Z_J)} \partial_{\mathbb{L}_0^-}(D_z \cap \mathbb{L}_0^-), k(\mathcal{O}_0^-(Z_J)) \geq k\big\}\big] \leq e^{-c(L_0^-)k} \tag{7.46}$$

for each $x \in \partial \tilde{C}_z$ via a union bound over x. To show (7.46), we will construct a sequence of 'good' random times $(\tau_l)_{l \geq 1}$ coupled to the exploration sequence $(w_n)_{n \geq 1}$ revealing the cluster $\mathscr{C}_J(x)$ (see Section 7.1) and satisfying the following two properties (cf. properties (7.3)–(7.5) in Proposition 7.1).

(7.47) If $\tau_l < \infty$, then $w_{\tau_l} \in \partial^{\text{out}} C_{y_l^-, L_0} \cap \mathcal{V}(Z_J)$ for some $y_l^- \in \bigcup_{1 \leq j \leq \ell} O_j'$. Conversely, if $y^- \in \bigcup_{1 \leq j \leq \ell} O_j'$ and $\mathscr{C}_J(x) \cap \partial^{\text{out}} C_{y^-, L_0} \neq \emptyset$, there exists $l \geq 1$ such that $\tau_l < \infty$ with $y^- = y_l^-$.

(7.48) For any $z^- \in \mathbb{L}_0^-$ and $l \geq 0$, the event $\cap_{1 \leq j \leq l}\{\tau_j < \infty, w_{\tau_j} \stackrel{S_j}{\not\longleftrightarrow} \square(y_j^-, L_0^-)\} \cap \{\tau_{l+1} < \infty, y_{l+1}^- = y^-\}$ is measurable relative to the σ-algebra $\overline{\mathcal{F}}_{y^-, L_0^-}(Z_J)$ defined in (3.18) in Section 3. Here and in the sequel S_j is the set $(\{w_{\tau_j}\} \cup C_{y_j^-, L_0}) \cap \mathcal{V}(Z_J)$.

Let us suppose for the moment that such a sequence of random times exists. Then it follows from the definition of $k(\mathcal{O}_0^-(Z_J))$ below (7.1) and property (7.47) above that

$$(\overline{\mathrm{V}}_{z,x}(Z_J))^c \cap \big\{(C_z \cap \mathbb{L}_0^-) \xleftrightarrow{\mathcal{O}_0^-(Z_J)} \partial_{\mathbb{L}_0^-}(D_z \cap \mathbb{L}_0^-), k(\tilde{D}_z \setminus \tilde{C}_z, \mathcal{O}_0^-(Z_J)) \geq k\big\} \subset \overline{\mathcal{E}}_k \tag{7.49}$$

where

$$\overline{\mathcal{E}}_k \stackrel{\text{def.}}{=} \big\{\tau_j < \infty, w_{\tau_j} \overset{S_j}{\not\leftrightarrow} \square(y_j^-, L_0^-) \text{ for all } j \leq k\big\}.$$

Now using similar arguments as used to derive (7.19) in the proof of Proposition 6.6 with property (7.48) and Lemma 3.3 in lieu of (7.5) and Proposition 3.1 respectively, we obtain $\mathbb{P}[\overline{\mathcal{E}}_k] \leq e^{-c(L_0^-)k}$ for any $k \geq 0$ thus yielding (7.46) in view of (7.49). Coming back to the sequence $(\tau_l)_{l\geq 1}$, let us recall the sequence $(w_n)_{n\geq 1}$ from the beginning of §7.1 and define for each $l \geq 1$, with $\tau_0 = 0$,

$$\tau_l = \inf\left\{ n > \tau_l : \begin{array}{l} w_n \in \mathcal{V}(Z_J) \text{ and } w_n \in \partial^{\text{out}} C_{y^-,L_0^-} \text{ for some } y^- \in \bigcup_{1\leq j\ell} O_j' \\ \text{such that } \bigcup_{1\leq i<n} \{w_i\} \cap \mathcal{V}(Z_J) \cap \partial^{\text{out}} C_{y^-,L_0^-} = \emptyset \end{array} \right\}.$$

Properties (7.47) and (7.48) follow from this definition and the definition of $(w_n)_{n\geq 1}$ in a straightforward manner.

It remains to verify (7.41). Since the event $\mathrm{W}_{y^-}^-(Z)$ in decreasing in Z (see (3.16)) and $2u_0 < u_4$ (see the statement of Proposition 6.7), we have

$$\mathrm{W}_{y^-}^-(Z_{z',L}^{u_4}) \cap \mathcal{F}_L^{\frac{3u_0}{2},u_4} \overset{(2.21)}{\subset} \mathrm{W}_{y^-}^-(\overline{Z}_z^{\frac{3u_0}{2}}) \tag{7.50}$$

for any $z' \in \mathbb{L}$ satisfying $D_{z',L_0} \subset D_z$ and $U_{z',L_0} \subset U_z$ (cf. (6.37)). Now recalling the definition of the event $\mathcal{G}_{z'}^{\mathrm{II}}(\widehat{Z}, u_4; a)$ from (6.38) and, as part of that, the event $\mathrm{W}_{z',y}^{\mathrm{II}}$ from (6.37) (see also Definition 5.1 for the generic events $\mathcal{G}_{z'}(\cdot)$), we can write in view of (7.50),

$$\mathcal{G}_{z'}^{\mathrm{II}}(Z_{\mathbb{L}}, u_4; a) \cap \mathcal{F}_L^{\frac{3u_0}{2},u_4} \subset \mathcal{G}_{z'}(\mathrm{V} = \{\Omega : z'' \in \mathbb{L}\}, \widetilde{\mathrm{W}}^{\mathrm{II}}, \mathscr{C} = \{\mathbb{Z}^d : z'' \in \mathbb{L}\}; a) \tag{7.51}$$

for any $z' \in \mathbb{L}$ such that $D_{z',L} \subset D_z$ and $U_{z',L} \subset U_z$ where $\widetilde{\mathrm{W}}^{\mathrm{II}} = \{\widetilde{\mathrm{W}}_{z',y'}^{\mathrm{II}} : z' \in \mathbb{L}, y' \in \mathbb{L}_0\}$ with $\widetilde{\mathrm{W}}_{z',y'}^{\mathrm{II}} = \mathcal{G}_{y'}^-(\overline{Z}_z^{\frac{3u_0}{2}})$. Hence from (5.8) in Proposition 5.2 and (6.40) we obtain that, on the event $\mathscr{G}_z^{\mathrm{II}}$,

> any crossing of $\tilde{D}_z \setminus \tilde{C}_z$ intersects at least cam-many boxes C_{y^-,L_0} such that $y^- \in \mathbb{L}_0^-$ and $\mathrm{W}_{y^-}^-(\overline{Z}_z^{\frac{3u_0}{2}})$ occurs

