INFINITESIMAL DEFORMATIONS AND BRAUER GROUP OF SOME GENERALIZED CALABI–ECKMANN MANIFOLDS

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Abstract. Let X be a compact connected Riemann surface. Let $\xi_1 : E_1 \longrightarrow X$ and $\xi_2 : E_2 \longrightarrow X$ be holomorphic vector bundles of rank at least two. Given these together with a $\lambda \in \mathbb{C}$ with positive imaginary part, we construct a holomorphic fiber bundle S^{ξ_1,ξ_2}_λ over X whose fibers are the Calabi–Eckmann manifolds. We compute the Picard group of the total space of S^{ξ_1,ξ_2}_λ. We also compute the infinitesimal deformations of the total space of S^{ξ_1,ξ_2}_λ. The cohomological Brauer group of S^{ξ_1,ξ_2}_λ is shown to be zero. In particular, the Brauer group of S^{ξ_1,ξ_2}_λ vanishes.

1. Introduction

Let X be compact connected Riemann surface. Let $\xi_1 : E_1 \longrightarrow X$ and $\xi_2 : E_2 \longrightarrow X$ be holomorphic vector bundles of rank m and n respectively, with $m, n \geq 2$. Let E^0_1 (respectively, E^0_2) be the complement of the image of the zero section in E_1 (respectively, E_2). Fix a complex number λ with positive imaginary part.

The group $\mathbb{C}$ acts on the fiber product $E^0_1 \times_X E^0_2$ as follows:

$$t \cdot (z, w) = (\exp(t) \cdot z, \exp(t(\lambda - 1)/\lambda) \cdot w), \quad t \in \mathbb{C}, (z, w) \in E^0_1 \times_X E^0_2.$$

The quotient for this action is a compact complex manifold; we denote this complex manifold by S^{ξ_1,ξ_2}_λ. Each fiber of the natural projection $p : S^{\xi_1,\xi_2}_\lambda \longrightarrow X$ is a Calabi–Eckmann manifold.

Define the elliptic curve $T := \mathbb{C}/(\mathbb{Z} \oplus \lambda \cdot \mathbb{Z})$.

We prove the following (see Theorem 3.6 and Corollary 5.2):

Theorem 1.1. *The Picard group of S^{ξ_1,ξ_2}_λ fits in a short exact sequence*

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(S^{\xi_1,\xi_2}_\lambda) \longrightarrow H^1(T, \mathcal{O}_T) \longrightarrow 0.$$

The injective homomorphism $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(S^{\xi_1,\xi_2}_\lambda)$ sends any holomorphic line bundle L to its pullback p^*L.

Theorem 1.2. *The cohomological Brauer group $\mathrm{Br}'(S^{\xi_1,\xi_2}_\lambda)$ vanishes. In particular, the Brauer group $\mathrm{Br}(S^{\xi_1,\xi_2}_\lambda)$ vanishes.*

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Assume that all endomorphisms of the holomorphic vector bundles E_1 and E_2 are scalar multiplications. Also, assume that the genus of X is at least two. We prove the following (see Corollary 4.4):

Theorem 1.3. *The dimension of the space of all infinitesimal deformations of the complex manifold $S_\lambda^{\xi_1,\xi_2}$ is $(m^2+n^2+2)(g-1)+2$, where g is the genus of X.*

In fact we compute the infinitesimal deformations of $S_\lambda^{\xi_1,\xi_2}$ explicitly.

The infinitesimal deformations of Calabi–Eckmann manifolds were computed by Akao in [1].

2. Generalized Calabi–Eckmann manifolds

We briefly recall the construction of the Calabi-Eckmann manifolds (see [2]). Take integers $m\,,n\,\geq\,2$, and take $\lambda\,\in\,\mathbb{C}$ with $\mathrm{Im}\,\lambda\,>\,0$. Consider $(\mathbb{C}^m\setminus\{0\})\times(\mathbb{C}^n\setminus\{0\})$. The additive group $\mathbb{C}$ acts on this product as follows:

$$t\cdot(z\,,w)\,=\,(\exp(t)z\,,\exp(t(\lambda-1)/\lambda)w)\,,\;\;t\,\in\,\mathbb{C}\,,\;\;(z\,,w)\,\in\,(\mathbb{C}^m\setminus\{0\})\times(\mathbb{C}^n\setminus\{0\})\,.$$