(condition (5.5) is satisfied owing to (7.51) and the observation that $D_{z',L} \subset D_z$ and $U_{z',L} \subset U_z$ as soon as $z' \in (\tilde{D}_z \setminus \tilde{C}_z) \cap \mathbb{L}$, a consequence of (6.16) and (5.1)). But the above statement directly implies (7.50) in view of the definition of the set $\mathcal{O}_0^-(\overline{Z}_z^{\frac{3u_0}{2}})$ in (3.17) and the functional $k(\Sigma)$ below (7.1). □

8 Seed estimates

In this section we prove the seed estimates in Lemma 6.9. Recall from (6.32) and (6.38) in §6.2 that the events $\mathcal{G}^{\mathrm{i}}$, $\mathrm{i} \in \{\mathrm{I}, \mathrm{II}\}$ involve, among others, the events $\mathrm{FE}_{y,L_0}(Z)$ (through $\mathrm{W}^{\mathrm{I}}_{z,y}$ in (6.29)) and $\mathcal{G}_{y,L_0,L_0^-}(Z)$ (through $\mathrm{W}^{\mathrm{II}}_{z,y}$ in (6.37)) for some suitable sequence of excursions Z. We need to show that both of these events are 'likely' for the proof of Lemma 6.9.

Lemma 8.1 (the event FE_{y,L_0} is likely)**.** *For any $u_0, u \in (0,\infty)$, $K \geq 100$ (from (5.1)) and $L_0 \in \mathbb{N}^*$, we have*

$$\mathbb{P}\big[\mathrm{FE}_{0,L_0}((\overline{Z}^u_{0,L_0})_+(N^{u_0}_{0,L_0}))\big] \geq 1 - C(u,u_0)\, e^{-c(u)L_0^c}. \tag{8.1}$$

Proof of Lemma 8.1. By (6.45) and (6.23), we can write

$$\mathrm{FE}_{0,L_0}\big((Z^{2u}_{0,L_0})_+(\tfrac{u_0}{2}\mathrm{cap}(D_{0,L_0}))\big) \cap \mathcal{F}^{u,2u}_{0,L_0} \cap \mathcal{F}^{u_0,\frac{u_0}{2}}_{0,L_0} \subset \mathrm{FE}_{0,L_0}((\overline{Z}^u_{0,L_0})_+(N^{u_0}_{0,L_0})). \tag{8.2}$$

Henceforth in this proof we will omit the subscript L_0 from all our notations like Z^{2u}_{0,L_0}, $\mathrm{FE}_{0,L_0}(\cdot)$ etc. Since the event $\mathrm{O}_0(Z)$ is monotonically decreasing in Z w.r.t. inclusion of $\{Z\}$ (revisit definition (3.3)), it follows from (6.23) that

$$\mathrm{O}_0\big((Z^{2u}_0)_+(\tfrac{u_0}{2}\mathrm{cap}(D_0))\big) = \mathrm{O}_0(Z^{2u}_0). \tag{8.3}$$

Now we come to the other part of the definition of the event $\mathrm{FE}_0(Z)$ in (6.30), namely $\mathrm{LU}_0(Z)$. From (6.44), we obtain that under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_0$,

$$\mathrm{LU}_0\big((\widetilde{Z}^{4u}_0)_+(\tfrac{u_0}{4}\mathrm{cap}(D_0))\big) \cap \mathrm{Incl}_0^{\frac{1}{10},\frac{u_0}{8}\,\mathrm{cap}(D_0)} \subset \mathrm{LU}_0\big((Z^{2u}_0)_+(\tfrac{u_0}{2}\mathrm{cap}(D_0))\big). \tag{8.4}$$

for all $L_0 \geq C(u_0)$. The main advantage of working with the LU event on the left-hand side is that it involves independent and identically distributed excursions. We now proceed to further *simplify* this event.

To this end, we first claim that for any two finite subsets J and J' of $\mathbb{N}^*$, we have the following inclusion of events.

$$\bigcap_{J'' \subset J',\, |J''| \leq 2} \mathrm{LU}_0\big(\widetilde{Z}^{D_0,U_0}_{J \cup J''}\big) \subset \mathrm{LU}_0\big(\widetilde{Z}^{D_0,U_0}_{J \cup J'}\big) \tag{8.5}$$

where $\widehat{Z}^{D_0,U_0}_J = (\widehat{Z}^{D_0,U_0}_j)_{j \in J}$ for any $J \subset \mathbb{N}^*$. To see this suppose that we are *on the event* at the left-hand side of (8.5) and consider $x', x'' \in \mathcal{I}\big(\widetilde{Z}^{D_0,U_0}_{J \cup J'}\big) \cap (\tilde{D}_0 \setminus \tilde{C}_0)$. If neither of x' and x'' lies in $\mathcal{I}(\widetilde{Z}^{D_0,U_0}_J)$, then there exist $j', j'' \in J'$ such that $x' \in \mathcal{I}(\widetilde{Z}^{D_0,U_0}_{j'})$ and $x'' \in \mathcal{I}(\widetilde{Z}^{D_0,U_0}_{j''})$. Since we are on the event $\mathrm{LU}_0\big(\widetilde{Z}^{D_0,U_0}_{J \cup \{j',j''\}}\big)$, we have in this case that x' and x'' are connected in

$$\mathcal{I}\big(\widetilde{Z}^{D_0,U_0}_{J \cup \{j',j''\}}\big) \cap (\mathring{D}_0 \setminus C_0) \subset \mathcal{I}\big(\widetilde{Z}^{D_0,U_0}_{J \cup J'}\big) \cap (\mathring{D}_0 \setminus C_0)$$

(recall the definition of $\mathrm{LU}_0(Z)$ from (6.31) and also that $\mathring{D}_0 = D_0 \setminus \partial D_0$). Similarly we can verify the cases $x', x'' \in \mathcal{I}(\widetilde{Z}^{D_0,U_0}_J)$, $x' \in \mathcal{I}(\widetilde{Z}^{D_0,U_0}_J)$ and $x'' \in \mathcal{I}(\widetilde{Z}^{D_0,U_0}_{J'})$ and finally, $x'' \in \mathcal{I}(\widetilde{Z}^{D_0,U_0}_J)$ and $x' \in \mathcal{I}(\widetilde{Z}^{D_0,U_0}_{J'})$. All in all, the inclusion in (8.5) follows.

Now recall from (6.19) that any $Z \in (\widetilde{Z}_0^{4u})_+(\frac{u_0}{4}\mathrm{cap}(D_0))$ must necessarily contain $\widetilde{Z}_0^{\frac{u_0}{4}}$ as a subsequence for some $j' \in \mathbb{N}$ (see (2.20) for notation). Therefore, we obtain from (8.5) that

$$\mathrm{LU}_0\big((\widetilde{Z}_0^{4u})_+(\tfrac{u_0}{4}\mathrm{cap}(D_0))\big) \subset \bigcap_{J \in \mathbf{J}_{u_0,u}} \mathrm{LU}_0(\widetilde{Z}_J^{D_0,U_0}) \tag{8.6}$$

where

$$\begin{aligned}\mathbf{J}_{u_0,u} \stackrel{\text{def.}}{=} &\text{the collection of all subsets of } \{1,\ldots,\lfloor 4u\,\mathrm{cap}(D_0)\rfloor\} \text{ of the} \\ &\text{form } \{1,\ldots,\lfloor \tfrac{u_0}{4}\,\mathrm{cap}(D_0)\rfloor\} \cup J'' \text{ for some } J'' \subset \mathbb{N}^* \text{ such that } |J''| \leq 2.\end{aligned} \tag{8.7}$$

The major gain from the inclusion in (8.6) is that the event on the right-hand side (or the complement thereof) is now amenable to a union bound argument as $|\mathbf{J}_{u_0,u}|$ is at most a power in L_0 (see below).