The quotient

$$M_\lambda^{m,n}\,:=\,((\mathbb{C}^m\setminus\{0\})\times(\mathbb{C}^n\setminus\{0\}))/\mathbb{C} \tag{2.1}$$

is a Calabi–Eckmann manifold. Let S^{2m-1} and S^{2n-1} be the unit spheres in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. The composition of maps

$$S^{2m-1}\times S^{2n-1}\,\hookrightarrow\,(\mathbb{C}^m\setminus\{0\})\times(\mathbb{C}^n\setminus\{0\})\,\longrightarrow\,M_\lambda^{m,n}$$

is a diffeomorphism. Let

$$T_\lambda\,:=\,\mathbb{C}/(\mathbb{Z}\oplus\lambda\cdot\mathbb{Z}) \tag{2.2}$$

be the complex elliptic curve. The natural projection

$$(\mathbb{C}^m\setminus\{0\})\times(\mathbb{C}^n\setminus\{0\})\,\longrightarrow\,\mathbb{CP}^{m-1}\times\mathbb{CP}^{n-1}$$

descends to a projection to $\mathbb{CP}^{m-1}\times\mathbb{CP}^{n-1}$ of the above quotient space $M_\lambda^{m,n}$. This projection $M_\lambda^{m,n}\,\longrightarrow\,\mathbb{CP}^{m-1}\times\mathbb{CP}^{n-1}$ makes $M_\lambda^{m,n}$ a holomorphic principal T_λ–bundle over $\mathbb{CP}^{m-1}\times\mathbb{CP}^{n-1}$. We will extend this construction to a family parametrized by a Riemann surface.

Let X be a compact connected Riemann surface of genus g. Let

$$\xi_1\,:\,E_1\,\longrightarrow\,X\;\;\text{and}\;\;\xi_2\,:\,E_2\,\longrightarrow\,X$$

be two holomorphic vector bundles over X of rank m and n respectively; as before, $m\,,n\,\geq\,2$. Let E_i^0, $i\,=\,1\,,2$, be the complement of the image of the zero section in the total space of E_i. Take $\lambda\,\in\,\mathbb{C}$ as above. The additive group $\mathbb{C}$ acts on the fiber product $E_1^0\times_X E_2^0$ as follow:

$$t\cdot(z\,,w)\,=\,(\exp(t)\cdot z\,,\exp(t(\lambda-1)/\lambda)\cdot w)\,,\;\;t\,\in\,\mathbb{C}\,,\,(z\,,w)\,\in\,E_1^0\times_X E_2^0\,.$$

It is easy to check that this $\mathbb{C}$-action is free and proper. Hence the corresponding quotient

$$S_\lambda^{\xi_1,\xi_2} := (E_1^0 \times_X E_2^0)/\mathbb{C} \tag{2.3}$$

is a compact complex manifold (see, for example, [5, Proposition 2.1.13]). The projection $(\xi_1\,,\xi_2)|_{E_1^0\times E_2^0} \,:\, E_1^0 \times E_2^0 \longrightarrow X$ descends to a holomorphic projection

$$p \,:\, S_\lambda^{\xi_1,\xi_2} \longrightarrow X\,. \tag{2.4}$$

This projection makes $S_\lambda^{\xi_1,\xi_2}$ a holomorphic fiber bundle over X with fiber $M_\lambda^{m,n}$ (constructed in (2.1)). The complex manifold $M_\lambda^{m,n}$ is not Kähler because $H^2(M_\lambda^{m,n}, \mathbb{R}) = 0$. Hence $S_\lambda^{\xi_1,\xi_2}$ is also not Kähler (any complex submanifold of a Kähler manifold is Kähler).

For $i = 1\,,2$, let $P(E_i)$ be the holomorphic projective bundles over X parametrizing all the lines in E_i. The natural projection of $E_1^0 \times_X E_2^0$ to $P(E_1) \times_X P(E_2)$ descends to a projection

$$\varphi \,:\, S_\lambda^{\xi_1,\xi_2} \longrightarrow P(E_1) \times_X P(E_2)\,. \tag{2.5}$$

We note that $P(E_1) \times_X P(E_2)$ is a complex projective manifold. The projection p in (2.4) is the composition of φ with the natural projection

$$q \,:\, P(E_1) \times_X P(E_2) \longrightarrow X\,. \tag{2.6}$$

The projection φ makes $S_\lambda^{\xi_1,\xi_2}$ a holomorphic principal T_λ bundle over $P(E_1) \times_X P(E_2)$, where T_λ is defined in (2.2). To see this, consider the action of the multiplicative group $\mathbb{C}^* = \mathbb{C}/(2\pi\sqrt{-1}\cdot\mathbb{Z})$ on $E_1^0 \times_X E_2^0$ defined by $t\cdot(z\,,w) = (t\cdot z\,, t\cdot w)$. This action commutes with the above action of $\mathbb{C}$ on $E_1^0 \times_X E_2^0$. Therefore, we get an action of $\mathbb{C}^*$ on the quotient $S_\lambda^{\xi_1,\xi_2}$. This action of $\mathbb{C}^*$ on $S_\lambda^{\xi_1,\xi_2}$ factors through the quotient group $T_\lambda = \mathbb{C}^*/\langle\exp(2\pi\sqrt{-1}\cdot\lambda)\rangle$. Using this action of T_λ, the projection φ is a holomorphic principal T_λ–bundle over $P(E_1) \times_X P(E_2)$.