Together with (8.3) and (8.4), (8.6) implies in view of definition (6.30) that under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_0$, the following inclusion holds.

$$\big(\mathrm{O}_0(Z_0^{2u}) \cap \bigcap_{J \in \mathbf{J}_{u_0,u}} \mathrm{LU}_0(\widetilde{Z}_J^{D_0,U_0})\big) \cap \mathrm{Incl}_0^{\frac{1}{10},\frac{u_0}{8}\,\mathrm{cap}(D_0)} \subset \mathrm{FE}_0\big((Z_0^{2u})_+(\tfrac{u_0}{2}\mathrm{cap}(D_0))\big).$$

Finally, plugging this into the left-hand side of (8.2) yields us

$$\begin{aligned}&\big(\mathrm{O}_0(Z_0^{2u}) \cap \bigcap_{J \in \mathbf{J}_{u_0,u}} \mathrm{LU}_0(Z_J^{D_0,U_0})\big) \cap \mathrm{Incl}_0^{\frac{1}{10},\frac{u_0}{8}\,\mathrm{cap}(D_0)} \cap \mathcal{F}_0^{u,2u} \cap \mathcal{F}_0^{u_0,\frac{u_0}{2}} \\ &\subset \mathrm{FE}_0\big((\overline{Z}_0^{u})_+(N_0^{u_0})\big).\end{aligned} \tag{8.8}$$

under any coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_0$ and for all $L_0 \geq C(u)$.

We will now bound from below the probabilities of each of the events on the left-hand side of (8.8) starting with $\mathrm{O}_y(Z_0^{2u})$. From the definition of this event in (3.3), property (2.5) of random interlacements, standard tail bound for a Possion variable X with mean λ, namely

$$\mathbb{P}[X \geq \lambda + x] \leq e^{-cx} \text{ whenever } x \geq \lambda/2$$

(see, e.g. [30, pp. 97-98]) and finally, the exponential decay of the tail of occupation time for transient random walks, we get

$$\mathbb{P}\big[\mathrm{O}_0(Z_0^{2u})\big] \geq 1 - e^{-c(u)L_0^c}.$$

Next we want to prove that for any $K \geq 100$ and $J \in \mathbf{J}_{u_0,u}$, one has

$$\mathbb{P}\big[\mathrm{LU}_0(\widetilde{Z}_J^{D_0,U_0})\big] \geq 1 - C(u_0)e^{-L_0^c}. \tag{8.9}$$

To this end, for any $C > 0$, $R' > R \geq 1$ and $x \in \mathbb{Z}^d$, let us consider the event

$$\overline{\mathrm{LU}}_{x,R,R'}(\widetilde{Z}_J^{D_0,U_0}) \stackrel{\text{def.}}{=} \bigcap_{x,x' \in B_R(x) \cap \mathcal{I}(\widetilde{Z}_J^{D_0,U_0})} \{x \xleftrightarrow{\mathcal{I}(\widetilde{Z}_J^{D_0,U_0}) \cap B_{R'}(x)} x'\},$$

(cf. (6.31)) where $B_R(x) \subset \mathbb{Z}^d$ denotes the closed ℓ^∞-ball of radius R centered at x for any $R \geq 0$ and $x \in \mathbb{Z}^d$. We will show that there exists $C > 0$ such that for any $R \geq 1$ and $x \in \mathbb{Z}^d$ satisfying $B_{CR}(x) \subset D_0$, we have

$$\mathbb{P}\big[\overline{\mathrm{LU}}_{x,R,CR}(Z_J^{D_0,U_0})\big] \geq 1 - C(u_0)e^{-L_0^c} \tag{8.10}$$

for any $K \geq 100$ and $J \in \mathbf{J}_{u_0,u}$. We can deduce (8.9) from this by a standard covering argument, see, e.g. the proof of Proposition 1 at the end of page 390 in [38].

We now outline the proof of (8.10) along the lines of [38]. In view of the definition of $\mathbf{J}_{u_0,u}$ from (8.7), we need to verify (8.10) when $J = \{1, \dots, \lfloor \frac{u_0}{4} \operatorname{cap}(D_0) \rfloor\}$, $\{1, \dots, \lfloor \frac{u_0}{4} \operatorname{cap}(D_0) \rfloor + 1\}$ or $\{1, \dots, \lfloor \frac{u_0}{4} \operatorname{cap}(D_0) \rfloor + 2\}$. We will only discuss the first case since the other two cases follow from similar arguments.

We assume, without any loss of generality, that $u_0 \operatorname{cap}(D_0) \geq 100d$ (notice the lower bound in (8.10) and that $\operatorname{cap}(D_0) \geq cL_0^{d-2}$ by (A.8)). Observe that by Lemma 2.1 along with (2.15), Proposition A.1 and the bounds (2.14) and (2.22), there exists a coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_0$ for any $v \in (0, \infty)$ and $j' \in \mathbb{N}$ which satisfies

$$\mathbb{Q}\Big[\mathcal{I}^v \cap D_0 \subset \mathcal{I}\big((\widetilde{Z}_j^{D_0,U_0})_{j'+1 \leq j \leq j'+2v \operatorname{cap}(D_0)}\big) \cap D_0\Big] \geq 1 - Ce^{-cvL_0^{d-2}} \tag{8.11}$$

as soon as $K \geq 100$ (we will implicitly work under this condition in the sequel). Since the excursions $(\widetilde{Z}_j^{D_0,U_0})_{j\geq 1}$ are i.i.d., it follows from the proofs of Lemma 9 and 10 in [38] together with (8.11) that for all $\varepsilon \in (0, \frac{2}{3})$, $x \in \mathbb{Z}^d$ and $R > 0$ such that $B_R(x) \subset D_0$, we have the following analogue of Lemma 10 in [38]. For any $v \in (0, \infty)$ and $j \in \mathbb{N}$,

$$\mathbb{P}\Big[x \in \widetilde{\mathcal{I}}_j^v, \operatorname{cap}(\mathcal{C}(x, R; j, v)) < c(v)R^{(d-2)(1-\varepsilon)}\Big] \leq C(v)e^{-c(v)R^{\frac{\varepsilon}{2}}}$$

where $\widetilde{\mathcal{I}}_j^v \overset{\text{def.}}{=} \mathcal{I}\big((\widetilde{Z}_j^{\check{D}_0,\check{U}_0})_{j+1 \leq j \leq j+v \operatorname{cap}(\check{D}_0)}\big)$ and $\mathcal{C}(x, R; j, v)$ is the component of $\mathcal{I}_j^v \cap B_R(x)$ containing x (we set $\mathcal{C}(x, R; j, v) = \emptyset$ if $x \notin \mathcal{I}_j^v$). Also from Lemma 12 in [38] and (8.11), we can deduce

$$\mathbb{P}\big[U \xleftrightarrow{\widetilde{\mathcal{I}}_j^v \cap B_{CR}(x)} V\big] \geq 1 - C(v) \exp\Big(-c(v)R^{2-d}\operatorname{cap}(U)\operatorname{cap}(V)\Big)$$

for some $C \in (0, \infty)$, any x and R such that $B_{CR}(x) \subset D_0$, any $j \in \mathbb{N}$ and $v \in (0, \infty)$ and all subsets U and V of $B_R(x)$. Since the excursions in $\widetilde{Z}^{D_0,U_0}$ are i.i.d., we can derive (8.10) from the previous two displays in exactly the same way as Lemma 13 was proved in [38] using Lemma 10 and Lemma 12.

Next note that Lemma 2.1 together with (2.15), Proposition A.1 and the bound in (2.14) gives us that there is a coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\widetilde{\mathbb{P}}_0$ such that

$$\mathbb{Q}\big[\mathrm{Incl}_0^{\frac{1}{10}, \frac{u_0}{4} \operatorname{cap}(\check{D}_0)}\big] \geq 1 - Ce^{-c(u_0)L_0^{d-2}}. \tag{8.12}$$

Finally, from (2.22) we have

$$\mathbb{P}\big[(\mathcal{F}_0^{u_0, \frac{u_0}{2}} \cap \mathcal{F}_0^{u,2u})^c\big] \leq 2e^{-c\,(u_0 \wedge u)L_0^{d-2}}. \tag{8.13}$$

Now plugging the estimates from the displays (8.9), (8.12) and (8.13) along with the bound $|\mathbf{J}_{u_0,u}| \leq C(u \vee 1)^3 L_0^C$ into (8.8) via a union bound yields us (8.1). □

As to the other event, i.e. $\mathcal{G}_{y,L_0,L_0^-}(Z)$, our next result is a straightforward adaptation of [16, Corollary 3.7].

Lemma 8.2 (the event $\mathcal{G}_{0,L_0,L_0^-}$ is likely)**.** *There exists a scale L_0^- and $c_9 \in (0, u_*)$ satisfying*

$$\mathbb{P}[\mathcal{G}_{0,L_0,L_0^-}(\overline{Z}_{0,L_0}^u)] \geq 1 - Ce^{-L_0^c}$$

for all $L_0 > 100L_0^-$ and $u \in [0, c_9]$.

We are now ready to give the

Proof of Lemma 6.9. We only prove the case $\mathrm{i} = \mathrm{I}$ and the proof for $\mathrm{i} = \mathrm{II}$ follows from similar (and simpler) arguments with Lemma 8.2 in place of Lemma 8.1.

The proof of (6.42) involves essentially the same arguments as used for proving Lemma 5.16 in [24] and, like in the proof of (6.12) in §6.1, we only highlight the necessary changes below. We apply the same renormalisation argument as in that proof replacing the events A_x^1 and A_x^2 in [24, (5.60), (5.61)] at scale $L_0 \geq 1$ by their analogues where the configurations $\{\varphi \geq h_3 + \varepsilon\}$, $\{\varphi \geq h_2 - \varepsilon\}$ and $\{\varphi \geq h_1 + \varepsilon\}$ are replaced with $\mathsf{N}_{2\delta}(\mathcal{V}^{u_3(1+\varepsilon)})$, $\mathsf{N}_{2\delta}(\mathcal{V}^{u_2(1-\varepsilon)})$ and $\mathsf{N}_{\delta}(\mathcal{V}^{u_1(1+\varepsilon)})$ respectively (cf. the definitions between (6.17)–(6.18)). On the other hand, we replace A_x^3 in [24, (5.62)], by the event

$$\bigcap_{x \in D_{y,L_0}} \{\mathsf{U}_x > \delta\} \cap \mathrm{W}^{\mathrm{I}}_{0,y}(\overline{Z}_{\mathbb{L}}, u_0(1-\varepsilon), u_1(1+\varepsilon)).$$

Choosing $\delta < L_0^{C(\lambda)}$ with $\lambda = \lambda(d)$ as below [24, (5.62)] and invoking Lemma 8.1 to bound the probability of $\mathrm{W}^{\mathrm{I}}_{0,y}$ (recall (6.29)), one obtains that $\lim_{L_0 \to \infty} \mathbb{P}[\widetilde{A}^k_{x,0}] = 1$ for $k = 3$ with $\widetilde{A}^3_{x,0}$ as in [24, (5.63)]. The cases of $k = 1, 2$ follow from the bounds (1.13) and (1.14) in [20, Theorem 1.3]. One then applies [12, Proposition 7.1] (which also concerns interlacements), using Lemma 2.1 and inequalities like (6.44) and (6.53) to decouple events instead of the decoupling inequalities in [12, Theorem 2.4], to deduce analogues of the bounds [24, (5.64)] upon choosing $L_0 = L_0(\boldsymbol{u})$ large enough, thereby also fixing $c_7(L_0) = L_0^{-C}$. The bound (6.42) (for $\mathrm{i} = \mathrm{I}$) now follows as in [24, Lemma 5.16] at the geometric scales $L = L_n = L_0 \ell_0^n$, $n \geq 1$, with ℓ_0 as chosen in [24]. For general $L \in [L_n, L_{n+1}]$, we obtain the corresponding bound by a straightforward covering argument using a bounded number of spatially shifted copies of the event $\mathcal{G}^{\mathrm{I}}_{0,L_n}$ followed by a union bound. □

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A Equilibrium measures and comparison

We gather here a few elements of potential theory for the random walk on $\mathbb{Z}^d$ that are used throughout. In particular, Propositions A.1 contains estimates concerning equilibrium measures that are used to obtain the key tail bounds in Proposition 4.2.