Fix Hermitian structures h_1 and h_2 on the vector bundles E_1 and E_2 respectively. Let

$$S(\xi_1) := \{v \in E_1 \,|\, h_1(v) = 1\} \text{ and } S(\xi_2) := \{v \in E_2 \,|\, h_2(v) = 1\}$$

be the corresponding unit sphere bundles over X. Let

$$S(\xi_i) \longrightarrow P(E_i) = E_i^0/\mathbb{C}^*$$

be the restriction of the quotient map $E_i^0 \longrightarrow P(E_i)$. It makes $S(\xi_i)$ a principal S^1–bundle over $P(E_i)$ (in particular, $S(\xi_i)$ is a circle bundle over $P(E_i)$). The composition of maps

$$S(\xi_1) \times_X S(\xi_2) \hookrightarrow E_1^0 \times_X E_2^0 \longrightarrow S_\lambda^{\xi_1,\xi_2}$$

is a diffeomorphism of fiber bundles over X. The complex structure on $S_\lambda^{\xi_1,\xi_2}$ produces a complex structure on $S(\xi_1) \times_X S(\xi_2)$ using this diffeomorphism.

3. The Picard group

For notational conveniences, T_λ, $M_\lambda^{m,n}$ and $S_\lambda^{\xi_1,\xi_2}$ will be denoted by T, M and S respectively. The fiber product $P(E_1)\times_X P(E_2)$ will be denoted by Y.

Fix a point of S. Let $i : T \hookrightarrow S$ be the orbit of this point (recall that S is a principal T–bundle over Y).

Proposition 3.1. *Let $T \stackrel{i}{\hookrightarrow} S \stackrel{\varphi}{\longrightarrow} Y$ be the principal bundle in* (2.5)*. Then we have the following short exact sequence:*

$$0 \longrightarrow H^1(Y,\, \mathcal{O}_Y) \stackrel{\varphi^*}{\longrightarrow} H^1(S,\, \mathcal{O}_S) \stackrel{i^*}{\longrightarrow} H^1(T,\, \mathcal{O}_T) \longrightarrow 0\,,$$

where φ^* and i^* are induced homomorphisms of cohomologies.

Proof. Consider the Borel spectral sequence (see Appendix 2 of [4]) associated with the above principal bundle

$$T \stackrel{i}{\hookrightarrow} S \stackrel{\varphi}{\longrightarrow} Y$$

for the trivial holomorphic line bundle over Y. We have

$$\begin{array}{lcll} {}^{0,1}E_2^{1,0} & \xrightarrow{\quad d_2 \quad} & {}^{0,2}E_2^{3,-1} & = 0 \\ {}^{0,1}E_2^{0,1} & \xrightarrow{\quad d_2 \quad} & {}^{0,2}E_2^{2,0} & = H^{0,2}(Y,\, \mathcal{O}_Y)\,. \end{array}$$

From the Leray–Hirsch theorem for the fiber bundle in (2.6) it follows that the cohomology algebra $H^*(Y,\, \mathbb{C})$ is generated by $H^2(X,\, \mathbb{C})$ together with $c_1(\mathcal{O}_{P(E_1)})$ and $c_1(\mathcal{O}_{P(E_2)})$ (see [3, p. 432, Theorem 4D.1] for the Leray–Hirsch theorem). Therefore, $H^2(Y,\, \mathbb{C}) = H^{1,1}(Y)$. In other words, $H^{0,2}(Y, \mathcal{O}_Y) = 0$.

As no element of ${}^{0,1}E_r^{1,0}$ and ${}^{0,1}E_r^{0,1}$ is d_r-boundary for $r \geq 2$, we have ${}^{0,1}E_2^{1,0} = {}^{0,1}E_\infty^{1,0}$ and ${}^{0,1}E_2^{0,1} = {}^{0,1}E_\infty^{0,1}$. We have a filtration

$$H^1(S,\, \mathcal{O}_S) = D^1 \supset D^0 \supset 0\,,$$

where $D^0 = {}^{0,1}E_\infty^{1,0}$ and $D^1/D^0 = {}^{0,1}E_\infty^{0,1}$. The corresponding graded object is

$$\mathrm{Gr}\, H^1(S,\, \mathcal{O}_S) = {}^{0,1}E_\infty^{1,0} \oplus {}^{0,1}E_\infty^{0,1}\,.$$

Hence, the natural homomorphism

$$\varphi^* : H^1(Y,\, \mathcal{O}_Y) = {}^{0,1}E_2^{1,0} \longrightarrow {}^{0,1}E_\infty^{1,0} = D^0 \subseteq H^1(S,\, \mathcal{O}_S)$$

is injective, and the natural homomorphism

$$i^* : H^1(S,\, \mathcal{O}_S) = D^1 \longrightarrow D^1/D^0 = {}^{0,1}E_\infty^{0,1} = {}^{0,1}E_2^{0,1} = H^1(T, \mathcal{O})$$

is surjective. So we have the exact sequence

$$0 \longrightarrow H^1(Y,\, \mathcal{O}_Y) \stackrel{\varphi^*}{\longrightarrow} H^1(S,\, \mathcal{O}_S) \stackrel{i^*}{\longrightarrow} H^1(T,\, \mathcal{O}_T) \longrightarrow 0\,.$$

This completes the proof. □

Lemma 3.2. *For the projection q in* (2.6)*, the homomorphism*

$$q^* : H^1(X, \mathcal{O}_X) \longrightarrow H^1(Y, \mathcal{O}_Y)$$

is an isomorphism. In particular, $\dim H^1(Y, \mathcal{O}_Y) = g$.

Proof. Since the fibers of q are connected and simply connected, the long exact sequence of homotopy groups for q gives that the homomorphism $\pi_1(Y) \longrightarrow \pi_1(X)$ induced by q is an isomorphism. Hence $q^* : H^1(X, \mathbb{Q}) \longrightarrow H^1(Y, \mathbb{Q})$ is an isomorphism. Since both X and Y are Kähler, this implies the lemma. □

Proposition 3.1 and Lemma 3.2 together have the following corollary:

Corollary 3.3. *The dimension of $H^1(S, \mathcal{O}_S)$ is $g+1$.*

Lemma 3.4. *For the projection p in* (2.4)*, the homomorphism*

$$p_* : \pi_1(S) \longrightarrow \pi_1(X)$$

is an isomorphism. In particular, the pullback homomorphism

$$p^* : H^1(X, \mathbb{Z}) \longrightarrow H^1(S, \mathbb{Z})$$

is an isomorphism.

Proof. The fiber $M_\lambda^{m,n}$ of p is connected and simply connected (it is a product of two spheres of dimensions at least three). Hence from the homotopy exact sequence it follows that the above homomorphism p_* is an isomorphism. Therefore, the homomorphism

$$H_1(S, \mathbb{Z}) \longrightarrow H_1(X, \mathbb{Z})$$

given by p is an isomorphism. Now from the universal coefficient theorem for cohomologies it follows that the homomorphism p^* in the lemma is an isomorphism. □

Proposition 3.5. *The pullback homomorphism*

$$p^* : H^2(X, \mathbb{Z}) \longrightarrow H^2(S, \mathbb{Z})$$

is an isomorphism.

Proof. Let

$$M \overset{\iota}{\hookrightarrow} S \overset{p}{\longrightarrow} X$$

be the fiber bundle in (2.4). Consider the Serre spectral sequence associated to this fiber bundle for the constant sheaf $\mathbb{Z}$. We will show that the local system $R^i p_* \mathbb{Z}$ is constant for all i. Recall that the fibers of p are $M = S^{2m-1} \times S^{2n-1}$. For the action of $\mathrm{U}(m)$ on $S^{2m-1} = \{v \in \mathbb{C}^m \mid ||v||^2 = 1\}$, the action of $\mathrm{U}(m)$ on $H^*(S^{2m-1}, \mathbb{Z})$ is trivial. Similarly, $\mathrm{U}(n)$ acts trivially on $H^*(S^{2n-1}, \mathbb{Z})$. Therefore, the local system $R^i p_* \mathbb{Z}$ is constant for all i.

Consequently, we have

$$\begin{aligned} E_2^{0,2} &= H^0(X, \mathbb{Z}) \otimes H^2(M, \mathbb{Z}) &&= 0, \\ E_2^{1,1} &= H^1(X, \mathbb{Z}) \otimes H^1(M, \mathbb{Z}) &&= 0, \\ E_2^{2,0} &= H^2(X, \mathbb{Z}) \otimes H^0(M, \mathbb{Z}) &&= H^2(X, \mathbb{Z}). \end{aligned}$$

Further,

$$d_2 \,:\, E_2^{0,1} \,=\, 0 \longrightarrow E_2^{2.0}\,.$$

is a zero map. This implies that

$$E_\infty^{0,2} \,=\, 0\,,\;\; E_\infty^{1,1} \,=\, 0 \;\;\text{and}\;\; E_\infty^{2,0} \,=\, H^2(X,\, \mathbb{Z})\,.$$

Hence the pullback homomorphism

$$p^* \,:\, H^2(X,\, \mathbb{Z}) \,=\, E_\infty^{2,0} \longrightarrow H^2(S,\, \mathbb{Z})$$

is an isomorphism. □

Theorem 3.6. *The Picard group of S fits in a short exact sequence*

$$0 \longrightarrow \text{Pic}(X) \longrightarrow \text{Pic}(S) \longrightarrow H^1(T,\, \mathcal{O}_T) \longrightarrow 0\,.$$