Recall from around (2.1) that $X = (X_t)_{t\geq 0}$ denotes the continuous-time simple random walk on $\mathbb{Z}^d$ with unit jump rate. With $T_U := H_{\mathbb{Z}^d\setminus U}$ (recall that $H_U = \inf\{t \geq 0 : X_t \in K\}$) the exit time from $U \subset \mathbb{Z}^d$, let

(A.1)
$$g_U(x,y) \overset{\text{def.}}{=} E_x\left[\int_0^{T_U} 1_{\{X_s=y\}}ds\right],\ x,y\in\mathbb{Z}^d.$$

the Green's function of X killed outside U, and set $g = g_{\mathbb{Z}^d}$. By translation invariance $g(x,y) = g(x+z, y+z)$ for all $x, y, z \in \mathbb{Z}^d$. By [27, Theorem 1.5.4], one has that $g(x) \coloneqq g(0,x)$ satisfies

$$g(x) \sim c_{10}|x|^{2-d}, \quad \text{as } |x| \to \infty \tag{A.2}$$

(where $\sim$ means that the ratio of both sides tends to 1 in the given limit), for an explicit constant $c_{10} \in (0, \infty)$ with $c_{10}(3) = \frac{3}{2\pi}$. For $K \subset U \subset \mathbb{Z}^d$ with K a finite set, the equilibrium measure of K relative to U is defined as

$$e_{K,U}(x) \stackrel{\text{def.}}{=} P_x[\widetilde{H}_K > T_U]1\{x \in K\} \tag{A.3}$$

with $\widetilde{H}_K = \inf\{t \geq \zeta_0 : X_t \in K\}$. It is a measure supported on ∂K. We denote by

$$\text{cap}_U(K) = \sum_x e_{K,U}(x) \tag{A.4}$$

its total mass, the (electrostatic) capacity of K (relative to U) and by $\bar{e}_{K,U} = \frac{e_{K,U}}{\text{cap}_U(K)}$ the normalized equilibrium measure. Naturally, we omit U from the notation whenever $U = \mathbb{Z}^d$. One has the last-exit decomposition, see, e.g. [27, Lemma 2.1.1] for a proof,

$$P_x[H_K < T_U] = \sum_y g_U(x,y)e_{K,U}(y), \text{ for all } x \in \mathbb{Z}^d, \tag{A.5}$$

valid for all $K \subset\subset \mathbb{Z}^d$ (with $\subset\subset$ denoting a subset of finite cardinality). One also has the following sweeping identity, see for instance [47, (1.12)], asserting that

$$e_K(y) = P_{e_{K'}}[H_K < \infty,\ X_{H_K} = y], \text{ for every } K \subset K' \subset\subset U \text{ and } y \in \mathbb{Z}^d. \tag{A.6}$$

In view of (A.4), summing over y in (A.6) gives

$$\text{cap}(K) = \text{cap}(K')P_{\bar{e}_{K'}}[H_K < \infty]. \tag{A.7}$$

In particular, it follows immediately from (A.7) that $\text{cap}(K)$ is increasing in K. One further knows, see for instance [20, (2.8)-(2.9)] for the argument, that for all $L > 0$,

$$cL^{d-2} \leq \text{cap}(B_L) \leq CL^{d-2}. \tag{A.8}$$

The following result is important in Section 5 for proving Proposition 4.2 and then again in Section 6 for coupling of excursions obtained from the interlacement trajectories with a collection of independent excursions (see Lemma 2.1). It is, in some sense, a *mixing* result for certain entrance distributions of the walk from afar and appeared, e.g. as Proposition 2.5 in [48]. Inspection of its proof yields the quantitative dependence on K stated below.

Proposition A.1. *There exist $C_7 \in [1, \infty)$, $C_8 \in (0, \infty)$ such that for all $K \geq C_7, L \geq 1$, non-empty $A \subset B_{4L}$ and $B \subset\subset \mathbb{Z}^d$ such that $B \cap B_{KL} = A$ with $\mathbb{Z}^d \setminus B$ connected, one has for any $y \in A$ and $x \in \mathbb{Z}^d \setminus (B \cup B_{KL})$,*

$$\left(1 - C_8 K^{-1}\right)\bar{e}_A(y) \leq P_x\left[X_{H_B} = y \,\middle|\, H_B < \infty, X_{H_B} \in A\right] \leq \left(1 - C_8 K^{-1}\right)\bar{e}_A(y) \tag{A.9}$$

and

$$\left(1 - C_8 K^{-1}\right)\bar{e}_A(y) \leq \frac{\bar{e}_B(y)}{\bar{e}_B(A)} \leq \left(1 + C_8 K^{-1}\right)\bar{e}_A(y). \tag{A.10}$$

The following result provides a comparison between V in (4.3) and e_Σ with Σ as in (4.2) (cf. [48, Proposition 4.1]).

Lemma A.2. *For all $\varepsilon \in (0,1)$, $L \geq 1$, $K \geq \frac{C_8}{\varepsilon}$ and $\mathcal{C}$ as in (4.1), one has*

$$(1-\varepsilon)e_\Sigma \leq V \leq (1+\varepsilon)e_\Sigma \tag{A.11}$$

pointwise on $\mathbb{Z}^d$.

Proof. We only need to verify the bound on Σ since both e_Σ and V are supported on Σ. Let $y \in D = D_z$ for some $z \in \mathcal{C}$. Owing to (4.1) and (4.2), the assumptions of Proposition A.1 are in force with the choice $A = \check{D}$, $B = \Sigma$ and $5L$ in place of L. Applying (A.10) with these choices, we get

$$(1-\varepsilon)\,\bar{e}_{\check{D}}(y) \leq \frac{e_\Sigma(y)}{e_\Sigma(\check{D})} \leq (1+\varepsilon)\,\bar{e}_{\check{D}}(y)$$

for all $K \geq \frac{C_8}{\varepsilon}$, which yields (A.11). □

B Coarse-graining of paths

In this appendix, we give the:

Proof of Proposition 4.3. Owing to the absence of the requiment $\lim_{K\to\infty}\lambda(K) = 1$ when $d \geq 4$, we can simply use the coarseing given by Proposition 4.3-(ii) in [24] for $d \geq 4$ as $B_{r/\sqrt{d}} \subset B_r^2 \subset B_r$ as subsets of $\mathbb{Z}^d$ for all $r \geq 1$, where B_r refers to the closed ℓ^∞-ball of radius r around the origin (recall from the beginning of Section 4.2 that the superscript 2 refers to Euclidean balls). Therefore, we only need to consider the case $d = 3$. Here too, we will present our arguments for the case of $\Lambda_N = B_N^2$, cf. (4.9). The other cases can be either easily adapted similarly to the way they were dealt with in the case of ℓ^∞-balls at the end of Section 4.1 in [24] or obtained directly from [24, Proposition 4.3] itself.

Now, with $\Lambda_N = B_N^2$ we define, for each $i = 1, \dots, n := \lfloor N/3\sqrt{d}KL \rfloor - 1$ (note that $n \geq 1$), the concentric shells $\mathsf{S}_i := \partial_{\mathbb{R}^d} B^2_{3\sqrt{d}KLi} \subset \mathbb{R}^d$ where $\partial_{\mathbb{R}^d}$ denotes the topological boundary and $B^2_{3\sqrt{d}KLi}$ is viewed as a Euclidean ball in $\mathbb{R}^d$. Clearly the collection of points $z \in \mathbb{L}$ such that $C_z \cap \mathsf{S}_i \neq \emptyset$ (see (2.19) for notation) is a separating set for Λ_N, meaning that any path crossing Λ_N, i.e. connecting 0 and $\partial\Lambda_N$, must intersect at least one box C_z. Let $S_i \subset \mathbb{L}$ denote the union of the points z with the above property. Now given any crossing γ of B_N^2, the successive first exits of γ from the sets $S_i + C_{0,L}$, $i = 1, \dots, n$, give a sequence of points $(z_i : 1 \leq i \leq n)$ in $\mathbb{L}$ such that

$$z_i \in S_i \text{ and } \gamma \text{ crosses } \tilde{D}_{z_i} \setminus C_{z_i}, \text{ for all } i = 1, \dots, n. \tag{B.1}$$

We then define $\mathcal{A} = \mathcal{A}^K_{N,L}(B^2_N)$ as the family consisting of all collections $\mathcal{C} := \{z_i : 1 \leq i \leq n\}$ that can be obtained in this manner.