The injective homomorphism $\text{Pic}(X) \longrightarrow \text{Pic}(S)$ *sends any holomorphic line bundle L to p^*L.*

Proof. Let $\mathcal{O}_S^*$ be the multiplicative sheaf on S of nowhere zero holomorphic functions. Consider the following short exact sequence of sheaves on S

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S^* \longrightarrow 0\,,$$

where the surjective homomorphism is $f \longmapsto \exp(2\pi\sqrt{-1}\cdot f)$. From the long exact sequence of cohomologies associated to it we conclude that $\text{Pic}(S)$ fits in the exact sequence

$$H^1(S,\, \mathbb{Z}) \longrightarrow H^1(S,\, \mathcal{O}_S) \longrightarrow \text{Pic}(S) \longrightarrow H^2(S,\, \mathbb{Z}) \longrightarrow H^1(S,\, \mathcal{O}_S)\,.$$

We have the exact sequence

$$H^1(X,\, \mathbb{Z}) \longrightarrow H^1(X,\, \mathcal{O}_X) \longrightarrow \text{Pic}(X) \longrightarrow H^2(X,\, \mathbb{Z}) \longrightarrow 0$$

which is constructed from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

on X.

Consider the pullback homomorphism $p^* \,:\, \text{Pic}(X) \longrightarrow \text{Pic}(S)$ defined by $L \longmapsto p^*L$. Since $H^1(X,\, \mathcal{O}_X) \,\subset\, H^1(S,\, \mathcal{O}_S)$ (see Proposition 3.1 and Lemma 3.2), $H^1(S,\, \mathbb{Z}) \,=\, H^1(X,\, \mathbb{Z})$ (see Lemma 3.4) and $H^2(S,\, \mathbb{Z}) \,=\, H^2(X,\, \mathbb{Z})$ (see Proposition 3.5) with the homomorphisms given by pullback, we conclude from the above two exact sequences that the homomorphism p^* makes $\text{Pic}(X)$ a subgroup of $\text{Pic}(S)$. Since $H^1(S,\, \mathcal{O}_S)/H^1(X,\, \mathcal{O}_X) \,=\, H^1(T,\, \mathcal{O}_T)$ by Proposition 3.1 and Lemma 3.2, we conclude that $\text{Pic}(S)/p^*(\text{Pic}(X)) \,=\, H^1(T,\, \mathcal{O}_T)$. (The argument is same as the proof of five lemma.) □

4. Infinitesimal deformations of the complex structure

In this section, we make the following assumptions:

(1) The two holomorphic vector bundles E_1 and E_2 are simple, meaning

$$H^0(X,\, End(E_1)) \,=\, \mathbb{C} \,=\, H^0(X,\, End(E_2))\,.$$

(2) $\text{genus}(X) \,=\, g \,\geq\, 2$.

We note that any stable holomorphic vector bundle is simple.

Lemma 4.1. *Let θ_Y be the holomorphic tangent bundles of $Y \,=\, P(E_1)\times_X P(E_2)$. Then $H^0(Y,\, \theta_Y) \,=\, 0$.*

Proof. For $i \,=\, 1\,, 2$, let $\text{ad}(E_i) \,\subset\, End(E_i)$ be the holomorphic subbundle of co-rank one defined by the sheaf of endomorphisms of E_i of trace zero. So, $End(E_i) \,=\, \text{ad}(E_i)\oplus \mathcal{O}_X$. We note that

$$H^0(X,\, \text{ad}(E_i)) \,=\, 0 \tag{4.1}$$

because E_i is simple.

Consider the projection q in (2.6). Let $\theta_{Y/X} \,\subset\, \theta_Y$ be the relative holomorphic tangent bundle for q. We note that

$$q_*\theta_{Y/X} \,=\, \text{ad}(E_1)\oplus \text{ad}(E_2)\,. \tag{4.2}$$

The short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \theta_{Y/X} \longrightarrow \theta_Y \longrightarrow q^*\theta_X \longrightarrow 0\,, \tag{4.3}$$

where θ_X is the holomorphic tangent bundle of X, produces a short exact sequence

$$0 \longrightarrow q_*\theta_{Y/X} \longrightarrow q_*\theta_Y \longrightarrow \theta_X \longrightarrow 0 \tag{4.4}$$

on X because $R^1q_*\theta_{Y/X} \,=\, 0$.