Let us now verify the conditions (4.12)-(4.10) of admissibility for $\mathcal{A}$. Property (4.12) (with $d = 3$) is immediate by construction and (4.13) follows from (B.1). The separation condition (4.1) inherent to admissibility follows from the triangle inequality and the observation that $d_\infty(\mathsf{S}_i, \mathsf{S}_{i+1}) \geq 3KL$. Regarding the cardinality of $\mathcal{A}$, one notes that any C_z intersecting S_i is necessarily a subset of $A_i := B^2_{4\sqrt{d}KLi} \setminus B^2_{2\sqrt{d}KLi}$. Therefore the number of choices for z_i in (B.1) is bounded by

$$\frac{\mathrm{Vol}_{\mathbb{R}^d}(A_i)}{\mathrm{Vol}_{\mathbb{R}^d}(C_0)} \leq C\,\frac{(4\sqrt{d}KLi)^d - (2\sqrt{d}KLi)^d}{L^d} \leq C\Big(\frac{N}{L}\Big)^{d-1}.$$

However, this immediately implies that $|\mathcal{A}| \leq (C\frac{N}{L})^{n(d-1)}$, from which (4.10) follows with $\Gamma(x) = C_2 K^{-1} x \log ex$ for a suitable choice of C_2.

It remains to verify (4.14) which is arguably the most delicate part. However, fortunately this part runs essentially parallel to the verification of the analogous property in the proof of Proposition 4.3 in [24] (see display (4.16) there). The only difference comes from the fact that in place of inequality (4.21) in [24], we have the following slightly modified version:

$$|\tau(z_i) - \tau(z_j)| \leq |z_i - z_j| + 2\sqrt{d}L1_{i\neq j} \text{ for all } z_i, z_j \in \mathcal{C} \in \mathcal{A}.$$

But this does not affect the subsequent computations there as a similar discrepancy is already accounted for through the upper bound on κ in the display right after (4.23). As to the factor $\big(1 - \frac{C_3}{K}\big)$ and the first condition in the second line of (4.15), which are both new, they follow respectively from the display (4.24) — this needs no further explanation — and the proof of Lemma 4.6 in [24], as we now explain. The proof of Lemma 4.6 unfolds as it is, by which the probability in the last display of the proof is bounded by application of Lemma 2.4 in [24] (see in particular display (2.17) in [24]), as $C(\log(1 + d_\infty(x, \widetilde{T}_N))/\log N)^{c(\gamma)}$, which needs to tend to 0 in the limit as $N \to \infty$; in the context of Lemma 4.6 of [24], the set of $\widetilde{T}_N \subset T_N$ is a perforated line and x is a point at ℓ^∞-distance at most CKL from $\widetilde{T}_N$. Moreover one chooses $\gamma = \frac{1}{13K}$ in that proof. The first condition in the second line of (4.15) now arises upon inspection of the proof of Lemma 2.4 in [24], which reveals that one can choose $c(\gamma) = c'\gamma$ for suitable $c' \in (0, 1)$. □

C Crossings and blocking interfaces

In this appendix section we exhibit a basic result, Proposition C.1 below, which is of topological nature and may have independent interest. It connects the minimum number of times any path crossing an annular region $V \setminus U$ (where $U \subset V \subset \mathbb{Z}^d$) intersects a set Σ to the density of certain dual 'blocking' interfaces in Σ. The existence of these interfaces is established as part of Proposition C.1. This result significantly refines Lemma 2.1 in [24] in order to suit the more sophisticated requirements of the current article (see Section 6). In particular, a key novel feature is a certain maximality property of blocking interfaces, see Proposition C.1, (c) below.

We first introduce the necessary notation. For any $U \subset\subset \mathbb{Z}^d$, we let $\text{Fill}(U)$ denote the union of U and all its holes, where a hole is any finite component of U^c. The set $\text{Fill}(U)$ is $(*$-)connected whenever the set U is $(*$-)connected. Since U is finite, there exists a unique infinite connected component U^c_∞ of U^c and we define the *exterior boundary* of U as $\partial^{\text{ext}} U := \partial U^c_\infty$, which equals $\partial^{\text{out}}\text{Fill}(U)$. For any two sets $U, \Sigma \subset\subset \mathbb{Z}^d$, we say U is surrounded by Σ, denoted as

(C.1) $\quad U \preceq \Sigma$, if any infinite connected set γ intersecting U also intersects Σ.

Clearly, each finite component of U^c is surrounded by U which is itself surrounded by $\partial^{\text{ext}} U$. It is also clear from the definition that the relation '$\preceq$' is a transitive and reflexive relation on finite subsets of $\mathbb{Z}^d$.

Following is the main result of this section. Property (c) is the most delicate of the three stated below; it is also the main feature of this result compared to [24, Lemma 2.1].

Proposition C.1 (Blocking interfaces)**.** *Let $V \subset \mathbb{Z}^d$ be a box, $U \subset V$ a $*$-connected set and $k \geq 1$ an integer. Suppose that $\Sigma \subset (V \setminus U)$ is such that any path γ connecting U and ∂V intersects Σ in at least $k(\geq 1)$ points. Then there exists an integer $\ell \geq 1$ and $*$-connected sets $O_1, \dots, O_\ell \subset \Sigma$ satisfying the following three properties:*

(a) $U \preceq O_1 \preceq \ldots \preceq O_\ell \preceq \partial V$.

(b) *Any path connecting* U *and* ∂V *intersects* $O := \bigcup_{1\le i\le \ell} O_i$ *in at least* k *points.*

(c) *The sets* $O_1, \ldots, O_\ell$ *are maximal in the following sense. If for some* $j \in \{1, \ldots, \ell\}$ *and* $j' \in \{j, (j+1)\wedge \ell\}$ *two points* $x_j \in \overline{O_j}$ *and* $x_{j'} \in \overline{O_{j'}}$ *are connected (by a path) in* $V \setminus O$*, then they are also connected in* $V \setminus \Sigma$*. Similarly if* $x_\ell \in \overline{O_\ell}$ *is connected to* ∂V *outside* O*, then* x_ℓ *is connected to* ∂V *outside* Σ*.*

The following lemma captures an essential feature that will be used to prove (c) above. For a path $\gamma = (\gamma(n))_{0\le n\le k}$, $k \ge 0$, we denote by $\gamma^\circ = \bigcup_{0<n<k}\{\gamma(n)\}$ its range with endpoints omitted, also referred to as the *interior* of γ.

Lemma C.2. *Let* $\Sigma \subset\subset \mathbb{Z}^d$ *and* $U \subset \Sigma$ *be a* $*$*-component of* Σ*. Let* $W \subset\subset \mathbb{Z}^d$ *be either i) a connected set, or ii) a* $*$*-component of* Σ*, and assume* W *is such that*

$$U \preceq W, \text{ and} \tag{C.2}$$

$$\overline{U} \text{ is connected to } \overline{W} \text{ in } \Sigma^c. \tag{C.3}$$

Then any point in U *that is connected to* W *by a path* π *with* $\pi^\circ \cap U = \emptyset$*, is also connected to* W *by a path* $\tilde{\pi}$ *with* $\tilde{\pi}^\circ \cap \Sigma = \emptyset$*.*

Proof. Throughout the proof, we refer to a point $z \in \mathbb{Z}^d$ as having *property NC* if z

$$\text{is } \textit{not} \text{ connected to } W \text{ by a path } \pi \text{ with } \pi^\circ \cap \Sigma = \emptyset. \tag{C.4}$$

Let $x \in U$ be connected to W by a path π with $\pi^\circ \cap U = \emptyset$. We assume that $\pi^\circ \neq \emptyset$ else choosing $\tilde{\pi} = \pi$ works. Our aim is to show that x does *not* have property NC. Suppose for the sake of contradiction that it did. Consider the component $\mathscr{C}_x$ of $\Sigma^c \cup \{x\}$ containing the point x. By definition, $\mathscr{C}_x$ is a connected set. Since x is assumed to satisfy (C.4), it necessarily holds that

$$\big((\mathscr{C}_x \cup \partial^{\text{ext}}\mathscr{C}_x) \cap W\big) \subset \big(\overline{\mathscr{C}_x} \cap W\big) = \emptyset; \tag{C.5}$$

indeed the inclusion is obvious since $\partial^{\text{ext}}A \subset \partial^{\text{out}}A$ for any set A, and a non-empty intersection of $\overline{\mathscr{C}_x} \cap W$ would imply the existence of a path π with the above (precluded) properties since $\mathscr{C}_x$ is connected. Next, since $\mathscr{C}_x \cup \partial^{\text{ext}}\mathscr{C}_x$ is a connected set containing x and $\{x\} \preceq W$ by (C.2) and the transitivity of $\preceq$ (using that $\{x\} \subset U$ which implies that $\{x\} \preceq U$), it follows from (C.5) in view of (C.1) that

$$(\mathscr{C}_x \cup \partial^{\text{ext}}\mathscr{C}_x) \preceq W. \tag{C.6}$$

Moreover, by definition of $\mathscr{C}_x$ and (C.5),

$$\text{every point in } \mathscr{C}_x \text{ has property NC.}$$

Now recall the definition of $\text{Fill}(\mathscr{C}_x)$ from the beginning of this section. We claim that, in fact,

$$\text{every point in } \text{Fill}(\mathscr{C}_x) \text{ has property NC} \tag{C.7}$$

and first finish the proof of the lemma assuming this claim by deriving the desired contradiction. By hypothesis in (C.3), U contains a point y (say) that does not have property NC: indeed with $\gamma' \subset \Sigma^c (\subset U^c)$ a path starting in $\overline{U}$ and ending in $\overline{W}$, any neighbor $y \in U$ of $\gamma'(0) (\in \partial^{\text{out}}U)$ will do (note that γ'

can always be extended by addition of at most one point so as to intersect W). Since U is $*$-connected, there is a $*$-path $\gamma \subset U$ connecting x and y. Also since $y \notin \text{Fill}(\mathscr{C}_x)$ on account of (C.7), it follows from the definition of $*$-connectivity that γ contains a point that either lies in $\partial^{\text{out}}\text{Fill}(\mathscr{C}_x) = \partial^{\text{ext}}\mathscr{C}_x \subset \Sigma$ or is a $*$-neighbor of $\partial^{\text{ext}}\mathscr{C}_x$. The inclusion in Σ is immediate since $\mathscr{C}_x$ is a component in $\Sigma^c \cup \{x\}$.

We will now argue that the stronger inclusion $\partial^{\text{ext}}\mathscr{C}_x \subset U(\subset \Sigma)$ holds true. Indeed, $\mathscr{C}_x$ is connected and therefore $\partial^{\text{ext}}\mathscr{C}_x$ is $*$-connected by [11, Lemma 2.1-(i)]. Consequently, the set $\gamma \cup \partial^{\text{ext}}\mathscr{C}_x$ is itself a $*$-connected subset of Σ. Since U is a $*$-component of Σ and $\gamma \subset U$, we thus obtain $\partial^{\text{ext}}\mathscr{C}_x \subset U$.

On the other hand, since x can be connected to W by a path π with $\pi^\circ \cap U = \emptyset$ and $\overline{\mathscr{C}_x} \cap W = \emptyset$, the latter on account of (C.5), it must be the case that $\pi \cap \partial^{\text{ext}}\mathscr{C}_x \neq \emptyset$ and hence $\partial^{\text{ext}}\mathscr{C}_x \cap U^c \neq \emptyset$ (recall that $\pi^\circ \neq \emptyset$) which leads to a contradiction.

It remains to prove (C.7). To this end let $w \in \text{Fill}(\mathscr{C}_x)$. We first argue that if w does not have property NC, then it is necessarily the case that

(C.8) $$W \text{ intersects some finite component of } \mathscr{C}_x^c.$$

Indeed since $w \in \text{Fill}(\mathscr{C}_x)$, any path γ connecting w to a point in $(\text{Fill}(\mathscr{C}_x) \cup \partial^{\text{ext}}\mathscr{C}_x)^c$ must satisfy $\gamma^\circ \cap \partial^{\text{ext}}\mathscr{C}_x \neq \emptyset$. But if (C.8) does not occur then, since $\overline{\mathscr{C}_x} \cap W = \emptyset$ in view of (C.5), it further holds that $(\text{Fill}(\mathscr{C}_x) \cup \partial^{\text{ext}}\mathscr{C}_x) \cap W = \emptyset$. The last two observations together imply that π° must intersect $\partial^{\text{ext}}\mathscr{C}_x$ and hence Σ for any path π connecting w to W, i.e. w has Property NC. All in all, (C.8) thus follows.