From (4.1) and (4.2) it follows that $H^0(X,\, q_*\theta_{Y/X}) \,=\, 0$. We also have $H^0(X,\, \theta_X) \,=\, 0$ because $g \,\geq\, 2$. Therefore, from the long exact sequence of cohomologies associated to (4.4) it follows that $H^0(X,\, q_*\theta_Y) \,=\, 0$. This implies that $H^0(Y,\, \theta_Y) \,=\, 0$. ☐

Lemma 4.2. *The cohomology $H^1(Y,\, \theta_Y)$ fits in a natural short exact sequence*

$$0 \longrightarrow H^1(X,\, \text{ad}(E_1))\oplus H^1(X,\, \text{ad}(E_2)) \longrightarrow H^1(Y,\, \theta_Y) \longrightarrow H^1(X,\, \theta_X) \longrightarrow 0.$$

Proof. Consider the short exact sequence in (4.3). We note that $R^iq_*\theta_{Y/X} \,=\, 0$ for all $i \,\geq\, 1$. From the projection formula, and the fact that $H^i(\mathbb{CP}^N,\, \mathcal{O}_{\mathbb{CP}^N}) \,=\, 0$ for all $i \,\geq\, 1$ and all N, we conclude that

$$R^iq_*q^*\theta_X \,=\, \theta_X\otimes R^iq_*\mathcal{O}_Y \,=\, 0$$

for all $i \,\geq\, 1$. Therefore,

$$H^j(Y,\, \theta_{Y/X}) \,=\, H^j(X,\, q_*\theta_{Y/X}) \,=\, H^j(X,\, \text{ad}(E_1)\oplus \text{ad}(E_2))$$

(see (4.2) for the second equality) and

$$H^j(Y,\, q^*\theta_X) \,=\, H^j(X,\, q_*q^*\theta_X) \,=\, H^j(X,\, \theta_X)$$

for all $j \,\geq\, 0$. In particular, $H^0(Y,\, q^*\theta_X) \,=\, H^0(X,\, \theta_X) \,=\, 0$ (because $g \,\geq\, 2$), and $H^2(Y,\, \theta_{Y/X}) \,=\, H^2(X,\, q_*\theta_{Y/X}) \,=\, 0$. Therefore, the long exact sequence of cohomologies for (4.3) gives the short exact sequence

$$0 \longrightarrow H^1(Y,\, \theta_{Y/X}) \,=\, H^1(X,\, \mathrm{ad}(E_1) \oplus \mathrm{ad}(E_2)) \longrightarrow H^1(Y,\, \theta_Y)$$
$$\longrightarrow H^1(Y,\, q^*\theta_X) \,=\, H^1(X,\, \theta_X) \longrightarrow 0\,.$$

From this the lemma follows because $H^1(X,\, \mathrm{ad}(E_1) \oplus \mathrm{ad}(E_2)) \,=\, H^1(X,\, \mathrm{ad}(E_1)) \oplus H^1(X,\, \mathrm{ad}(E_2))$. □

Proposition 4.3. *Let θ_S be the holomorphic tangent bundles of S. Then $H^0(S,\, \theta_S) \,=\, \mathbb{C}$.*

The cohomology $H^1(S,\, \theta_S)$ fits in a natural short exact sequence

$$0 \longrightarrow H^1(S,\, \mathcal{O}_S) \longrightarrow H^1(S,\, \theta_S) \longrightarrow H^1(Y,\, \theta_Y) \longrightarrow 0\,.$$

Proof. Consider the Borel spectral sequence associated to φ for the tangent bundle TY. We have

$${}^{0,0}E_\infty^{0,0} \,=\, {}^{0,0}E_2^{0,0} \,=\, H^0(Y,\, \theta_Y)\,.$$

Now, Lemma 4.1 says that $H^0(Y, \theta_Y) \,=\, 0$. Hence

$$H^0(S,\, \varphi^*\theta_Y) \,=\, {}^{0,0}E_\infty^{0,0} \,=\, 0\,. \tag{4.5}$$

Let $\theta_{S/Y} \,\subset\, \theta_S$ be the relative tangent bundle for the projection φ. We note that $\theta_{S/Y} \,=\, \mathcal{O}_S$ using the action of T on S. Consider the long exact sequence of cohomologies associated to the short exact sequence of vector bundles

$$0 \longrightarrow \theta_{S/Y} \,=\, \mathcal{O}_S \longrightarrow \theta_S \longrightarrow \varphi^*\theta_Y \longrightarrow 0\,. \tag{4.6}$$

Since $H^0(S,\, \varphi^*\theta_Y) \,=\, 0$, we conclude that the homomorphism

$$H^0(S,\, \theta_{S/Y}) \,=\, H^0(S,\, \mathcal{O}_S) \longrightarrow H^0(S,\, \theta_S)$$

in the long exact sequence is an isomorphism. Therefore, the first statement of the proposition is proved.

To prove the second part of the proposition, first note that

$${}^{0,1}E_2^{0,1} \,=\, H^{0,0}(Y,\, \theta_Y) \otimes H^{0,1}(T,\, \mathcal{O}_T) \,=\, 0$$

because $H^0(Y,\, \theta_Y) \,=\, 0$. Hence ${}^{0,1}E_\infty^{0,1} \,=\, 0$. Further, since

$${}^{0,1}E_2^{1,0} \,=\, H^{0,1}(Y,\, \theta_Y) \xrightarrow{\ d_2\ } {}^{0,2}E_2^{3,-1} = 0\,,$$

we conclude that ${}^{0,1}E_\infty^{1,0} \,=\, H^{0,1}(Y, \theta_Y)$.