Now notice that (C.6) and (C.5) together imply

(C.9) $$(\text{Fill}(\mathscr{C}_x) \cup \partial^{\text{ext}}\mathscr{C}_x)^c \cap W \neq \emptyset.$$

Hence if W is connected and (C.8) holds, then W must intersect $\partial^{\text{ext}}\mathscr{C}_x$ as any path between a point in some finite component of $\mathscr{C}_x^c$ and $(\text{Fill}(\mathscr{C}_x) \cup \partial^{\text{ext}}\mathscr{C}_x)^c$ has to intersect $\partial^{\text{ext}}\mathscr{C}_x$. But this contradicts (C.5) and thus (C.8) is not possible in this case. On the other hand, if W is a $*$- connected set and (C.8) holds, then it follows from (C.9) and the definition of $*$-connectivity that W contains a point which is either an element of $\partial^{\text{ext}}\mathscr{C}_x$ or a $*$-neighbor of $\partial^{\text{ext}}\mathscr{C}_x$. Therefore if W is a $*$-component of Σ, we get $\partial^{\text{ext}}\mathscr{C}_x \subset W$ since $\partial^{\text{ext}}\mathscr{C}_x \subset \Sigma$ as we already noted above. But this also contradicts (C.5). Thus the conclusion holds for both types of W considered. □

We now turn to the:

Proof of Proposition C.1. Since $k \geq 1$, U is not connected to ∂V in $\mathbb{Z}^d \setminus \Sigma\,(\supset U)$. Then by [11, Lemma 2.1-(i)], the exterior boundary $\partial^{\text{ext}}\mathscr{C}_U$ of the $*$-component $\mathscr{C}_U$ of U in $\mathbb{Z}^d \setminus \Sigma$ is a non-empty $*$-connected subset of Σ. Let O_1 be defined as the $*$-component of $\partial^{\text{ext}}\mathscr{C}_U$ in Σ. Thus by definition, $U \preceq O_1$ and $O_1 \subset \Sigma \subset V$. Also observe that $\mathscr{C}_U \cup O_1 \subset V$ is $*$-connected.

Now the hypothesis of the proposition is clearly satisfied with $U_1 = \mathscr{C}_U \cup O_1$, $\Sigma_1 = \Sigma \setminus O_1$ and k_{U_1,Σ_1} (in case the latter is ≥ 1) substituting for U, Σ and k respectively, where for any two disjoint subsets U' and Σ' of V, we denote

$$k_{U',\Sigma'} = \min\{|\gamma \cap \Sigma'| : \gamma \text{ is any path between } U' \text{ and } \partial V\} \geq 0.$$

Iterating the construction in the previous paragraph over successive rounds until the first ℓ such that $k_{U_\ell,\Sigma_\ell} = 0$ by letting O_k be the $*$-component of $\partial^{\text{ext}}U_{k-1}$ in Σ_{k-1}, $U_k = \mathscr{C}_{U_{k-1}} \cup O_k$ and $\Sigma_k = \Sigma_{k-1} \setminus O_k$ in each round $2 \leq k \leq \ell$, we obtain a sequence of $*$-components $(U \preceq)\, O_1 \preceq \ldots \preceq O_\ell \preceq \partial V$ in Σ. This can be verified inductively in a straightforward manner. In particular, the collection $\{O_1, \ldots, O_\ell\}$ satisfies (a).

Next, consider any path γ connecting U and ∂V with exactly one (end-)point in ∂V. Let $\gamma_1, \ldots, \gamma_m$ denote the maximal non-trivial segments of γ whose interiors lie outside O. By our construction of the sets $O_1, \ldots, O_\ell$, any such segment must necessarily have either both its endpoints in $\{O_j, O_{j+1}\}$ for some $j \in \{1, \ldots, \ell - 1\}$ or one endpoint in O_ℓ and the other in ∂V. Now assuming the sets $O_1, \ldots, O_\ell$ also satisfy (c), we can replace each γ_i with a suitable segment whose interior is disjoint from Σ such that the resulting sequence γ' is also a path between U and ∂V satisfying $\gamma' \cap \Sigma = \gamma' \cap O = \gamma \cap O$. Hence $|\gamma \cap O| \geq k$ by the hypothesis of our proposition applied to the path γ', yielding item (b).

It only remains to verify (c). Let us suppose that some $x \in O_j$ is connected to some $y \in O_j \cup O_{j+1}$, or $O_j \cup \partial V$ in case $j = \ell$, by a path whose interior lies in $V \setminus O$.

We will first consider the case $y \in O_j$. Let $\mathscr{C}_x$ denote the component of x in $(V \setminus \Sigma) \cup \{x\}$. If x and y can *not* be connected by a path whose interior lies in $V \setminus \Sigma$, then y necessarily lies in a component, say $\mathscr{C}_y^x$, of $V \setminus \mathscr{C}_x$ such that $y \in \mathscr{C}_y^x \setminus \partial_V \mathscr{C}_y^x$ and

$$\partial_V \mathscr{C}_y^x \subset \Sigma \tag{C.10}$$

Also since x, y are connected by a path whose interior lies in $V \setminus O$ and $y \in \mathscr{C}_y^x \setminus \partial_V \mathscr{C}_y^x$, it follows that

$$\partial_V \mathscr{C}_y^x \cap O^c \neq \emptyset. \tag{C.11}$$

On the other hand, x and y are connected by a $*$-path γ in Σ as $O_j \subset \Sigma$ is $*$-connected by definition. Hence by the definition of $*$-connectivity, γ either intersects $\partial_V \mathscr{C}_y^x$ or contains a point that is a $*$-neighbor of $\partial_V \mathscr{C}_y^x$. Therefore the set $\gamma \cup \partial_V \mathscr{C}_y^x$ is $*$-connected *provided* $\partial_V \mathscr{C}_y^x$ is also $*$-connected which turns out to be a consequence of [11, Lemma 2.1-(ii)] as the set $\mathscr{C}_x$ is a connected subset of V. But $\gamma \cup \partial_V \mathscr{C}_y^x \subset \Sigma$ (recall (C.10) as well as that $\gamma \subset \Sigma$) and intersects $\{x, y\} \subset O_j$. Since O_j is a $*$-component by construction, the previous two observations imply that $\partial_V \mathscr{C}_y^x \subset \gamma \cup \partial_V \mathscr{C}_y^x \subset O_j$. However, this contradicts (C.11) and thus property (b) is satisfied in this case.

Next let us consider the case $y \in O_{j+1}$ where $j < \ell$ and let $\mathscr{C}_x$ denote the component of x in $(V \setminus \Sigma) \cup \{x\}$, as before. Since $O_j \subset \Sigma$ and $(\{x\} \subset) \, O_j \preceq O_{j+1}$ by (a), it follows from Lemma C.2 applied with $(U, \Sigma, W) = (O_j, \Sigma, O_{j+1})$ that there is a point $z \in \partial_{V,\text{out}} \mathscr{C}_x \cap O_{j+1}$. Since $z \in \partial_V^{\text{out}} \mathscr{C}_x$, it must be the case that $z \in \partial_V \mathscr{C}_y^x$ if $z \in \partial_V \mathscr{C}_y^x$. Thus any $*$-path contained in O_{j+1} connecting y and z, which necessarily exists as O_{j+1} is $*$-connected, must either intersect or be a $*$-neighbor of $\partial_V \mathscr{C}_y^x$. Now we repeat the same argument as in the previous case with such a $*$-path γ.

Finally the case where $x \in O_\ell$ and $y \in \partial V$ follows almost immediately from Lemma C.2 with $(U, \Sigma, W) = (O_\ell, \Sigma, \partial V')$ where V' is any box containing V in its interior. □