Now, let

$$H^1(S, \varphi^*\theta_Y) \,=\, D^1 \,\supset\, D^0 \,\supset\, 0$$

be the natural filtration for which the corresponding graded object is

$$\mathrm{Gr}\, H^1(S, \varphi^*\theta_X) \,=\, {}^{0,1}E_\infty^{1,0} \oplus {}^{0,1}E_\infty^{0,1}\,,$$

more precisely, $D^0 = {}^{0,1}E_\infty^{1,0}$ and $D^1/D^0 = {}^{0,1}E_\infty^{0,1}$. Since ${}^{0,1}E_\infty^{0,1} = 0$, we have $D^1 = D^0$. This implies that the natural homomorphism

$$\varphi^* : H^1(Y,\, \theta_Y) = {}^{0,1}E_2^{1,0} \longrightarrow {}^{0,1}E_\infty^{1,0} = D^0 = D^1 = H^1(S,\, \varphi^*\theta_Y) \tag{4.7}$$

is an isomorphism.

Consider the long exact sequence of cohomologies

$$H^0(S,\, \varphi^*\theta_Y) \longrightarrow H^1(S,\, \mathcal{O}_S) \longrightarrow H^1(S,\, \theta_S) \xrightarrow{\phi} H^1(S,\, \varphi^*\theta_Y) \tag{4.8}$$

associated to the short exact sequence in (4.6). Since $H^0(S,\, \varphi^*\theta_Y) = 0$ (see (4.5)) and $H^1(S,\, \varphi^*\theta_Y) = H^1(Y,\, \theta_Y)$ (see (4.7)), to prove the second part of the proposition it suffices to show that the homomorphism ϕ in (4.8) is surjective.

From Lemma 4.2 we know that all the infinitesimal deformations of Y are given by the infinitesimal deformations of the two vector bundles E_1 and E_2 and the infinitesimal deformations of the Riemann surface X. The subspaces

$$H^1(X,\, \mathrm{ad}(E_1)) \subset H^1(Y,\, \theta_Y) \quad \text{and} \quad H^1(X,\, \mathrm{ad}(E_2)) \subset H^1(Y,\, \theta_Y)$$

in Lemma 4.2 correspond to the infinitesimal deformations of the projective bundle $P(E_1)$ and $P(E_2)$ respectively (keeping the Riemann surface X fixed). The infinitesimal deformations of E_1 (respectively, E_2) is given by $H^1(X,\, End(E_1))$ (respectively, $H^1(X,\, End(E_2))$). The natural map from the infinitesimal deformations of E_i to the infinitesimal deformations of $P(E_i)$ corresponds to the projection $H^1(X,\, End(E_i)) \longrightarrow H^1(X,\, \mathrm{ad}(E_i))$ given by the decomposition $End(E_i) = \mathrm{ad}(E_i) \oplus \mathcal{O}_X$. The projection $H^1(Y,\, \theta_Y) \longrightarrow H^1(X,\, \theta_X)$ corresponds to the infinitesimal deformations of X. All these infinitesimal deformations give rise to infinitesimal deformations of S. Hence the homomorphism ϕ in (4.8) is surjective. □

Corollary 4.4. *The dimension of* $H^1(S,\, \theta_S)$ *is* $(m^2 + n^2 + 2)(g-1) + 2$.

Proof. Since $H^0(X,\, \mathrm{ad}(E_1)) = 0 = H^0(X,\, \mathrm{ad}(E_2))$ (recall that E_1 and E_2 are both simple), from the Riemann–Roch theorem we have

$$\dim H^1(X,\, \mathrm{ad}(E_1)) = (m^2-1)(g-1) \quad \text{and} \quad \dim H^1(X,\, \mathrm{ad}(E_2)) = (n^2-1)(g-1)\,.$$

Therefore, Proposition 4.3 and Lemma 4.2,

$$\dim H^1(S,\, \theta_S) = (m^2+n^2+1)(g-1) + \dim H^1(S,\, \mathcal{O}_S)\,.$$

Now the corollary follows from Corollary 3.3. □

5. Computation of the Brauer group

Let M be a compact connected complex manifold. Let $\mathcal{O}_M^*$ be the multiplicative sheaf on M of nowhere zero holomorphic functions. The *cohomological Brauer group* $\mathrm{Br}'(M)$ is the group of torsion elements in $H^2(M,\, \mathcal{O}_M^*)$.

To define the Brauer group of M, consider all holomorphic principal $\mathrm{PGL}(r, \mathbb{C})$–bundles on M for all $r \geq 1$. Let

$$\mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(r', \mathbb{C}) \longrightarrow \mathrm{GL}(rr', \mathbb{C})$$

be the homomorphism given by the natural action of any $A \times B \in \mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(r', \mathbb{C})$ on $\mathbb{C}^r \otimes \mathbb{C}^{r'}$. This homomorphism descends to a homomorphism

$$\gamma : \mathrm{PGL}(r, \mathbb{C}) \times \mathrm{PGL}(r', \mathbb{C}) \longrightarrow \mathrm{PGL}(rr', \mathbb{C}) .$$

Given a holomorphic principal $\mathrm{PGL}(r, \mathbb{C})$–bundle $\mathcal{A}$ on M and a holomorphic principal $\mathrm{PGL}(r', \mathbb{C})$–bundle $\mathcal{B}$ on M, the homomorphism γ produces a holomorphic principal $\mathrm{PGL}(rr', \mathbb{C})$–bundle on M by extension of structure group. This holomorphic principal $\mathrm{PGL}(rr', \mathbb{C})$–bundle will be denoted by $\mathcal{A} \otimes \mathcal{B}$. The two principal bundles $\mathcal{A}$ and $\mathcal{B}$ will be called *equivalent* if there are holomorphic vector bundles V and W on M such that $\mathcal{A} \otimes P(V)$ is holomorphically isomorphic to $\mathcal{B} \otimes P(W)$.

The equivalence classes of projective bundles form a group. The addition operation is given by the tensor product, and the inverse is given by the automorphism $A \longmapsto (A^t)^{-1}$ of $\mathrm{PGL}(r, \mathbb{C})$ (it corresponds to taking the dual projective bundles). (See [6, Section 1] for the details.) This group is called the *Brauer group* of M, and it is denoted by $\mathrm{Br}(M)$.

The Brauer group $\mathrm{Br}(M)$ is a subgroup of the cohomological Brauer group $\mathrm{Br}'(M)$ [6, p. 878].

Let T denote the torsion part of $H^3(M, \mathbb{Z})$. Let

$$\gamma : H^1(M, \mathcal{O}^*_M) \longrightarrow H^2(M, \mathbb{Z})$$

be the homomorphism that sends any holomorphic line bundle on M to its first Chern class. Let

$$A := H^2(M, \mathbb{Z})/\gamma(H^1(M, \mathcal{O}^*_M))$$

be the quotient. The cohomological Brauer group $\mathrm{Br}'(M)$ fits in a short exact sequence

$$0 \longrightarrow A \otimes (\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Br}'(M) \longrightarrow T \longrightarrow 0 \tag{5.1}$$

(see [6, p. 878, Proposition 1.1]).

Proposition 5.1. *Let $M \stackrel{\iota}{\hookrightarrow} S \stackrel{p}{\longrightarrow} X$ be the holomorphic fiber bundle in* (2.4). *Then the cohomology group $H^3(S, \mathbb{Z})$ is torsionfree.*

Proof. The proof is similar to the proof of Proposition 3.5. Consider the Serre spectral sequence associated to the fiber bundle

$$M \stackrel{\iota}{\hookrightarrow} S \stackrel{p}{\longrightarrow} X$$

for the constant sheaf $\mathbb{Z}$. We have seen in the proof of Proposition 3.5 that the local system $R^i p_* \mathbb{Z}$ is constant for all i.

We have

$$\begin{array}{lll} E_2^{0,3} & = H^0(X, \mathbb{Z}) \otimes H^3(M, \mathbb{Z}) & = H^3(M, \mathbb{Z}), \\ E_2^{2,1} & = H^2(X, \mathbb{Z}) \otimes H^1(M, \mathbb{Z}) & = 0, \\ E_2^{1,2} & = H^1(X, \mathbb{Z}) \otimes H^2(M, \mathbb{Z}) & = 0, \\ E_2^{3,0} & = H^3(X, \mathbb{Z}) \otimes H^0(M, \mathbb{Z}) & = 0. \end{array}$$

With a similar argument as above, we can conclude that

$$H^3(S, \mathbb{Z}) = E_\infty^{0,3} = E_2^{0,3} = H^3(M, \mathbb{Z}) .$$

Since $M = S^{2m-1} \times S^{2n-1}$ with $m, n \geq 2$, it thus follows that $H^3(S, \mathbb{Z})$ is torsionfree. □

Corollary 5.2. *The cohomological Brauer group* $\mathrm{Br}'(S)$ *vanishes. The Brauer group* $\mathrm{Br}(S)$ *vanishes.*

Proof. Every element of $H^2(X, \mathbb{Z})$ is the first Chern class of a holomorphic line bundle on X. Therefore, from Proposition 3.5 it follows that each element of $H^2(S, \mathbb{Z})$ is the first Chern class of a holomorphic line bundle on S. Now the first statement follows from (5.1) and Proposition 5.1. The second statement follows from the first statement because $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$. □

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